2. Mean-variance portfolio theory

(2.1) Markowitz's mean-variance formulation

(2.2) Two-fund theorem

(2.3) Inclusion of the riskfree asset

2.1 Markowitz mean-variance formulation

Suppose there are N risky assets, whose rates of returns are given by the random variables $R_1, \cdots, R_N$, where

$$R_n = \frac{S_n(1) - S_n(0)}{S_n(0)}, n = 1, 2, \cdots, N.$$

Let $\boldsymbol{w} = (w_1 \cdots w_N)^T, w_n$ denotes the proportion of wealth invested in asset n, with $\sum_{n=1}^{N} w_n = 1$. The rate of return of the portfolio is

$$R_P = \sum_{n=1}^{N} w_n R_n.$$

Assumptions

1. There does not exist any asset that is a combination of other assets in the portfolio, that is, non-existence of redundant security.
2. $\boldsymbol{\mu} = (\overline{R}_1 \quad \overline{R}_2 \cdots \overline{R}_N)$ and $\mathbf{1} = (1 \quad 1 \cdots 1)$ are linearly independent, otherwise R_P is a constant irrespective of any choice of portfolio weights.

The first two moments of R_P are

$$\mu_P = E[R_P] = \sum_{n=1}^{N} E[w_n R_n] = \sum_{n=1}^{N} w_n \mu_n, \text{ where } \mu_n = \overline{R}_n,$$

and

$$\sigma_P^2 = \text{var}(R_P) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \text{cov}(R_i, R_j) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i \sigma_{ij} w_j.$$

Let Ω denote the covariance matrix so that

$$\sigma_P^2 = \boldsymbol{w}^T \Omega \boldsymbol{w}.$$

For example when $n = 2$, we have

$$(w_1 \quad w_2)\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_1^2\sigma_1^2 + w_1 w_2(\sigma_{12} + \sigma_{21}) + w_2^2\sigma_2^2.$$

Remark

1. The portfolio risk of return is quantified by σ_P^2. In mean-variance analysis, only the first two moments are considered in the portfolio model. Investment theory prior to Markowitz considered the maximization of μ_P but without σ_P.

2. The measure of risk by variance would place equal weight on the upside deviations and downside deviations.

3. In the mean-variance model, it is assumed that μ_i, σ_i and σ_{ij} are all known.

Two-asset portfolio

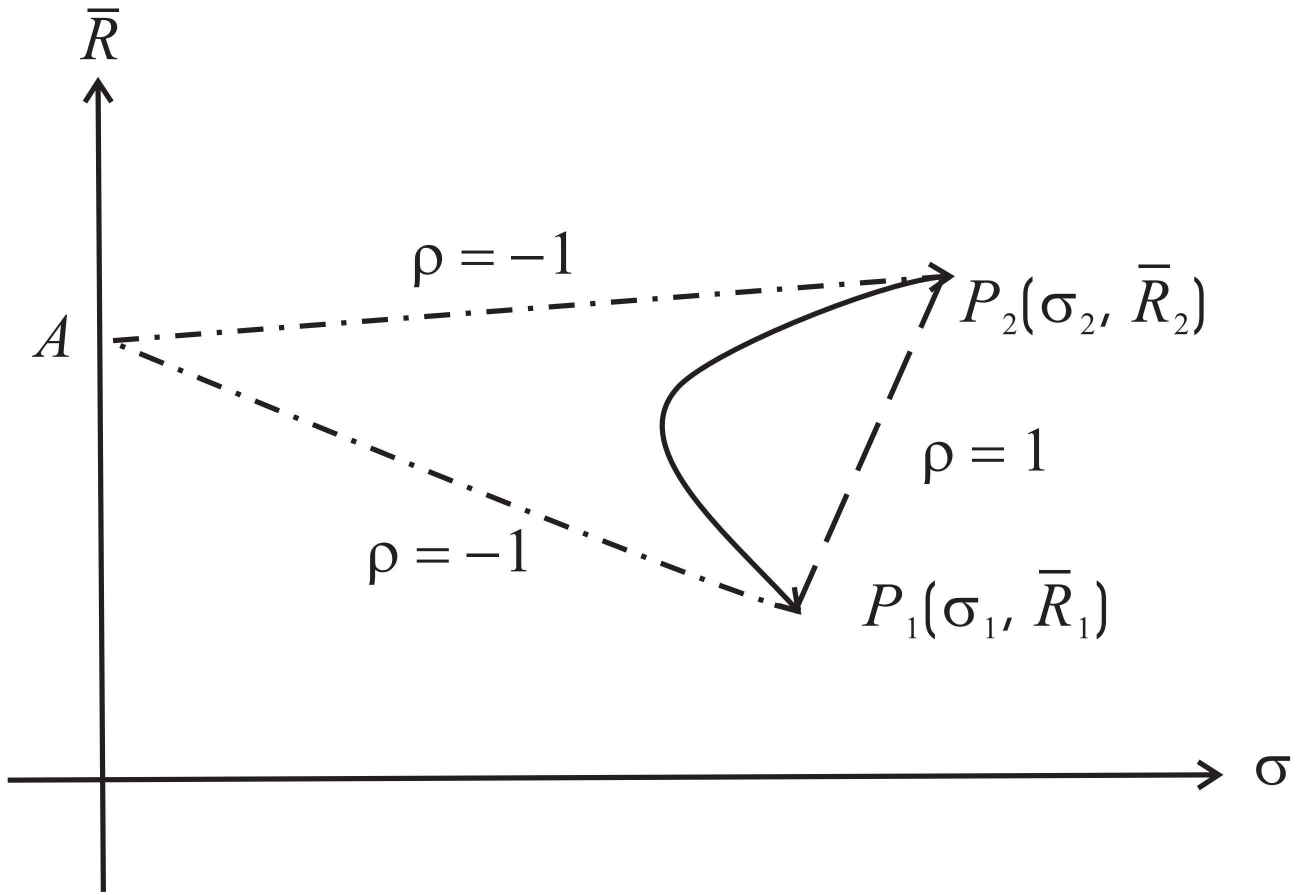
Consider two risky assets with known means $\overline{R}_1$ and $\overline{R}_2$, variances σ_1^2 and σ_2^2, of the expected rates of returns R_1 and R_2, together with the correlation coefficient ρ.

Let $1-\alpha$ and α be the weights of assets 1 and 2 in this two-asset portfolio.

Portfolio mean: $\overline{R}_P = (1-\alpha)\overline{R}_1 + \alpha\overline{R}_2, 0 \leq \alpha \leq 1$

Portfolio variance: $\sigma_P^2 = (1-\alpha)^2\sigma_1^2 + 2\rho\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2$.

We represent the two assets in a mean-standard deviation diagram (recall: standard deviation $= \sqrt{\text{variance}}$)



As α varies, $(\sigma_P, \overline{R}_P)$ traces out a conic curve in the $\sigma - \overline{R}$ plane. With $\rho = -1$, it is possible to have $\sigma = 0$ for some suitable choice of weight. In general, putting two assets whose returns are negatively correlated has the desirable effect of lowering the portfolio risk.

In particular, when $\rho = 1$,

$$\begin{aligned}\sigma_P(\alpha; \rho = 1) &= \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2} \\ &= (1-\alpha)\sigma_1 + \alpha\sigma_2.\end{aligned}$$

This is the straight line joining $P_1(\sigma_1, \overline{R}_1)$ and $P_2(\sigma_2, \overline{R}_2)$.

