MATH304, Spring 2007

Solution to Test Two

1. (a) $w = \cos z = \cosh y \cos x - i \sinh y \sin x = u + iv$

$$\Rightarrow \begin{cases} u = \cosh y \cos x \\ v = -\sinh y \sin x \end{cases}$$

Let $x = \alpha$, $\begin{cases} u = \cosh y \cos \alpha \\ v = -\sinh y \sin \alpha \end{cases}$. Eliminating y, we get

$$\left(\frac{u}{\cos \alpha}\right)^2 - \left(\frac{v}{\sin \alpha}\right)^2 = 1.$$

If $\cos \alpha = 0$, $\begin{cases} v = \pm \sinh y \\ u = 0 \end{cases} \Rightarrow \begin{cases} u = 0 \\ v \in \mathbb{R} \end{cases}$.

If $\sin \alpha = 0$, $\begin{cases} u = \pm \cosh y \\ v = 0 \end{cases} \Rightarrow \begin{cases} u \in (-\infty, 1) \cup (1, \infty) \\ v = 0 \end{cases}$.

2. Let $w = \coth^{-1} \dfrac{z}{a}$ so that $\dfrac{z}{a} = \coth w = \dfrac{e^{2w} + 1}{e^{2w} - 1}$. Solving for e^{2w}, we obtain $e^{2w} = \dfrac{z+a}{z-a}$. Taking the logarithmic on both sides, we have

$$w = \coth^{-1} \frac{z}{z} = \frac{1}{2} \log \frac{z+a}{z-a} = \frac{1}{2}\left[\ln \left|\frac{z+a}{z-a}\right| + i\arg\left(\frac{z+a}{z-a}\right)\right].$$

3. $\left(\dfrac{1-i}{\sqrt{2}}\right)^{\sqrt{2}i} = \left(e^{-i\frac{\pi}{4} + 2k\pi i}\right)^{\sqrt{2}i} = e^{\sqrt{2}\frac{\pi}{4} - 2\sqrt{2}k\pi}$, k is any integer. The imaginary part is always zero so that all values lie on the real axis. The equation of the line that contains all these point is $\text{Im}\, z = 0$.

4. (a) The Riemann surface of $= (z-1)^{1/3}$ consists of 3 *sheets* superimposed over each other. They are joined together along the *branch cut* taken to be along the negative real axis starting from $z = 1$. The *branch points* are $z = 1$ and $z = \infty$.

(b) $i = e^{i\pi/2} = e^{i(\pi/2 + 2\pi)} = e^{i(\pi/2 + 4\pi)}$. Hence, $i^{1/3} = \begin{cases} e^{i\pi/6} \\ e^{i5\pi/6} \\ e^{i3\pi/2} = -i \end{cases}$.

Note that $-2 = 2e^{i(\pi + 2\pi)}$ if the branch $f(1+i) = e^{(5\pi/6)i}$ is taken. We then have $f(-1) = 2^{1/3} e^{i3\pi/3} = -2^{1/3}$.

5. (a) Since $\cosh z$ is entire, $\displaystyle\int_{C_1} \cosh z \, dz$ is path independent, we have

$$\int_{\text{Log}2}^{i\pi/2} \cosh z \, dz = \sinh z \Big]_{\text{Log}\, 2}^{i\pi/2} = \sin \frac{\pi}{2} - \frac{e^{\text{Log}\, 2} - e^{-\text{Log}\, 2}}{2} = i - \left(\frac{2 - \frac{1}{2}}{2}\right) = -\frac{3}{4} + i.$$

(b) Since $\sinh ix = i\sin x$, the maximum value of $\left|\frac{1}{\sinh z}\right|$ along the line segment joining $i\dfrac{\pi}{4}$ and $i\dfrac{\pi}{2}$ is

$$\max_{x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]}\left|\frac{1}{\sin x}\right| = \sqrt{2}.$$

Length of the line segment $= \dfrac{\pi}{4}$. Hence

$$\left|\int_{C_2}\frac{1}{\sinh z}\,dz\right| \le \frac{\pi}{2\sqrt{2}}.$$

6. (a) Let $C: x^2+y^2 = 2x$ or $|z-1| = 1$; here C is a circle with centre at $(1,0)$ and radius equals 1.

$$\begin{aligned}\oint_C \frac{\sin\frac{\pi z}{4}}{z^2-1}\,dz &= \oint_C \frac{\sin\frac{\pi z}{4}}{(z-1)(z+1)}\,dx\\ &= \frac{1}{2}\left[\oint_C \frac{\sin\frac{\pi z}{4}}{z-1}\,dz - \oint_C \frac{\sin\frac{\pi z}{4}}{z+1}\,dz\right].\end{aligned}$$

Since $\dfrac{\sin\frac{\pi z}{4}}{z+1}$ is analytic on and inside C, so $\displaystyle\oint_C \frac{\sin\frac{\pi z}{4}}{z+1}\,dz = 0$.

$$\frac{1}{2\pi i}\oint_C \frac{\sin\frac{\pi z}{4}}{z-1}\,dz = \sin\frac{\pi}{4} \quad \text{by Cauchy's integral formula}$$

so that

$$\oint_C \frac{\sin\frac{\pi z}{4}}{z-1}\,dz = \frac{2\pi i\sqrt{2}}{2} = \sqrt{2}\pi i.$$

Finally,

$$\oint_C \frac{\sin\frac{\pi z}{4}}{z^2-1}\,dz = \frac{\sqrt{2}\pi i}{2}.$$

(b) Note that $\dfrac{1}{z+1}$ is analytic on and inside the circle. By the Maximum Modulus Theorem, the maximum value of $\left|\dfrac{1}{z+1}\right|$ occurs on the circumference of the disc. The parametric form of the circle is $z = 1+e^{i\theta}, 0\le\theta\le 2\pi$ so that

$$\left|\frac{1}{z+1}\right|^2 = \frac{1}{|2+e^{i\theta}|^2} = \frac{1}{5+4\cos\theta}$$

and its maximum value is attained at $\cos\theta = -1$, that is, $z = 0$. This gives the maximum value of $\left|\dfrac{1}{z+1}\right|$ on and inside the disc to be 1.

7. If f is entire, so is e^f, because $(e^f)' = f'e^f$ exists. Now if $\mathrm{Re}f(z) \le M$ for all z, then $e^f = e^{\mathrm{Re}f}e^{i\mathrm{Im}f}$. So $|e^f| = e^{\mathrm{Re}f} \le e^M$ for all z. By Liouville's Theorem, $e^f = K$ which is a constant function. Therefore, $f = \ln\, K =$, a constant function.

8. Let f be entire and $|f(z)| \leq A|z|$ for all z. By Cauchy's inequality, $|f^{(n)}(z_0)| \leq \dfrac{n!M_R}{R^n}$, where $M_R = \max\limits_{|z-z_0|=R} A|z| \leq A|z-z_0| + A|z_0| = A(R+|z_0|)$.

   Hence, $|f^{(n)}(z_0)| \leq \dfrac{n!A(R+|z_0|)}{R^n}, \forall R > 0$. For $n \geq 2$, and take R to be sufficiently large, the inequality is valid for any sufficiently large R only if $f^{(n)}(z_0) = 0$. This result holds for all $z_0 \in \mathbb{C}$, so $f(z) = a_1 z + a_0$. But $|f(0)| \leq A|0| = 0 \Rightarrow f(0) = 0$ giving $a_0 = 0$, so $f(z) = a_1 z$. Obviously, $|a_1| \leq A$ as $|f(z)| = |a_1| \cdot |z| \leq A|z|$.