Available online at www.sciencedirect.com
SCIENCE DIRECT®
Insurance: Mathematics and Economics 34 (2004) 273–295
INSURANCE MATHEMATICS & ECONOMICS
www.elsevier.com/locate/econbase

Reset and withdrawal rights in dynamic fund protection

Chi Chiu Chu*, Yue Kuen Kwok

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, China

Received October 2003; received in revised form December 2003; accepted 23 December 2003

Abstract

We analyze the nature of the dynamic fund protection which provides an investment fund with a floor level of protection against a reference stock index (or stock price). The dynamic protection feature entitles the investor the right to reset the value of his investment fund to that of the reference stock index. The reset may occur automatically whenever the investment fund value falls below that of the reference stock index, or only allowed at pre-determined time instants. The protected funds may allow a finite number of resets throughout the life of the fund, where the reset times are chosen optimally by the investor. We examine the relation between the finite-reset funds and automatic-reset funds. We also analyze the premium and the associated exercise policy of the embedded withdrawal right in protected funds, where the investor has the right to withdraw the fund prematurely. The impact of proportional fees on the optimal withdrawal policies is also analyzed. The holder should optimally withdraw at a lower critical fund value when the rate of proportional fees increases. Under the assumption that the fund value and index value follow the Geometric Brownian processes, we compute the grant-date and mid-contract valuation of these protected funds. Pricing properties of the protected fund value and the cost to the sponsor are also discussed.

JEL classification: G220; 130

Keywords: Optimal reset; Withdrawal rights; Dynamic funds

1. Introduction

Equity-indexed annuities have generated much interest since their first launch by Keyport Life Insurance Co. in 1995. Tiong (2000) provided a comprehensive summary of the design of different types of equity-indexed annuities and their pricing properties. In a number of research articles (Gerber and Pafumi, 2000; Imai and Boyle, 2001; Gerber and Shiu, 2003), the concept of dynamic protection (applied to equity-indexed annuities) has been proposed. The dynamic protection feature entitles the investor the right to reset the fund value to that of the reference stock index. In this paper, we consider finite-lived investment funds with the dynamic guarantee feature where the value of the investment (protected) fund is upgraded to the value of the reference stock index whenever the investor exercises his reset right. We also analyze the withdrawal right embedded in the protected funds and the impact of payment of proportional fees on the optimal withdrawal policy.

* Corresponding author.
E-mail addresses: maccc@ust.hk (C.C. Chu), maykwok@ust.hk (Y.K. Kwok).

0167-6687/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2003.12.004

Let $F(t)$ and $I(t)$ denote the value of the primary (without the dynamic protection) fund and the reference stock index, respectively, and let $\tilde{F}(t)$ denote the value of the protected (modified) fund. When the investor makes his reset decision (he does so only when the protected fund value falls below the reference index value), the sponsor of the fund has to purchase additional units of the primary fund so that the protected fund value is upgraded to that of the reference index. At the reset instant ξ_i, $i = 1, 2, \dots,$ we have

$$\tilde{F}(\xi_i) = I(\xi_i) = n(\xi_i)F(\xi_i), \tag{1.1}$$

where $n(\xi_i) = I(\xi_i)/F(\xi_i) > 1$ is the new number of units of the primary fund in the investment fund. It is obvious that $n(\xi_1) < n(\xi_2) < \cdots$ since each reset should lead to acquisition of more units of the primary fund. The investor chooses to exercise the reset right each time whenever the primary fund value falls to some historical low value. The obvious challenge in the pricing of the protected fund is how to determine such threshold fund value following the optimal policy of reset.

With the embedded withdrawal right in the investment fund, the holder may withdraw the fund prematurely at his optimal choice. Upon withdrawal, the investor holds the primary fund directly but forfeits the dynamic protection offered in the remaining life of the investment fund. However, he is compensated by receiving the dividends paid by the primary fund. Also, the investor may be required to pay proportional fees throughout the life of the fund. Intuitively, the investor exercises the withdrawal right only when the primary fund value has reached sufficiently high threshold value. Such withdrawal-threshold value should decrease when the calendar time is approaching the maturity date of the investment fund or the proportional fees increase or both.

Gerber and Pafumi (2000) considered an investment fund that is guaranteed not to fall below a predetermined constant level K at all times. In their model, the investment fund value $\tilde{F}(t)$ and the primary fund value $F(t)$ are related by

$$\tilde{F}(t) = F(t) \max\left(1, \max_{0\le u\le t} \frac{K}{F(u)}\right). \tag{1.2}$$

Assuming that $F(t)$ follows the Geometric Brownian process, they claimed that the stochastic process of the modified fund $\tilde{F}(t)$ can be obtained from the stochastic process of the primary fund $F(t)$ by placing a reflecting barrier at K. They obtained the price function of the investment fund at the grant-date, where the number of units of the primary fund equals one. By relating the protected fund value to the payoff of a lookback option, Imai and Boyle (2001) derived the mid-contract valuation of the protected fund. They also considered the withdrawal right in funds with dynamic protection and argued that it is never optimal to withdraw if the fund does not pay dividends. Gerber and Shiu (2003) considered *perpetual* equity-indexed annuities with dynamic protection and withdrawal right, where the guarantee level is another stock index. Fung and Li (2003) proposed an efficient numerical scheme to compute the value of protected funds under discrete monitoring. In their algorithm, they allow the underlying fund value process to be a lognormal process or a constant elasticity of variance process. To fund the guarantee, the sponsor may charge the holder proportional fees over the life of the fund. Windcliff et al. (2002) examined the impact of the proportional fees on the hedging strategies adopted by the sponsor in Canadian segregated funds.

This paper extends the previous results in several aspects. By taking the number of allowable resets to be infinite, we show that the dynamic protection becomes that of automatic reset, whereby the upgrade occurs whenever the investment fund value falls below the reference index value. We provide a justification why the stochastic process of the protected fund can be obtained by enforcing a reflecting barrier at the protected level in the stochastic process of the primary fund. We obtain price functions of mid-contract valuation of the investment fund with and without withdrawal right. We also derive an analytic valuation formula for the cost to the sponsor for funding the guarantee. Our pricing models include the consideration of the withdrawal right and proportional fees. The characterization of the optimal withdrawal policies adopted by the investor is discussed.

The paper is organized as follows. The next section presents the pricing formulation of an investment fund under dynamic protection with respect to another reference stock index. The fund is entitled to have a finite number of allowable resets but without the right to withdraw the fund before maturity. We illustrate how to obtain the automatic-reset model by taking the number of resets to be infinite, and then derive the price function of the finite-lived protected fund with automatic reset. We also derive an integral representation of the price function using the concept of rollover hedging strategy. The relations between the price functions at the grant-date and mid-contract valuation are examined. In Section 3, we compute the integral representation of the premium of the embedded withdrawal right in the protected fund, and also consider the optimal withdrawal policies adopted by the investor. We also formulate the valuation model for the cost to the sponsor of the protection. Finite difference scheme and recursive integration method are used to solve for the price function and the critical withdrawal threshold value. The impact of proportional fees on the optimal withdrawal policies is also examined. Summaries and conclusions are presented in the last section.

2. Valuation of the dynamic fund protection

First, we present the pricing formulation of the investment fund with dynamic protection with respect to a reference stock index, where the holder has at most n resets. Thereafter, we take the limit $n \to \infty$ and obtain the price function of the protected fund with automatic reset. We perform our valuation of the protected fund using the Black–Scholes pricing paradigm. Under the risk neutral valuation framework, we assume that the primary fund value $F(t)$ and the reference index value $I(t)$ follow the Geometric Brownian processes:

$$\frac{\mathrm{d}F}{F} = (r - q_p)\,\mathrm{d}t + \sigma_p\,\mathrm{d}Z_p, \tag{2.1a}$$

$$\frac{\mathrm{d}I}{I} = (r - q_i)\,\mathrm{d}t + \sigma_i\,\mathrm{d}Z_i, \tag{2.1b}$$

where r is the riskless interest rate, q_p and q_i are the dividend yield of the primary fund and stock index, respectively, σ_p and σ_i are the volatility of the primary fund value and reference index value, respectively, and $\mathrm{d}Z_p\,\mathrm{d}Z_i = \rho\,\mathrm{d}t$. Here, ρ is the correlation coefficient between the primary fund process and reference index process.

2.1. Pricing formulation of the protected fund with n resets

Let $V_n(F, I, t)$ denote the value of the investment fund with dynamic protection with respect to a reference stock index, where the investor has n reset rights outstanding. We first consider the simpler case, where there has been no prior reset. That is, the number of units of the primary fund is equal to one at current time t.

The dimension of the pricing model can be reduced by one if F is chosen as the numeraire. We define the stochastic state variable

$$x = \frac{I}{F}, \tag{2.2a}$$

which also follows the Geometric Brownian process

$$\frac{\mathrm{d}x}{x} = (q_p - q_i)\,\mathrm{d}t + \sigma\,\mathrm{d}Z, \tag{2.2b}$$

where $\sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_i + \sigma_i^2$. Accordingly, we define the normalized fund value function with F as the numeraire by

$$W_n(x, t) = \frac{V_n(F, I, t)}{F}. \tag{2.2c}$$

The investor should never reset when $F(t)$ stays above $I(t)$. With only a finite number of reset rights, he also does not reset immediately when $F(t)$ just hits the level of $I(t)$. When $F(t)$ falls below $I(t)$ to certain threshold level, it may become optimal for the investor to exercise the first reset right. The optimal reset policy is similar to that of put option with finite number of reset rights (Dai et al., 2003).

With reference to the variable x, the investor resets when x reaches some sufficiently high threshold value (denoted by x_n^*). The value of x_n^* is not known in advance, but has to be solved as part of the solution to the pricing model. Upon reset at $x = x_n^*$, the sponsor has to increase the number of units of the primary fund so that the new value of the investment fund equals I. The corresponding number of units should then be x_n^*, which is the ratio of the reference index value to the primary fund value at the reset moment. After one reset, the number of resets outstanding is reduced by one, and the value of x becomes one since the ratio of the reference index value to the *newly upgraded* investment fund value is one. Hence, we obtain the boundary condition:

$$W_n(x_n^*, t) = x_n^* W_{n-1}(1, t). \tag{2.3a}$$

Since the reset decision is made optimally by the investor, by Bellman's principle of optimality (Dixit and Pindyck, 1994), we should have the smooth pasting condition at $x = x_n^*$, namely,

$$W_n'(x_n^*, t) = W_{n-1}(1, t). \tag{2.3b}$$

This extra smooth pasting condition determines the value of x_n^* such that the investment fund value is maximized. The terminal payoff of the investment fund is simply equal to F, if no reset has occurred throughout the life of the fund.

