Counterparty risk for credit default swaps: Markov chain interacting intensities model with stochastic intensity

KWAI SUN LEUNG
Department of Systems Engineering and Engineering Management,
Chinese University of Hong Kong, Shatin, Hong Kong, China

YUE KUEN KWOK[1]
Department of Mathematics,
Hong Kong University of Science and Technology,
Clear Water Bay, Hong Kong, China

We analyze the counterparty risk for credit default swaps using the Markov chain model of portfolio credit risk of multiple obligors with interacting default intensity processes. The default correlation between the protection seller and underlying entity is modeled by an increment in default intensity upon the occurrence of an external shock event. The arrival of the shock event is a Cox process whose stochastic intensity is assumed to follow an affine diffusion process with jumps. We examine how the correlated default risks between the protection seller and the underlying entity may affect the credit default premium in a credit default swap.

Keywords: credit default swaps, counterparty risk, Markov chain model, default correlation

1 Introduction

A credit default swap (CDS) is a financial contract between the buyer of default protection on a reference risky entity and the seller of that protection. The protection seller receives fixed periodic payments (CDS premium) from the protection buyer, typically expressed in basis points per annum on a notional amount, in return for making a single payment covering losses on

[1]Corresponding author. Email: maykwok@ust.hk

the reference entity following a credit event. The CDS terminates prior to maturity upon the default of the reference entity.

In this paper, we would like to consider the impact of default risk of the protection seller (counterparty risk) on the CDS premium of a CDS. We examine how the correlated default risks between the protection seller and reference entity may affect the CDS premium. We propose a Markov chain model of portfolio credit risk of multiple obligors with interacting default intensity processes. To generate the default correlation, the model assumes that the default intensity processes of the protection seller and reference entity are subject to an increment in intensity upon the arrival of an external shock event. The shock event is modeled as a Cox process whose intensity follows an affine diffusion process with jumps. For example, let us consider a CDS on a risky Korean bond whose protection seller is a Korean financial institution. Compared to a non-Korean financial institution, the Korean financial institution serving as the protection seller may exhibit a higher level of correlated risk with the Korean reference entity upon the arrival of a country wide shock, like the 1997 economic melt down in Korea. With higher correlated default risks, we expect that the CDS premium paid to the Korean protection seller should be lower than that paid to a non-Korean protection seller.

We model the arrival of a default event using the reduced form approach, where default occurs unpredictably at an exogenous intensity rate. The framework of interacting intensities (so called contagion effects of default events) is adopted so as to create the default correlation among the risky obligors in a credit portfolio. Under the interacting intensities model (Jarrow and Yu, 2001), the default contagion effect is introduced via a positive jump in the default intensity of an obligor when there is an occurrence of default of another obligor in the credit portfolio. A general Markovian formulation of a portfolio credit risk model with interacting intensities can be found in Leung and Kwok (2007) and Frey and Backhaus (2008). Using the total hazard construction approach,Yu (2007) presents the numerical simulation technique for calculating the joint distribution of the default times of risky obligors in a credit portfolio. Zheng and Jiang (2008) manage to derive closed form analytic expressions for the joint distribution of default times of risky obligors with interacting default intensities under stochastic intensity processes.

There have been several papers that discuss the impact of counterparty risk on credit default swap valuation. Hull and White (2001) and Kim and

Kim (2001) examine the counterparty risk for CDS using the default barrier correlated models, where the default time is modeled by the first passage time that the credit index hits the default barrier. In Hull and White (2001), the loss due to default of the protection seller is modeled by the replacement cost of entering into a new contract that demands a higher credit default premium. Kim and Kim (2001) model the loss due to counterparty risk via the loss of the compensation payment. Leung and Kwok (2005) use Jarrow-Yu's (2001) interacting intensity model to analyze the impact on the credit default swap premium due to replacement cost and loss of contingent payment. The replacement cost incurs when the protection seller defaults earlier than the reference entity. The loss of contingent compensation payment occurs when the protection seller defaults during the settlement period after the occurrence of default of the reference entity. Brigo and Chourdakis (2008) consider counterparty risk for a CDS in the presence of default correlation of the protection seller and reference entity. In their model, stochastic intensity models are adopted for the default events, and defaults are connected through a copula function. Walker (2006) presents a continuous time Markov approach for the risk neutral pricing of a CDS with counterparty risk. The dependence between the counterparty and the reference entity is introduced through the transition rates. Our model is similar to Walker's Markov chain model. However, we provide a more structural specification of the hazard rate change. The contagion effect is modeled by an external shock event, the arrival of which leads to a positive jump on the default intensities of the protection seller and reference entity.

The paper is organized as follows. In the next section, we present the model formulation of the Markov chain model with stochastic default intensity. We illustrate how to apply the three-firm interacting intensities model to analyze the impact of counterparty risk of the protection seller on the CDS premium. In Section 3, the credit default swap premium is derived with and without the default risk of the protection seller. We present the numerical results that illustrate the impact of various parameters in the stochastic intensity model on the CDS premium. We also illustrate how to calibrate the time dependent parameter function in the default intensity of a risky firm using the market prices of traded bonds issued by the firm. The paper is ended with conclusive remarks in the last section.

