7. Barrier options, lookback options and Asian options

Path dependent options: payouts are related to the underlying asset price path history during the whole or part of the life of the option.

- The *barrier option* is either nullified, activated or exercised when the underlying asset price breaches a *barrier* during the life of the option.

- The payoff of a *lookback option* depends on the minimum or maximum price of the underlying asset attained during certain period of the life of the option

- The payoff of an *average option* (usually called an *Asian option*) depends on the average asset price over some period within the life of the option.

- An interesting example is the *Russian option*, which is in fact a perpetual American lookback option. The owner of a Russian option on a stock receives the historical maximum value of the asset price when the option is exercised and the option has no pre-set expiration date.

Discrete and continuous monitoring of asset price process

- Due to the path dependent nature of these options, the asset price process is monitored over the life of the option contract either for breaching of a barrier level, observation of new extremum value or sampling of asset prices for computing average value.

- In actual implementation, these monitoring procedures can only be performed at discrete time instants rather than continuously at all times.

- When the asset price path is monitored at discrete time instants, the analytic forms of the price formulas become quite daunting since they involve multi-dimensional cumulative normal distribution functions and the dimension is equal to the number of monitoring instants.

Barrier options

1. An *out-barrier option* (or knock-out option) is one where the option is nullified prior to expiration if the underlying asset price touches the barrier. The holder of the option may be compensated by a rebate payment for the cancellation of the option. An *in-barrier option* (or knock-in option) is one where the option only comes in existence if the asset price crosses the in-barrier, though the holder has paid the option premium up front.

2. When the barrier is upstream with respect to the asset price, the barrier option is called an *up-option*; otherwise, it is called a *down-option*.

   One can identify eight types of European barrier options, such as down-and-out calls, up-and-out calls, down-and-in puts, down-and-out puts, etc.

How both buyer and writer benefit from the up-and-out call?

With an appropriate rebate paid upon breaching the upside barrier, this type of barrier options provide the upside exposure for option buyer but at a lower cost.

- The option writer is not exposed to unlimited liabilities when the asset price rises acutely.
- Barrier options are attractive since they give investors more flexibility to express their view on the asset price movement in the option contract design.

Discontinuity at the barrier (circuit breaker effect upon knock-out)

- Pitched battles often erupt around popular knock-out barriers in currency barrier options and these add much unwanted volatility to the markets.
- George Soros once said "knock-out options relate to ordinary options the way crack relates to cocaine."

European down-and-out call

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc, \qquad S > B \text{ and } \tau \in (0, T],$$

subject to

$$\begin{array}{ll} \text{knock-out condition:} & c(B,\tau) = R(\tau) \\ \text{terminal payoff:} & c(S,0) = \max(S - X, 0), \end{array}$$

Set $y = \ln S$, the barrier becomes the line $y = \ln B$.

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial y^2} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial c}{\partial y} - rc$$

defined in the semi-infinite domain: $y > \ln B$ and $\tau \in (0, T]$.

The auxiliary conditions become

$$c(\ln B, \tau) = R(\tau) \text{ and } c(y, 0) = \max(e^y - X, 0).$$

Write $\mu = r - \dfrac{\sigma^2}{2}$, the Green function in the infinite domain: $-\infty < y < \infty$ is given by

$$G_0(y, \tau; \xi) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{(y + \mu\tau - \xi)^2}{2\sigma^2\tau}\right),$$

where $G_0(y, \tau; \xi)$ satisfies the initial condition:

$$\lim_{\tau \to 0^+} G_0(y, \tau; \xi) = \delta(y - \xi).$$

Method of images

Assuming that the Green function in the semi-infinite domain takes the form

$$G(y,\tau;\xi) = G_0(y,\tau;\xi) - H(\xi)G_0(y,\tau;\eta),$$

we are required to determine $H(\xi)$ and η (in terms of ξ) such that the zero Dirichlet boundary condition $G(\ln B,\tau;\xi) = 0$ is satisfied.

Note that both $G_0(y,\tau;\xi)$ and $H(\xi)G_0(y,\tau;\eta)$ satisfy the differential equation. Also, provided that $\eta \notin (\ln B, \infty)$, then

$$\lim_{\tau\to 0^+} G_0(y,\tau;\eta) = 0 \text{ for all } y > \ln B.$$

By imposing the boundary condition, $H(\xi)$ has to satisfy

$$H(\xi) = \frac{G_0(\ln B,\tau;\xi)}{G_0(\ln B,\tau;\eta)} = \exp\left(\frac{(\xi-\eta)[2(\ln B + \mu\tau) - (\xi+\eta)]}{2\sigma^2\tau}\right).$$

The assumed form of $G(y,\tau;\xi)$ is feasible only if the right hand side becomes a function of ξ only. This can be achieved by the judicious choice of

$$\eta = 2\ln B - \xi,$$

so that

$$H(\xi) = \exp\left(\frac{2\mu}{\sigma^2}(\xi - \ln B)\right).$$

- This method works only if μ/σ^2 is a constant, independent of τ.

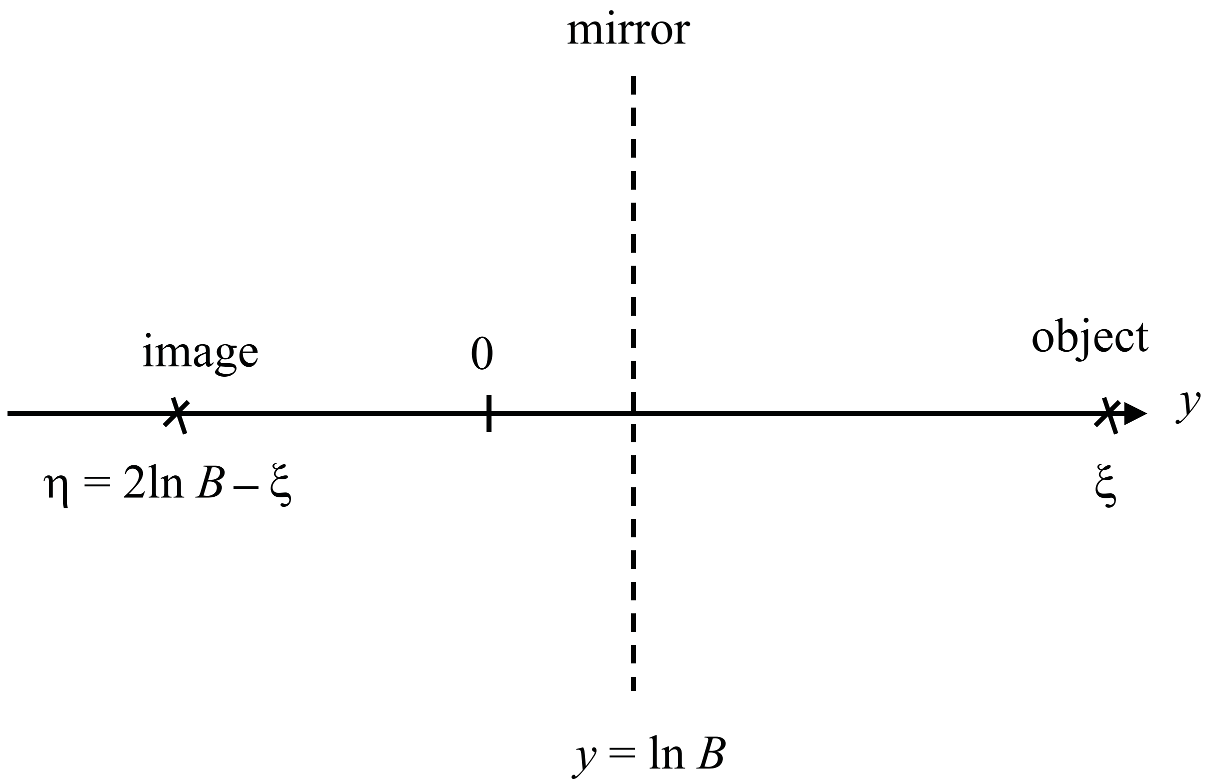
- The parameter η can be visualized as the *mirror image* of ξ with respect to the barrier $y = \ln B$.

$$
\begin{aligned}
& H(\xi)G_0(y,\tau;\eta) \\
= \;& \exp\left(\frac{2\mu}{\sigma^2}(\xi - \ln B)\right)\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{[y+\mu\tau-(2\ln B-\xi)]^2}{2\sigma^2\tau}\right) \\
= \;& \left(\frac{B}{S}\right)^{2\mu/\sigma^2}\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{[(y-\xi)+\mu\tau-2(y-\ln B)]^2}{2\sigma^2\tau}\right).
\end{aligned}
$$

The Green function in the specified semi-infinite domain: $\ln B < y < \infty$ becomes

$$
\begin{aligned}
G(y,\tau;\xi) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\Bigg\{ & \exp\left(-\frac{(u-\mu\tau)^2}{2\sigma^2\tau}\right) \\
& - \left(\frac{B}{S}\right)^{2\mu/\sigma^2}\exp\left(-\frac{(u-2\beta-\mu\tau)^2}{2\sigma^2\tau}\right)\Bigg\},
\end{aligned}
$$

where $u = \xi - y$ and $\beta = \ln B - y = \ln \dfrac{B}{S}$.



