MATH304, Spring 2007

Test One

Time allowed: 75 minutes **Course instructor:** *Prof. Y.K. Kwok*

[points]

1. When $|z| \leq 1$, find the maximum value of $|z^n + \alpha|$, where n is a positive real integer and α is complex. **[2]**

2. An isometry is a complex function $f : \mathbb{C} \to \mathbb{C}$ such that
$$|f(z_1) - f(z_2)| = |z_1 - z_2|$$
for all z_1 and z_2 in $\mathbb{C}$. Define
$$g(z) = \frac{f(z) - f(0)}{f(1) - f(0)}.$$
   (a) Given that f is an isometry, show that $g(z)$ is also an isometry. **[1]**

   (b) By observing the properties: $g(1) = 1$ and $g(0) = 0$, show that

      (i) the real parts of $g(z)$ and z are equal for all z in $\mathbb{C}$, **[2]**

      (ii) $g(i) = i$ or $-i$. **[2]**

3. A set is said to be closed if its contains all its boundary points. A set is said to be open if it consists of interior points only. Show that a closed set contains all its limit points. **[3]**

4. Let $f(z) = (x - y)^2 + 2i(x + y)$.

   (a) Show that the Cauchy-Riemann equations are satisfied only along the curve $x - y = 1$. **[1]**

   (b) Deduce that $f(z)$ has a derivative along that curve and find the derivative value. Then explain why $f(z)$ is nowhere analytic. **[2]**

5. Given the complex function: $f(z) = 1/z$, and write
$$w = f(z) = u(x, y) + iv(x, y), \quad z = x + iy.$$
Find the preimage of $u(x, y) = 1$ in the x-y plane under the mapping $w = f(z)$. **[2]**

6. Determine the set on which

$$f(z) = \begin{cases} z & \text{if } |z| \leq 1 \\ z^2 & \text{if } |z| > 1 \end{cases},$$

is analytic and compute its derivative. Justify your answer. **[4]**

7. If v is a harmonic conjugate of u in a domain $\mathcal{D}$, is uv harmonic in $\mathcal{D}$? Give an explanation to your answer. **[3]**

8. Suppose the isothermal lines of a steady state temperature field are the family of the straight lines through the origin

$$y/x = \alpha, \quad \alpha \text{ is any real number.}$$

(a) Find the general solution of the temperature function. **[4]**

(b) Find the equation of the family of flux lines. **[4]**

— *End* —