2 Markov chain model with stochastic default intensity

Considering a portfolio of N risky firms, we associate a random default time τ_i with firm i in the credit portfolio. The default status of the portfolio is characterized by the default process

$$\boldsymbol{H}_t = \begin{pmatrix} H_t^1 & H_t^2 \cdots H_t^N \end{pmatrix} \in \{0,1\}^N = S, \tag{1}$$

where

$$H_t^i = \mathbf{1}_{\{\tau_i \le t\}} = \begin{cases} 1 & \text{if } \tau_i \le t \\ 0 & \text{if } \tau_i > t \end{cases}, \quad i = 1, 2, \cdots, N.$$

Here, $\boldsymbol{H}$ is visualized as a finite state Markov chain and S is the state space of $\boldsymbol{H}$. The macroeconomic variables are described by the d-dimensional stochastic process $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_t)_{t\in[0,T]}$ with state space $D \subseteq \mathbb{R}^d$. The information available to the investor in the market at time t includes the history of macroeconomic variables and default status of the portfolio up to time t. The filtration $(\mathcal{F}_t)_{t\ge0}$ is given by

$$\mathcal{F}_t = \mathcal{F}_t^{\boldsymbol{\Psi}} \bigvee \mathcal{F}_t^1 \bigvee \mathcal{F}_t^2 \bigvee \cdots \bigvee \mathcal{F}_t^N,$$

where

$$\begin{aligned} \mathcal{F}_t^{\boldsymbol{\Psi}} &= \sigma\left(\boldsymbol{\Psi}_s : 0 \le s \le t\right) \\ \mathcal{F}_t^i &= \sigma\left(H_s^i : 0 \le s \le t\right), \quad i = 1, 2, \cdots, N. \end{aligned}$$

The martingale default intensity $\lambda_i(\boldsymbol{\Psi}_t, \boldsymbol{H}_t)$ of firm i is defined by the property that

$$H_t^i - \int_0^{t\wedge\tau_i} \lambda_i\left(\boldsymbol{\Psi}_s, \boldsymbol{H}_s\right) ds \text{ is a } \{\mathcal{F}_t\}\text{-martingale.}$$

Let τ_C and τ_R denote the random default time of the counterparty and reference entity, respectively, and let τ_S be the random time of arrival of the external shock S. It is assumed that τ_S is independent of τ_C and τ_R.

We model the arrival of the shock event as a Cox process with stochastic intensity process $\{\lambda_t^S : t \ge 0\}$. Prior to the arrival of the shock event S, the default intensities λ_t^C and λ_t^R of the counterparty and reference entity, respectively, are assumed to be $a_C(t)$ and $a_R(t)$. Here, both $a_C(t)$ and

$a_R(t)$ are assumed to be deterministic functions of t. Upon the arrival of S, λ_t^C jumps from $a_C(t)$ to $\alpha_C a_C(t), \alpha_C > 1$, while λ_t^R jumps from $a_R(t)$ to $\alpha_R a_R(t), \alpha_R > 1$. Here, α_C and α_R represent the proportional upward jump of λ_t^C and λ_t^R, respectively, in response to the shock event. In summary, the default intensities of C and R can be expressed as

$$\begin{aligned} \lambda_t^R &= a_R(t)\left[(\alpha_R - 1)\mathbf{1}_{\{\tau_S \le t\}} + 1\right] \\ \lambda_t^C &= a_C(t)\left[(\alpha_C - 1)\mathbf{1}_{\{\tau_S \le t\}} + 1\right]. \end{aligned} \tag{2}$$

The proportional jump factors α_C and α_R are taken to be exogenously given. Later, we illustrate how we can calibrate $a_C(t)$ and $a_R(t)$ from the term structures of market prices of traded defaultable bonds of C and R, respectively.

The state space S of $\boldsymbol{H}_t = (H_t^R \quad H_t^C \quad H_t^S)$ is defined by

$$S = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}.$$

The various Markov states are labelled as follows:

State 1 $(0,0,0)$	State 2 $(1,0,0)$	State 3 $(0,1,0)$	State 4 $(0,0,1)$
State 5 $(1,1,0)$	State 6 $(1,0,1)$	State 7 $(0,1,1)$	State 8 $(1,1,1)$

Let $\psi^S = (\lambda_t^S)_{t\in[0,T]}$, and conditional on the given state ψ^S, the entries of the infinitesimal generator $\bigwedge_{[\psi^S]}(t) = (\wedge_{ij}(t|\psi^S))_{8\times 8}$ for the Markov chain $\boldsymbol{H} : \{\boldsymbol{H}_t : t \ge 0\}$ are given by

$$\begin{aligned}
\wedge_{11}(t|\psi^S) &= -[a_R(t) + a_C(t) + \lambda_t^S], \quad \wedge_{12}(t|\psi^S) = a_R(t), \\
\wedge_{13}(t|\psi^S) &= a_C(t), \quad \wedge_{14}(t|\psi^S) = \lambda_t^S, \\
\wedge_{22}(t|\psi^S) &= -[a_C(t) + \lambda_t^S], \quad \wedge_{25}(t|\psi^S) = a_C(t), \quad \wedge_{26}(t|\psi^S) = \lambda_t^S, \\
\wedge_{33}(t|\psi^S) &= -[a_R(t) + \lambda_t^S], \quad \wedge_{35}(t|\psi^S) = a_R(t), \quad \wedge_{37}(t|\psi^S) = \lambda_t^S, \\
\wedge_{44}(t|\psi^S) &= -[\alpha_R a_R(t) + \alpha_C a_C(t)], \quad \wedge_{46}(t|\psi^S) = \alpha_R a_R(t), \\
\wedge_{47}(t|\psi^S) &= \alpha_C a_C(t), \quad \wedge_{55}(t|\psi^S) = -\lambda_t^S, \quad \wedge_{58}(t|\psi^S) = \lambda_t^S, \\
\wedge_{66}(t|\psi^S) &= -\alpha_C a_C(t), \quad \wedge_{68}(t|\psi^S) = \alpha_C a_C(t) \\
\wedge_{77}(t|\psi^S) &= -\alpha_R a_R(t), \quad \wedge_{78}(t|\psi^S) = \alpha_R a_R(t).
\end{aligned}$$