When $\rho = -1$, we have

$$\sigma_P(\alpha; \rho = -1) = \sqrt{[(1-\alpha)\sigma_1 - \alpha\sigma_2]^2} = |(1-\alpha)\sigma_1 - \alpha\sigma_2|.$$

When α is small (close to zero), the corresponding point is close to $P_1(\sigma_1, \overline{R}_1)$. The line AP_1 corresponds to

$$\sigma_P(\alpha; \rho = -1) = (1-\alpha)\sigma_1 - \alpha\sigma_2.$$

The point A (with zero σ) corresponds to $\alpha = \dfrac{\sigma_1}{\sigma_1 + \sigma_2}$.

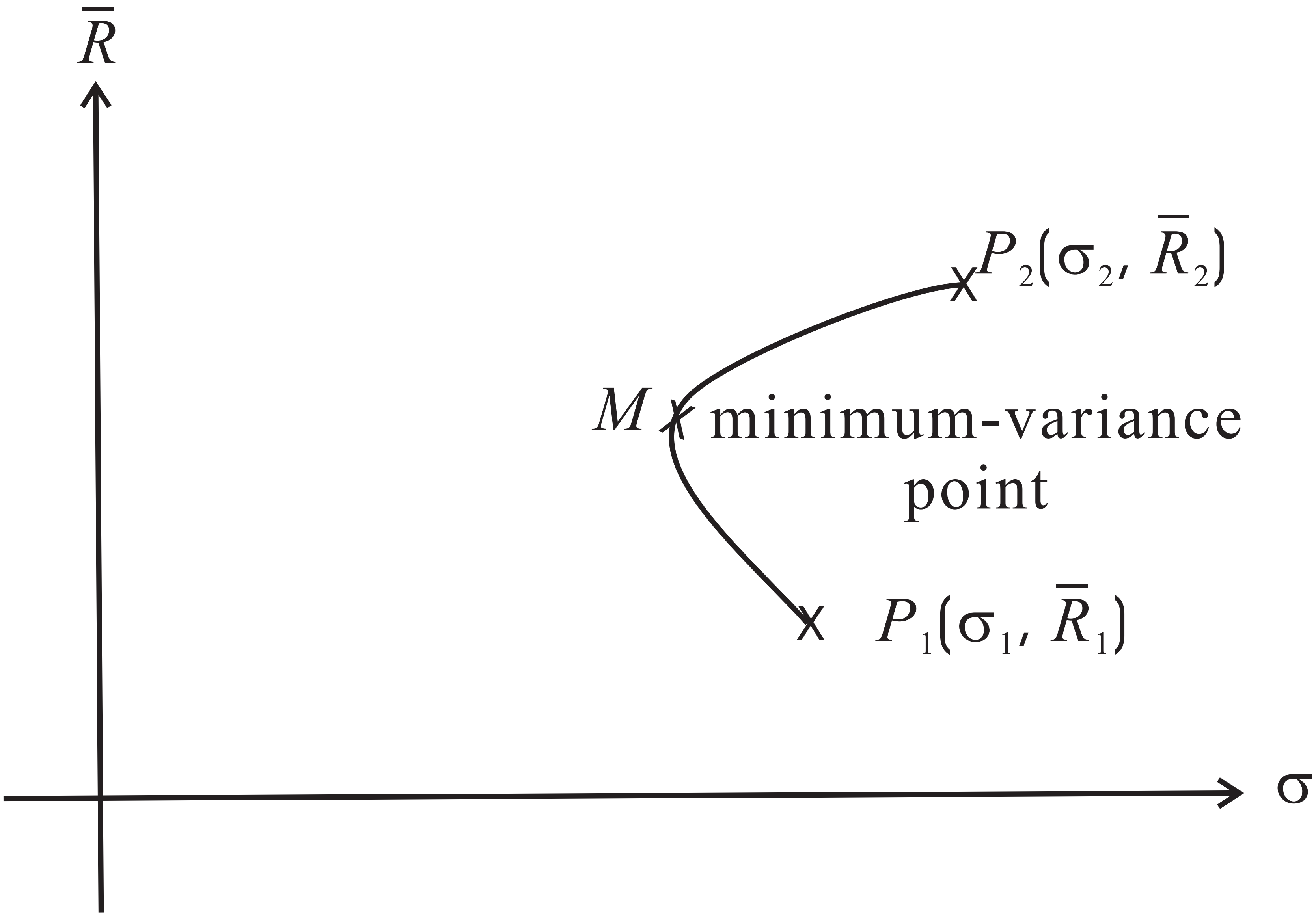
The quantity $(1-\alpha)\sigma_1 - \alpha\sigma_2$ remains positive until $\alpha = \dfrac{\sigma_1}{\sigma_1 + \sigma_2}$. When $\alpha > \dfrac{\sigma_1}{\sigma_1 + \sigma_2}$, the locus traces out the upper line AP_2.

Suppose $-1 < \rho < 1$, the minimum variance point on the curve that represents various portfolio combinations is determined by

$$\frac{\partial \sigma_P^2}{\partial \alpha} = -2(1-\alpha)\sigma_1^2 + 2\alpha\sigma_2^2 + 2(1-2\alpha)\rho\sigma_1\sigma_2 \underset{\substack{\uparrow \\ \text{set}}}{=} 0$$

giving

$$\alpha = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.$$


$\bar{R}$
$P_2(\sigma_2, \bar{R}_2)$
M
minimum-variance point
$P_1(\sigma_1, \bar{R}_1)$
σ

Mathematical formulation of Markowitz's mean-variance analysis

$$\text{minimize } \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

subject to $\sum_{i=1}^{N} w_i \overline{R}_i = \mu_P$ and $\sum_{i=1}^{N} w_i = 1$. Given the target expected rate of return of portfolio μ_P, find the portfolio strategy that minimizes σ_P^2.

Solution

We form the Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} - \lambda_1 \left(\sum_{i=1}^{N} w_i - 1\right) - \lambda_2 \left(\sum_{i=1}^{N} w_i \overline{R}_i - \mu_P\right)$$

where λ_1 and λ_2 are Lagrangian multipliers.

We then differentiate L with respect to w_i and the Lagrangian multipliers, and set the derivative to zero.

$$\begin{aligned}
\frac{\partial L}{\partial w_i} &= \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \overline{R}_i = 0, i = 1, 2, \cdots, N. && (1) \\
\frac{\partial L}{\partial \lambda_1} &= \sum_{i=1}^{N} w_i - 1 = 0; && (2) \\
\frac{\partial L}{\partial \lambda_2} &= \sum_{i=1}^{N} w_i \overline{R}_i - \mu_P = 0. && (3)
\end{aligned}$$

From Eq. (1), the portfolio weight admits solution of the form

$$\boldsymbol{w}^* = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu}) \tag{4}$$

where $\mathbf{1} = (1 \quad 1 \cdots 1)^T$ and $\boldsymbol{\mu} = (\overline{R}_1 \quad \overline{R}_2 \cdots \overline{R}_N)^T$.