Pictorial representation of the method of image. The mirror is placed at $y = \ln B$.

We consider the barrier option with zero rebate, where $R(\tau) = 0$, and let $K = \max(B, X)$. The price of the zero-rebate European down-and-out call can be expressed as

$$
\begin{aligned}
c_{do}(y,\tau) &= \int_{\ln B}^{\infty} \max(e^{\xi} - X, 0) G(y,\tau;\xi)\, d\xi \\
&= \int_{\ln K}^{\infty} (e^{\xi} - X) G(y,\tau;\xi)\, d\xi \\
&= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K/S}^{\infty} (Se^{u} - X) \left[\exp\left(-\frac{(u-\mu\tau)^2}{2\sigma^2\tau} \right) \right. \\
&\qquad \left. - \left(\frac{B}{S} \right)^{2\mu/\sigma^2} \exp\left(-\frac{(u - 2\beta - \mu\tau)^2}{2\sigma^2\tau} \right) \right] du.
\end{aligned}
$$

The direct evaluation of the integral gives

$$\begin{aligned} c_{do}(S,\tau) &= S\left[N(d_1) - \left(\frac{B}{S}\right)^{\delta+1} N(d_3)\right] \\ &\quad -Xe^{-r\tau}\left[N(d_2) - \left(\frac{B}{S}\right)^{\delta-1} N(d_4)\right], \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}, \\ d_3 &= d_1 + \frac{2}{\sigma\sqrt{\tau}}\ln\frac{B}{S}, \qquad d_4 = d_2 + \frac{2}{\sigma\sqrt{\tau}}\ln\frac{B}{S}, \quad \delta = \frac{2r}{\sigma^2}. \end{aligned}$$

Suppose we define

$$\widetilde{c}_E(S, \tau; X, K) = SN(d_1) - Xe^{-r\tau}N(d_2),$$

then $c_{do}(S, \tau; X, B)$ can be expressed in the following succinct form

$$c_{do}(S, \tau; X, B) = \widetilde{c}_E(S, \tau; X, K) - \left(\frac{B}{S}\right)^{\delta-1} \widetilde{c}_E\left(\frac{B^2}{S}, \tau; X, K\right).$$

One can show by direct calculation that the function $\left(\frac{B}{S}\right)^{\delta-1} \widetilde{c}_E\left(\frac{B^2}{S}, \tau\right)$ satisfies the Black-Scholes equation identically. The above form allows us to observe readily the satisfaction of the boundary condition: $c_{do}(B, \tau) = 0$, and terminal payoff condition.

Remarks

1. Closed form analytic price formulas for barrier options with exponential time dependent barrier, $B(\tau) = Be^{-\gamma\tau}$, can also be derived. However, when the barrier level has arbitrary time dependence, the search for analytic price formula for the barrier option fails. Roberts and Shortland (1997) show how to derive the analytic approximation formula by estimating diffusion process boundary hitting times via the Brownian bridge technique.

2. Closed form price formulas for barrier options can also be obtained for other types of diffusion processes followed by the underlying asset price; for example square root constant elasticity of variance process the double exponential jump diffusion process.

3. Assuming $B < X$, the price of a down-and-in call option can be deduced to be

$$c_{di}(S, \tau; X, B) = \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S}, \tau; X\right).$$

4. The method of images can be extended to derive the density functions of restricted multi-state diffusion processes where barrier occurs in only one of the state variables.

5. The monitoring period for breaching of the barrier may be limited to only part of the life of the option.

Transition density function and first passage time density

The realized maximum and minimum value of the asset price from time zero to time t (under continuous monitoring) are defined by

$$\begin{aligned} m_0^t &= \min_{0 \le u \le t} S_u \\ M_0^t &= \max_{0 \le u \le t} S_u, \end{aligned}$$

respectively. The terminal payoffs of various types of barrier options can be expressed in terms of m_0^T and M_0^T. For example, consider the down-and-out call and up-and-out put, their respective terminal payoff can be expressed as

$$\begin{aligned} c_{do}(S_T, T; X, B) &= \max(S_T - X, 0)\mathbf{1}_{\{m_0^T > B\}} \\ p_{uo}(S_T, T; X, B) &= \max(X - S_T, 0)\mathbf{1}_{\{M_0^T < B\}}. \end{aligned}$$

Suppose B is the down-barrier, we define τ_B to be the stopping time at which the underlying asset price crosses the barrier for the first time:

$$\tau_B = \inf\{t | S_t \leq B\}, \quad S_0 = S.$$

Assume $S > B$ and due to path continuity, we may express τ_B (commonly called the *first passage time*) as

$$\tau_B = \inf\{t | S_t = B\}.$$

In a similar manner, if B is the up-barrier and $S < B$, we have

$$\tau_B = \inf\{t | S_t \geq B\} = \inf\{t | S_t = B\}.$$

- It is easily seen that $\{\tau_B > T\}$ and $\{m_0^T > B\}$ are equivalent events if B is a down-barrier. By virtue of the risk neutral valuation principle, the price of a down-and-out call at time zero is given by

$$
\begin{aligned}
c_{do}(S, 0; X, B) &= e^{-rT} E_Q[\max(S_T - X, 0)\mathbf{1}_{\{m_0^T > B\}}] \\
&= e^{-rT} E_Q[(S_T - X)\mathbf{1}_{\{S_T > \max(X,B)\}}\mathbf{1}_{\{\tau_B > T\}}].
\end{aligned}
$$

- The determination of the price function $c_{d_0}(S, 0; X, B)$ requires the determination of the joint distribution function of S_T and m_0^T.

Reflection principle

Let W_t^0 (W_t^μ) denote the Brownian motion that starts at zero, with constant volatility σ and zero drift rate (constant drift rate μ). We would like to find $P[m_0^T < m, W_T^\mu > x]$, where $x \geq m$ and $m \leq 0$. First, we consider the zero-drift Brownian motion W_t^0.

Given that the minimum value m_0^T falls below m, then there exists some time instant $\xi, 0 < \xi < T$, such that ξ is the first time that W_ξ^0 equals m. As Brownian paths are continuous, there exist some times during which $W_t^0 < m$. In other words, W_t^0 decreases at least below m and then increases at least up to level x (higher than m) at time T.

Suppose we define a random process

$$\widetilde{W}_t^0 = \begin{cases} W_t^0 & \text{for } t < \xi \\ 2m - W_t^0 & \text{for } \xi \le t \le T, \end{cases}$$

that is, $\widetilde{W}_t^0$ is the mirror reflection of W_t^0 at the level m within the time interval between ξ and T. It is then obvious that $\{W_T^0 > x\}$ is equivalent to $\{\widetilde{W}_T^0 < 2m - x\}$. Also, the reflection of the Brownian path dictates that