All other entries in the infinitesimal generator are zero.

By using the forward Kolmogorov equation, the conditional transition probability matrix $P(t,u|\psi^S) = (P_{ij}(t,u|\psi^S))_{8\times 8}$ is governed by

$$\frac{dP(t,u|\psi^S)}{du} = P(t,u|\psi^S) \bigwedge_{[\psi^S]}(u), \quad 0 \le t \le u, \tag{3}$$

with $P(t,t|\psi^S) = I$. Since the matrix $\bigwedge_{[\psi^S]}(u)$ is upper triangular, the conditional transition probabilities $P_{ij}(t,u|\psi^S)$ can be solved successively in a sequential manner. The solution to some of the conditional transition probabilities are obtained as follows:

$$\begin{aligned}
P_{11}(t,T|\psi^S) &= e^{-\int_t^T [a_R(u)+a_C(u)+\lambda_u^S]\,du};\\
P_{12}(t,T|\psi^S) &= e^{-\int_t^T [a_C(u)+\lambda_u^S]\,du}\left[1-e^{-\int_t^T a_R(u)\,du}\right];\\
P_{13}(t,T|\psi^S) &= e^{-\int_t^T [a_R(u)+\lambda_u^S]\,du}\left[1-e^{-\int_t^T a_C(u)\,du}\right];\\
P_{14}(t,T|\psi^S) &= e^{-\int_t^T [a_R(u)+a_C(u)]\,du}\\
&\quad \int_t^T \lambda_v^S e^{-\int_v^T [(\alpha_R-1)a_R(u)+(\alpha_C-1)a_C(u)]\,du-\int_t^v \lambda_u^S\,du}\,dv;\\
P_{16}(t,T|\psi^S) &= e^{-\int_t^T \alpha_C a_C(v)\,dv}\int_t^T \lambda_u^S e^{\int_t^u [(\alpha_C-1)a_C(v)-\lambda_v^S]}\,dv\\
&\qquad \left[1-e^{-\int_t^u a_R(v)\,dv}\right]du\\
&\quad + \alpha_R e^{-\int_t^T \alpha_C a_C(v)\,dv}\int_t^T a_R(u)e^{-\int_t^u \alpha_R a_R(v)\,dv}\\
&\quad \left(\int_t^u \lambda_v^S e^{\int_t^v [(\alpha_R-1)a_R(w)+(\alpha_C-1)a_C(w)-\lambda_w^S]\,dw}\,dv\right)du;\\
P_{17}(t,T|\psi^S) &= e^{-\int_t^T \alpha_R a_R(v)\,dv}\int_t^T \lambda_u^S e^{\int_t^u [(\alpha_R-1)a_R(v)-\lambda_v^S]\,dv}\\
&\quad \left[1-e^{-\int_t^u a_C(v)\,dv}\right]du\\
&\quad + \alpha_C e^{-\int_t^T \alpha_R a_R(v)\,dv}\int_t^T a_C(u)e^{-\int_t^u \alpha_C a_C(v)\,dv}\\
&\quad \left\{\int_t^u \lambda_v^S e^{\int_t^v [(\alpha_R-1)a_R(w)+(\alpha_C-1)a_C(w)-\lambda_w^S]\,dw}\,dv\right\}du. \qquad (4)
\end{aligned}$$

By taking the expectation operation $E_{\psi^S}[\,\cdot\,]$, which is the expectation taken over the path of $\left(\lambda_t^S\right)_{t\in[0,T]}$, we obtain the following unconditional transition probabilities:

$$
\begin{aligned}
P_{11}(t,T) &= e^{-\int_t^T [a_R(u)+a_C(u)]\,du} E_{\psi^S}\left[e^{-\int_t^T \lambda_u^S\,du}\right];\\
P_{12}(t,T) &= e^{-\int_t^T a_C(u)\,du}\left[1-e^{-\int_t^T a_R(u)\,du}\right] E_{\psi^S}\left[e^{-\int_t^T \lambda_u^S\,du}\right]\\
P_{13}(t,T) &= e^{-\int_t^T a_R(u)\,du}\left[1-e^{-\int_t^T a_C(u)\,du}\right] E_{\psi^S}\left[e^{-\int_t^T \lambda_u^S\,du}\right];\\
P_{14}(t,T) &= e^{-\int_t^T [a_R(u)+a_C(u)]\,du}\int_t^T e^{-\int_v^T [(\alpha_R-1)a_R(u)+(\alpha_C-1)a_C(u)]\,du}\\
&\quad E_{\psi^S}\left[\lambda_v^S e^{-\int_t^v \lambda_u^S\,du}\right] dv;\\
P_{16}(t,T) &= e^{-\int_t^T \alpha_C a_C(v)\,dv}\\
&\quad \int_t^T e^{\int_t^u (\alpha_C-1)a_C(v)\,dv}\left[1-e^{-\int_t^u a_R(v)\,dv}\right] E_{\psi^S}\left[\lambda_u^S e^{-\int_t^u \lambda_v^S\,dv}\right] du\\
&\quad + \alpha_R e^{-\int_t^T \alpha_C a_C(v)\,dv}\int_t^T a_R(u) e^{-\int_t^u \alpha_R a_R(v)\,dv}\\
&\quad \left\{\int_t^u e^{\int_t^v [(\alpha_R-1)a_R(w)+(\alpha_C-1)a_C(w)]\,dw} E_{\psi^S}\left[\lambda_v^S e^{-\int_t^v \lambda_w^S\,dw}\right] dv\right\} du\\
P_{17}(t,T) &= e^{-\int_t^T \alpha_R a_R(v)\,dv}\int_t^T e^{\int_t^u (\alpha_R-1)a_R(v)\,dv}\left[1-e^{-\int_t^u \alpha_C(v)\,dv}\right]\\
&\quad E_{\psi^S}\left[\lambda_u^S e^{-\int_t^u \lambda_v^S\,dv}\right] du\\
&\quad + \alpha_C e^{-\int_t^T \alpha_R a_R(v)\,dv}\int_t^T a_C(u) e^{-\int_t^u \alpha_C a_C(v)\,dv}\\
&\quad \left\{\int_t^u e^{\int_t^v [(\alpha_R-1)a_R(w)+(\alpha_C-1)a_C(w)]\,dw}\right.\\
&\quad \left. E_{\psi^S}\left[\lambda_v^S e^{-\int_t^v \lambda_w^S\,dw}\right] dv\right\} du. \qquad (5)
\end{aligned}
$$

Once these transition probabilities are known, one can deduce the joint distribution of the default times (Leung and Kwok, 2007). The marginal distribution for τ_R and τ_C are given by

$$
\begin{aligned}
P[\tau_R > T|\mathcal{F}_t] &= P_{11}(t,T) + P_{13}(t,T) + P_{14}(t,T) + P_{17}(t,T)\\
P[\tau_C > T|\mathcal{F}_t] &= P_{11}(t,T) + P_{12}(t,T) + P_{14}(t,T) + P_{16}(t,T). \qquad (6)
\end{aligned}
$$

It is necessary to prescribe the dynamic of λ_t^S in order to compute

$$
E_{\psi^S}\left[e^{-\int_t^v \lambda_u^S\,du}\right] \quad \text{and} \quad E_{\psi^S}\left[\lambda_v^S e^{-\int_t^v \lambda_u^S\,du}\right].
$$

We adopt the same affine diffusion process with jump for λ_t^S as that proposed by Duffie and Gârleanu (2001). The governing stochastic differential equation of λ_t^S takes the form:

$$d\lambda_t^S = k(\theta - \lambda_t^S)\,dt + \sigma\sqrt{\lambda_t^S}\,dZ_t + \Delta J_t, \tag{7}$$

where ΔJ_t denotes any jump that occurs at time t of a pure jump process J_t. Here, J_t is taken to be independent of Z_t, with jump sizes that are independent and exponentially distributed with mean μ and whose jump times are those of an independent Poisson processes with jump arrival rate ℓ. It can be shown that (Duffie and Gârleanu, 2001)

$$E_{\psi^S}\left[e^{-\int_0^t \lambda_u^S\,du}\right] = e^{\alpha(t)+\beta(t)\lambda_0^S},$$

where

$$\begin{aligned}
\alpha(t) &= -\frac{k\theta(c_1+d_1)}{b_1c_1d_1}\ln\frac{c_1+d_1e^{b_1t}}{c_1+d_1}+\frac{k\theta}{c_1}t \\
&\quad + \frac{\ell(a_2c_2-d_2)}{b_2c_2d_2}\ln\frac{c_2+d_2e^{b_2t}}{c_2+d_2}+\left(\frac{\ell}{c_2}-\ell\right)t, \\
\beta(t) &= \frac{1-e^{b_1t}}{c_1+d_1e^{b_1t}}; \\
b_1 &= -\sqrt{k^2+2\sigma^2}, \quad c_1 = \frac{k+\sqrt{k^2+2\sigma^2}}{-2}, \\
d_1 &= \frac{-k+\sqrt{k^2+2\sigma^2}}{-2}, \quad a_2 = \frac{-k+\sqrt{k^2+2\sigma^2}}{k+\sqrt{k^2+2\sigma^2}}, \\
b_2 &= b_1, \quad d_2 = \frac{d_1+\mu}{c_1}, \quad c_2 = 1-\frac{\mu}{c_1}.
\end{aligned}$$