To determine λ_1 and λ_2, we apply the two constraints

$$\begin{aligned} 1 &= \mathbf{1}^T\Omega^{-1}\Omega \boldsymbol{w}^* = \lambda_1 \mathbf{1}^T\Omega^{-1}\mathbf{1} + \lambda_2 \mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}. & (5) \\ \mu_P &= \boldsymbol{\mu}^T\Omega^{-1}\Omega \boldsymbol{w}^* = \lambda_1 \boldsymbol{\mu}^T\Omega^{-1}\mathbf{1} + \lambda_2 \boldsymbol{\mu}^T\Omega^{-1}\boldsymbol{\mu}. & (6) \end{aligned}$$

Write $a = \mathbf{1}^T\Omega^{-1}\mathbf{1}, b = \mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}$ and $c = \boldsymbol{\mu}^T\Omega^{-1}\boldsymbol{\mu}$, we have

$$1 = \lambda_1 a + \lambda_2 b \text{ and } \mu_P = \lambda_1 b + \lambda_2 c.$$

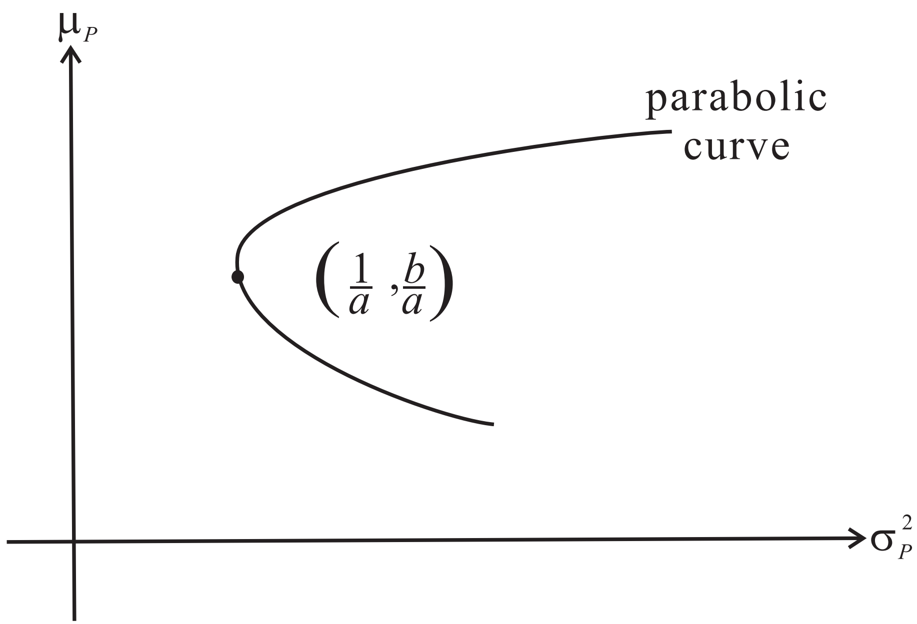
Solving for λ_1 and λ_2 : $\lambda_1 = \dfrac{c - b\mu_P}{\Delta}$ and $\lambda_2 = \dfrac{a\mu_P - b}{\Delta}$, where $\Delta = ac - b^2$.

Note that λ_1 and λ_2 have dependence on μ_P, which is the target mean prescribed in the variance minimization problem.

Assume $\boldsymbol{\mu} \neq h\mathbf{1}$, and Ω^{-1} exists. Since Ω is positive definite, so $a > 0, c > 0$. By virtue of the Cauchy-Schwarz inequality, $\Delta > 0$. The minimum portfolio variance for a given value of μ_P is given by

$$\begin{aligned}\sigma_P^2 &= \boldsymbol{w}^{*T}\Omega\boldsymbol{w}^* = \boldsymbol{w}^{*T}\Omega(\lambda_1\Omega^{-1}\mathbf{1} + \lambda_2\Omega^{-1}\boldsymbol{\mu}) \\ &= \lambda_1 + \lambda_2\mu_P = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}.\end{aligned}$$

The set of minimum variance portfolios is represented by a parabolic curve in the $\sigma_P^2 - \mu_P$ plane. The parabolic curve is generated by varying the value of the parameter μ_P.



Alternatively, when μ_P is plotted against σ_P, the set of minimum variance portfolio is a hyperbolic curve.

What are the asymptotic values of $\lim_{\mu \to \pm\infty} \frac{d\mu_P}{d\sigma_P}$?

$$\begin{aligned}
\frac{d\mu_P}{d\sigma_P} &= \frac{d\mu_P}{d\sigma_P^2}\frac{d\sigma_P^2}{d\sigma_P} \\
&= \frac{\Delta}{2a\mu_P - 2b}2\sigma_P \\
&= \frac{\sqrt{\Delta}}{a\mu_P - b}\sqrt{a\mu_P^2 - 2b\mu_P + c}
\end{aligned}$$

so that

$$\lim_{\mu \to \pm\infty} \frac{d\mu_P}{d\sigma_P} = \pm\sqrt{\frac{\Delta}{a}}.$$

Summary

Given μ_P, we obtain $\lambda_1 = \dfrac{c - b\mu_P}{\Delta}$ and $\lambda_2 = \dfrac{a\mu_P - b}{\Delta}$, and the optimal weight $\boldsymbol{w}^* = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu})$.

To find the global minimum variance portfolio, we set

$$\frac{d\sigma_P^2}{d\mu_P} = \frac{2a\mu_P - 2b}{\Delta} = 0$$

so that $\mu_P = b/a$ and $\sigma_P^2 = 1/a$. Correspondingly, $\lambda_1 = 1/a$ and $\lambda_2 = 0$. The weight vector that gives the global minimum variance is found to be

$$\boldsymbol{w}_g = \frac{\Omega^{-1}\mathbf{1}}{a} = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T \Omega^{-1}\mathbf{1}}.$$

Another portfolio that corresponds to $\lambda_1 = 0$ is obtained when μ_P is taken to be $\dfrac{c}{b}$. The value of the other Lagrangian multiplier is given by

$$\lambda_2 = \frac{a\left(\frac{c}{b}\right) - b}{\Delta} = \frac{1}{b}.$$

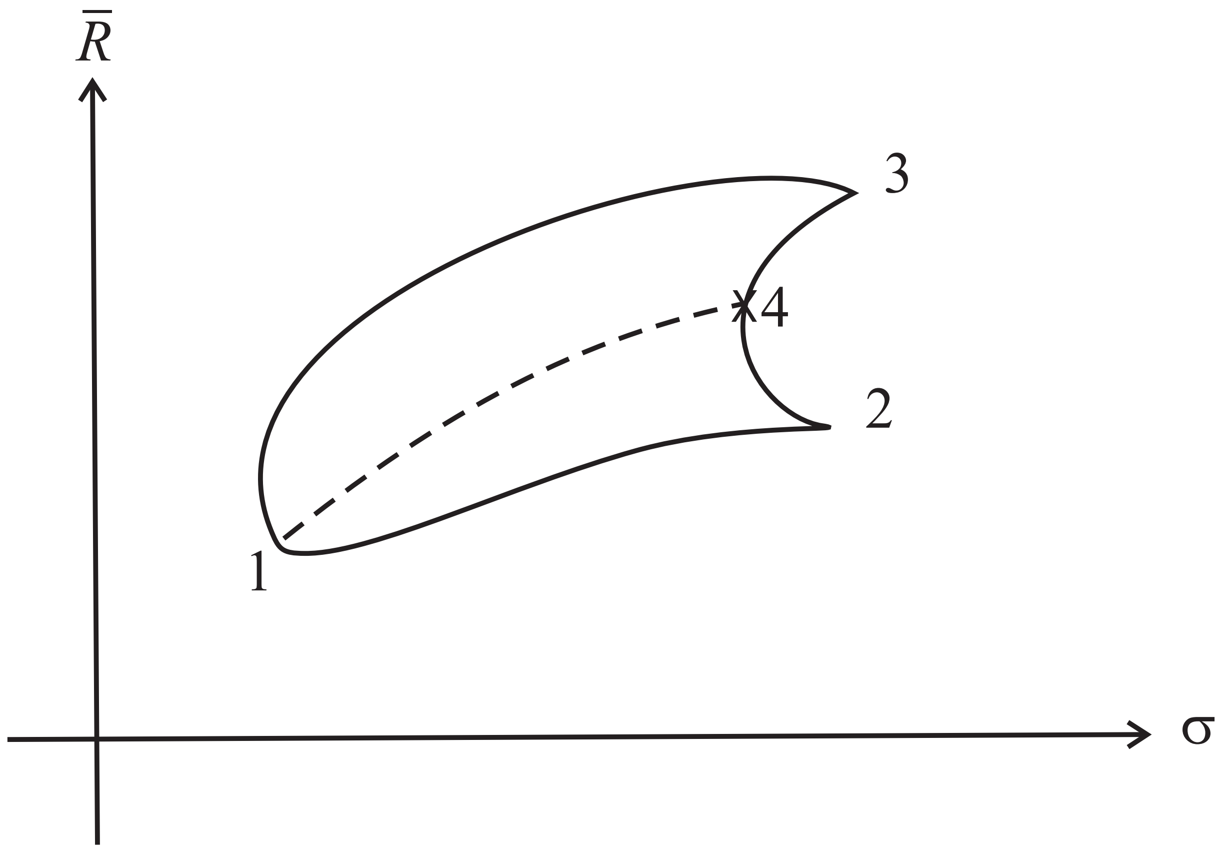
The weight vector of this particular portfolio is