In the continuation region, inside which the investor chooses not to exercise the reset right, the value function $V_n(F, I, t)$ satisfies a Black–Scholes equation with two state variables, F and I. In terms of x, the governing equation and the associated auxiliary conditions for $W_n(x, t)$ are given by (Gerber and Shiu, 2003; Chu and Kwok, 2003)

$$\begin{gathered}\frac{\partial W_n}{\partial t} + \left[\frac{\sigma^2}{2}x^2\frac{\partial^2 W_n}{\partial x^2} + (q_p - q_i)x\frac{\partial W_n}{\partial x} - q_p W_n\right] = 0, \quad t < T, \ \ x < x_n^*(t), \\ W_n(x_n^*, t) = x_n^* W_{n-1}(1, t) \quad \text{and} \quad W_n'(x_n^*, t) = W_{n-1}(1, t), \qquad W_n(x, T) = 1,\end{gathered} \tag{2.4}$$

where $x_n^*(t)$ is the time-dependent threshold value at which the investor optimally exercises the reset right. The pricing model leads to a free boundary value problem with the free boundary $x_n^*(t)$ separating the continuation region $\{(x, t) : x < x^*(t), t < T\}$ and the stopping region $\{(x, t) : x \geq x^*(t), t < T\}$. The free boundary is not known in advance but has to be determined as part of the solution of the pricing model.

At times close to expiry, the investor should choose to reset even when $F(t)$ is only slightly below $I(t)$, so we deduce that $x_n^*(T) = 1$. When the time to expiry is infinite, the threshold x_n^* has been determined by the analytic procedures proposed by Chu and Kwok (2003). It is obvious from intuition that $x_n^*(t)$ should be a monotonically increasing function of time t since the holder should reset at a lower threshold fund value as time is approaching maturity.

The detailed solution of $W_n(x, t)$ and $x_n^*(t)$ can be pursued by following the technique developed by Dai et al. (2003). In this paper, we would like to deduce the pricing model for the investment fund that allows infinite number of resets by taking $n \to \infty$ in pricing formulation (2.4).

2.2. Limit of infinite resets—automatic reset

If there were no limit on the number of resets, then the investor should reset whenever the value of the investment fund falls to that of the reference stock index. We call this scenario "automatic reset" since the reset policy becomes automatic. Mathematically, this corresponds to $x_\infty^*(t) = 1$ for all $t < T$. This is seen to be a solution to the equation: $W_\infty(x_\infty^*, t) = x_\infty^* W_\infty(1, t)$ [see Eq. (2.4)]. Interestingly, in the limit of $n \to \infty$, the free boundary value problem posed in Eq. (2.4) becomes a fixed boundary value problem.

We let $V_\infty(F, I, t)$ denote the price function of the finite-lived investment fund at the grant-date that allows infinite number of resets. When the time to expiry tends to infinity, the price function of the perpetual counterpart has been determined by Gerber and Shiu (2003). When the reference index becomes a constant, the corresponding price function has been obtained by Gerber and Pafumi (2000). In this paper, we extend these two previous results by generalizing the pricing model to stochastic guarantee and finite time horizon.

For convenience, we define $W_\infty(y, \tau) = V_\infty(F, I, t)/F$, where $y = \ln x = \ln(I/F)$ and $\tau = T - t$. From Eq. (2.4), the governing equation and auxiliary conditions for $W_\infty(y, \tau)$ are deduced to be

$$\frac{\partial W_\infty}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 W_\infty}{\partial y^2} + \mu\frac{\partial W_\infty}{\partial y} - q_p W_\infty, \quad \tau > 0,\ y < 0;$$
$$\frac{\partial W_\infty}{\partial y}(0, \tau) = W_\infty(0, \tau), \qquad W_\infty(y, 0) = 1, \tag{2.5}$$

where $\mu = q_p - q_i - (\sigma^2/2)$. Note that the free boundary $x_n^*(t)$ becomes the fixed boundary $y = \ln x_\infty^*(t) = \ln 1 = 0$.

The Robin boundary condition at $y = 0$ leads to a slight complication in the solution procedure (the outline of which is presented in Appendix A). The analytic representation of the solution $W_\infty(y, \tau)$ admits different forms, depending on whether $q_p \neq q_i$ or $q_p = q_i$ [see Eqs. (A.2a,b)]. Let $\alpha = 2(q_i - q_p)/\sigma^2$ and $\tilde{\mu} = \mu + \sigma^2$, the price function $V_\infty(F, I, t)$ is found to be

(i) $q_p \neq q_i$

$$V_\infty(F, I, t) = I\,\mathrm{e}^{-q_i\tau}\left(1 - \frac{1}{\alpha}\right)N\left(\frac{\ln(I/F) + \tilde{\mu}\tau}{\sigma\sqrt{\tau}}\right) + \frac{I}{\alpha}\left(\frac{I}{F}\right)^{\alpha}\mathrm{e}^{-q_p\tau}N\left(\frac{\ln(I/F) - \mu\tau}{\sigma\sqrt{\tau}}\right)$$
$$+ F\,\mathrm{e}^{-q_p\tau}N\left(\frac{-\ln(I/F) - \mu\tau}{\sigma\sqrt{\tau}}\right), \quad F > I. \tag{2.6a}$$

(ii) $q_p = q_i$ (write the common dividend yield as q)

$$V_\infty(F, I, t) = I\,\mathrm{e}^{-q\tau}\sigma\sqrt{\tau}n\left(\frac{\ln(I/F) + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right) + I\,\mathrm{e}^{-q\tau}\left(\ln\frac{I}{F} + 1 + \frac{\sigma^2\tau}{2}\right)N\left(\frac{\ln(I/F) + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right)$$
$$+ F\,\mathrm{e}^{-q\tau}N\left(\frac{-\ln(I/F) + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right), \quad F > I. \tag{2.6b}$$

Remarks.

1. When we set $I = K, q_i = r, q_p = 0$ and $\sigma_i = 0$, Eq. (2.6a) reduces to the price formula (2.10) in Gerber–Pafumi's paper (2000). Their formula corresponds to a constant guarantee level K instead of dynamic protection against a stochastic stock index. Also, we include the modified representation of the price formula [see Eq. (2.6b)] corresponding to the special case when $q_p = q_i$.
2. The Robin boundary condition: $(\partial W_\infty/\partial y)(0, \tau) = W_\infty(0, \tau)$ in Eq. (2.5) can be expressed as $(\partial V_\infty/\partial F)(I, F, t)|_{F=I} = 0$. If the index value is taken to be the constant value K, then the Neumann condition: $(\partial V_\infty/\partial F)(F, t)|_{F=K} = 0$ is equivalent to the reflecting boundary condition placed at the guarantee level K. This gives a stronger version of justification on the claim made by Gerber and Pafumi (2000) that the protected (modified) fund process can be obtained by placing a reflecting boundary at the guarantee level K on the primary fund process. The value of the protected fund can be visualized to be the same as that of the European barrier call option with zero strike and a reflecting down-barrier at $F = K$.
3. When $q_p = 0$, we observe that

$$\lim_{F\to\infty} V_\infty(F, I, t) = F \tag{2.7a}$$

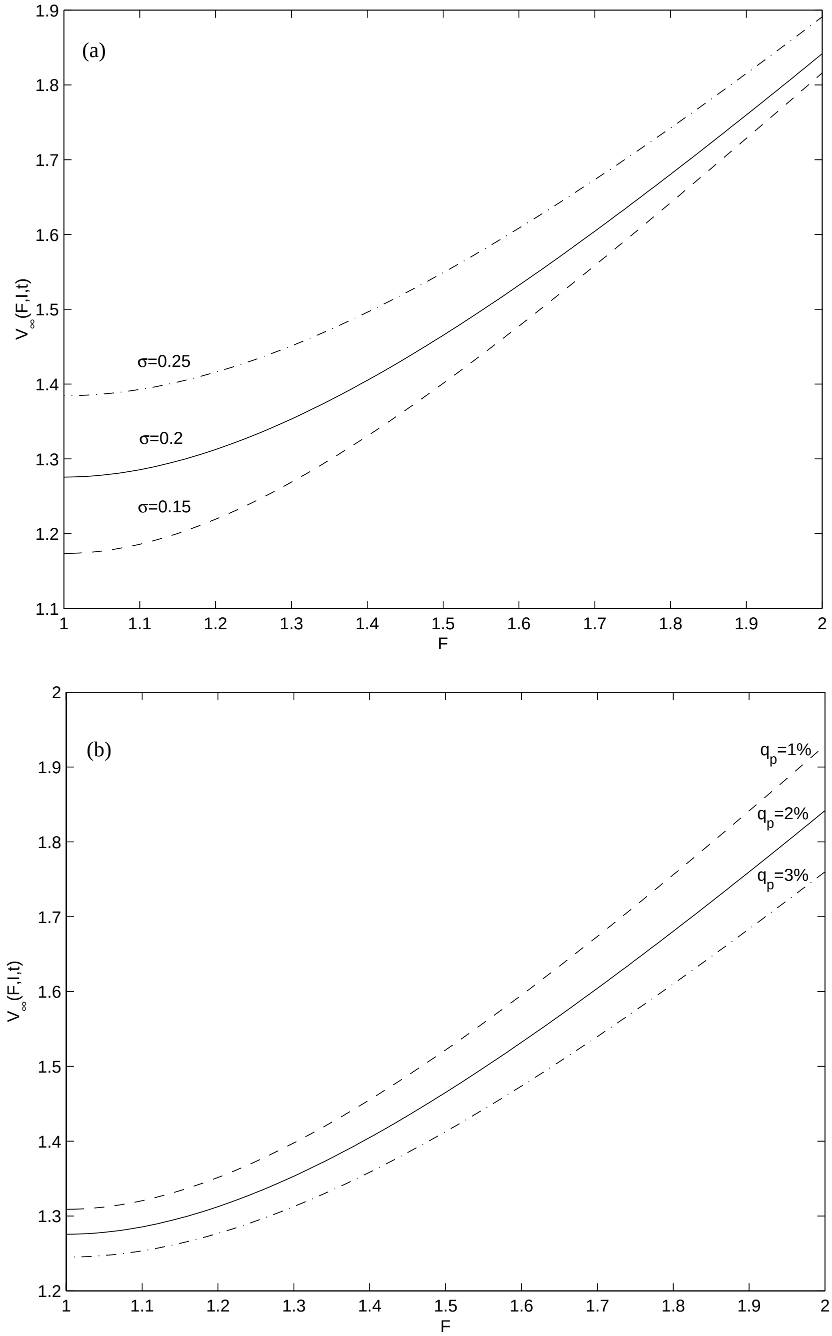

Fig. 1. (a) The price function $V_\infty(F, I, t)$ is an increasing function of fund value F and volatility σ, but the sensitivity to σ decreases as F increases. (b) The price function $V_\infty(F, I, t)$ is a decreasing function of fund's dividend yield q_p, and the sensitivity to q_p increases as the fund value F increases. (c) The delta $\partial V_\infty/\partial F$ has higher value for the shorter-lived fund. At a lower value of F, the longer-lived fund is more expensive since the insurance value provided by the guarantee is higher.