Also, we obtain

$$E_{\psi^S}\left[\lambda_v^S e^{-\int_0^v \lambda_u^S\,du}\right] = -\frac{\partial}{\partial v}E_{\psi^S}\left[e^{-\int_0^v \lambda_u^S\,du}\right] = -[\overline{\alpha}(v)+\overline{\beta}(v)\lambda_0^S]e^{\alpha(v)+\beta(v)\lambda_0^S},$$

where

$$\begin{aligned}
\overline{\alpha}(v) &= -\frac{k\theta(c_1+d_1)}{c_1}\frac{e^{b_1v}}{c_1+d_1e^{b_1v}}+\frac{k\theta}{c_1} \\
&\quad + \frac{\ell(a_2c_2-d_2)}{c_2}\frac{e^{b_2v}}{c_2+d_2e^{b_2v}}+\left(\frac{\ell}{c_2}-\ell\right), \\
\overline{\beta}(v) &= \frac{-(k^2+2\sigma^2)e^{b_1v}}{(c_1+d_1e^{b_1v})^2}.
\end{aligned}$$

3 Calculations of the credit default swap premium

We would like to compute the fair credit default swap premium with and without the default risk of the protection seller. For simplicity, we take the notional of the CDS to be unity and let the valuation date of the CDS premium to be time zero. Under our continuous model assumption, the swap premium payments are assumed to be made continuously at a constant CDS premium rate. Let T denote the maturity date of the CDS. We let $C(T)$ denote the CDS premium rate that is paid by the protection buyer subject to potential default risk of the seller. Similarly, we let $\widehat{C}(T)$ denote the CDS premium rate without the default risk of the protection seller. The impact on the CDS premium rate with the presence of the counterparty risk is then measured by $\widehat{C}(T) - C(T)$.

There are two possible scenarios during $[0, t]$: non-occurrence of the shock event S or occurrence of S. Given that there has been no default of the underlying entity during the time interval $[0, t]$ and default of the reference entity occurs during the next infinitesimal time interval $(t, t + dt]$, the probability of such occurrence is given by

$$[P_{11}(0, t)a_R(t) + P_{14}(0, t)\alpha_R a_R(t)]\, dt.$$

The first term $P_{11}(0, t)a_R(t)\, dt$ corresponds to "no prior arrival" of S while the second term $P_{14}(0, t)\alpha_R a_R(t)\, dt$ corresponds to the case otherwise. Let ρ be the deterministic recovery rate of the reference entity upon default. The expected present value of contingent compensation payment made by the protection seller that is paid within $(t, t + dt]$ is given by

$$(1 - \rho)e^{-rt} [P_{11}(0, t)a_R(t) + P_{14}(0, t)\alpha_R a_R(t)]\ dt,$$

where r is the constant interest rate. Hence, the expected present value of the contingent payment paid by the protection seller over the whole period $[0, T]$ is given by

$$\int_0^T (1 - \rho)e^{-rt} [P_{11}(0, t)a_R(t) + P_{14}(0, t)\alpha_R a_R(t)]\ dt.$$

The swap premium payment stream continues when the Markov chain state is either in State 1 or State 4. The swap premium payment paid by the protection buyer over $(t, t+dt]$ is given by

$$C(T)e^{-rt}\left[P_{11}(0,t) + P_{14}(0,t)\right] dt,$$

so the expected present value of the premium payment over the whole time period $[0, T]$ is found to be

$$C(T)\int_0^T e^{-rt}\left[P_{11}(0,t) + P_{14}(0,t)\right] dt.$$

By equating the expected present value of the payments from the two counterparties, the fair CDS premium with counterparty risk is obtained as

$$C(T) = \frac{(1-\rho)\int_0^T e^{-rt}\left[P_{11}(0,t)a_R(t) + P_{14}(0,t)\alpha_R a_R(t)\right] dt}{\int_0^T e^{-rt}\left[P_{11}(0,t) + P_{14}(0,t)\right] dt}. \tag{8}$$

Since the corresponding transition probabilities $P_{11}(0,t)$ and $P_{14}(0,t)$ have been obtained for the three-firm Markov chain interacting intensities model with stochastic intensity [see Eq. (4)], it then becomes quite straightforward to compute $C(T)$. To compute the CDS premium rate $\overline{C}(T)$ without the default risk of the protection seller, we simply set $a_C(t)$ to be zero in the above calculations.

Numerical calculations of the CDS premium rates and default probability

We performed the numerical calculations of the CDS premium rates $C(T)$ and $\overline{C}(T)$, and the default probability of the protection seller $P[\tau_C \leq 5]$ of a 5-year CDS contract. The basic set of parameter values used in our calculations are listed below:

$$\begin{aligned}
&\rho = 0.4, \quad r = 0.04, \quad \alpha_R = 1.15, \quad \alpha_C = 1.15,\\
&\lambda_0^S = 0.05, \quad \sigma = 0.2, \quad k = 0.3,\\
&\theta = 0.02, \quad \ell = 0.3, \quad \mu = 0.15.
\end{aligned}$$

We let $a_R(t)$ and $a_C(t)$ assume the constant value 0.2. In Figure 1(a–e), we plot the CDS premium rates $C(T)$ (with counterparty risk of the protection seller) and $\overline{C}(T)$ (without counterparty risk) against varying values of the different parameters: $\alpha_C, \alpha_R, \ell, k$ and θ.

A higher value of α_C means a higher proportional jump in the default intensity of the counterparty upon the arrival of S. The dotted curve in

Figure 1(a) shows that $C(T)$ is a decreasing function of α_C. This agrees with the intuition that a higher counterparty risk leads to a lower value of the fair CDS premium rate. Without counterparty risk, the parameter α_C becomes irrelevant, so $\overline{C}(T)$ stays at the same constant value [as shown by the horizontal solid line in Figure 1(a)].