$$\boldsymbol{w}_d^* = \frac{\Omega^{-1}\boldsymbol{\mu}}{b} = \frac{\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}.$$

Also, $\sigma_d^2 = \dfrac{a\left(\frac{c}{b}\right)^2 - 2b\left(\frac{c}{b}\right) + c}{\Delta} = \dfrac{c}{b^2}$.

Feasible set

Given N risky assets, we form various portfolios from these N assets. We plot the point $(\sigma_P, \overline{R}_P)$ representing the portfolios in the $\sigma - \overline{R}$ diagram. The collection of these points constitutes the feasible set or feasible region.



Consider a 3-asset portfolio, the various combinations of assets 2 and 3 sweep out a curve between them (the particular curve taken depends on the correlation coefficient ρ_{12}).

A combination of assets 2 and 3 (labelled 4) can be combined with asset 1 to form a curve joining 1 and 4. As 4 moves between 2 and 3, the curve joining 1 and 4 traces out a solid region.

Properties of feasible regions

1. If there are at least 3 risky assets (not perfectly correlated and with different means), then the feasible set is a solid two-dimensional region.

2. The feasible region is *convex to the left*. That is, given any two points in the region, the straight line connecting them does not cross the left boundary of the feasible region. This is because the minimum variance curve in the mean-variance plot is a parabolic curve.

Minimum variance set and efficient funds

The left boundary of a feasible region is called the *minimum variance set*. The most left point on the minimum variance set is called the *minimum variance point*. The portfolios in the minimum variance set are called *frontier funds*.

For a given level of risk, only those portfolios on the *upper half* of the efficient frontier are desired by investors. They are called *efficient funds*.

A portfolio $\boldsymbol{w}^*$ is said to be mean-variance efficient if there exists no portfolio $\boldsymbol{w}$ with $\mu_P \geq \mu_P^*$ and $\sigma_P^2 \leq {\sigma_P^*}^2$, except itself. That is, you cannot find a portfolio that has a higher return and lower risk than those for an efficient portfolio.

2.2 Two-fund theorem

Two frontier funds (portfolios) can be established so that any frontier portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking frontier portfolios need only invest in combinations of these two funds.

Remark

Any convex combination (that is, weights are non-negative) of efficient portfolios is an efficient portfolio. Let $\alpha_i \geq 0$ be the weight of Fund i whose rate of return is R_f^i. Since $E\left[R_f^i\right] \geq \dfrac{b}{a}$ for all i, we have

$$\sum_{i=1}^{n} \alpha_i E\left[R_f^i\right] \geq \sum_{i=1}^{n} \alpha_i \frac{b}{a} = \frac{b}{a}.$$

Proof

Let $\boldsymbol{w}^1 = (w_1^1 \cdots w_n^1), \lambda_1^1, \lambda_2^1$ and $\boldsymbol{w}^2 = (w_1^2 \cdots w_n^2)^T, \lambda_1^2, \lambda_2^2$ are two known solutions to the minimum variance formulation with expected rates of return μ_P^1 and μ_P^2, respectively.

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \overline{R}_i = 0, \quad i = 1, 2, \cdots, n \tag{1}$$

$$\sum_{i=1}^{n} w_i \overline{r}_i = \mu_P \tag{2}$$

$$\sum_{i=1}^{n} w_i = 1. \tag{3}$$

It suffices to show that $\alpha \boldsymbol{w}_1 + (1-\alpha)\boldsymbol{w}_2$ is a solution corresponds to the expected rate of return $\alpha\mu_P^1 + (1-\alpha)\mu_P^2$.

1. $\alpha \boldsymbol{w}^1 + (1-\alpha)\boldsymbol{w}^2$ is a legitimate portfolio with weights that sum to one.

2. Eq. (1) is satisfied by $\alpha \boldsymbol{w}^1 + (1-\alpha)\boldsymbol{w}^2$ since the system of equations is linear.

3. Note that

$$\begin{aligned} & \sum_{i=1}^{n} \left[\alpha w_i^1 + (1-\alpha) w_i^2\right] \overline{R}_i \\ = \;& \alpha \sum_{i=1}^{n} w_i^1 \overline{R}_i + (1-\alpha) \sum_{i=1}^{n} w_i^2 \overline{R}_i \\ = \;& \alpha \mu_P^1 + (1-\alpha) \mu_P^2. \end{aligned}$$

Proposition

Any minimum variance portfolio with target mean μ_P can be uniquely decomposed into the sum of two portfolios

$$\boldsymbol{w}_P^* = A\boldsymbol{w}_g + (1 - A)\boldsymbol{w}_d$$

where $A = \dfrac{c - b\mu_P}{\Delta}a$.

Proof

For a minimum-variance portfolio whose solution of the Lagrangian multipliers are λ_1 and λ_2, the optimal weight is

$$\boldsymbol{w}_P^* = \lambda_1 \Omega^{-1}\mathbf{1} + \lambda_2 \Omega^{-1}\boldsymbol{\mu} = \lambda_1(a\boldsymbol{w}_g) + \lambda_2(b\boldsymbol{w}_d).$$

Observe that the sum of weights is

$$\lambda_1 a + \lambda_2 b = a\frac{c - \mu_P b}{\Delta} + b\frac{\mu_P a - b}{\Delta} = \frac{ac - b^2}{\Delta} = 1.$$

We set $\lambda_1 a = A$ and $\lambda_2 b = 1 - A$, where

$$\lambda_1 = \frac{c - \mu_P b}{\Delta} \quad \text{and} \quad \lambda_2 = \frac{\mu_P a - b}{\Delta}.$$

Indeed, any two minimum-variance portfolios can be used to substitute for $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$. Suppose

$$\begin{aligned}\boldsymbol{w}_u &= (1-u)\boldsymbol{w}_g + u\boldsymbol{w}_d \\ \boldsymbol{w}_v &= (1-v)\boldsymbol{w}_g + v\boldsymbol{w}_d\end{aligned}$$

we then solve for $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$ in terms of $\boldsymbol{w}_u$ and $\boldsymbol{w}_v$. Then

$$\begin{aligned}\boldsymbol{w}_P^* &= \lambda_1 a\boldsymbol{w}_g + (1-\lambda_1 a)\boldsymbol{w}_d \\ &= \frac{\lambda_1 a + v - 1}{v-u}\boldsymbol{w}_u + \frac{1-u-\lambda_1 a}{v-u}\boldsymbol{w}_v,\end{aligned}$$

where sum of coefficients $= 1$.