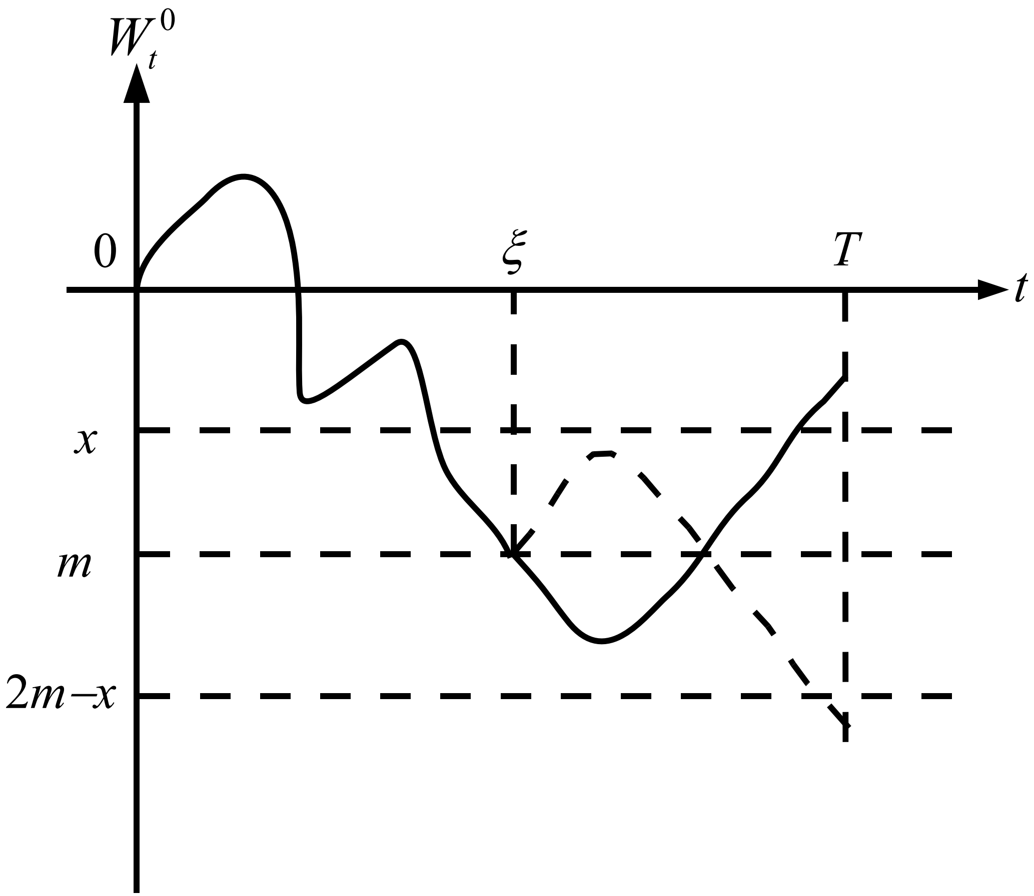
$$\widetilde{W}_{\xi+u}^0 - \widetilde{W}_\xi^0 = -(W_{\xi+u}^0 - W_\xi^0), \quad u > 0.$$

- The stopping time ξ only depends on the path history $\{W_t^0 : 0 \leq t \leq \xi\}$ and it will not affect the Brownian motion at later times.

- By the strong Markov property of Brownian motions, we argue that the two Brownian increments have the same distribution, and the distribution has zero mean and variance $\sigma^2 u$. For every Brownian path that starts at 0, travels at least m units (downward, $m \leq 0$) before T and later travels at least $x - m$ units (upward, $x \geq m$), there is an equally likely path that starts at 0, travels m units (downward, $m \leq 0$) some time before T and travels at least $m - x$ units (further downward, $m \leq x$).

- Suppose $W_T^0 > x$, then $\widetilde{W}_T^0 < 2m - x$,

$$\begin{aligned} P[W_T^0 > x, m_0^T < m] &= P[\widetilde{W}_T^0 < 2m - x] = P[W_T^0 < 2m - x] \\ &= N\left(\frac{2m - x}{\sigma\sqrt{T}}\right), \quad m \leq \min(x, 0). \end{aligned}$$



Pictorial representation of the reflection principle of the Brownian motion W_t^0. The dotted path after time ξ is the mirror reflection of the Brownian path at the level m. Suppose W_T^0 ends up at a value higher than x, then the reflected path at time T has a value lower than $2m - x$.

Next, we apply the Girsanov Theorem to effect the change of measure for finding the above joint distribution when the Brownian motion has non-zero drift. Suppose under the measure Q, W_t^μ is a Brownian motion with drift rate μ. We change the measure from Q to $\widetilde{Q}$ such that W_t^μ becomes a Brownian motion with zero drift under $\widetilde{Q}$. Consider the following joint distribution

$$
\begin{aligned}
& P[W_T^\mu > x, m_0^T < m] \\
= \; & E_Q[\mathbf{1}_{\{W_T^\mu > x\}} \mathbf{1}_{\{m_0^T < m\}}] \\
= \; & E_{\widetilde{Q}}\left[\mathbf{1}_{\{W_T^\mu > x\}} \mathbf{1}_{\{m_0^T < m\}} \exp\left(\frac{\mu W_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)\right],
\end{aligned}
$$

where the Radon-Nikodym derivative term $\exp\left(\frac{\mu W_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)$ is appended.

By applying the reflection principle and observing that W_T^μ is a zero-drift Brownian motion under $\widetilde{Q}$, we obtain

$$
\begin{aligned}
& P[W_T^\mu > x, m_0^T < m] \\
= \;& E_{\widetilde{Q}}\left[\mathbf{1}_{\{2m - W_T^\mu > x\}} \exp\left(\frac{\mu}{\sigma^2}(2m - W_T^\mu) - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\
= \;& e^{\frac{2\mu m}{\sigma^2}} E_{\widetilde{Q}}\left[\mathbf{1}_{\{W_T^\mu < 2m - x\}} \exp\left(-\frac{\mu}{\sigma^2} W_T^\mu - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\
= \;& e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{z^2}{2\sigma^2 T}} e^{-\frac{\mu z}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}} \, dz \\
= \;& e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{(z + \mu T)^2}{2\sigma^2 T}\right) dz \\
= \;& e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right), \quad m \le \min(x, 0).
\end{aligned}
$$

Suppose the Brownian motion W_t^μ has a downstream barrier m over the period $[0,T]$ so that $m_0^T > m$, we would like to derive the joint distribution

$$P[W_T^\mu > x, m_0^T > m], \quad \text{and} \quad m \leq \min(x,0).$$

By applying the law of total probabilities, we obtain

$$\begin{aligned}
& P[W_T^\mu > x, m_0^T > m] \\
= \quad & P[W_T^\mu > x] - P[W_T^\mu > x, m_0^T < m] \\
= \quad & N\left(\frac{-x+\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{2m-x+\mu T}{\sigma\sqrt{T}}\right), \quad m \leq \min(x,0).
\end{aligned}$$

Under the special case $m = x$, since $W_T^\mu > m$ is implicitly implied from $m_0^T > m$, we have

$$P[m_0^T > m] = N\left(\frac{-m+\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m+\mu T}{\sigma\sqrt{T}}\right).$$

Extension to upstream barrier

When the Brownian motion W_t^μ has an upstream barrier M over the period $[0, T]$ so that $M_0^T < M$, the joint distribution function of W_T^μ and M_0^T can be deduced using the following relation between M_0^T and m_0^T:

$$M_0^T = \max_{0 \le t \le T} (\sigma Z_t + \mu t) = - \min_{0 \le t \le T} (-\sigma Z_t - \mu t),$$

where Z_t is the standard Brownian motion. Since $-Z_t$ has the same distribution as Z_t, the distribution of the maximum value of W_t^μ is the same as that of the negative of the minimum value of $W_t^{-\mu}$.

By swapping $-\mu$ for $\mu, -M$ for m and $-y$ for x, we obtain

$$P[W_T^\mu < y, M_0^T > M] = e^{\frac{2\mu M}{\sigma^2}} N\left(\frac{y - 2M - \mu T}{\sigma\sqrt{T}}\right), M \geq \max(y, 0).$$

In a similar manner, we obtain

$$\begin{aligned} & P[W_T^\mu < y, M_0^T < M] \\ = \; & P[W_T^\mu < y] - P[W_T^\mu < y, M_0^T > M] \\ = \; & N\left(\frac{y - \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(\frac{y - 2M - \mu T}{\sigma\sqrt{T}}\right), \quad M \geq \max(y, 0), \end{aligned}$$

and by setting $y = M$, we obtain

$$P[M_0^T < M] = N\left(\frac{M - \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(-\frac{M + \mu T}{\sigma\sqrt{T}}\right).$$

Density functions of restricted Brownian processes

We define $f_{down}(x, m, T)$ to be the density function of W_T^μ with the downstream barrier m, where $m \leq \min(x, 0)$, that is,

$$f_{down}(x, m, T)\, dx = P[W_T^\mu \in dx, m_0^T > m].$$

By differentiating with respect to x and swapping the sign, we obtain

$$\begin{aligned} & f_{down}(x, m, T) \\ = \quad & \frac{1}{\sigma\sqrt{T}} \left[n\left(\frac{x - \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} n\left(\frac{x - 2m - \mu T}{\sigma\sqrt{T}}\right) \right] \mathbf{1}_{\{m \leq \min(x,0)\}}. \end{aligned}$$

