Enhanced Equity-Credit Modeling for Contingent Convertibles

Tsz-Kin Chung*
Mitsubishi UFJ Trust Investment Technology Institute Co., Ltd.

Yue-Kuen Kwok†
Department of Mathematics, Hong Kong University of Science and Technology

January, 2016

Abstract

Contingent convertible (CoCo) bonds are characterized by forced equity conversion under either accounting or regulatory trigger. Accounting trigger occurs when the capital ratio of the issuing bank falls below some contractual threshold. Under the regulatory trigger, sometimes called the point-of-non-viability (PONV) trigger, the regulatory authority may enforce equity conversion when the financial health of the bank deteriorates to certain distressed level. In this paper, we propose an equity-credit modeling of the joint process of the stock price and capital ratio that integrates both the structural approach of accounting trigger and reduced form approach of PONV trigger of equity conversion. We also construct effective Fortet algorithms and finite difference schemes for numerical pricing of CoCo bonds under various forms of equity conversion payoff. The pricing properties of the CoCo bonds under different assumptions of the state dependent intensity of PONV trigger, contractual specifications and market conditions are examined.

Keywords: Contingent convertibles, equity-credit modeling, Fortet algorithms

1 Introduction

The contingent convertible (CoCo) bond provides a higher coupon rate compared to its non-convertible counterpart since it is embedded with a loss absorption mechanism that is triggered when the capital of the issuing bank falls close to the regulatory level as required by the Basel Committee on Banking Supervision (BCBS). At a triggering event, the bond is automatically converted into equity of the issuing bank (or an equivalent amount of cash). The equity conversion is meant to provide a timely recapitalization of the issuing bank and boost its capital adequacy when it is under financial distress. This would help mitigate the chance of a systemic banking crisis, thus minimizing the use of taxpayers' money to bail out distressed financial institutions.

Since the first issuance of the Enhanced Capital Notes by the Lloyds Banking Group in December 2009, there has been an active discussion on the triggering mechanism and loss absorption design of CoCo bonds. In a typical contractual design of a CoCo bond, there are two possible triggering mechanisms: accounting trigger and regulatory trigger. In an accounting trigger, the capital ratio is chosen as the indicator on a bank's financial health. For example, when the core tier-1 capital

*E-mail: btkchung@gmail.com
†E-mail: maykwok@ust.hk

ratio falls below a predetermined level, the bond is converted automatically into equity. In a regulatory trigger, sometimes called the point-of-non-viability (PONV) trigger, the banking supervisory authority holds the discretion based on the financial health of the bank to decide whether equity conversion should be activated. As noted in a recent survey on the contractual features of CoCo bonds (Avdjiev *et al.*, 2013), most of the CoCo bonds include the provision of accounting trigger.

There have been a number of papers that address pricing and risk management of a CoCo bond using various modeling approaches. The structural approach starts with the modeling of a bank's balance sheet dynamics that allows one to analyze the impact of the issuance of contingent convertibles on the capital structure (Albul *et al.*, 2010; Brigo *et al.*, 2015; Glasserman and Nouri, 2012; Pennacchi, 2011). On the other hand, Cheridito and Xu (2015) apply the reduced form approach to price CoCo bonds. The reduced form approach, commonly used for pricing of credit derivatives, has better flexibility to perform calibration of the model parameters using the market prices of traded derivatives, like the credit default swap (CDS) spreads. Also, Spiegeleer and Schoutens (2012) propose an easy-to-use equity derivative approach by approximating the accounting trigger of the capital ratio by the first passage time of the stock price process to an implied barrier level. Following a similar approach, Corcuera *et al.* (2013) pursue a smile conforming model that assumes the stock price to follow a Lévy process. Gupta *et al.* (2013) discuss pricing issues related to various contractual features of CoCo bonds and resort to numerical methods for pricing CoCo bonds under a mean reversion process of the capital ratio. Wilkens and Bethke (2014) report the empirical assessment of the aforementioned approaches and find that the equity derivative approach implies a hedging ratio that may be practically useful with reference to the risk management of CoCo bonds during the sample period of 2011. Leung and Kwok (2015) model the regulatory trigger using the Parisian feature where equity conversion is activated when the capital ratio stays under the non-viable state cumulatively over a certain period of time.