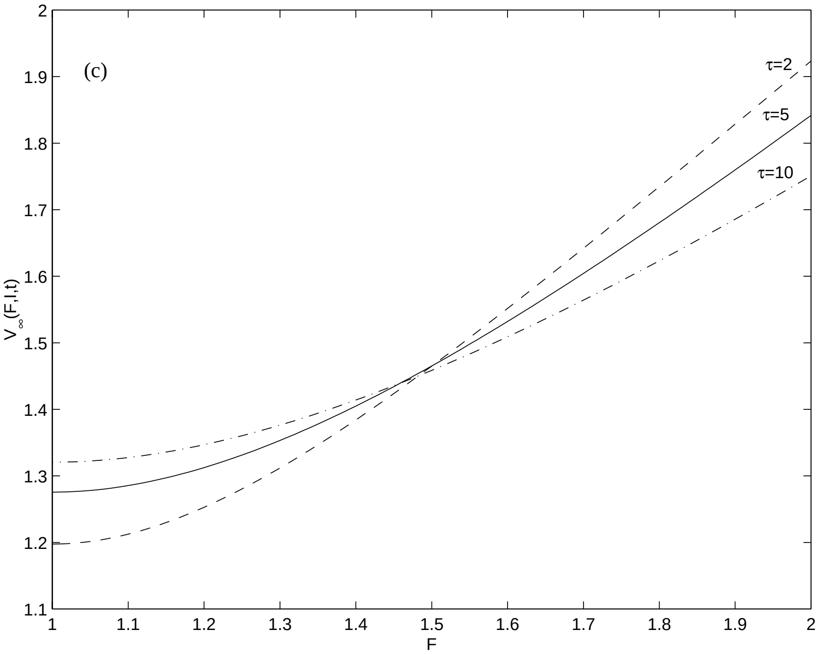

Fig. 1. (*Continued*).

or equivalently,

$$\lim_{y\to-\infty} W_\infty(y,t) = 1. \tag{2.7b}$$

When the primary fund value is very high and there are no dividends paid by the primary fund, then the benefit of the reset protection has zero value and there is no loss on dividends. In this case, holding the protected fund is equivalent to holding the primary fund.

Pricing behaviors of the protected funds on the grant-date. The plots in Fig. 1a–c reveal the dependence of the grant-date price function of the protected fund on different parameters of the pricing model. In Fig. 1a, we plot $V_\infty(F, I, t)$ against F for varying value of volatility σ. The other parameter values used in the calculations are: $I = 1$, $q_p = q_i = 0.02$, $\tau = 5$. The price function $V_\infty(F, I, t)$ is seen to be an increasing function of F and σ, but the sensitivity to σ decreases as F increases. At a higher fund value, the chance of taking advantage of the fund protection is less so the insurance value associated with the protection becomes less sensitive to volatility. The delta $\partial V_\infty/\partial F$ tends to zero as F approaches I ($I = 1$) and tends to one as F stays further away from I. These behaviors on $\partial V_\infty/\partial F$ agree with the prescription of boundary conditions of the pricing model. Fig. 1b shows the plot of $V_\infty(F, I, t)$ against F for varying value of fund's dividend yield q_p. The values of other parameters are: $I = 1$, $q_i = 0.02$, $\sigma = 0.2$, $\tau = 5$. The price function $V_\infty(F, I, t)$ is a decreasing function of q_p and the sensitivity to q_p increases as the fund value increases. This is because higher value of dividend yield q_p leads to slower expected rate of growth of the fund, but the holder cannot receive the dividend payouts. At a lower value of F, the drop in expected rate of growth of the fund is likely to be compensated by the fund guarantee clause so the price function becomes less sensitive to q_p. The plots of $V_\infty(F, I, t)$ against F in Fig. 1c reveals the sensitivity of the price function to varying length of time to maturity τ. The delta $\partial V_\infty/\partial F$ has a higher value for the shorter-lived fund. At lower value of F, the longer-lived fund is more expensive since the insurance value provided by the guarantee clause is higher.

2.3. Mid-contract valuation

Let M denote the path-dependent state variable which represents the realized maximum value of the state variable x from the grant-date to the mid-contract time t, that is, $M = \max_{0\le u\le t}(I(u)/F(u))$. At any mid-contract time, the number of units of primary fund held in the investment fund is given by

$$n(t) = \begin{cases} 1 & \text{if } M \le 1, \\ M & \text{if } M > 1. \end{cases} \tag{2.8}$$

If the primary fund value has been staying above or at the reference stock index so far (corresponding to $M \le 1$), then upgrade has never occurred, so the number of units of primary fund remains at one. Otherwise, the number of units is upgraded to M.

Let $V_{\text{mid}}(F, I, M, t)$ denote the mid-contract investment fund value at time t, with dependence on the state variable M. Both $V_{\text{mid}}(F, I, M, t)$ and $V_\infty(F, I, t)$ satisfy the same two-state Black–Scholes equation, namely,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{F,I}\right) V_{\text{mid}}(F, I, M, t) = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} + \mathcal{L}_{F,I}\right) V_\infty(F, I, t) = 0, \tag{2.9a}$$

where

$$\mathcal{L}_{F,I} = \frac{\sigma_p^2}{2}F^2\frac{\partial^2}{\partial F^2} + \rho\sigma_p\sigma_i FI\frac{\partial^2}{\partial F\partial I} + \frac{\sigma_i^2}{2}I^2\frac{\partial^2}{\partial I^2} + (r - q_p)F\frac{\partial}{\partial F} + (r - q_i)I\frac{\partial}{\partial I} - r. \tag{2.9b}$$

The terminal payoff of the investment fund value at maturity T is given by $F\max(M, 1)$, a payoff structure that involves both F and M. The valuation of the mid-contract value may seen to be quite involved, but economic intuition may help us to express the mid-contract value $V_{\text{mid}}(F, I, M, t)$ in terms of the grant-date value $V_\infty(F, I, t)$ [see Eqs. (2.6a,b)].

When $M > 1$, the number of units of primary fund is increased to M so that the investment fund is equivalent to one unit of "new" primary fund having fund value MF. When $M \le 1$, V_{mid} is insensitive to M since the terminal payoff value will not be dependent on the current realized maximum value M. That is, V_{mid} remains constant at different values of M, for all $M \le 1$. By continuity of the price function with respect to the variable M, V_{mid} at $M = 1$ is equal to the limiting value of V_{mid} (corresponding to the regime: $M > 1$) as $M \to 1^+$. In summary, we have

$$V_{\text{mid}}(F, I, M, t) = V_\infty(\max(M, 1)F, I, t) = \begin{cases} V_\infty(F, I, t) & M \le 1, \\ V_\infty(MF, I, t) & M > 1. \end{cases} \tag{2.10}$$

The details of the theoretical justification to the above formula are given in Appendix B.

2.3.1. Cost to the sponsor

Let $U_{\text{grant}}(F, I, t)$ and $U_{\text{mid}}(F, I, M, t)$ denote the cost to the sponsor that offers the dynamic protection at the grant-date and at mid-contract time, respectively. The terminal payoff $U_{\text{mid}}(F, I, M, T)$ is given by

$$U_{\text{mid}}(F, I, M, T) = V_{\text{mid}}(F, I, M, T) - F = \max(M - 1, 0)F. \tag{2.11}$$

Note that both $U_{\text{mid}}(F, I, M, t)$ and $V_{\text{mid}}(F, I, M, t)$ satisfy the same Black–Scholes equation and auxiliary condition. We claim that

$$U_{\text{mid}}(F, I, M, t) = V_{\text{mid}}(F, I, M, t) - F\,e^{-q_p(T-t)}, \tag{2.12}$$

since both $V_{\text{mid}}(F, I, M, t)$ and the term $F\,e^{-q_p(T-t)}$ satisfy the Black–Scholes equation and the terminal payoff condition (2.11) is satisfied. In terms of financial interpretation, a factor $e^{-q_p(T-t)}$ appears in front of F since the holder of the protected fund does not receive the dividends paid by the primary fund.

At the grant-date, we have $F \geq I$ so that $M = I/F \leq 1$. By virtue of Eqs. (2.10) and (2.12), we obtain

$$U_{\text{grant}}(F, I, t) = U_{\text{mid}}(F, I, M, t) = V_{\infty}(F, I, t) - F\,\mathrm{e}^{-q_p(T-t)}. \tag{2.13}$$

Furthermore, by combining Eqs. (2.10), (2.12) and (2.13), the two cost functions $U_{\text{mid}}(F, I, M, t)$ and $U_{\text{grant}}(F, I, t)$ are related by

$$U_{\text{mid}}(F, I, M, t) = U_{\text{grant}}(\max(M, 1)F, I, t) + \max(M - 1, 0)F\,\mathrm{e}^{-q_p(T-t)}. \tag{2.14}$$

The last term in Eq. (2.14) gives the present value of additional units of primary fund supplied by the sponsor due to the protection clause. The sponsor has to add $M - 1$ units of primary fund when $M > 1$, but supplements nothing when $M \leq 1$.

2.3.2. Rollover hedging strategy—replenishment premium

We can derive an integral representation of the price function $V_{\text{mid}}(F, I, M, t)$ using the concept of rollover hedging strategy and replenishment premium (Wong and Kwok, 2003). First, we define the stochastic process

$$\hat{M}_t^{t'} = \max\left(\max_{t \leq u \leq t'} \frac{I(u)}{F(u)}, 1\right) \tag{2.15a}$$

and at the current time t, the following quantity

$$\hat{M}_0^{t} = \max\left(\max_{0 \leq u \leq t} \frac{I(u)}{F(u)}, 1\right) = \max(M, 1) \tag{2.15b}$$

is known. The terminal payoff of the protected fund can be expressed as $\max(\hat{M}_0^t, \hat{M}_t^T)F_T$, and we write F as the current value of the primary fund.

At the current time t, we hold a replicating portfolio that contains $\mathrm{e}^{-q_p(T-t)}\hat{M}_0^t$ units of the primary fund. This portfolio will grow to $\hat{M}_0^t$ units of fund at maturity. Suppose $\hat{M}_t^T \leq \hat{M}_0^t$, then this portfolio can fully replicate the terminal payoff of the protected fund. However, if $\hat{M}_t^T > \hat{M}_0^t$, then the terminal payoff is higher than $\hat{M}_0^t F_T$; and correspondingly, the replicating portfolio becomes sub-replication. By sub-replication, we mean that the terminal payoff of the replicating portfolio may fall short of the terminal payoff of the derivative instrument being replicated under certain scenarios.