Similarly, we define $f_{up}(x, M, T)$ to be the density function of W_T^μ with the upstream barrier M, where $M > \max(y, 0)$, then

$$
\begin{aligned}
& P[W_T^\mu \in dy, M_0^T < M] \\
= \; & f_{up}(y, M, T)\, dy \\
= \; & \frac{1}{\sigma\sqrt{T}} \left[n\left(\frac{y - \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} n\left(\frac{y - 2M - \mu T}{\sigma\sqrt{T}}\right)\right] dy \\
& \mathbf{1}_{\{M > \max(y,0)\}}.
\end{aligned}
$$

Suppose the asset price S_t follows the lognormal process under the risk neutral measure such that $\ln \dfrac{S_t}{S} = W_t^\mu$, where S is the asset price at time zero and the drift rate $\mu = r - \dfrac{\sigma^2}{2}$. Let $\psi(S_T; S, B)$ denote the transition density of the asset price S_T at time T given the asset price S at time zero and conditional on $S_t > B$ for $0 \leq t \leq T$. Here, B is the downstream barrier. By Eq. (4.1.27b), we deduce that $\psi(S_T; S, B)$ is given by

$$\begin{aligned}\psi(S_T; S, B) \;=\; & \frac{1}{\sigma\sqrt{T}S_T}\left[n\left(\frac{\ln\frac{S_T}{S} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right. \\ & \left. - \left(\frac{B}{S}\right)^{\frac{2r}{\sigma^2}-1} n\left(\frac{\ln\frac{S_T}{S} - 2\ln\frac{B}{S} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right].\end{aligned}$$

First passage time density functions

Let $Q(u;m)$ denote the density function of the first passage time at which the downstream barrier m is first hit by the Brownian path W_t^μ, that is, $Q(u;m)\,du = P[\tau_m \in du]$. First, we determine the distribution function $P[\tau_m > u]$ by observing that $\{\tau_m > u\}$ and $\{m_0^u > m\}$ are equivalent events.

$$
\begin{aligned}
P[\tau_m > u] &= P[m_0^u > m] \\
&= N\left(\frac{-m+\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m+\mu u}{\sigma\sqrt{u}}\right).
\end{aligned}
$$

The density function $Q(u;m)$ is then given by

$$
\begin{aligned}
Q(u;m)\,du &= P[\tau_m \in du] \\
&= -\frac{\partial}{\partial u}\left[N\left(\frac{-m+\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m+\mu u}{\sigma\sqrt{u}}\right)\right] du\, \mathbf{1}_{\{m<0\}} \\
&= \frac{-m}{\sqrt{2\pi\sigma^2 u^3}} \exp\left(-\frac{(m-\mu u)^2}{2\sigma^2 u}\right) du\, \mathbf{1}_{\{m<0\}}.
\end{aligned}
$$

Let $Q(u;M)$ denote the first passage time density associated with the upstream barrier M.

$$
\begin{aligned}
Q(u;M) &= -\frac{\partial}{\partial u}\left[N\left(\frac{M-\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(-\frac{M+\mu u}{\sigma\sqrt{u}}\right)\right] \mathbf{1}_{\{M>0\}} \\
&= \frac{M}{\sqrt{2\pi\sigma^2 u^3}} \exp\left(-\frac{(M-\mu u)^2}{2\sigma^2 u}\right) \mathbf{1}_{\{M>0\}}.
\end{aligned}
$$

We write B as the barrier, either upstream or downstream. When the barrier is downstream (upstream), we have $\ln \dfrac{B}{S} < 0 \left(\ln \dfrac{B}{S} > 0 \right)$.

$$Q(u; B) = \frac{\left| \ln \frac{B}{S} \right|}{\sqrt{2\pi\sigma^2 u^3}} \exp\left(-\frac{\left[\ln \frac{B}{S} - \left(r - \frac{\sigma^2}{2} \right) u \right]^2}{2\sigma^2 u} \right).$$

Suppose a rebate $R(t)$ is paid to the option holder upon breaching the barrier at level B by the asset price path at time t. Since the expected rebate payment over the time interval $[u, u + du]$ is given by $R(u)Q(u; B)\, du$, then the expected present value of the rebate is given by

$$\text{rebate value } = \int_0^T e^{-ru} R(u) Q(u; B)\, du.$$

When $R(t) = R_0$, a constant value, direct integration of the above integral gives

$$\begin{aligned}\text{rebate value} \quad = \quad & R_0 \left[\left(\frac{B}{S}\right)^{\alpha_+} N\left(\delta \frac{\ln \frac{B}{S} + \beta T}{\sigma\sqrt{T}}\right)\right. \\ & \left. + \left(\frac{B}{S}\right)^{\alpha_-} N\left(\delta \frac{\ln \frac{B}{S} - \beta T}{\sigma\sqrt{T}}\right)\right],\end{aligned}$$

where

$$\beta = \sqrt{\left(r - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}, \quad \alpha_\pm = \frac{r - \frac{\sigma^2}{2} \pm \beta}{\sigma^2}$$

$$\text{and} \quad \delta = \text{sign}\left(\ln \frac{S}{B}\right).$$

Here, δ is a binary variable indicating whether the barrier is downstream ($\delta = 1$) or upstream ($\delta = -1$).

Correction formula for discretely monitored barrier options

Let $V(B;m)$ be the price of a discretely monitored knock-in or knock-out down call or up put option with constant barrier B. Let $V(B)$ be the price of the corresponding continuously monitored barrier option. We have

$$V(B;m) = V(Be^{\pm\beta\sigma\sqrt{\delta t}}) + o\left(\frac{1}{\sqrt{m}}\right),$$

where $\beta = -\xi\left(\frac{1}{2}\right) \Big/ \sqrt{2\pi} \approx 0.5826$, ξ is the Riemann zeta function, σ is the volatility.

The "$+$" sign is chosen when $B > S$, while the "$-$" sign is chosen when $B < S$.

One observes that the correction shifts the barrier away from the current underlying asset price by a factor of $e^{\beta\sigma\sqrt{\delta t}}$.

Lookback options

Let T denote the time of expiration of the option and $[T_0, T]$ be the lookback period. We denote the minimum value and maximum value of the asset price realized from T_0 to the current time t ($T_0 \leq t \leq T$) by

$$m_{T_0}^t = \min_{T_0 \leq \xi \leq t} S_\xi$$

and

$$M_{T_0}^t = \max_{T_0 \leq \xi \leq t} S_\xi$$

- A floating strike lookback call gives the holder the right to buy at the lowest realized price while a floating strike lookback put allows the holder to sell at the highest realized price over the lookback period. Since $S_T \geq m_{T_0}^T$ and $M_{T_0}^T \geq S_T$ so that the holder of a floating strike lookback option always exercise the option. Hence, the respective terminal payoff of the lookback call and put are given by $S_T - m_{T_0}^T$ and $M_{T_0}^T - S_T$.

- A fixed strike lookback call (put) is a call (put) option on the maximum (minimum) realized price. The respective terminal payoff of the fixed strike lookback call and put are $\max(M_{T_0}^T - X, 0)$ and $\max(X - m_{T_0}^T, 0)$, where X is the strike price.

- Under the risk neutral measure, the process for the stochastic variable $U_\xi = \ln \dfrac{S_\xi}{S}$ is a Brownian process with drift rate $\mu = r - \dfrac{\sigma^2}{2}$ and variance rate σ^2, where r is the riskless interest rate and S is the asset price at current time t (dropping subscript t for brevity).

We define the following stochastic variables

$$
\begin{aligned}
y_T &= \ln \frac{m_t^T}{S} = \min\{U_\xi, \xi \in [t,T]\} \\
Y_T &= \ln \frac{M_t^T}{S} = \max\{U_\xi, \xi \in [t,T]\},
\end{aligned}
$$

and write $\tau = T - t$. For $y \leq 0$ and $y \leq u$, we can deduce the following joint distribution function of U_T and y_T from the transition density function of the Brownian process with the presence of a downstream barrier

$$
P[U_T \geq u, y_T \geq y] = N\left(\frac{-u + \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-u + 2y + \mu\tau}{\sigma\sqrt{\tau}}\right).
$$

Here, U_ξ is visualized as a restricted Brownian process with drift rate μ and downstream absorbing barrier y.