Example

Mean, variances and covariances of the rates of return of 5 risky assets are listed:

Security	covariance					$\overline{R}_i$
1	2.30	0.93	0.62	0.74	-0.23	15.1
2	0.93	1.40	0.22	0.56	0.26	12.5
3	0.62	0.22	1.80	0.78	-0.27	14.7
4	0.74	0.56	0.78	3.40	-0.56	9.02
5	-0.23	0.26	-0.27	-0.56	2.60	17.68

Solution procedure to find the two funds in the minimum variance set:

1. Set $\lambda_1 = 1$ and $\lambda_2 = 0$; solve the system of equations

$$\sum_{j=1}^{5} \sigma_{ij} v_j^1 = 1, \quad i = 1, 2, \cdots, 5.$$

   The actual weights w_i should be summed to one. This is done by normalizing v_k^1's so that they sum to one

$$w_i^1 = \frac{v_i^1}{\sum_{j=1}^{n} v_j^1}.$$

   After normalization, this gives the solution to $\boldsymbol{w}_g$, where $\lambda_1 = \frac{1}{a}$ and $\lambda_2 = 0$.

2. Set $\lambda_1 = 0$ and $\lambda_2 = 1$; solve the system of equations:

$$\sum_{j=1}^{5} \sigma_{ij} v_j^2 = \overline{R}_i, \quad i = 1, 2, \cdots, 5.$$

Normalize v_i^2's to obtain w_i^2.

After normalization, this gives the solution to $\boldsymbol{w}_d$, where $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{b}$.

The above procedure avoids the computation of $a = \mathbf{1}^T \Omega^{-1} \mathbf{1}$ and $b = \mathbf{1}^T \Omega^{-1} \boldsymbol{\mu}$.

security	$\boldsymbol{v}^1$	$\boldsymbol{v}^2$	$\boldsymbol{w}^1$	$\boldsymbol{w}^2$
1	0.141	3.652	0.088	0.158
2	0.401	3.583	0.251	0.155
3	0.452	7.284	0.282	0.314
4	0.166	0.874	0.104	0.038
5	0.440	7.706	0.275	0.334
mean			14.413	15.202
variance			0.625	0.659
standard deviation			0.791	0.812

* Note that $\boldsymbol{w}^1$ corresponds to the global minimum variance point.

We know that $\mu_g = b/a$; how about μ_d?

$$\mu_d = \boldsymbol{\mu}^T \boldsymbol{w}_d = \boldsymbol{\mu}^T \frac{\Omega^{-1}\boldsymbol{\mu}}{b} = \frac{c}{b}.$$

Difference in expected returns $= \mu_d - \mu_g = \dfrac{c}{b} - \dfrac{b}{a} = \dfrac{\Delta}{ab} > 0.$

Also, difference in variances $= \sigma_d^2 - \sigma_g^2 = \dfrac{c}{b^2} - \dfrac{1}{a} = \dfrac{\Delta}{ab^2} > 0.$

What is the covariance of portfolio returns for any two minimum variance portfolios?

Write

$$R_P^u = \boldsymbol{w}_u^T \boldsymbol{R} \text{ and } R_P^v = \boldsymbol{w}_v^T \boldsymbol{R}$$

where $\boldsymbol{R} = (R_1 \cdots R_N)^T$. Recall that $\boldsymbol{w}_g = \dfrac{\Omega^{-1}\mathbf{1}}{a}$ and $\boldsymbol{w}_d = \dfrac{\Omega^{-1}\boldsymbol{\mu}}{b}$ so that

$$\begin{aligned}
\sigma_{gd} &= \text{cov}\left(\frac{\Omega^{-1}\mathbf{1}}{a}\boldsymbol{R}, \frac{\Omega^{-1}\boldsymbol{\mu}}{b}\boldsymbol{R}\right) \\
&= \left(\frac{\Omega^{-1}\mathbf{1}}{a}\right)^T \Omega \left(\frac{\Omega^{-1}\boldsymbol{\mu}}{b}\right) \\
&= \frac{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}{ab} = \frac{1}{a} \quad \text{since} \quad b = \mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}.
\end{aligned}$$

In general,

$$
\begin{aligned}
\text{cov}(R_P^u, R_P^v) &= (1-u)(1-v)\sigma_g^2 + uv\sigma_d^2 + [u(1-v) + v(1-u)]\sigma_{gd} \\
&= \frac{(1-u)(1-v)}{a} + \frac{uvc}{b^2} + \frac{u+v-2uv}{a} \\
&= \frac{1}{a} + \frac{uv\Delta}{ab^2}.
\end{aligned}
$$

In particular,

$$\text{cov}(R_g, R_P) = \boldsymbol{w}_g^T \Omega \boldsymbol{w}_P = \frac{\mathbf{1}\Omega^{-1}\Omega \boldsymbol{w}_P}{a} = \frac{1}{a} = \text{var}(R_g)$$

for any portfolio $\boldsymbol{w}_P$.

For any Portfolio u, we can find another Portfolio v such that these two portfolios are uncorrelated. This can be done by setting

$$\frac{1}{a} + \frac{uv\Delta}{ab^2} = 0.$$

The mean-variance criterion can be reconciled with the expected utility approach in either of two ways: (1) using a quadratic utility function, or (2) making the assumption that the random returns are normal variables.

Quadratic utility

The quadratic utility function can be defined as $U(x) = ax - \frac{b}{2}x^2$, where $a > 0$ and $b > 0$. This utility function is really meaningful only in the range $x \leq a/b$, for it is in this range that the function is increasing. Note also that for $b > 0$ the function is strictly concave everywhere and thus exhibits risk aversion.

$$\text{mean-variance analysis} \quad \Leftrightarrow \quad \begin{array}{l}\text{maximum expected utility criterion}\\ \text{based on quadratic utility}\end{array}$$

Suppose that a portfolio has a random wealth value of y. Using the expected utility criterion, we evaluate the portfolio using

$$\begin{aligned} E[U(y)] &= E\left[ay - \frac{b}{2}y^2\right] \\ &= aE[y] - \frac{b}{2}E[y^2] \\ &= aE[y] - \frac{b}{2}(E[y])^2 - \frac{b}{2}\text{var}(y). \end{aligned}$$

Note that we choose the range of the quadratic utility function such that $aE[y] - \dfrac{b}{2}(E[y])^2$ is increasing in $E[y]$. Maximizing $E[y]$ for a given var(y) or minimizing var(y) for a given $E[y]$ is equivalent to maximizing $E[U(y)]$.

Normal Returns

When all returns are normal random variables, the mean-variance criterion is also equivalent to the expected utility approach for any risk-averse utility function.

To deduce this, select a utility function U. Consider a random wealth variable y that is a normal random variable with mean value M and standard deviation σ. Since the probability distribution is completely defined by M and σ, it follows that the expected utility is a function of M and σ. If U is risk averse, then

$$E[U(y)] = f(M, \sigma), \quad \text{with } \frac{\partial f}{\partial M} > 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma} < 0.$$

- Now suppose that the returns of all assets are normal random variables. Then any linear combination of these assets is a normal random variable. Hence any portfolio problem is therefore equivalent to the selection of combination of assets that maximizes the function $f(M, \sigma)$ with respect to all feasible combinations.