We adopt the following rollover hedging strategy to achieve full replication. We increase the number of units of fund to $\hat{M}_t^u\,\mathrm{e}^{-q_p(T-u)}$ whenever a higher realized maximum value of $\hat{M}_t^u$ occurs at time u, where $t \leq u \leq T$. This rollover strategy would guarantee that the number of units of fund at maturity T is $\max(\hat{M}_0^t, \hat{M}_t^T)$. Throughout the replenishment procedure for achieving full replication, some costs would be incurred to acquire additional units of fund. The corresponding present value of the cost or replenishment premium is

$$\mathrm{e}^{-q_p(T-t)}FE[\max(\hat{M}_t^T - \hat{M}_0^t, 0)] = \mathrm{e}^{-q_p(T-t)}F\int_{\hat{M}_0^t}^{\infty} P[\hat{M}_t^T \geq \xi]\,\mathrm{d}\xi. \tag{2.16}$$

The last formula is obtained by using the well-known result that the expectation of a positive random variable is the integral of its tail probabilities. The value of the protected fund is the sum of the value of the sub-replicating portfolio and the replenishment premium. We then obtain an alternative analytic representation of the mid-contract price function

$$V_{\text{mid}}(F, I, M, t) = \max(M, 1)\,\mathrm{e}^{-q_p(T-t)}F + \mathrm{e}^{-q_p(T-t)}F\int_{\max(M,1)}^{\infty} P[\hat{M}_t^T \geq \xi]\,\mathrm{d}\xi. \tag{2.17}$$

Suppose I/F follows the process defined by Eq. (2.2b); then for $\xi \geq 1$, we have

$$P[\hat{M}_t^T \geq \xi] = P\left[\max_{t \leq u \leq T} \frac{I(u)}{F(u)} \geq \xi\right] = \mathrm{e}^{2\mu\xi/\sigma^2} N\left(-\frac{\xi + \mu\tau}{\sigma\sqrt{\tau}}\right) + N\left(-\frac{\xi - \mu\tau}{\sigma\sqrt{\tau}}\right), \tag{2.18}$$

where $\mu = q_p - q_i - (\sigma^2/2)$ and $\sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_i + \sigma_i^2$.

3. Protected funds with withdrawal right

In this section, we consider the protected funds with the embedded withdrawal right to receive the primary fund prematurely. The withdrawal right resembles the early exercise feature of an American option where the decision of early withdrawal is optimally determined by the holder. The pricing model now becomes an optimal stopping problem. The threshold value F^* at which the holder chooses to exercise optimally is not known in advance, but has to be determined as part of the solution to the optimal stopping model. In general, F^* depends on the reference index fund value I and time t.

3.1. Pricing formulation and analytic properties of the price functions

Let $\hat{V}_{\text{grant}}(F, I, t)$ and $\hat{V}_{\text{mid}}(F, I, M, t)$ denote the price function of the protected fund with withdrawal right at the grant-date and during mid-contract time, respectively. At the withdrawal threshold F^*, the payoff of the investment fund is equal to $\max(M, 1)F^*$. Also, from Bellman's optimality condition, we impose the smooth pasting condition at $F = F^*$. The formulation of $\hat{V}_{\text{grant}}(F, I, t)$ and $\hat{V}_{\text{mid}}(F, I, M, t)$ are presented as follows:

(i) grant-date price function: $\hat{V}_{\text{grant}}(F, I, t)$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{F,I}\right)\hat{V}_{\text{grant}} = 0, \quad 0 < I < \infty, \ \ 0 < I < F < F^*_{\text{grant}}(I, t), \ \ t < T,$$

with auxiliary conditions:

$$\hat{V}_{\text{grant}}(F, I, T) = F, \qquad \left.\frac{\partial \hat{V}_{\text{grant}}}{\partial F}\right|_{F=I} = 0, \qquad \hat{V}_{\text{grant}}|_{F=F^*_{\text{grant}}} = F^*_{\text{grant}} \quad \text{and} \quad \left.\frac{\partial \hat{V}_{\text{grant}}}{\partial F}\right|_{F=F^*_{\text{grant}}} = 1. \tag{3.1}$$

The model resembles an American option with a downside reflecting barrier at $F = I$. Like the non-withdrawal counterpart, the factor $\max(M, 1)$ does not appear in the terminal payoff and exercise payoff once the reflecting barrier at $F = I$ is imposed.

(ii) mid-contract price function: $\hat{V}_{\text{mid}}(F, I, M, t)$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{F,I}\right)\hat{V}_{\text{mid}} = 0, \quad 0 < I, M < \infty, \ \ 0 < \frac{I}{M} < F < F^*_{\text{mid}}(I, M, t), \ \ t < T,$$

with auxiliary conditions:

$$\hat{V}_{\text{mid}}(F, I, M, T) = \max(M, 1)F, \qquad \left.\frac{\partial \hat{V}_{\text{mid}}}{\partial M}\right|_{M=I/F} = 0,$$

$$\hat{V}_{\text{mid}}|_{F=F^*_{\text{mid}}} = \max(M, 1)F^*_{\text{mid}} \quad \text{and} \quad \left.\frac{\partial \hat{V}_{\text{mid}}}{\partial F}\right|_{F=F^*_{\text{mid}}} = \max(M, 1). \tag{3.2}$$

Remarks.

1. Compared to V_∞ and V_{mid}, the pricing formulation for $\hat{V}_{\text{grant}}$ and $\hat{V}_{\text{mid}}$ have the additional imposition of the value matching and smooth paste conditions at the withdrawal threshold F^*. The pricing model for $\hat{V}_{\text{mid}}$ resembles an American lookback option model.

2. The derivative boundary condition: $(\partial \hat{V}_{\text{mid}}/\partial M)|_{M=I/F} = 0$ is typical for option models with lookback payoff. It arises from the observation that if the current value of I/F equals the current realized maximum value M then $\hat{V}_{\text{mid}}$ is insensitive to M. This is because the probability that M is smaller than the realized maximum value of I/F over the whole contract period is one almost surely.

3.1.1. *Relation between the mid-contract and grant-date price functions*

One can show that the price functions $\hat{V}_{\text{mid}}$ and $\hat{V}_{\text{grant}}$ are related by

$$\hat{V}_{\text{mid}}(F, I, M, t) = \hat{V}_{\text{grant}}(\max(M, 1)F, I, t) = \begin{cases} \hat{V}_{\text{grant}}(F, I, t) & M \le 1, \\ \hat{V}_{\text{grant}}(MF, I, t) & M > 1 \end{cases} \tag{3.3}$$

and the withdrawal thresholds F^*_{mid} and F^*_{grant} are related by

$$F^*_{\text{mid}}(I, M, t) = \frac{F^*_{\text{grant}}(I, t)}{\max(M, 1)}. \tag{3.4}$$

The proofs of the above two relations (3.3) and (3.4) are presented in Appendix C. The protected fund at mid-contract can be visualized to be identical to the grant-date protected fund contract but having $\max(M, 1)$ units of the original primary fund as the underlying primary fund.

3.1.2. *Valuation of the grant-date price function*

Similar to $V_\infty(F, I, t)$, the two-state pricing model of $\hat{V}_{\text{grant}}$ can be reduced to a one-state model by choosing F as the numeraire. We define

$$y = \ln \frac{I}{F} \quad \text{and} \quad \hat{W}(y, \tau) = \frac{\hat{V}_{\text{grant}}(F, I, t)}{F}, \quad \tau = T - t, \tag{3.5}$$

the formulation in Eq. (3.1) can be written as

$$\frac{\partial \hat{W}}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 \hat{W}}{\partial y^2} + \mu \frac{\partial \hat{W}}{\partial y} - q_p \hat{W}, \quad y^*(\tau) < y < 0, \quad \tau > 0,$$

with auxiliary conditions:

$$\hat{W}(y, 0) = 1, \qquad \frac{\partial \hat{W}}{\partial y}(0, \tau) = \hat{W}(0, \tau), \qquad \hat{W}(y^*(\tau), \tau) = 1, \qquad \frac{\partial \hat{W}}{\partial y}(y^*(\tau), \tau) = 0, \tag{3.6}$$

where $y^*(\tau)$ is the free boundary. By solving the above optimal stopping problem, we obtain the following decomposition formula:

$$\hat{W}(y, \tau) = W_\infty(y, \tau) + W_E(y, \tau; y^*(\tau)), \tag{3.7}$$

where $W_\infty(y, \tau)$ is given by Eqs. (2.6a,b) and $W_E(y, \tau; y^*(\tau))$ is given by

$$W_E(y, \tau; y^*(\tau)) = q_p \int_0^\tau \mathrm{e}^{-q_p u} G(y, u; y^*(\tau - u))\, \mathrm{d}u, \tag{3.8}$$

where the kernel function $G(y, \tau; \xi)$ satisfies the following equation:

$$\frac{\partial G}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 G}{\partial y^2} + \mu \frac{\partial G}{\partial y}, \quad y < 0, \quad \tau > 0, \qquad G(y, 0; \xi) = \mathbf{1}_{\{y<\xi\}} \text{ and } G(0, \tau; \xi) = \frac{\partial G}{\partial y}(0, \tau; \xi), \quad \xi < 0. \tag{3.9}$$

The quantity $FW_E(\ln(I/F), T-t)$ gives the integral representation of the early withdrawal premium. When $q_p = 0$, the withdrawal premium becomes zero implying that it is never optimal for the holder to exercise the withdrawal right.

The solution to $G(y, \tau; \xi)$ takes different forms depending on $q_p \neq q_i$ or $q_p = q_i$ (see Appendix D for the details of derivation):

(i) $q_p \neq q_i$

$$G(y,\tau;\xi) = \exp(y-\xi+(q_p-q_i)\tau)\frac{\tilde{\mu}}{q_p-q_i}\left[N\left(\frac{\tilde{\mu}\tau-\xi}{\sigma\sqrt{\tau}}\right)-1\right]+\frac{1}{\alpha}\mathrm{e}^{-2\mu y/\sigma^2}N\left(\frac{\xi-\mu\tau}{\sigma\sqrt{\tau}}\right) + N\left(\frac{\xi-y-\mu\tau}{\sigma\sqrt{\tau}}\right)+\exp(y-\xi+(q_p-q_i)\tau)\frac{\xi\tilde{\mu}}{\sigma(q_i-q_p)}\int_0^\tau u^{-3/2}n\left(\frac{\tilde{\mu}u-\xi}{\sigma\sqrt{u}}\right)\,\mathrm{d}u + \frac{\tilde{\mu}}{q_i-q_p}\exp\left(y-\alpha\left(\xi+\frac{\sigma^2\tau}{2}\right)\right)\left[N\left(\frac{\xi+\tilde{\mu}\tau}{\sigma\sqrt{\tau}}\right)-N\left(\frac{y+\xi+\tilde{\mu}\tau}{\sigma\sqrt{\tau}}\right)\right], \tag{3.10a}$$

(ii) $q_p = q_i$

$$G(y,\tau;\xi) = \mathrm{e}^y\left\{(2-\xi)N\left(\frac{\xi+(\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right)+\int_0^\tau\left[\frac{(4+\sigma^2 u)\xi}{4\sigma u^{3/2}}-\frac{\sigma^3\sqrt{u}}{8}\right]n\left(\frac{\xi+(\sigma^2 u/2)}{\sigma\sqrt{u}}\right)\mathrm{d}u + \left(1+\xi+y+\frac{\sigma^2\tau}{2}\right)N\left(\frac{\xi+y+(\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right) + \sigma\sqrt{\tau}\left[n\left(\frac{\xi+y+(\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right)-n\left(\frac{\xi+(\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right)\right]\right\}+N\left(\frac{\xi-y+(\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right). \tag{3.10b}$$

Remark. Suppose we set $\xi = 0$ in Eq. (3.9), $G(y, \tau; 0)$ is seen to be $\mathrm{e}^{q_p\tau}W_\infty(y, \tau)$. Note that $G(y, \tau; 0)$ cannot be obtained directly by taking the limit $\xi \to 0^-$ in $G(y, \tau; \xi)$.