Similarly, for $y \geq 0$ and $y \geq u$, the corresponding joint distribution function of U_T and Y_T is given by

$$P[U_T \leq u, Y_T \leq y] = N\left(\frac{u - \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{u - 2y - \mu\tau}{\sigma\sqrt{\tau}}\right).$$

By taking $y = u$ in the above two joint distribution functions, we obtain the distribution functions for y_T and Y_T

$$\begin{aligned} P(y_T \geq y) &= N\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right), \quad y \leq 0, \\ P(Y_T \leq y) &= N\left(\frac{y - \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right), \quad y \geq 0. \end{aligned}$$

The density functions of y_T and Y_T can be obtained by differentiating the above distribution functions.

European fixed strike lookback options

Consider a European fixed strike lookback call option whose terminal payoff is $\max(M_{T_0}^T - X, 0)$. The value of this lookback call option at the current time t is given by

$$c_{fix}(S, M, t) = e^{-r\tau} E\left[\max(\max(M, M_t^T) - X, 0)\right],$$

where $S_t = S, M_{T_0}^t = M$ and $\tau = T - t$, and the expectation is taken under the risk neutral measure. The payoff function can be simplified into the following forms, depending on $M \leq X$ or $M > X$:

(i) $M \leq X$

$$\max(\max(M, M_t^T) - X, 0) = \max(M_t^T - X, 0)$$

(ii) $M > X$

$$\max(\max(M, M_t^T) - X, 0) = (M - X) + \max(M_t^T - M, 0).$$

Define the function H by

$$H(S,\tau;K) = e^{-r\tau}E[\max(M_t^T - K, 0)],$$

where K is a positive constant. Once $H(S,\tau;K)$ is determined, then

$$\begin{aligned} c_{fix}(S,M,\tau) &= \begin{cases} H(S,\tau;X) & \text{if } M \leq X \\ e^{-r\tau}(M-X) + H(S,\tau;M) & \text{if } M > X \end{cases} \\ &= e^{-r\tau}\max(M-X,0) + H(S,\tau;\max(M,X)). \end{aligned}$$

- $c_{fix}(S,M,\tau)$ is independent of M when $M \leq X$ because the terminal payoff is independent of M when $M \leq X$.
- When $M > X$, the terminal payoff is guaranteed to have the floor value $M - X$. If we subtract the present value of this guaranteed floor value, then the remaining value of the fixed strike call option is equal to a new fixed strike call but with the strike being increased from X to M.

Since $\max(M_t^T - K, 0)$ is a non-negative random variable, its expected value is given by the integral of the tail probabilities where

$$
\begin{aligned}
& e^{-r\tau} E[\max(M_t^T - K, 0)] \\
= \quad & e^{-r\tau} \int_0^\infty P[M_t^T - K \geq x] \, dx \\
= \quad & e^{-r\tau} \int_K^\infty P\left[\ln \frac{M_t^T}{S} \geq \ln \frac{z}{S}\right] dz \quad z = x + K \\
= \quad & e^{-r\tau} \int_{\ln \frac{K}{S}}^\infty S e^y P[Y_T \geq y] \, dy \qquad y = \ln \frac{z}{S} \\
= \quad & e^{-r\tau} \int_{\ln \frac{K}{S}}^\infty S e^y \left[N\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) + e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right)\right] dy,
\end{aligned}
$$

$$\begin{aligned} H(S,\tau;K) &= SN(d) - e^{-r\tau}KN(d-\sigma\sqrt{\tau}) \\ &\quad + e^{-r\tau}\frac{\sigma^2}{2r}S\left[e^{r\tau}N(d) - \left(\frac{S}{K}\right)^{-\frac{2r}{\sigma^2}} N\left(d - \frac{2r}{\sigma}\sqrt{\tau}\right)\right], \end{aligned}$$

where

$$d = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

The European fixed strike lookback put option with terminal payoff $\max(X - m_{T_0}^T, 0)$ can be priced in a similar manner. Write $m = m_{T_0}^t$ and define the function

$$h(S,\tau;K) = e^{-r\tau}E[\max(K - m_t^T, 0)].$$

The value of this lookback put can be expressed as

$$p_{fix}(S,m,\tau) = e^{-r\tau}\,\mathrm{max}(X-m,0) + h(S,\tau;\mathrm{min}(m,X)),$$

where

$$\begin{aligned}
h(S,\tau;K) &= e^{-r\tau}\int_0^{\infty} P[\mathrm{max}(K-m_t^T,0)\geq x]\,dx \\
&= e^{-r\tau}\int_0^{K} P[K-m_t^T\geq x]\,dx \quad \text{since } 0\leq \mathrm{max}(K-m_t^T,0)\leq K \\
&= e^{-r\tau}\int_0^{K} P[m_t^T\leq z]\,dz \qquad z=K-x \\
&= e^{-r\tau}\int_0^{\ln\frac{K}{S}} Se^y P[y_T\leq y]\,dy \qquad y=\ln\frac{y}{S} \\
&= e^{-r\tau}\int_0^{\ln\frac{K}{S}} Se^y \left[N\left(\frac{y-\mu\tau}{\sigma\sqrt{\tau}}\right)+e^{\frac{2\mu y}{\sigma^2}}N\left(\frac{y+\mu\tau}{\sigma\sqrt{\tau}}\right)\right]dy \\
&= e^{-r\tau}KN(-d+\sigma\sqrt{\tau})-SN(-d)+e^{-r\tau}\frac{\sigma^2}{2r}S \\
&\quad \left[\left(\frac{S}{K}\right)^{-2r/\sigma^2}N\left(-d+\frac{2r}{\sigma}\sqrt{\tau}\right)-e^{r\tau}N(-d)\right].
\end{aligned}$$

European floating strike lookback options

By exploring the pricing relations between the fixed and floating lookback options, we can deduce the price functions of floating strike lookback options from those of fixed strike options. Consider a European floating strike lookback call option whose terminal payoff is $S_T - m_{T_0}^T$, the present value of this call option is given by

$$\begin{aligned}
c_{f\ell}(S, m, \tau) &= e^{-r\tau} E[S_T - \min(m, m_t^T)] \\
&= e^{-r\tau} E[(S_T - m) + \max(m - m_t^T, 0)] \\
&= S - me^{-r\tau} + h(S, \tau; m) \\
&= SN(d_m) - \ e^{-r\tau} mN(d_m - \sigma\sqrt{\tau}) + e^{-r\tau} \frac{\sigma^2}{2r} S \\
&\qquad \left[\left(\frac{S}{m} \right)^{-\frac{2r}{\sigma^2}} N\left(-d_m + \frac{2r}{\sigma}\sqrt{\tau} \right) - \ e^{r\tau} N(-d_m) \right],
\end{aligned}$$

where

$$d_m = \frac{\ln \frac{S}{m} + \left(r + \frac{\sigma^2}{2} \right) \tau}{\sigma\sqrt{\tau}}.$$

In a similar manner, consider a European floating strike lookback put option whose terminal payoff is $M_{T_0}^T - S_T$, the present value of this put option is given by

$$
\begin{aligned}
p_{f\ell}(S, M, \tau) &= e^{-r\tau} E[\max(M, M_t^T) - S_T] \\
&= e^{-r\tau} E[\max(M_t^T - M, 0) - (S_T - M)] \\
&= H(S, \tau; M) - (S - Me^{-r\tau}) \\
&= e^{-r\tau} MN(-d_M + \sigma\sqrt{\tau}) - SN(-d_M) + \ e^{-r\tau}\frac{\sigma^2}{2r}S \\
&\qquad \left[e^{r\tau}N(d_M) - \left(\frac{S}{M}\right)^{-\frac{2r}{\sigma^2}} N\left(d_M - \frac{2r}{\sigma}\sqrt{\tau}\right)\right],
\end{aligned}
$$

where

$$
d_M = \frac{\ln \frac{S}{M} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.
$$

Boundary condition at $S = m$

- Consider the particular situation when $S = m$, that is, the current asset price happens to be at the minimum value realized so far. The probability that the current minimum value remains to be the realized minimum value at expiration is expected to be zero.

- We can argue that the value of the floating strike lookback call should be insensitive to infinitesimal changes in m since the change in option value with respect to marginal changes in m is proportional to the probability that m will be the realized minimum at expiry

$$\left.\frac{\partial c_{f\ell}}{\partial m}(S, m, \tau)\right|_{S=m} = 0.$$

Rollover strategy and strike bonus premium

- The sum of the first two terms in $c_{f\ell}$ can be seen as the price function of a European vanilla call with strike price m, while the third term can be interpreted as the strike bonus premium.