- For a risky-averse utility, this again implies that the variance should be minimized for any given value of the mean. In other words, the solution must be mean-variance efficient.

- Portfolio problem is to find $\boldsymbol{w}^*$ such that $f(M, \sigma)$ is maximized with respect to all feasible combinations.

2.3 Inclusion of the riskfree asset

Consider a portfolio with weight α for a risk free asset and $1-\alpha$ for a risky asset. The mean of the portfolio is

$$\overline{R}_P = \alpha \overline{R}_f + (1-\alpha)\overline{R}_j \quad \text{(note that } R_f = \overline{R}_f\text{)}.$$

The covariance σ_{fj} between the risk free asset and any risky asset is zero since

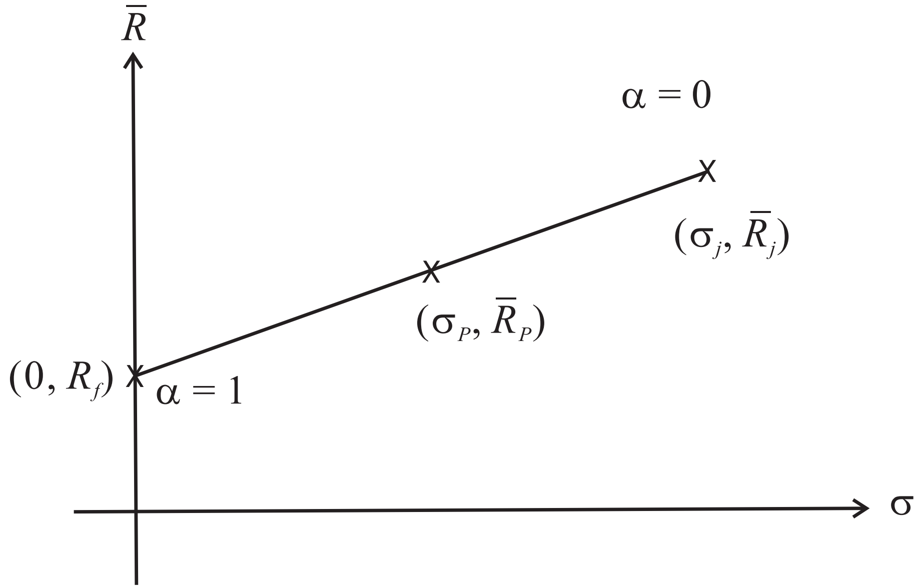
$$E[(R_j - \overline{R}_j)\underbrace{(R_f - \overline{R}_f)}_{\text{zero}}] = 0.$$

Therefore, the variance of portfolio σ_P^2 is

$$\sigma_P^2 = \alpha^2 \underbrace{\sigma_f^2}_{\text{zero}} + (1-\alpha)^2\sigma_j^2 + 2\alpha(1-\alpha)\underbrace{\sigma_{fj}}_{\text{zero}}$$

so that $\sigma_P = |1-\alpha|\sigma_j$.

The points representing $(\sigma_P, \overline{R}_P)$ for varying values of α lie on a straight line joining $(0, R_f)$ and $(\sigma_j, \overline{R}_j)$.



If borrowing of risk free asset is allowed, then α can be negative. In this case, the line extends beyond the right side of $(\sigma_j, \overline{R}_j)$ (possibly up to infinity).

Consider a portfolio with N risky assets originally, what is the impact of the inclusion of a risk free asset on the feasible region?

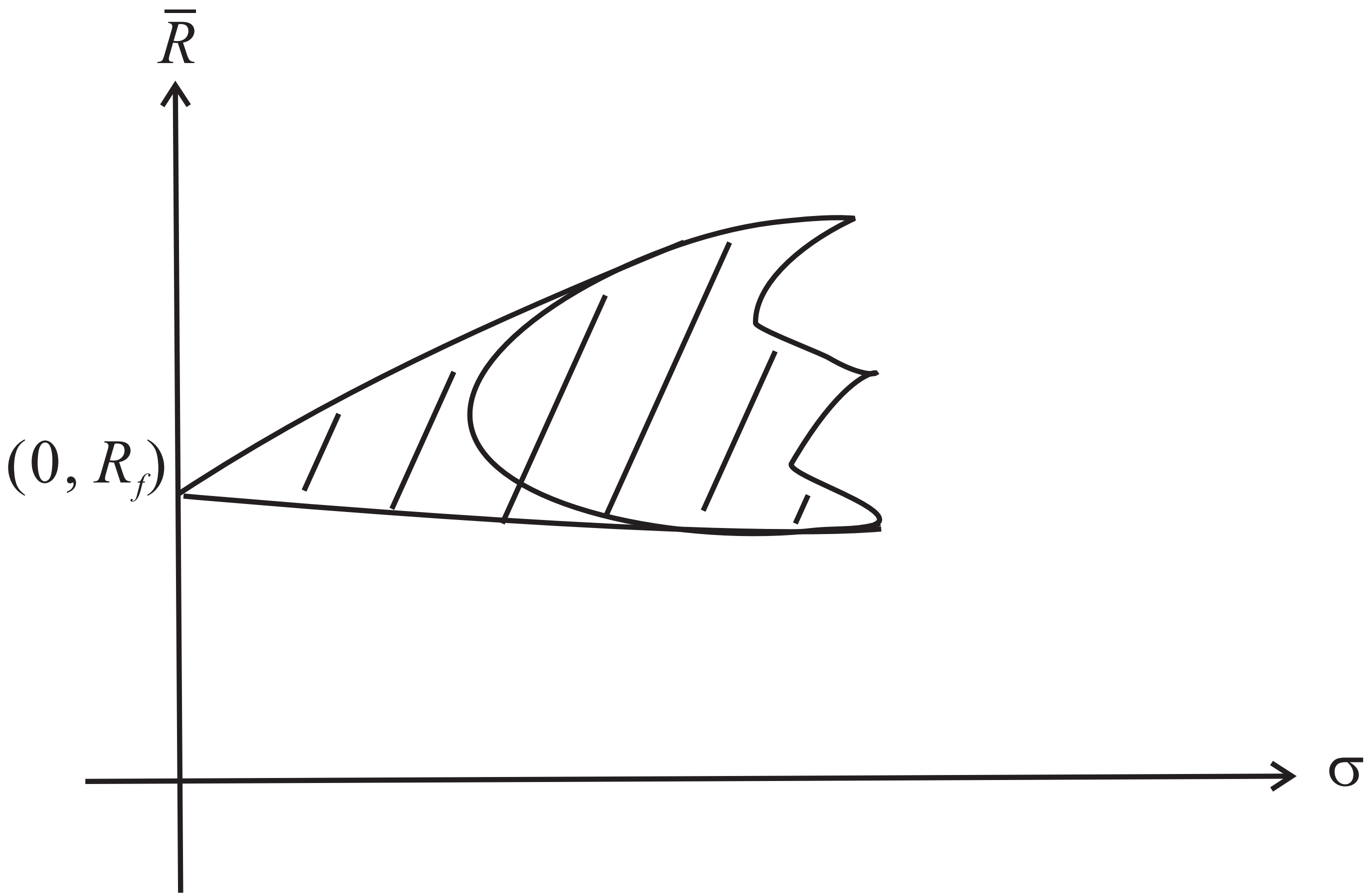
Lending and borrowing of risk free asset is allowed

For each original portfolio formed using the N risky assets, the new combinations with the inclusion of the risk free asset trace out the infinite straight line originating from the risk free point and passing through the point representing the original portfolio.