3.1.3. Integral equation for the free boundary

By setting $\hat{W}^*(y^*(\tau), \tau) = 1$, we obtain the following integral equation for the free boundary $y^*(\tau)$:

$$1 = W_\infty(y^*(\tau),\tau) + q_p\int_0^\tau \mathrm{e}^{-q_p u}G(y^*(\tau), u; y^*(\tau-u))\,\mathrm{d}u. \tag{3.11}$$

There is no known closed form solution for $y^*(\tau)$. One has to resort to numerical calculations to compute $y^*(\tau)$.

By virtue of Eq. (2.7b), the solution to $y^*(\tau)$ when $q_p = 0$ is given by

$$y^*(\tau) = -\infty \quad \text{for all } \tau, \tag{3.12}$$

implying $F^*(I, t) = \infty$ for all τ. This gives another theoretical justification why it is never optimal to withdraw prematurely when the primary fund does not pay dividend. Indeed, we have seen earlier that the early withdrawal premium equals zero when $q_p = 0$ (see Imai and Boyle, 2001; Gerber and Shiu, 2003 for alternative arguments to arrive at the same conclusion).

3.1.4. Cost to the sponsor

We consider the cost to the sponsor, denoted by the function $\hat{U}_{\text{mid}}(F, I, M, t)$, of providing the protection guarantee and the withdrawal right. Unlike $\hat{V}_{\text{mid}}$, the formulation for $\hat{U}_{\text{mid}}$ is not an optimal stopping problem since the withdrawal threshold is determined by the holder but not by the sponsor. To solve for $\hat{U}_{\text{mid}}$, the withdrawal threshold $F^*_{\text{mid}}(I, M, t)$ should be obtained as the first step by solving the optimal stopping problem (3.2), which is then imposed as a barrier condition in the pricing model of $\hat{U}_{\text{mid}}$. The formulation of $\hat{U}_{\text{mid}}(F, I, M, t)$ is given by

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{F,I}\right)\hat{U}_{\text{mid}} = 0, \quad 0 < I, \quad M < \infty, \quad 0 < \frac{I}{M} < F < F^*_{\text{mid}}(I, M, t), \quad t < T;$$
$$\hat{U}_{\text{mid}}(F, I, M, T) = \max(M-1, 0)F, \qquad \left.\frac{\partial \hat{U}_{\text{mid}}}{\partial M}\right|_{M=I/F} = 0 \text{ and } \hat{U}_{\text{mid}}(F^*_{\text{mid}}, I, M, t) = \max(M-1, 0)F^*_{\text{mid}}. \tag{3.13}$$

Note that there is no smooth pasting condition at $F = F^*_{\text{mid}}$ since the threshold F^*_{mid} is exogenously imposed as a barrier. As there is no explicit formula for $F^*_{\text{mid}}(I, M, t)$, we cannot obtain closed form analytic representation of $\hat{U}_{\text{mid}}(F, I, M, t)$. To solve for $\hat{U}_{\text{mid}}(F, I, M, t)$, one has to solve a two-state option pricing model. The inclusion of the state variable M is necessary since the early withdrawal right is path-dependent. The solution to $\hat{U}_{\text{mid}}$ can be obtained via numerical calculations alongside the computation that solves for $\hat{V}_{\text{mid}}$ and F^*_{mid}. Note that $\hat{U}_{\text{grant}}(F, I, t)$ is obtained by setting $M = I/F$ in $\hat{U}_{\text{mid}}(F, I, M, t)$.

Since the holder forfeits the guaranteed protection upon withdrawal, $\hat{U}_{\text{mid}}$ and $\hat{V}_{\text{mid}}$ do not have relation similar to that given in Eq. (2.12), but rather they are related by

$$\hat{U}_{\text{mid}}(F, I, M, t) < \hat{V}_{\text{mid}}(F, I, M, t) - F\,\mathrm{e}^{-q_p(T-t)}. \tag{3.14}$$

The above relation can be proved by applying the comparison principle in partial differential equation theory, which involves the comparison of the formulations of the two functions: $\hat{V}_{\text{mid}}$ and $\hat{U}_{\text{mid}} + F\,\mathrm{e}^{-q_p(T-t)}$. On the other hand, $\hat{U}_{\text{mid}}(F, I, M, t)$ should always be greater than $U_{\text{mid}}(F, I, M, t)$ due to the withdrawal premium.

3.2. Impact of proportional fees

We consider the scenario where the investor has to pay proportional fees at the rate p to the sponsor throughout the life of the fund. That is, the investor pays an amount $p \max(M, 1)F\,dt$ over the time interval $(t, t + dt)$, where $\max(M, 1)F$ is the *modified* fund value at time t. Let $V_p(F, I, t)$ denote the protected fund value at the grant-date with withdrawal right and payment of proportional fees at the rate p. The formulation for $V_p(F, I, t)$ resembles that of $\hat{V}_{\text{grant}}(F, I, t)$ [see Eq. (3.1)] except that the governing differential equation has to append on extra depletion term $-pF$. Even with the presence of the depletion terms, pricing relation like that in Eq. (3.3) still holds between the grant-date and mid-contract value functions. Suppose we define

$$W_p(y, \tau) = \frac{V_p(F, I, t)}{F}, \quad \text{where } y = \ln\frac{I}{F} \text{ and } \tau = T - t, \tag{3.15}$$

then the formulation for $W_p(y, \tau)$ can be expressed as

$$\frac{\partial W_p}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 W_p}{\partial y^2} + \mu\frac{\partial W_p}{\partial y} - q_p W_p - p, \quad y^*_p(\tau) < y < 0, \quad \tau > 0,$$

with auxiliary conditions:

$$W_p(y, 0) = 1, \qquad \frac{\partial W_p}{\partial y}(0, \tau) = W_p(0, \tau), \qquad W_p(y^*_p(\tau), \tau) = 1, \qquad \frac{\partial W_p}{\partial y}(y^*_p(\tau), \tau) = 0. \tag{3.16}$$

Here, $y^*_p(\tau)$ is the free boundary and $-p$ is the depletion term reflecting the payment of proportional fees. The withdrawal threshold $F^*(I, t; p)$ and $y^*_p(\tau)$ are related by $y^*_p(\tau) = \ln(I/F^*(I, t; p))$. Similarly, the solution to $W_p(y, \tau)$ admits the following decomposition form:

$$W_p(y,\tau) = \mathrm{e}^{-q_p\tau}G(y,\tau;0) + q_p\int_0^\tau \mathrm{e}^{-q_pu}G(y,u;y_p^*(\tau-u))\,\mathrm{d}u - p\int_0^\tau \mathrm{e}^{-q_pu}[G(y,u;0) - G(y,u;y_p^*(\tau-u))]\,\mathrm{d}u, \tag{3.17}$$

where the first integral represents the early withdrawal premium while the second integral represents the loss in value due to payment of proportional fees. The integrand in the second integral contains the difference of two functions: $G(y,u;0) - G(y,u;y_p^*(\tau-u))$. This is because proportional fees payment will terminate upon the early withdrawal. In terms of the price functions $W_\infty(y,\tau)$ and $W_E(y,\tau;y^*(\tau))$ defined earlier [see Eqs. (A.2a,b) and Eq. (3.8)], we can express $W_p(y,\tau;p)$ in the following succinct form:

$$W_p(y,\tau) = W_\infty(y,\tau) + (q_p+p)W_E(y,\tau;y_p^*(\tau)) - p\int_0^\tau W_\infty(y,u)\,\mathrm{d}u. \tag{3.18}$$

Similar to Eq. (3.11), we can obtain an integral equation for $y_p^*(\tau)$ by setting $y = y_p^*(\tau)$ and $W_p(y_p^*(\tau),\tau) = 1$ in Eq. (3.18).

3.2.1. Properties of the free boundary, $y_p^*(\tau)$

1. From financial intuition, we expect that the fund value $V_p(F,I,t)$ decreases and the investor optimally withdraws at a lower threshold $F^*(I,t)$ when the rate of proportional fees is higher. The decreasing property of $V_p(F,I,t)$ with respect to p can be established by applying the comparison principle in partial differential equation theory and taking into account the increasing magnitude of the depletion term $-pF$ in the governing equation. The threshold $F^*(I,t;p)$ is obtained by the intersection of the value function curve $V_p(F,I,t)$ with the intrinsic value line: $V = F$. At a higher rate of proportional fees, the lowering of the value function curve causes its intersection with the intrinsic value line at a lower threshold $F^*(I,t)$ (see Fig. 2a). Since $y_p^*(\tau) = \ln(I/F^*(I,t;p))$, the free boundary $y_p^*(\tau)$ is monotonically increasing with respect to p.
2. Let $F^*_{\mathrm{mid}}(I,M,t;p)$ denote the withdrawal threshold value for the primary fund during mid-contract time. By following a similar argument as presented in Appendix C, we can establish

$$F^*(I,t;p) = \max(M,1)F^*_{\mathrm{mid}}(I,M,t;p). \tag{3.19}$$

3. Similar to a usual American option, a longer-lived protected fund with withdrawal right should be worth more than its shorter-lived counterpart. Hence, $V_p(F,I,t)$ is monotonically increasing with respect to the time to expiry, $\tau = T - t$. Also, a higher value function curve intersects the intrinsic value line at a higher withdrawal threshold so $F^*(I,t;p)$ is increasing with respect to τ. Correspondingly, $y_p^*(\tau)$ is a decreasing function of τ.
4. At time close to expiry, $\tau \to 0^+$, the investor should optimally withdraw at any fund value, for all $p \geq 0$ and $q_p \geq 0$ but p and q_p not both equal zero. This means that (see Appendix E for its proof)

$$y_p^*(0^+) = 0 \quad \text{for} \quad p \geq 0,\ q_p \geq 0 \text{ but } p \text{ and } q_p \text{ not both equal zero.} \tag{3.20}$$

5. In Appendix F, we explore the asymptotic behaviors of the value function $W_p(y,\tau)$ and the free boundary $y_p^*(\tau)$ at $\tau \to \infty$. The asymptotic solutions $W_p(y,\infty)$ and $y_p^*(\infty)$ exhibit a wide range of solution behaviors, depending on the choices of values for the parameters, q_i, q_p and p. In those cases where the free boundary $y_p^*(\tau)$ has a finite asymptotic limit as $\tau \to \infty$, the free boundary $y_p^*(\tau)$ exists for all $\tau > 0$. Some of the important properties of the asymptotic solutions are highlighted below:
   (a) When $q_i = p = 0$ and $q_p \geq 0$, $W_p(y,\infty)$ becomes infinite in value. The same result has been observed by Gerber and Shiu (2003).
   (b) When $q_p = p = 0$, the holder never withdraws prematurely, so that $F^*(I,t;p) = \infty$ or $y_p^*(\tau) = -\infty$ for all values of τ. At $\tau \to \infty$, the value function $W_p(y,\infty)$ is finite when $q_i > 0$ but becomes infinite in value when $q_i = 0$.