- We consider the hedging of the floating strike lookback call by the following rollover strategy. At any time, we hold a European vanilla call with the strike price set at the current realized minimum asset value. In order to replicate the payoff of the floating strike lookback call at expiry, whenever a new realized minimum value of the asset price is established at a later time, one should sell the original call option and buy a new call with the same expiration date but with the strike price set equal to the newly established minimum value.

- Since the call with a lower strike is always more expensive, an extra premium is required to adopt the rollover strategy. The sum of these expected costs of rollover is termed the strike bonus premium.

- We would like to show how the strike bonus premium can be obtained by integrating a joint probability distribution function involving m_t^T and S_T. First, we observe that

$$\begin{aligned}\text{strike bonus premium} &= h(S,\tau;m) + S - me^{-r\tau} - c_E(S,\tau;m) \\ &= h(S,\tau;m) - p_E(S,\tau;m),\end{aligned}$$

  where $c_E(S,\tau,m)$ and $p_E(S,\tau;m)$ are the price functions of European vanilla call and put, respectively. The last result is due to put-call parity relation.

Recall

$$h(S,\tau;m) = e^{-r\tau}\int_0^m P[m_t^T \leq \xi]\,d\xi$$

and

$$\begin{aligned} p_E(S,\tau;m) &= e^{-r\tau}\int_0^\infty P[\max(m - S_T, 0) \geq x]\,dx \\ &= e^{-r\tau}\int_0^m P[S_T \leq \xi]\,d\xi. \end{aligned}$$

Since the two stochastic state variables satisfies $0 \leq m_t^T \leq S_T$, we have

$$P[m_t^T \leq \xi] - P[S_T \leq \xi] = P[m_t^T \leq \xi < S_T]$$

so that

$$\text{strike bonus premium } = e^{-r\tau}\int_0^m P[m_t^T \leq \xi \leq S_T]\,d\xi.$$

Differential equation formulation

We would like to illustrate how to derive the governing partial differential equation and the associated auxiliary conditions for the European floating strike lookback put option. First, we define the quantity

$$M_n = \left[\int_{T_0}^{t} (S_\xi)^n d\xi \right]^{1/n}, \quad t > T_0,$$

the derivative of which is given by

$$dM_n = \frac{1}{n} \frac{S^n}{(M_n)^{n-1}} dt$$

so that dM_n is deterministic. Taking the limit $n \to \infty$, we obtain

$$M = \lim_{n\to\infty} M_n = \max_{T_0 \le \xi \le t} S_\xi,$$

giving the realized maximum value of the asset price process over the lookback period $[T_0, t]$.

- We attempt to construct a hedged portfolio which contains one unit of a put option whose payoff depends on M_n and $-\triangle$ units of the underlying asset. Again, we choose $\triangle$ so that the stochastic components associated with the option and the underlying asset cancel.

- Let $p(S, M_n, t)$ denote the value of the lookback put option and let Π denote the value of the above portfolio. We then have

$$\Pi = p(S, M_n, t) - \triangle S.$$

- The dynamics of the portfolio value is given by

$$d\Pi = \frac{\partial p}{\partial t}dt + \frac{1}{n}\frac{S^n}{(M_n)^{n-1}}\frac{\partial p}{\partial M_n}dt + \frac{\partial p}{\partial S}dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 p}{\partial S^2}dt - \triangle dS$$

by virtue of Ito's lemma. Again, we choose $\triangle = \dfrac{\partial p}{\partial S}$ so that the stochastic terms cancel.

- Using usual no-arbitrage argument, the non-stochastic portfolio should earn the riskless interest rate so that

$$d\Pi = r\Pi\, dt,$$

where r is the riskless interest rate. Putting all equations together,

$$\frac{\partial p}{\partial t} + \frac{1}{n}\frac{S^n}{(M_n)^{n-1}}\frac{\partial p}{\partial M_n} + \frac{\sigma^2}{2}S^2\frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} - rp = 0.$$

- Now, we take the limit $n \to \infty$ and note that $S \leq M$. When $S < M$, $\lim_{n\to\infty} \frac{1}{n}\frac{S^n}{(M_n)^{n-1}} = 0$; and when $S = M$, $\frac{\partial p}{\partial M} = 0$. Hence, the second term becomes zero as $n \to \infty$.

- The governing equation for the floating strike lookback put is given by

$$\frac{\partial p}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} - rp = 0, \quad 0 < S < M, t > T_0$$

The domain of the pricing model has an upper bound M on S. The variable M does not appear in the equation, though M appears as a parameter in the auxiliary conditions. The final condition is

$$p(S, M, T) = M - S.$$

In this European floating strike lookback put option, the boundary conditions are applied at $S = 0$ and $S = M$. Once S becomes zero, it stays at the zero value at all subsequent times and the payoff at expiry is certain to be M.

Discounting at the riskless interest rate, the lookback put value at the current time t is

$$p(0, M, t) = e^{-r(T-t)}M.$$

The boundary condition at the other end $S = M$ is given by

$$\frac{\partial p}{\partial M} = 0 \quad \text{at} \quad S = M.$$

Asian options

- Asian options are averaging options whose terminal payoff depends on some form of averaging of the price of the underlying asset over a part or the whole of option's life.

- There are frequent situations where traders may be interested to hedge against the average price of a commodity over a period rather than, say, end-of-period price.

- Averaging options are particularly useful for business involved in trading on thinly-traded commodities. The use of such financial instruments may avoid the price manipulation near the end of the period.

The most common averaging procedures are the discrete arithmetic averaging defined by

$$A_T = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}$$

and the discrete geometric averaging defined by

$$A_T = \left[\prod_{i=1}^{n} S_{t_i}\right]^{1/n}.$$

Here, S_{t_i} is the asset price at discrete time $t_i, i = 1, 2, \cdots, n$.

In the limit $n \to \infty$, the discrete sampled averages become the continuous sampled averages. The continuous arithmetic average is given by

$$A_T = \frac{1}{T}\int_0^T S_t\,dt,$$

while the continuous geometric average is defined to be

$$A_T = \exp\left(\frac{1}{T}\int_0^T \ln S_t\,dt\right).$$

Partial differential equation formulation

Suppose we write the average of the asset price as

$$A = \int_0^t f(S,u)\,du,$$

where $f(S,t)$ is chosen according to the type of average adopted in the Asian option. For example, $f(S,t) = \frac{1}{t}S$ corresponds to continuous arithmetic average, $f(S,t) = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\delta(t-t_i)\ln S\right)$ corresponds to discrete geometric average, etc.

Suppose $f(S,t)$ is a continuous time function, then by the mean value theorem

$$dA = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} f(S,u)\,du \;=\; \lim_{\Delta t \to 0} f(S,u^*)\,dt = f(S,t)\,dt,$$
$$t < u^* < t + \Delta t,$$

so dA is deterministic. Hence, a riskless hedge for the Asian option requires only eliminating the asset-induced risk.

Consider a portfolio which contains one unit of the Asian option and $-\Delta$ units of the underlying asset. We then choose Δ such that the stochastic components associated with the option and the underlying asset cancel off each other.

Assume the asset price dynamics to be given by

$$dS = [\mu S - D(S,t)]\, dt + \sigma S\, dZ,$$

where Z is the standard Wiener process, $D(S,t)$ is the dividend yield on the asset, μ and σ are the expected rate of return and volatility of the asset price, respectively. Let $V(S,A,t)$ denote the value of the Asian option and let Π denote the value of the above portfolio. The portfolio value is given by

$$\Pi = V(S,A,t) - \Delta S,$$

and its differential is found to be

$$d\Pi = \frac{\partial V}{\partial t}\, dt + f(S,t)\frac{\partial V}{\partial A}\, dt + \frac{\partial V}{\partial S}\, dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}\, dt - \Delta\, dS - \Delta D(S,t)\, dt.$$

The last term in the above equation corresponds to the contribution of the dividend from the asset to the portfolio's value. As usual, we choose $\Delta = \dfrac{\partial V}{\partial S}$ so that the stochastic terms containing dS cancel. The absence of arbitrage dictates

$$d\Pi = r\Pi \; dt,$$

where r is the riskless interest rate. Putting the results together, we obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + [rS - D(S,t)] \frac{\partial V}{\partial S} + f(S,t) \frac{\partial V}{\partial A} - rV = 0.$$

The equation is a degenerate diffusion equation since it contains diffusion term corresponding to S only but not for A. The auxiliary conditions in the pricing model depend on the specific details of the Asian option contract.

Continuously monitored geometric averaging options

- We take time zero to be the time of initiation of the averaging period, t is the current time and T denotes the expiration time.