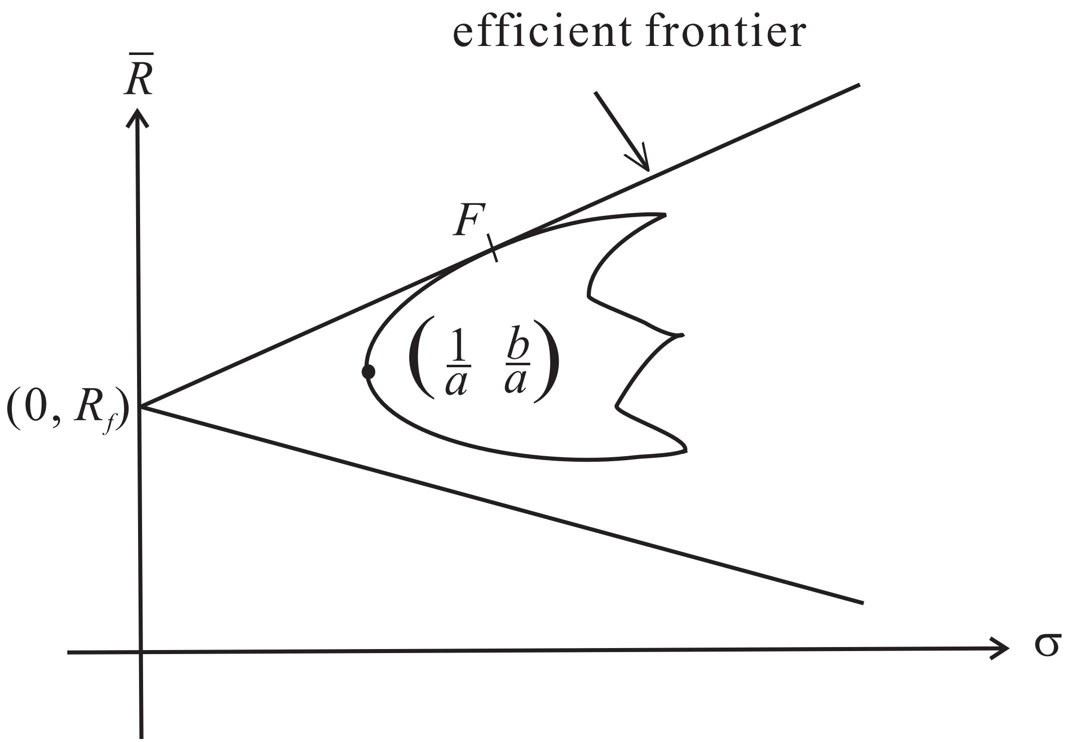
The totality of these lines forms an infinite triangular feasible region bounded by the two tangent lines through the risk free point to the original feasible region.

No shorting of the riskfree asset

The line originating from the risk free point cannot be extended beyond points in the original feasible region (otherwise entail borrowing of the risk free asset). The new feasible region has straight line front edges.



The new *efficient set* is the single straight line on the top of the new triangular feasible region. This tangent line touches the original feasible region at a point F, where F lies on the efficient frontier of the original feasible set.



Here, $R_f < \dfrac{b}{a}$. This assumption is reasonable since the risk free asset should earn a rate of return less than the expected rate of return of the global minimum variance portfolio.

One fund theorem

Any efficient portfolio (any point on the upper tangent line) can be expressed as a combination of the risk free asset and the portfolio (or fund) represented by F.

"There is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk free asset."

Under the assumptions that

- every investor is a mean-variance optimizer
- they all agree on the probabilistic structure of the assets
- unique risk free asset

Then everyone will purchase a single fund, which is the *market portfolio*.

Now, the proportion of wealth invested in the risk free asset is $1 - \sum_{i=1}^{N} w_i$.

Modified Lagrangian formulation

minimize $$\frac{\sigma_P^2}{2} = \frac{1}{2}\boldsymbol{w}^T \Omega \boldsymbol{w}$$

subject to $$\boldsymbol{w}^T \boldsymbol{\mu} + (1 - \boldsymbol{w}^T \mathbf{1})r = \mu_P.$$

Define the Lagrangian: $L = \frac{1}{2}\boldsymbol{w}^T \Omega \boldsymbol{w} + \lambda[\mu_P - r - (\boldsymbol{\mu} - r\mathbf{1})^T \boldsymbol{w}]$

$$\frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda(\boldsymbol{\mu} - r\mathbf{1}) = 0, \quad i = 1, 2, \cdots, N \tag{1}$$

$$\frac{\partial L}{\partial \lambda} = 0 \quad \text{giving} \quad (\boldsymbol{\mu} - r\mathbf{1})^T \boldsymbol{w} = \mu_P - r. \tag{2}$$

Solving (1): $\boldsymbol{w}^* = \lambda\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})$. Substituting into (2)

$$\mu_P - r = \lambda(\boldsymbol{\mu} - r\mathbf{1})^T\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}) = \lambda(c - 2rb + r^2a).$$

By eliminating λ, the relation between μ_P and σ_P is given by the following pair of half lines

$$\begin{aligned} \sigma_P^2 &= {\boldsymbol{w}^*}^T\Omega\boldsymbol{w}^* = \lambda({\boldsymbol{w}^*}^T\boldsymbol{\mu} - r{\boldsymbol{w}^*}^T\mathbf{1}) \\ &= \lambda(\mu_P - r) = (\mu_P - r)^2/(c - 2rb + r^2a). \end{aligned}$$

With the inclusion of the riskfree asset, the set of minimum variance portfolios are represented by portfolios on the two half lines

$$L_{up} : \mu_P - r = \sigma_P \sqrt{ar^2 - 2br + c} \qquad (3a)$$

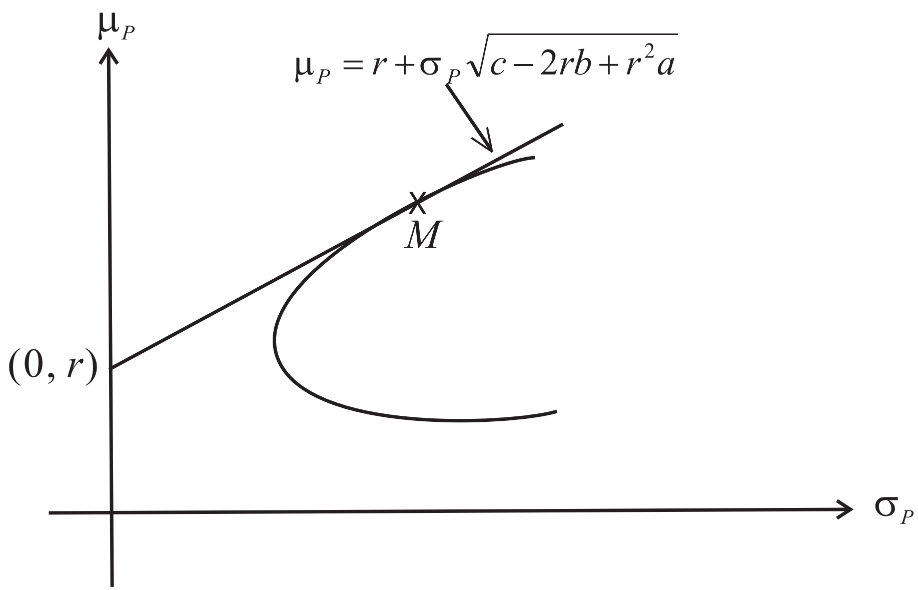
$$L_{low} : \mu_P - r = -\sigma_P \sqrt{ar^2 - 2br + c}. \qquad (3b)$$

Recall that $ar^2 - 2br + c > 0$ for all values of r since $\Delta = ac - b^2 > 0$.