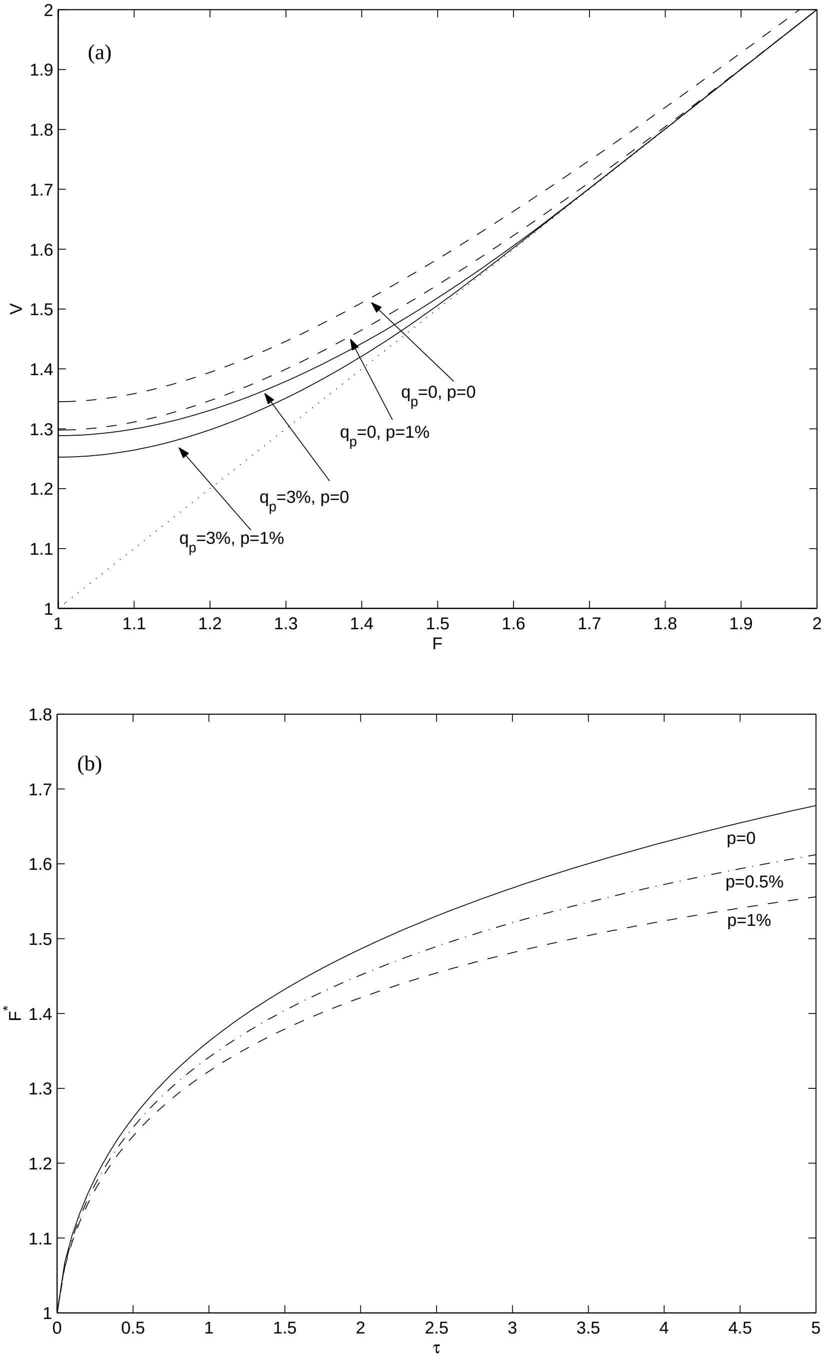

Fig. 2. (a) The value function V of the protected fund with withdrawal right and payment of proportional fees is plotted against the primary fund value F. When $q_p \geq 0$, $p \geq 0$ but q_p and p not both equal zero, the value curves touch tangentially the dotted line, which is the intrinsic value line. Other parameter values used in the calculations are: $I = M = 1$, $\tau = 5$, $q_i = 0.02$ and $\sigma = 0.2$. (b) The withdrawal threshold F^* is plotted against time to expiry τ for varying value of rate of proportional fee p. The holder should withdraw at a lower threshold when p increases.

(c) When $q_p = 0$ the behaviors of asymptotic solutions depend on $q_i = 0$ or $q_i > 0$.
   (i) When $q_i > 0$, both $W_p(y, \infty)$ and $y_p^*(\infty)$ exist for $p > 0$. However, for $p = 0$, only $W_p(y, \infty)$ exists but $y_p^*(\infty) = -\infty$.
   (ii) When $q_i = 0$, $W_p(y, \infty)$ and $y_p^*(\infty)$ exist for $p > 0$. For $p = 0$, both $W_p(y, \infty)$ and $y_p^*(\infty)$ do not exist.

(d) When $q_i > 0$ and $q_p > 0$, both $W_p(y, \infty)$ and $y_p^*(\infty)$ exist for all $p \geq 0$.

3.3. Numerical calculations

With embedded withdrawal right, the price of the protected fund does not admit a closed-form analytic formula since the free boundary $y_p^*(\tau)$ is not known in advance. We performed numerical calculations to obtain the value function $W_p(y, \tau)$ and $y_p^*(\tau)$ via the finite difference scheme and the recursive integration method. The recursive integration method solves for the free boundary $y_p^*(\tau)$ directly from the integral equation. The numerical procedure involves the numerical integration of the integral withdrawal premium term, then followed by solving recursively for $y^*(\tau)$ at successive time steps. In the finite difference calculations, in order to take care of the early withdrawal right, we incorporate the dynamic programming procedure of comparing the continuation value and value upon withdrawal at each lattice node. The purpose of adopting two different numerical approaches is to verify the analytic expressions appearing in the integral representation formulas of the early withdrawal premium and the cost of payment of proportional fees. The details on the construction of these two numerical algorithms can be found in Kwok's text (1998).

3.3.1. Comparison of numerical accuracy

In our numerical experiment to compare the numerical accuracy of the finite difference scheme and recursive integration method for computing $V_p(F, I, t)$, we chose the following set of parameter values in our pricing model: $\tau = 5$, $q_i = 0.02$, $q_p = 0.03$, $p = 0.01$, $\sigma = 0.2$. We take the numerical results obtained from the finite difference calculations using 2560 time steps as the "exact" solution and compare the numerical solution to $V_p(F, I, t)$ obtained using a fewer number of time steps or via recursive integration method at selected lattice points in the computational domain. The root mean square error (RMSE) of numerical results is computed based on the formula

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(V_{p,i} - V_{p,i}^{\text{exact}})^2}, \tag{3.21}$$

where N is the total number of lattice points chosen for comparison and $V_{p,i}$ the numerical solution to $V_p(F, I, t)$ at the ith lattice point.

In Table 1, we list the RMSE of the calculated results of the value function obtained from the finite difference calculations and recursive integration method with varying number of time steps. We observe reasonably good accuracy of the numerical results using either numerical algorithm. Since the finite difference scheme computes the numerical solution based on the dynamic programming procedure, which does not require the knowledge of the analytic integral representation formulas of the withdrawal premium and the cost of the payment of proportional fees, the consistency of numerical results from both methods serves to verify the accuracy of the integral formulas.

Table 1
Comparison of numerical accuracy of the finite difference scheme and recursive integration method for computing the value function $V_p(F, I, t)$. The parameter values used in the calculations are: $\tau = 5$, $q_i = 0.02$, $q_p = 0.03$, $p = 0.01$ and $\sigma = 0.2$

Number of time steps	40	160	640
RMSE in finite difference	1.8546×10^{-1}	2.1871×10^{-2}	6.8376×10^{-3}
Number of time steps	10	20	30
RMSE in recursive integration	2.0147×10^{-2}	9.1786×10^{-3}	5.5493×10^{-3}

3.3.2. Behaviors of the price function and withdrawal threshold

We performed numerical calculations to compute $V_p(F, I, t)$ and $F^*(I, t;\ p)$ of the protected fund with withdrawal right and payment of proportional fees. In Fig. 2a, we show the plot of $V_p(F, I, t)$ against F for varying values of q_p and p. The dotted line is the intrinsic value line: $V = F$. The set of parameter values used in the calculations are: $I = 1$, $\tau = 5$, $q_i = 0.02$ and $\sigma = 0.2$. When $q_p = p = 0$, the value curve always lies above the intrinsic value line showing that it is never optimal to withdraw prematurely. When $q_p \geq 0$, $p \geq 0$ but q_p and p not both equal zero, we observe that the value curve cuts the intrinsic value line tangentially (smooth paste property) at sufficiently high fund value F (withdrawal threshold fund value). Fig. 2(b) shows the plot of the withdrawal threshold F^* against time to expiry τ for varying value of rate of proportional fee p. The threshold value curves clearly reveal the two monotonicity properties of F^*, namely, F^* is an increasing function of τ and a decreasing function of p.

4. Conclusion

We have constructed and analyzed the pricing models for protected funds with reset and withdrawal rights. When the number of allowable resets tends to infinity, the protection clause becomes "automatic reset", that is, the fund value is upgraded automatically to the reference index value whenever a drop of fund value below the index value occurs. The pricing model of the protected fund with automatic reset resembles that of a lookback option model. We provide the justification why the protected (modified) fund process can be obtained by placing a reflecting boundary at the guarantee level on the primary fund process. We also derive the relation between the price functions of the grant-date value and mid-contract value of the protected fund. Concerning valuation of the protected fund value during mid-contract time, though the number of units of primary fund may have been increased by virtue of the automatic-reset clause, one can use the grant-date price function for valuation by taking the "modified" fund as the primary fund. We also illustrate how to apply the rollover hedging strategy to derive an analytic representation of the mid-contract price function.

With embedded withdrawal right, the pricing model of the protected fund with automatic reset becomes a free boundary value problem. This is because the optimal withdrawal policy adopted by the fund holder is not known in advance. Rather, the withdrawal threshold has to be determined as part of the solution procedure. We derive the integral representation of the withdrawal premium in terms of the withdrawal threshold values. We also obtain the integral equation that determines the withdrawal threshold values at different times. When the primary fund is non-dividend paying, we provide the mathematical justification why it is never optimal to exercise the withdrawal right prematurely.