- We define the continuously monitored geometric averaging of the asset price S_u over the time period $[0, t]$ by

$$G_t = \exp\left(\frac{1}{t}\int_0^t \ln S_u \, du\right).$$

  The terminal payoff of the fixed strike call option and floating strike call option are, respectively, given by

$$\begin{aligned} c_{fix}(S_T, G_T, T; X) &= \max(G_T - X, 0) \\ c_{f\ell}(S_T, G_T, T) &= \max(S_T - G_T, 0), \end{aligned}$$

  where X is the fixed strike price.

European fixed strike Asian call option

We assume the existence of a risk neutral pricing measure Q under which discounted asset prices are martingales, implying the absence of arbitrage. Under the measure Q, the asset price follows

$$\frac{dS_t}{S_t} = (r-q)\,dt + \sigma\,dZ_t,$$

where Z_t is a Q-Brownian motion. For $0 < t < T$, the solution of the above stochastic differential equation is given by

$$\ln S_T = \ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)(T-t) + \sigma(Z_T - Z_t).$$

By integrating $\ln S_t$, we obtain

$$\begin{aligned}\ln G_T &= \frac{t}{T}\ln G_t + \frac{1}{T}\left[(T-t)\ln S_t + \left(r-q-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2}\right] \\ &\quad + \frac{\sigma}{T}\int_t^T (Z_u - Z_t)\,du.\end{aligned}$$

The stochastic term $\dfrac{\sigma}{T}\displaystyle\int_t^T (Z_u - Z_t)\,du$ can be shown to be Gaussian with zero mean and variance $\dfrac{\sigma^2}{T^2}\dfrac{(T-t)^3}{3}$. By the risk neutral valuation principle, the value of the European fixed strike Asian call option is given by

$$c_{fix}(S_t, G_t, t) = e^{-r(T-t)} E[\max(G_T - X, 0)],$$

where E is the expectation under Q conditional on $S_t = S, G_t = G$. We assume the current time t to be within the averaging period.

By defining

$$\overline{\mu} = \left(r - q - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T} \quad \text{and} \quad \overline{\sigma} = \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}},$$

G_T can be written as

$$G_T = G_t^{t/T} S_t^{(T-t)/T} \exp(\overline{\mu} + \overline{\sigma}\widehat{Z}),$$

where $\widehat{Z}$ is the standard normal random variable.

Recall

$$
\begin{aligned}
& E[\max(F\exp(\overline{\mu}+\overline{\sigma}\widehat{Z})-X,0] \\
= \quad & Fe^{\overline{\mu}+\overline{\sigma}^2/2}N\left(\frac{\ln\frac{F}{X}+\overline{\mu}+\overline{\sigma}^2}{\overline{\sigma}}\right)-XN\left(\frac{\ln\frac{F}{X}+\overline{\mu}}{\overline{\sigma}}\right),
\end{aligned}
$$

we then deduce that

$$
c_{fix}(S,G,t) = e^{-r(T-t)}\left[G^{t/T}S^{(T-t)/T}e^{\overline{\mu}+\overline{\sigma}^2/2}N(d_1)-XN(d_2)\right],
$$

where

$$
\begin{aligned}
d_2 &= \left(\frac{t}{T}\ln G+\frac{T-t}{T}\ln S+\overline{\mu}-\ln X\right)\Big/\overline{\sigma}, \\
d_1 &= d_1+\overline{\sigma}.
\end{aligned}
$$

European floating strike Asian call option

Since the terminal payoff of the floating strike Asian call option involves S_T and G_T, pricing by the risk neutral expectation approach would require the joint distribution of S_T and G_T. For floating strike Asian options, the partial differential equation method provides the more effective approach to derive the price formula for $c_{f\ell}(S,G,t)$. This is because the similarity reduction technique can be applied to reduce the dimension of the differential equation.

When continuously monitored geometric averaging is adopted, the governing equation for $c_{f\ell}(S,G,t)$ can be expressed as

$$\frac{\partial c_{f\ell}}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c_{f\ell}}{\partial S^2} + (r-q)S\frac{\partial c_{f\ell}}{\partial S} + \frac{G}{t}\ln\frac{S}{G}\frac{\partial c_{f\ell}}{\partial G} - rc_{f\ell} = 0, \quad 0 < t < T.$$

Next, we define the similarity variables

$$y = t \ln \frac{G}{S} \quad \text{and} \quad W(y,t) = \frac{c_{f\ell}(S,G,t)}{S}.$$

This is equivalent to choose S as the numeraire. In terms of the similarity variables, the governing equation for $c_{f\ell}(S,G,t)$ becomes

$$\frac{\partial W}{\partial t} + \frac{\sigma^2 t^2}{2}\frac{\partial^2 W}{\partial y^2} - \left(r - q + \frac{\sigma^2}{2}\right) t \frac{\partial W}{\partial y} - qW = 0, \quad 0 < t < T,$$

with terminal condition: $W(y,T) = \max(1 - e^{y/T}, 0)$.

We write $\tau = T - t$ and let $F(y, \tau; \eta)$ denote the fundamental solution to the following parabolic equation with time dependent coefficients

$$\frac{\partial F}{\partial \tau} = \frac{\sigma^2 (T-\tau)^2}{2} \frac{\partial^2 F}{\partial y^2} - \left(r - q + \frac{\sigma^2}{2} \right) (T - \tau) \frac{\partial F}{\partial y}, \quad \tau > 0,$$

with initial condition at $\tau = 0$ (corresponding to $t = T$) given as

$$F(y, 0; \eta) = \delta(y - \eta).$$

Though the differential equation has time dependent coefficients, the fundamental solution is readily found to be

$$F(y,\tau;\eta) = n\left(\frac{y-\eta-\left(r-q+\frac{\sigma^2}{2}\right)\int_0^\tau (T-u)\,du}{\sigma\sqrt{\int_0^\tau (T-u)^2\,du}}\right).$$

The solution to $W(y,\tau)$ is then given by

$$W(y,\tau) = e^{-q\tau}\int_{-\infty}^{\infty} \max(1-e^{\eta/T},0)F(y,\tau;\eta)\,d\eta.$$

The direct integration of the above integral gives

$$c_{f\ell}(S,G,t) = \; Se^{-q(T-t)}N(\widehat{d}_1) - G^{t/T}S^{(T-t)/T}e^{-q(T-t)}e^{-\widehat{Q}}N(\widehat{d}_2),$$

where

$$\begin{aligned}
\widehat{d}_1 &= \frac{t \ln \frac{S}{G} + \left(r - q + \frac{\sigma^2}{2}\right)\frac{T^2-t^2}{2}}{\sigma\sqrt{\frac{T^3-t^3}{3}}}, \quad \widehat{d}_2 = \widehat{d}_1 - \frac{\sigma}{T}\sqrt{\frac{T^3-t^3}{3}}, \\
\widehat{Q} &= \frac{r - q + \frac{\sigma^2}{2}}{2}\frac{T^2-t^2}{T} - \frac{\sigma^2}{6}\frac{T^3-t^3}{T^2}.
\end{aligned}$$

Continuously monitored arithmetic averaging options

We consider a European fixed strike European Asian call based on continuously monitored arithmetic averaging. The terminal payoff is defined by

$$c_{fix}(S_T, A_T, T; X) = \max(A_T - X, 0).$$

To motivate the choice of variable transformation, we consider the following expectation representation of the price of the Asian call at time t

$$\begin{aligned} c_{fix}(S_t, A_t, t) &= e^{-r(T-t)} E\left[\max(A_T - X, 0)\right] \\ &= e^{-r(T-t)} E\left[\max\left(\frac{1}{T}\int_0^t S_u\,du - X + \frac{1}{T}\int_t^T S_u\,du, 0\right)\right] \\ &= \frac{S_t}{T} e^{-r(T-t)} E\left[\max\left(x_t + \int_t^T \frac{S_u}{S_t}\,du, 0\right)\right], \end{aligned}$$

where the state variable x_t is defined by

$$x_t = \frac{1}{S_t}(I_t - XT), \qquad I_t = \int_0^t S_u\,du = tA_t.$$

In subsequent exposition, it is more convenient to use I_t instead of A_t as the averaging state variable. Since $S_u/S_t, u > t$, is independent of the history of the asset price up to time t, one argues that the conditional expectation is a function of x_t only. We then deduce that

$$c_{fix}(S_t, I_t, t) = S_t f(x_t, t)$$

for some function of f. In other words, $f(x_t, t)$ is given by

$$f(x_t, t) = \frac{e^{-r(T-t)}}{T} E\left[\max\left(x_t + \int_t^T \frac{S_u}{S_t}\, du, 0\right)\right].$$