On the other hand, both $C(T)$ and $\overline{C}(T)$ are increasing functions of α_R since the reference entity becomes riskier with a higher value of α_R [see Figure 1(b)]. When $\alpha_R = 1$, there is no jump in the default intensity of the reference entity upon the arrival of S. In this case, there is no default correlation between the entity R and protection seller C, so $C(T)$ and $\overline{C}(T)$ have the same value. With an increasing value of α_R, the spread $\overline{C}(T) - C(T)$ widens since the default correlation between R and C increases.

We also explore the impact of the different parameters that characterize the stochastic intensity of the shock event S on the CDS premium rates. As expected, a higher arrival rate ℓ of the jump process J_t leads to higher CDS premium rates since the reference entity becomes riskier [see Figure 1(c)]. Since the drift rate of λ_t^S decreases with increasing k, so the CDS premium rates are decreasing functions of k [see Figure 1(d)]. Lastly, a higher value of θ means a higher drift rate of λ_t^S, so the CDS premium rates are increasing functions of θ.

Though $C(T)$ is a decreasing function of α_C, the default probability $P[\tau_C \leq 5]$ is an increasing function of α_C [see Figure 2(a)]. This is intuitive since a higher proportional jump in the default intensity upon the arrival of S leads to a higher default risk of the protection seller C. In a similar manner, a higher arrival rate of jump in λ_t^S makes C to become riskier, so $P[\tau_C \leq 5]$ is an increasing function of ℓ [see Figure 2(b)].

Remark - Calibration of the time dependent default intensities

In our model, the default intensities of the reference entity and the protection seller prior to the arrival of the shock event are assumed to be deterministic functions of time. Provided that the term structure of the prices of defaultable bonds $B_R(t,T)$ issued by the firm of the reference entity is available, one can calibrate the time dependent default intensity $a_R(t)$ in terms of $B_R(t,T)$ and other known parameter values. Under the risk neutral measure Q, the defaultable bond price $B_R(t,T)$ is given by

$$\begin{aligned} B_R(t,T) &= e^{-r(T-t)} E_Q\left[\mathbf{1}_{\{\tau_R > T\}} | \mathcal{F}_t\right] \\ &= e^{-r(T-t)} P\left[\tau_R > T | \mathcal{F}_t\right]. \end{aligned} \tag{9}$$

Recall that the marginal distribution for τ_R can be expressed in terms of the transition probabilities of the Markov chain model [see Eq. (6)]. One can derive the following integral equation for $a_R(t)$:

$$\begin{aligned}\frac{\partial B_R}{\partial T}(t,T) &= -re^{-r(T-t)}B_R(t,T) - \alpha_R a_R(T) B_R(t,T) \\ &\quad + e^{-r(T-t)}(\alpha_R - 1) a_R(T) e^{-\int_t^T a_R(u)\,du} E_{\psi^S}\left[e^{-\int_t^T \lambda_u^S\,du}\right] \quad (10)\end{aligned}$$

Similar calibration procedure can be performed for the default intensity $a_C(t)$ of the counterparty (protection seller).

4 Conclusion

We apply the three-firm interacting intensities Markov chain model with stochastic intensity to analyze the counterparty risk of the protection seller in a credit default swap. The default correlation between the protection seller and the underlying entity is modeled by an increment of the default intensity upon the arrival of an external shock event. Our analysis indicates that the impact of correlated risks between the protection seller and the underlying reference entity on the fair credit default swap premium rates can be quite substantial under a high arrival rate of the external shock and the subsequent high proportional jumps in the default intensities of the various parties. This study provides insight in assessing the importance of correlated counterparty risk in structuring a credit default swap. Should we seek protection from a Korean financial institution on a Korean risky bond, though the Korean institution demands a lower swap premium, or seek protection from a European institution instead?

References

[1] Brigo D. and Chourdakis K., Counterparty risk for credit default swaps: Impact of spread volatility and default correlation, Working paper of Fitch Solution, 2008.

[2] Duffie D. and Gârleanu N., Risk and valuation of collateralized debt obligations, *Financial Analysts Journal*, 2001, 41–59.

[3] Frey R. and Backhaus J., Credit derivatives in models with interacting default intensities: a Markovian approach. Working paper of University of Leipzig, 2008.

[4] Hull J. and White A., Valuing credit default swap II: Modeling default correlations, *Journal of Derivatives*, 2001, **8**(3), 12–22.

[5] Jarrow R. and Yu F., Counterparty risk and the pricing of defaultable securities, *Journal of Finance*, 2001, **56**(5), 1765–1799.

[6] Kim M.A. and Kim T.S., Credit default swap valuation with counterparty default risk and market risk, *Journal of Risk*, 2001, **6**(2), 49–80.

[7] Leung K.S. and Kwok Y.K., Contagion models with interacting default intensity processes, *Proceedings of the International Congress of Chinese Mathematicians*, vol. III, 748–758.

[8] Leung S.Y. and Kwok Y.K., Credit default swap valuation with counterparty risk, *Kyoto Economics Review*, 2005, **74**(1), 25–45.

[9] Walker M.B., Credit default swaps with counterparty risk: A calibrated Markov model, *Journal of Credit Risk*, 2006, **2**(1), 31–49.