We have considered the impact of payment of proportional fees by the fund holder to the sponsor. We examine the characterization of the early withdrawal policy under the combination of withdrawal right and payment of proportional fees, in particular, the behaviors of the free exercise boundary at times close to expiration and infinitely far from expiration. The withdrawal threshold values exhibit two monotonicity properties, namely, the holder should optimally withdraw at a higher threshold value with increasing value of time to expiry and at a lower threshold value with increasing rate of proportional fees.

Finite difference scheme and recursive integral method were employed to compute the price functions and withdrawal threshold values. The comparison of numerical accuracy revealed good agreement between the numerical results obtained from both numerical methods. The consistency of the numerical results also serve to verify the accuracy of the analytic representation formulas of the value functions.

Appendix A. Derivation of $W_\infty(y, \tau)$

We perform the continuation of the initial condition to the whole domain $(-\infty, \infty)$, where

$$W_\infty(y, 0) = \begin{cases} 1 & \text{if } y < 0, \\ \psi(y) & \text{if } y \geq 0. \end{cases}$$

The function $\psi(y)$ is determined such that the Robin boundary condition is satisfied. Let $g(y, \tau; \xi)$ denote the fundamental solution to the governing differential equation, where

$$g(y, \tau; \xi) = \frac{\mathrm{e}^{-q_p\tau}}{\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(\xi - y - \mu\tau)^2}{2\sigma^2\tau}\right).$$

The solution to $W_\infty(y, \tau)$ can be formally expressed as

$$W_\infty(y, \tau) = \int_{-\infty}^{\infty} W_\infty(\xi, 0)g(y, \tau; \xi)\,\mathrm{d}\xi = \mathrm{e}^{-q_p\tau}\left[1 - N\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right)\right] + \int_0^{\infty} \psi(\xi)g(y, \tau; \xi)\,\mathrm{d}\xi. \tag{A.1}$$

Performing the differentiation with respect to y on both sides of Eq. (A.1), we obtain

$$\frac{\partial W_\infty}{\partial y}(y, \tau) = -g(y, \tau; 0) - \int_0^{\infty} \psi(\xi)\frac{\partial g}{\partial \xi}(y, \tau; \xi)\,\mathrm{d}\xi = \int_0^{\infty} \psi'(\xi)g(y, \tau; \xi)\,\mathrm{d}\xi.$$

Next, we apply the Robin boundary condition in Eq. (2.5) to obtain

$$\int_0^{\infty} \{[\psi'(\xi) - \psi(\xi)]g(0, \tau; \xi) - g(0, \tau; -\xi)\}\,\mathrm{d}\xi = 0$$

so that $\psi(\xi)$ has to satisfy the following differential equation

$$\psi'(\xi) - \psi(\xi) = \frac{g(0, \tau; -\xi)}{g(0, \tau; \xi)} = \mathrm{e}^{(\alpha+1)\xi}, \quad \text{where } \alpha = \frac{2(q_i - q_p)}{\sigma^2}.$$

The auxiliary condition for $\psi(\xi)$ is obtained by observing continuity of $W(y, 0)$ at $y = 0$, giving $\psi(0) = 1$. The solution of $\psi(\xi)$ depends on $\alpha \neq 0$ or $\alpha = 0$, namely,

(i) when $\alpha \neq 0$,

$$\psi(\xi) = \mathrm{e}^{\xi}\left(\frac{\mathrm{e}^{\alpha\xi}}{\alpha} + 1 - \frac{1}{\alpha}\right);$$

(ii) when $\alpha = 0$, $\psi(\xi) = (1 + \xi)\,\mathrm{e}^{\xi}$.

By substituting the known solution of $\psi(\xi)$ into the integral in Eq. (A.1), we obtain

(i) when $\alpha \neq 0$,

$$W_\infty(y, \tau) = \mathrm{e}^{y-q_i\tau}\left(1 - \frac{1}{\alpha}\right)N\left(\frac{y + \tilde{\mu}\tau}{\sigma\sqrt{\tau}}\right) + \frac{1}{\alpha}\,\mathrm{e}^{(1+\alpha)y-q_p\tau}N\left(\frac{y - \mu\tau}{\sigma\sqrt{\tau}}\right) + \mathrm{e}^{-q_p\tau}N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right), \quad y < 0; \tag{A.2a}$$

(ii) when $\alpha = 0$ (write q as the common dividend yield),

$$W_\infty(y, \tau) = \mathrm{e}^{-q\tau}N\left(\frac{-y + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right) + \mathrm{e}^{y-q\tau}\sigma\sqrt{\tau}n\left(\frac{y + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right) + \mathrm{e}^{y-q\tau}\left(y + 1 + \frac{\sigma^2\tau}{2}\right)N\left(\frac{y + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right), \quad y < 0. \tag{A.2b}$$

Appendix B. Theoretical justification of price formula (2.10)

Consider $V_{\text{mid}}(F, I, M, t)$, due to the lookback payoff structure, we impose the usual auxiliary condition for a lookback option:

$$\left.\frac{\partial V_{\text{mid}}}{\partial M}\right|_{M=I/F} = 0. \tag{B.1}$$

It is obvious that $V_{\text{mid}}(F, I, M, t)$ given in Eq. (2.10) satisfies the governing differential equation (2.8). It suffices to show that $V_{\text{mid}}(F, I, M, t)$ satisfies the terminal payoff condition and the above auxiliary condition.

First, when $M > 1$, it is guaranteed that M at maturity must be greater than one so the terminal payoff becomes $F\max(1, M) = MF$. Since $V_\infty(F, I, T) = F$ so that $V_\infty(MF, I, T) = MF$, hence the terminal payoff condition is satisfied.

To show the satisfaction of the auxiliary condition (B.1) at $M = I/F$, we write $V_\infty(MF, I, t)$ in the form $MFW_\infty(\ln(I/MF), \tau)$. We obtain

$$\frac{\partial V_\infty}{\partial M}(MF, I, t) = F\left[W_\infty(y, \tau) - \frac{\partial W_\infty}{\partial y}(y, \tau)\right],$$

where $y = \ln(I/MF)$. When $M = I/F$, we have $y = 0$ so that

$$\left.\frac{\partial V_{\text{mid}}}{\partial M}(F, I, M, t)\right|_{M=I/F} = \left.\frac{\partial V_\infty}{\partial M}(MF, I, t)\right|_{M=I/F} = F\left[W_\infty(0, \tau) - \frac{\partial W_\infty}{\partial y}(0, \tau)\right] = 0$$

by virtue of the Robin boundary condition stated in Eq. (2.5).

Second, when $M \le 1$, we have $\partial V_{\text{mid}}/\partial M = 0$; and together with the continuity of V_{mid} at $M = 1$, we deduce that

$$V_{\text{mid}}(F, I, M, t) = V_\infty(F, I, t), \quad M \le 1.$$

Appendix C. Proofs of relations (3.3) and (3.4)

First, we consider $M > 1$, it suffices to verify that the price function $\hat{V}_{\text{grant}}(MF, I, t)$ and the exercise boundary $F^*_{\text{grant}}(I, t)/M$ satisfy the pricing formulation posed in Eq. (3.2). With $F^*_{\text{mid}} = F^*_{\text{grant}}/M$, we see that the pricing models of $\hat{V}_{\text{grant}}(F, I, t)$ and $\hat{V}_{\text{mid}}(F, I, M, t)$ share the same continuation region and stopping region. It is quite obvious that $V_{\text{grant}}(MF, I, t)$ satisfies both the differential equation and the terminal payoff condition. We would like to check whether (i) Neumann boundary condition at $M = I/F$, (ii) value matching and smooth paste conditions at $F = F^*_{\text{mid}}$ are satisfied. The verification procedures for $M > 1$ are presented below:

(i) $$\left.\frac{\partial \hat{V}_{\text{mid}}}{\partial M}\right|_{M=I/F} = F\left.\frac{\partial \hat{V}_{\text{grant}}}{\partial \tilde{F}}\right|_{\tilde{F}=I}, \quad \text{where } \tilde{F} = MF$$

$$\left.\frac{\partial \hat{V}_{\text{mid}}}{\partial M}\right|_{M=I/F} = 0,$$

(ii) $$\hat{V}_{\text{mid}}(F^*_{\text{mid}}, I, M, t) = \hat{V}_{\text{grant}}(MF^*_{\text{mid}}, I, t) = \hat{V}_{\text{grant}}(F^*_{\text{grant}}, I, t) = F^*_{\text{grant}} = MF^*_{\text{mid}} = \max(M, 1)F^*_{\text{mid}},$$

(iii) $$\left.\frac{\partial \hat{V}_{\text{mid}}}{\partial F}\right|_{F=F^*_{\text{mid}}} = M\left.\frac{\partial \hat{V}_{\text{grant}}}{\partial \tilde{F}}\right|_{\tilde{F}=MF^*_{\text{mid}}}, \quad \text{where } \tilde{F} = MF,$$

$$\left.\frac{\partial \hat{V}_{\text{mid}}}{\partial F}\right|_{F=F^*_{\text{mid}}} = M \left.\frac{\partial \hat{V}_{\text{grant}}}{\partial \tilde{F}}\right|_{\tilde{F}=F^*_{\text{grant}}} = M = \max(M, 1).$$

The payoff takes the form $\max(M, 1)F$, so $\hat{V}_{\text{mid}}$ becomes insensitive to the current realized maximum value M when $M \le 1$. As $\hat{V}_{\text{mid}}$ has no dependence on M, we then have

$$\frac{\partial \hat{V}_{\text{mid}}}{\partial M} = 0 \quad \text{for} \quad M \le 1.$$

Furthermore, by the continuity of the price function at $M = 1$, we deduce that for $M \le 1$

$$\hat{V}_{\text{mid}}(F, I, M, t) = \hat{V}_{\text{mid}}(F, I, 1, t) = \hat{V}_{\text{grant}}(F, I, t)$$

and

$$F^*_{\text{mid}}(I, M, t) = F^*_{\text{mid}}(I, 1, t) = F^*_{\text{grant}}(I, t).$$

Appendix D. Derivation of $G(y, \tau; \xi)$

Suppose we let

$$H(y, \tau; \xi) = G(y, \tau; \xi) - \frac{\partial G}{\partial y}(y, \tau; \xi), \tag{D.1}$$

then the boundary condition along $y = 0$ changes from the Robin type for G to the Dirichlet type for H. The differential equation formulation for $H(y, \tau; \xi)$ is given by

$$\frac{\partial H}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 H}{\partial y^2} + \mu\frac{\partial H}{\partial y}, \quad y < 0, \quad \tau > 0, \qquad H(0, \tau; \xi) = 0, \qquad H(y, \tau; \xi) = \mathbf{1}_{\{y<\xi\}} + \delta(y - \xi).$$

