MATH304, Spring 2007

Test Two

Time allowed: 75 minutes **Course instructor:** *Prof. Y.K. Kwok*

[points]

1. Consider the mapping associated with the complex function

$$w = \cos z, \quad z = x + iy,$$

find the image curve of $x = \alpha, \alpha$ is a constant, under the above mapping in the w-plane. In particular, examine the special cases where $\cos\alpha = 0$ and $\sin\alpha = 0$. **[4]**

Hint: $\cos z = \cosh y \cos x - i \sinh y \sin x$.

2. Show that, if a is a positive real constant, then

$$\coth^{-1}\frac{z}{a} = \frac{1}{2}\log\frac{z+a}{z-a} = \frac{1}{2}\left[\ln\left|\frac{z+a}{z-a}\right| + i\arg\left(\frac{z+a}{z-a}\right)\right].$$

[2]

Hint: $\sinh z = \dfrac{e^z - e^{-z}}{2}, \quad \cosh z = \dfrac{e^z + e^{-z}}{2}, \quad \coth z = \dfrac{\cosh z}{\sinh z}$.

3. Show that all the values of

$$\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^{\sqrt{2}i}$$

lie on a straight line in the complex plane. Find the equation of this line. **[3]**

4. Consider the multi-valued function: $f(z) = (z-1)^{1/3}$.

   (a) Describe the Riemann surface of the function. [Specify the branch cut, branch points and the number of sheets.] **[1]**

   (b) Suppose we choose the branch such that $f(1+i) = e^{\frac{5\pi}{6}i}$, compute $f(-1)$. **[2]**

5. (a) Evaluate

$$\int_{C_1} \cosh z \, dz$$

where C_1 is the line segment joining $\mathrm{Log}\, 2$ and $i\pi/2$ in the complex plane. **[2]**

   (b) Estimate an upper bound on

$$\left|\int_{C_2} \frac{1}{\sinh z}\, dz\right|$$

where C_2 is the line segment joining $i\dfrac{\pi}{4}$ and $i\dfrac{\pi}{2}$ in the complex plane.

Hint: $\sinh iy = i \sin y$. **[3]**

6. (a) Evaluate

$$\oint_{x^2+y^2=2x} \frac{\sin \frac{\pi z}{4}}{z^2-1}\, dz$$

using Cauchy's integral formula. **[2]**

(b) Find the maximum value of $\left|\dfrac{1}{z+1}\right|$ on and inside the circle: $x^2 + y^2 = 2x$. **[3]**

Hint: Use the Maximum Modulus Theorem or other judicious method.

7. Let f be entire and suppose $\text{Re}f(z) \leq M$ for all z, where M is a fixed real constant. Prove that f must be a constant function.

Hint: Apply Liouville's Theorem to the function e^f. It is necessary to show that e^f is also entire. **[3]**

8. Let f be an entire function such that

$$|f(z)| \leq A|z| \quad \text{for all } z,$$

where A is a fixed positive number.

(a) Let $f^{(n)}(z)$ denote the n^{th} order derivative of $f(z)$. Recall Cauchy's inequality:

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}, \quad n = 1, 2, \cdots,$$

where M_R denotes an upper bound of $|f(z)|$ on $C_R : |z - z_0| = R$. Use it to show that

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! A(R + |z_0|)}{R^n}, \quad R > 0.$$

[2]

(b) Hence, show that **[3]**

$$f(z) = a_1 z, \quad \text{where } a_1 \text{ is a complex constant such that } |a_1| \leq A\,.$$

Hint: Show that $f^{(n)}$ is zero everywhere in the plane, for $n \geq 2$, and $f(0) = 0$.

— *End* —