MATH304

Homework 6

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1. Let $f(z)$ be analytic in a domain containing the whole real axis, and write $z_n = \left(n + \frac{1}{2}\right)\pi$, n is any integer. Show that
$$\text{Res}\left(\frac{f(z)}{\cos^2 z}, z_n\right) = f'(z_n), \ \ n \text{ is any integer.}$$

2. Compute
   (a) $\text{Res}\left(\tan z, \frac{\pi}{2} + k\pi\right)$, k is any integer;
   (b) $\text{Res}\left(\frac{z^{2n}}{(z-1)^n}, 1\right)$.

3. What is the Laurent expansion around $z = 0$ for the function
$$f(z) = \begin{cases} \sin z & \text{if } z \neq 0 \\ 5 & \text{if } z = 0 \end{cases} \ ?$$
Compute $\text{Res}(f, 0)$.

4. What is the order of the pole of
$$f(z) = \frac{1}{(2\cos z - 2 + z^2)^2}$$
at zero? Compute $\text{Res}(f, 0)$.
Hint: $\text{Res}(f, \alpha) = -\text{Res}(f, -\alpha)$ if $f(z)$ is an even function.

5. Use the Residue Theorem to evaluate the following integrals:
   (a) $\oint_{|z|=2} \frac{z^4 + z}{(z-1)^2} \, dz$;
   (b) $\oint_{|z|=2} \frac{z^3 + 3z + 1}{z^4 - 5z^2} \, dz$;
   (c) $\oint_{|z|=2} \frac{\sinh^2 z}{z^4} \, dz$;
   (d) $\oint_{|z-i|=2} \frac{e^z + z}{(z-1)^4} \, dz$.

6. Find the residue of

$$f(z) = \frac{e^z - 1}{\sin^3 z}$$

at $z = 0$.

7. Evaluate the following definite integrals:

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$, a is a complex number and $a \neq \pm 1$,

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1}\, dx$;

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2 + b^2}\, dx$, a and b are positive constants.

8. Evaluate

$$PV \int_{-\infty}^{\infty} \frac{\sin x}{x + i}\, dx.$$

9. Find

$$\int_0^{\infty} \frac{\ln x}{x^2 + 4}\, dx.$$

Note that $\ln x$ has a singularity at $x = 0$. The improper integral is thus defined as

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon}^{R} \frac{\ln x}{x^2 + 4}\, dx.$$

10. Evaluate

$$\int_{-\infty}^{\infty} \frac{xe^x}{e^{4x} + 1}\, dx.$$

Hint: Choose the closed rectangular contour whose 4 sides are

$$\begin{aligned} \ell_1 &: \{(x, y) : -R \leq x \leq R, y = 0\} \\ \ell_2 &: \left\{(x, y) : x = R, 0 \leq y \leq \frac{\pi}{2}\right\} \\ \ell_3 &: \left\{(x, y) : -R \leq x \leq R, y = \frac{\pi}{2}\right\} \\ \ell_4 &: \left\{(x, y) : x = -R, 0 \leq y \leq \frac{\pi}{2}\right\}. \end{aligned}$$