The solution to $H(y, \tau; \xi)$ is given by

$$\begin{aligned} H(y, \tau; \xi) = {} & \frac{1}{\sigma\sqrt{\tau}}\left[n\left(\frac{y-\xi+\mu\tau}{\sigma\sqrt{\tau}}\right) - \mathrm{e}^{2\mu\xi/\sigma^2} n\left(\frac{y+\xi+\mu\tau}{\sigma\sqrt{\tau}}\right)\right] \\ & + 1 - N\left(\frac{y-\xi+\mu\tau}{\sigma\sqrt{\tau}}\right) - \mathrm{e}^{-2\mu y/\sigma^2} N\left(\frac{y+\xi-\mu\tau}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

Once $H(y, \tau; \xi)$ is known, we can use Eq. (D.1) to solve for $G(y, \tau; \xi)$. This leads to

$$G(y, \tau; \xi) = \psi(\tau; \xi)\,\mathrm{e}^y + \int_y^0 \mathrm{e}^{y-\eta} H(\eta, \tau; \xi)\,\mathrm{d}\eta, \tag{D.2}$$

where $\psi(\tau; \xi)$ is an arbitrary function to be determined. By substituting the above solution (D.2) into the governing equation for G [see Eq. (3.9)] and setting $y = 0$, we obtain the following differential equation for $\psi(\tau; \xi)$

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau}(\tau; \xi) - \left(\frac{\sigma^2}{2} + \mu\right)\psi(\tau; \xi) + \frac{\sigma^2}{2}\frac{\partial H}{\partial y}(0, \tau; \xi) = 0, \quad \xi < 0, \quad \tau > 0, \tag{D.3}$$

with initial condition: $\psi(0; \xi) = 0$. The solution to $\psi(\tau; \xi)$ is found to be

$$\psi(\tau; \xi) = -\frac{\sigma^2}{2}\int_0^\tau \mathrm{e}^{(\mu+(\sigma^2/2))(\tau-u)}\frac{\partial H}{\partial y}(0, u; \xi)\,\mathrm{d}u, \tag{D.4}$$

where

$$\frac{\partial H}{\partial y}(0,\tau;\xi) = \frac{2(\xi-\sigma^2\tau)}{(\sigma\sqrt{\tau})^3} n\left(\frac{\xi-\mu\tau}{\sigma\sqrt{\tau}}\right) + \frac{2\mu}{\sigma^2} N\left(\frac{\xi-\mu\tau}{\sigma\sqrt{\tau}}\right).$$

The remaining steps amount to tedious integration of the two integrals in Eqs. (D.2) and (D.4).

Appendix E. Asymptotic behavior of $y_p^*(\tau)$ at $\tau \to 0^+$

We assume $p \ge 0$, $q_p \ge 0$ but p and q_p not both equal zero. Since $F^*(I, t; p) \ge I$ for all t, it is obvious that $y_p^*(0^+) \le 0$. To show that $y_p^*(0^+) = 0$, we prove by contradiction. Suppose $y_p^*(0^+) < 0$, then there exists y belonging to the interval $(y_p^*(0^+), 0)$ such that the point $(y, 0^+)$ in the y–τ plane lies in the continuation region. Since $W_p(y, 0^+) = 1$, for $y^*(0^+) < y < 0$, we have

$$\left.\frac{\partial W_p}{\partial \tau}\right|_{\tau=0^+} = \left.\left[\frac{\sigma^2}{2}\frac{\partial^2 W_p}{\partial y^2} + \mu\frac{\partial W_p}{\partial y} - q_p W_p - p\right]\right|_{\tau=0^+} = -q_p - p.$$

Since $p \ge 0$, $q_p \ge 0$ but p and q_p not both equal zero, we obtain

$$\left.\frac{\partial W_p}{\partial \tau}\right|_{\tau=0^+} < 0$$

so that $W_p(y, \tau) < 1$ for some point (y, τ) in the continuation region. This leads to a contradiction since $W_p(y, \tau) > 1$ for all points (y, τ) in the continuation region. We then deduce that

$$y_p^*(0^+) = 0 \quad \text{for} \quad p \ge 0, q_p \ge 0 \text{ but } p \text{ and } q_p \text{ not both equal zero.}$$

Recall that when $p = q_p = 0$, we have shown that [see Eq. (3.12)]

$$y_p^*(\tau) = -\infty \quad \text{for all} \quad \tau.$$

Appendix F. Asymptotic behavior of $W_p(y, \tau)$ and $y_p^*(\tau)$ at $\tau \to \infty$

With infinite time to expiration, $W_p(y, \tau)$ becomes insensitive to τ so that $\partial W_p/\partial \tau = 0$. Without dependence on τ, the formulation for $W_p = W_p(y, \infty)$ reduces to

$$\frac{\sigma^2}{2}\frac{\mathrm{d}^2 W_p}{\mathrm{d}y^2} + \mu\frac{\mathrm{d}W_p}{\mathrm{d}y} - q_p W_p - p = 0, \qquad y_p^*(\infty) < y < 0,$$

with auxiliary conditions:

(i) at $y = 0$, $\mathrm{d}W_p/\mathrm{d}y = W_p$,
(ii) at $y = y_p^*(\infty)$, $W_p = 1$ and $\mathrm{d}W_p/\mathrm{d}y = 0$.

The nature of the solution to the above formulation depends on the properties of the two roots of the following indicial equation: $(\sigma^2/2)\lambda^2 + \mu\lambda - q_p = 0$. In terms of the parameters q_i, q_p and σ^2, these two roots are found to be

$$\lambda_\pm = \frac{1}{2} + \frac{q_i - q_p}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} + \frac{q_i - q_p}{\sigma^2}\right)^2 + \frac{2q_p}{\sigma^2}}.$$

Depending on different choices of values for the parameters: q_i, q_p and p, $W_p(y,\infty)$ and $y_p^*(\infty)$ exhibit a wide range of solution behaviors:

(i) $q_i = q_p = 0$

(a) $p > 0$

$$W_p(y,\infty) = \frac{2p}{\sigma^2}(\mathrm{e}^{(\sigma^2/2p)+y} - y - 1) \quad \text{and} \quad y_p^*(\infty) = -\frac{\sigma^2}{2p}.$$

(b) Solution does not exist when $p = 0$.

(ii) $q_i = 0, q_p > 0$

(a) $p > 0$

$$W_p(y,\infty) = \frac{p}{q_p + (\sigma^2/2)}\left[\frac{\sigma^2}{2q_p}\,\mathrm{e}^{-(2q_p/\sigma^2)y} + \left(1 + \frac{q_p}{p}\right)^{(\sigma^2/2q_p)+1}\mathrm{e}^{y}\right] - \frac{p}{q_p}$$

and

$$y_p^*(\infty) = -\frac{\sigma^2}{2q_p}\ln\left(1 + \frac{q_p}{p}\right).$$

(b) Solution does not exist when $p = 0$.

(iii) $q_i > 0, q_p = 0$

(a) $p = 0$

$$W_p(y,\infty) = 1 + \frac{\sigma^2}{2q_i}\,\mathrm{e}^{((2q_i/\sigma^2)+1)y} \quad \text{and} \quad y_p^*(\infty) = -\infty.$$

(b) $p > 0$

$$W_p(y,\infty) = 1 + \frac{2p(y^* - y)}{\sigma^2 + 2q_i} + \frac{2p\sigma^2}{(\sigma^2 + 2q_i)^2}\left[\exp\left(\left(\frac{2q_i}{\sigma^2} + 1\right)(y - y^*)\right) - 1\right],$$

where $y^* = y_p^*(\infty)$ is the solution to the algebraic equation

$$1 + \frac{2p}{\sigma^2 + 2q_i}\left[y^* - \frac{1}{\lambda_+} - \left(1 - \frac{1}{\lambda_+}\right)\mathrm{e}^{-\lambda_+ y^*} + 1\right] = 0.$$

(iv) $q_i > 0, q_p > 0$

(a) $p = 0$

$$W_p(y,\infty) = A_+\,\mathrm{e}^{\lambda_+ y} + A_-\,\mathrm{e}^{\lambda_- y} - \frac{p}{q_p},$$

where

$$A_+ = \frac{\lambda_-}{\lambda_- - \lambda_+}\left[\frac{\lambda_+(1-\lambda_-)}{\lambda_-(1-\lambda_+)}\right]^{\lambda_+/(\lambda_+-\lambda_-)}, \qquad A_- = \frac{\lambda_+}{\lambda_+ - \lambda_-}\left[\frac{\lambda_-(1-\lambda_+)}{\lambda_+(1-\lambda_-)}\right]^{\lambda_-/(\lambda_--\lambda_+)},$$

$$y_p^*(\infty) = \frac{1}{\lambda_+ - \lambda_-}\ln\frac{\lambda_-(1-\lambda_+)}{\lambda_+(1-\lambda_-)}.$$

(b) $p > 0$

$$W_p(y,\infty) = \frac{1 + (p/q_p)}{\lambda_+ - \lambda_-}[\lambda_+\,\mathrm{e}^{\lambda_-(y-y^*)} - \lambda_-\,\mathrm{e}^{\lambda_+(y-y^*)}] - \frac{p}{q_p},$$

where $y^* = y_p^*(\infty)$ is the solution to the algebraic equation

$$\lambda_+(1-\lambda_-)\,\mathrm{e}^{-\lambda_- y^*} - \lambda_-(1-\lambda_+)\,\mathrm{e}^{-\lambda_+ y^*} + \frac{(\lambda_- - \lambda_+)p}{p+q_p} = 0.$$

References

Chu, C.C., Kwok, Y.K., 2003. Discussion on pricing perpetual fund protection with withdrawal option. North American Actuarial Journal 7 (2), 77–81.

Dai, M., Kwok, Y.K., Wu, L., 2003. Options with multiple reset rights. International Journal of Theoretical and Applied Finance 6 (6), 637–653.

Dixit, A.K., Pindyck, R.S., 1994. Investment Under Uncertainty. Princeton University Press, Princeton.

Fung, M.K., Li, L.K., 2003. Pricing discrete dynamic fund protections. North American Actuarial Journal 7 (4), 23–31.

Gerber, H.U., Pafumi, G., 2000. Pricing dynamic investment fund protection. North American Actuarial Journal 4 (2), 28–41.

Gerber, H.U., Shiu, E., 2003. Pricing perpetual fund protection with withdrawal option. North American Actuarial Journal 7 (2), 60–77.

Imai, J., Boyle, P., 2001. Dynamic fund protection. North American Actuarial Journal 5 (3), 31–51.

Kwok, Y.K., 1998. Mathematical Models of Financial Derivatives. Springer, Singapore.

Tiong, S., 2000. Valuing equity-indexed annuities. North American Actuarial Journal 4 (4), 149–170.

Windcliff, H., Le Roux, M., Forsyth, P., Vetzal, K., 2002. Understanding the behavior and hedging of segregated funds offering the reset feature. North American Actuarial Journal 6 (2), 107–124.

Wong, H.Y., Kwok, Y.K., 2003. Sub-replication and replenishing premium: efficient pricing of multi-state lookbacks. Review of Derivatives Research 6, 83–106.