[10] Yu F., Correlated defaults in intensity-based models, *Mathematical Finance*, 2007, **17**(2), 155–173.

[11] Zheng H. and Jiang L., Basket CDS pricing with interacting intensities, Working paper of Imperial College, 2008.

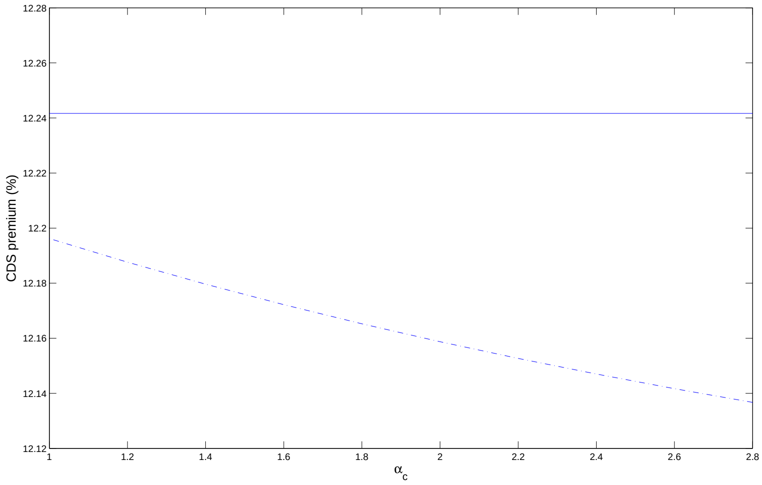

Figure 1(a) Plot of the CDS premium rates $C(T)$ (dotted curve) and $\overline{C}(T)$ (solid curve) against α_C.

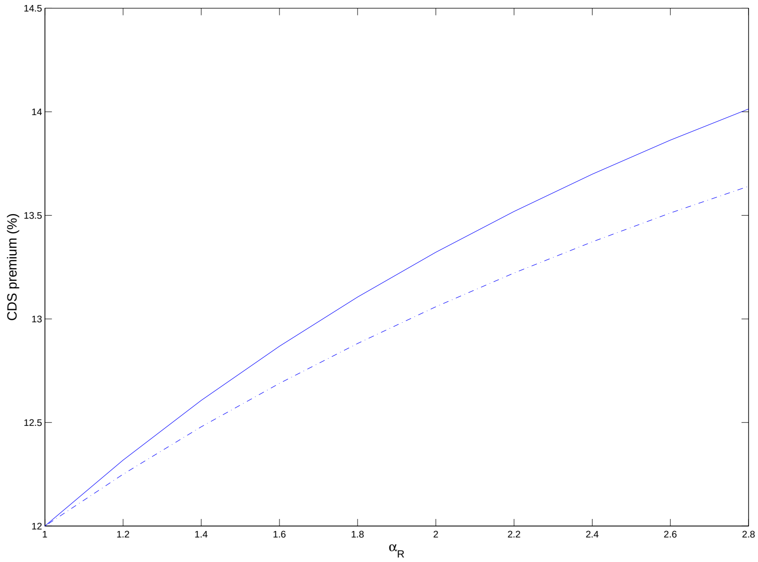

Figure 1(b) Plot of the CDS premium rates $C(T)$ (dotted curve) and $\overline{C}(T)$ (solid curve) against α_R.

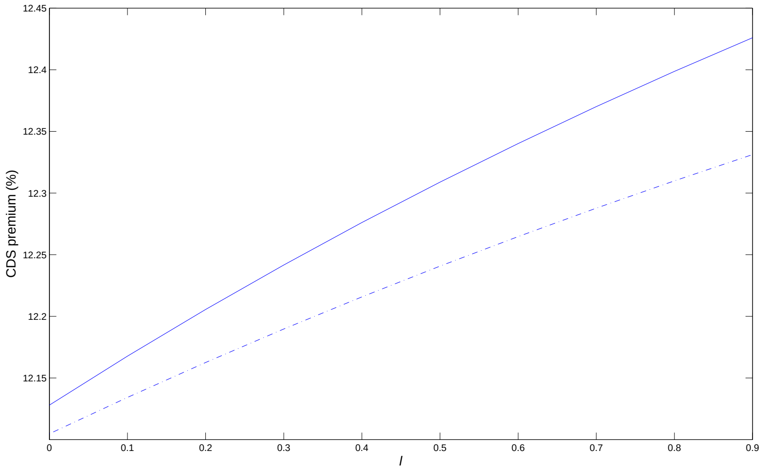

Figure 1(c) Plot of the CDS premium rates $C(T)$ (dotted curve) and $\overline{C}(T)$ (solid curve) against ℓ.