If we write the price function of the fixed strike call as $c_{fix}(S, I, t)$, then the governing equation for $c_{fix}(S, I, t)$ is given by

$$\frac{\partial c_{fix}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c_{fix}}{\partial S^2} + (r-q) S \frac{\partial c_{fix}}{\partial S} + S \frac{\partial c_{fix}}{\partial I} - r c_{fix} = 0.$$

Suppose we define the following transformation of variables:

$$x = \frac{1}{S}(I - XT) \quad \text{and} \quad f(x,t) = \frac{c_{fix}(S,I,t)}{S},$$

then the governing differential equation for $f(x,t)$ becomes

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2} + [1-(r-q)x]\frac{\partial f}{\partial x} - qf = 0, \quad -\infty < x < \infty, t > 0.$$

The terminal condition is given by

$$f(x,T) = \frac{1}{T}\max(x,0).$$

When $x_t \geq 0$, which corresponds to $\frac{1}{T}\int_0^t S_u\,du \geq X$, it is possible to find closed form analytic solution to $f(x,t)$. Since x_t is an increasing function of t so that $x_T \geq 0$, the terminal condition $f(x,T)$ reduces to x/T. In this case, $f(x,t)$ admits solution of the form

$$f(x,t) = a(t)x + b(t).$$

By substituting the assumed form of solution into the governing equation, we obtain the following pair of governing equations for $a(t)$ and $b(t)$:

$$\begin{aligned} &\frac{da(t)}{dt} - ra(t) = 0, && a(T) = \frac{1}{T}, \\ &\frac{db(t)}{dt} - a(t) - qb(t) = 0, && b(T) = 0. \end{aligned}$$

When $r \neq q, a(t)$ and $b(t)$ are found to be

$$a(t) = \frac{e^{-r(T-t)}}{T} \quad \text{and} \quad b(t) = \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)}.$$

Hence, the option value for $I \geq XT$ is given by

$$c_{fix}(S, I, t) = \left(\frac{I}{T} - X\right) e^{-r(T-t)} + \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)} S.$$

Though the volatility σ does not appear explicitly in the above price formula, it appears implicitly in S and A. The gamma is easily seen to be zero while the delta is a function of t and $T - t$ but not S or A.

For $I < XT$, there is no closed form analytic solution available.

Put-call parity relation

Let $c_{fix}(S, I, t)$ and $p_{fix}(S, I, t)$ denote the price function of the fixed strike arithmetic averaging Asian call option and put option, respectively. Their terminal payoff functions are given by

$$c_{fix}(S, I, T) = \max\left(\frac{I}{T} - X, 0\right)$$

$$p_{fix}(S, I, T) = \max\left(X - \frac{I}{T}, 0\right),$$

where $I = \int_0^T S_u \, du$. Let $D(S, I, t)$ denote the difference of c_{fix} and p_{fix}. Since both c_{fix} and p_{fix} are governed by the same equation so does $D(S, I, t)$.

The terminal condition of $D(S,I,t)$ is given by

$$D(S,I,T) = \max\left(\frac{I}{T} - X, 0\right) - \max\left(X - \frac{I}{T}, 0\right) = \frac{I}{T} - X.$$

The terminal condition $D(S,I,T)$ is the same as that of the continuously monitored arithmetic averaging option with $I \geq XT$. Hence, when $r \neq q$, the put-call parity relation between the prices of fixed strike Asian options under continuously monitored arithmetic averaging is given by

$$\begin{aligned} & c_{fix}(S,I,t) - p_{fix}(S,I,t) \\ = \; & \left(\frac{I}{T} - X\right) e^{-r(T-t)} + \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)} S. \end{aligned}$$

Similar techniques can be used to derive the put-call parity relations between other types of Asian options (floating/fixed strike and geometric/arithmetic averaging).

Fixed-floating symmetry relations

By applying a change of measure and identifying a time-reversal of a Brownian motion, it is possible to establish the symmetry relations between the prices of floating strike and fixed strike arithmetic averaging Asian options at the start of the averaging period.

Suppose we write the price functions of various continuously monitored arithmetic averaging option at the start of the averaging period (taken to be time zero) as

$$
\begin{aligned}
c_{fl}(S_0, \lambda, r, q, T) &= e^{-rT} E\left[\max\left(\lambda S_T - A_T, 0\right)\right] \\
p_{fl}(S_0, \lambda, r, q, T) &= e^{-rT} E\left[\max\left(A_T - \lambda S_T, 0\right)\right] \\
c_{fix}(X, S_0, r, q, T) &= e^{-rT} E\left[\max\left(A_T - X, 0\right)\right] \\
p_{fix}(X, S_0, r, q, T) &= e^{-rT} E\left[\max\left(X - A_T, 0\right)\right].
\end{aligned}
$$

Assume that the asset price S_t follows the Geometric Brownian process under the risk neutral measure Q, where

$$\frac{dS_t}{S_t} = (r - q)\,dt + \sigma\,dZ_t.$$

Here, Z_t is a Q-Brownian process. Suppose the asset price is used as the numeraire, then

$$\begin{aligned} c^*_{fl} = \frac{c_{fl}}{S_0} &= \frac{e^{-rT}}{S_0} E\left[\max\left(\lambda S_T - A_T, 0\right)\right] \\ &= E\left[\frac{S_T e^{-rT}}{S_0} \frac{\max\left(\lambda S_T - A_T, 0\right)}{S_T}\right]. \end{aligned}$$

To effect the change of numeraire, we define the measure Q^* by

$$\frac{dQ^*}{dQ} = e^{-\frac{\sigma^2}{2}T + \sigma Z_T} = \frac{S_T e^{-rT}}{S_0 e^{-qT}}.$$

By virtue of the Girsanov Theorem, $Z_t^* = Z_t - \sigma t$ is a Q^*-Brownian process. If we write $A_T^* = A_T/S_T$, then

$$c_{fl}^* = e^{-qT} E^* \left[\max\left(\lambda - A_T^*, 0\right)\right],$$

where E^* denotes the expectation under Q^*. Now, we consider

$$A_T^* = \frac{1}{T}\int_0^T \frac{S_u}{S_T}\, du = \frac{1}{T}\int_0^T S_u^*(T)\, du,$$

where

$$S_u^*(T) = \exp\left(-\left(r - q - \frac{\sigma^2}{2}\right)(T-u) - \sigma(Z_T - Z_u)\right).$$

In terms of the Q^*-Brownian process Z_t^*, where $Z_T - Z_u = \sigma(T - u) + Z_T^* - Z_u^*$, we can write

$$S_u^*(T) = \exp\left(\left(r - q + \frac{\sigma^2}{2}\right)(u - T) + \sigma(Z_u^* - Z_T^*)\right).$$

Furthermore, we define a reflected Q^*-Brownian process starting at zero by $\widehat{Z}_t$, where $\widehat{Z}_t = -Z_t^*$, then $\widehat{Z}_{T-u}$ equals in law to $Z_u^* - Z_T^*$ due to the stationary increment property of a Brownian process. Hence, we establish

$$A_T^* \stackrel{\text{law}}{=} \widehat{A}_T = \frac{1}{T}\int_0^T e^{\sigma \widehat{Z}_{T-u} + \left(r-q+\frac{\sigma^2}{2}\right)(u-T)}\, du,$$

and via time-reversal of $\widehat{Z}_{T-u}$, we obtain

$$\widehat{A}_T = \frac{1}{T}\int_0^T e^{\sigma \widehat{Z}_\xi + \left(q-r-\frac{\sigma^2}{2}\right)\xi}\, d\xi.$$

Note that $\widehat{A}_T S_0$ is the arithmetic average of the price process with drift rate $q - r$.

Summing the results together, we have

$$c_{fl} = S_0 c^*_{fl} = e^{-qT} E^* \left[\max\left(\lambda S_0 - \widehat{A}_T S_0, 0 \right) \right],$$

and from which we deduce the following fixed-floating symmetry relation

$$c_{fl}(S_0, \lambda, r, q, T) = p_{fix}(\lambda S_0, S_0, q, r, T).$$

By combining the put-call parity relations for floating and fixed Asian options and the above symmetry relation, we can derive the following fixed-floating symmetry relation between c_{fix} and p_{fl}.

$$c_{fix}(X, S_0, r, q, T) = p_{fl}\left(S_0, \frac{X}{S_0}, q, r, T \right).$$