The minimum variance portfolios without the riskfree asset lie on the hyperbola

$$\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}.$$

When $r < \mu_g = \dfrac{b}{a}$, the upper half line is a tangent to the hyperbola. The tangency portfolio is the tangent point to the efficient frontier (upper part of the hyperbolic curve) through the point $(0, r)$.



The tangency portfolio M is represented by the point $(\sigma_{P,M}, \mu_P^M)$, and the solution to $\sigma_{P,M}$ and μ_P^M are obtained by solving simultaneously

$$\begin{aligned}\sigma_P^2 &= \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta} \\ \mu_P &= r + \sigma_P\sqrt{c - 2rb + r^2a}.\end{aligned}$$

Once μ_P^M is obtained, the corresponding values for λ_M and $\boldsymbol{w}_M^*$ are

$$\lambda_M = \frac{\mu_P^M - r}{c - 2rb + r^2a} \quad \text{and} \quad \boldsymbol{w}_M^* = \lambda_M \Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}).$$

The tangency portfolio M is shown to be

$$\boldsymbol{w}_M^* = \frac{\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{b - ar}, \quad \mu_P^M = \frac{c - br}{b - ar} \quad \text{and} \quad \sigma_{P,M}^2 = \frac{c - 2rb + r^2a}{(b - ar)^2}.$$

When $r < \frac{b}{a}$, it can be shown that $\mu_P^M > r$. Note that

$$\begin{aligned}
\left(\mu_P^M - \frac{b}{a}\right)\left(\frac{b}{a} - r\right) &= \left(\frac{c - br}{b - ar} - \frac{b}{a}\right)\frac{b - ar}{a} \\
&= \frac{c - br}{a} - \frac{b^2}{a^2} + \frac{br}{a} \\
&= \frac{ca - b^2}{a^2} = \frac{\Delta}{a^2} > 0,
\end{aligned}$$

so we deduce that $\mu_P^M > \frac{b}{a} > r$, where $\mu_g = \frac{b}{a}$.

On the other hand, we can deduce that $(\sigma_{P,M}, \mu_P^M)$ does not lie on the upper half line if $r \geq \frac{b}{a}$.

When $r < \frac{b}{a}$, we have the following properties on the minimum variance portfolios.

1. *Efficient portfolios*

   Any portfolio on the upper half line

   $$\mu_P = r + \sigma_P\sqrt{ar^2 - 2br + c}$$

   within the segment FM joining the two points $(0, r)$ and M involves long holding of the market portfolio and riskfree asset, while those outside FM involves short selling of the riskfree asset and long holding of the market portfolio.
2. Any portfolio on the lower half line

   $$\mu_P = r - \sigma_P\sqrt{ar^2 - 2br + c}$$

   involves short selling of the market portfolio and investing the proceeds in the riskfree asset. This represents non-optimal investment strategy since the investor faces risk but gains no extra expected return above r.

What happens when $r = b/a$? The half lines become

$$\mu_P = r \pm \sigma_P \sqrt{c - 2\left(\frac{b}{a}\right) b + \frac{b^2}{a}} = r \pm \sigma_P \sqrt{\frac{\Delta}{a}},$$

which correspond to the asymptotes of the feasible region with risky assets only.

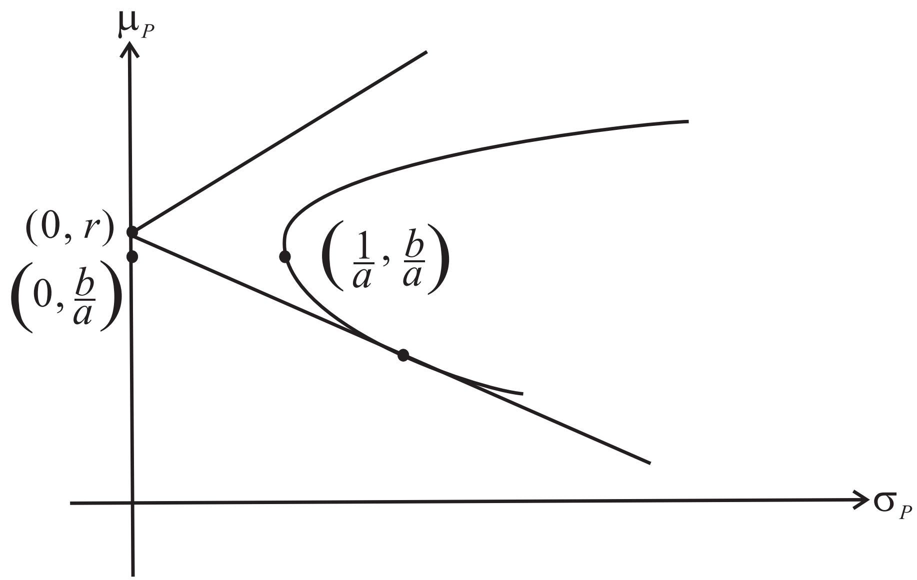
Even when $r = \dfrac{b}{a}$, efficient funds still lie on the upper half line, though μ_P^M does not exist. Recall that

$$\begin{aligned} &\boldsymbol{w}^* = \lambda \Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}) \text{ so that} \\ &\mathbf{1}^T \boldsymbol{w} = \lambda(\mathbf{1}\Omega^{-1}\boldsymbol{\mu} - r\mathbf{1}\Omega^{-1}\mathbf{1}) = \lambda(b - ra). \end{aligned}$$

When $r = b/a, \mathbf{1}^T \boldsymbol{w} = 0$ as λ is finite.

Any minimum variance portfolio involves investing everything in the riskfree asset and holding a portfolio of risky assets whose weights sum to zero.

When $r > \dfrac{b}{a}$, the lower half line touches the feasible region with risky assets only.



- Any portfolio on the upper half line involves short selling of the tangency portfolio and investing the proceeds in the riskfree asset.

- It makes sense to short sell the tangency portfolio since it has an expected rate of return lower than the risk free asset.

Interpretation of the tangency portfolio (market portfolio)

- One-fund theorem states that everyone will purchase a single fund of risky assets and borrow or lend at the risk free rate.

- If everyone purchases the same fund of risky assets, what must that fund be? This fund must equal the *market portfolio*.

- The market portfolio is the summation of all assets. If everyone buys just one fund, and their purchases add up to the market, then that one fund must be the market as well.

- In the situation where everyone follows the mean-variance methodology with the same estimates of parameters, the efficient fund of risky assets will be the market portfolio.

How does this happen? The answer is based on the equilibrium argument.

- If everyone else (or at least a large number of people) solves the problem, we do not need to. The return on an asset depends on both its initial price and its final price. The other investors solve the mean-variance portfolio problem using their common estimates, and they place orders in the market to acquire their portfolios.

- If orders placed do not match what is available, the prices must change. The prices of assets under heavy demand will increase; the prices of assets under light demand will decrease. These price changes affect the estimates of asset returns directly, and hence investors will recalculate their optimal portfolio.

- This process continues until demand exactly matches supply; that is, it continues until there is equilibrium.

Summary

- In the idealized world, where every investor is a mean-variance investor and all have the same estimates, everyone buys the same portfolio, and that must be equal to the market portfolio.

- Prices adjust to drive the market to efficiency. Then after other people have made the adjustments, we can be sure that the efficient portfolio is the market portfolio, so we need not make any calculations.