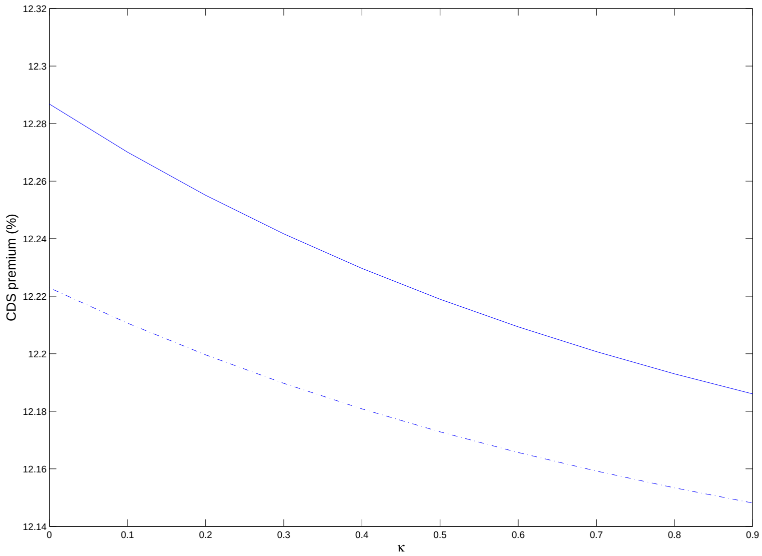

Figure 1(d) Plot of the CDS premium rates $C(T)$ (dotted curve) and $\overline{C}(T)$ (solid curve) against k.

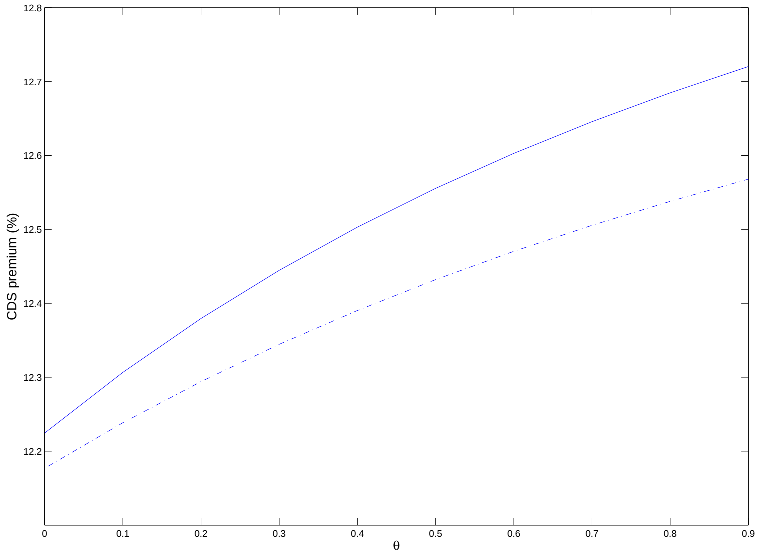

Figure 1(e) Plot of the CDS premium rates $C(T)$ (dotted curve) and $\overline{C}(T)$ (solid curve) against θ.

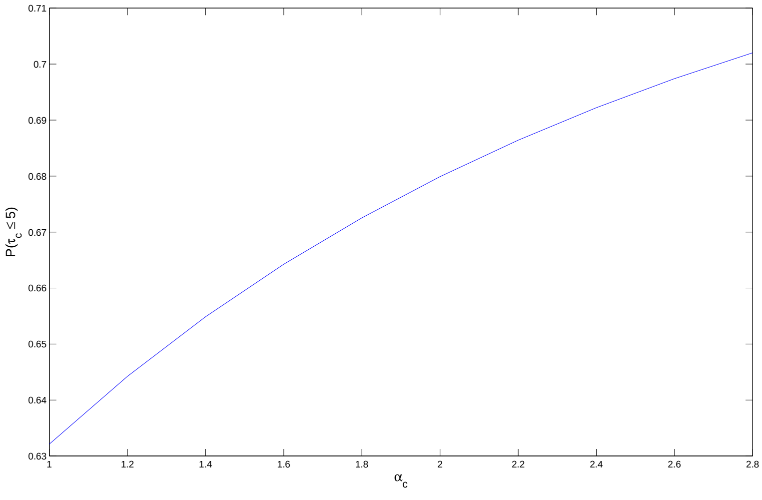

Figure 2(a) Plot of the default probability of the protection seller $P(\tau_C \leq 5)$ against α_C.

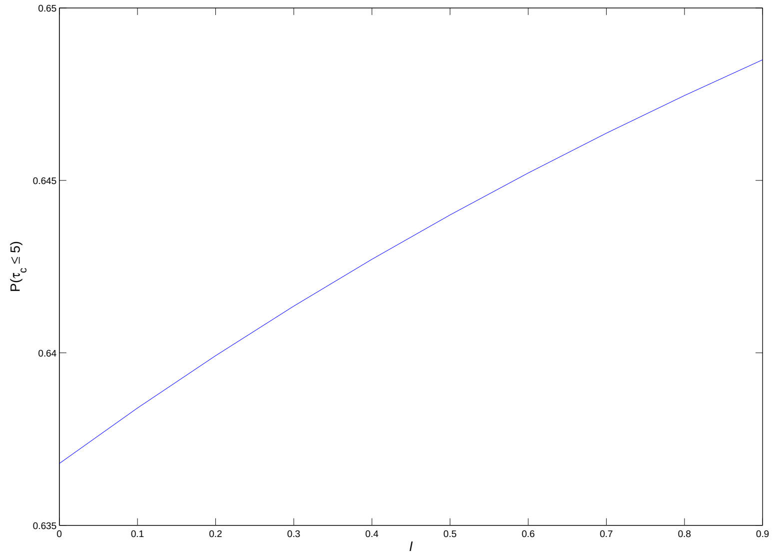

Figure 2(b) Plot of the default probability of the protection seller $P(\tau_C \leq 5)$ against ℓ.