A proper modeling of the capital ratio process is crucial with regard to the contractual design of a CoCo bond with accounting trigger. Since the capital ratio is a balance sheet quantity, the structural modeling approach provides a natural starting point for pricing CoCos. The structural approach, however, does not usually possess the flexibility for calibration to traded security prices and fails to generate a reasonable shape of the credit spread (Eom *et al.*, 2004). Furthermore, since equity is priced as a contingent claim on the bank asset value, the joint dynamics of the stock price and capital ratio is less tractable under the structural framework (Chen and Kou, 2009). It is also not straightforward to incorporate a jump in the stock price, which further restricts its ability to reflect the potential write-down of the CoCo bond value upon equity conversion. On the other hand, the reduced form approach has been found to be efficient for pricing corporate bonds (Duffie and Singleton, 1999; Jarrow *et al.*, 2010). When used in pricing CoCos, the reduced form approach only requires the specification of the conversion intensity and the jump magnitude of the stock price at the conversion time (Cheridito and Xu, 2015). Some earlier empirical studies show that the capital ratio may be subject to manipulation by a distressed bank, where the capital ratio remains at healthy level even the bank is in the distressed state. The reduced form approach remains feasible to capture the possibility of regulatory trigger even when the capital ratio is not quite close to the trigger threshold. The equity conversion under regulatory trigger can be treated as a random event that can be modeled by a single Poisson jump. In this model framework, the CoCo bond is seen to consist of a straight bond component while the equity component is only a residual. A criticism of the reduced form approach is that it completely ignores the contractual feature of an accounting trigger and neglects the interaction between stock price and capital ratio (Brigo *et al.*, 2015).

In this paper, we propose a bivariate equity-credit modeling of the stock price and capital ratio together with a jump-to-non-viability (JtNV) feature. This framework of a joint modeling is particularly important for pricing CoCos since the conversion value depends on the joint distribution of the first passage time of the capital ratio hitting the trigger threshold and the stock price at conversion. Since we model the bivariate process of capital ratio and stock price for pricing CoCos directly, the parametrization procedure is easier than a bottom-up structural model that requires a judicious choice of the proxy conversion epoch linked to other balance sheet quantities (Brigo *et al.*, 2015; Pennacchi, 2011). We demonstrate that the equity derivative approach proposed by Spiegeleer and Schoutens (2012) can be recast under our bivariate framework when the stock price and capital ratio are perfectly correlated, and explain the justification for the equity derivative approach that replicates the CoCo components using barrier options. In the aftermath of the Lehman Brothers financial crisis, it is not uncommon to see substantial write downs by major investment banks due to unexpected trading losses and bleaching of regulation, which might erode a significant part of the bank's capital. In our equity-credit model, we add the reduced form feature that is important to capture JtNV that can be used to model the sudden deterioration of a bank's financial health leading to a PONV trigger. The reduced form approach has been popular in the modeling of rare financial events such as company default, mortgage prepayment and stock market crash (Bates, 1991; Dai *et al.*, 2007; Jarrow *et al.*, 2010). We may allow the intensity of PONV trigger to be dependent on the state variables that are linked to the financial health of the bank. In summary, our proposed approach of integrating the reduced form approach and structural approach is natural for pricing a CoCo bond that has multiple sources of risk.

For the sake of practical implementation, we adopt a model that is tractable and contains essential structural features that determine the market prices of CoCos. We postulate the mean reversion capital ratio process and model the accounting trigger by the first passage time of the capital ratio bleaching a predetermined threshold. Before the PONV trigger, the capital ratio and the stock price follow a bivariate diffusion process. Furthermore, we assume the stock price process to include the jump feature in order to capture the abrupt change in the stock price upon PONV trigger. When the conversion intensity of the JtNV trigger is constant, we may employ an integral equation approach known as the Fortet method to compute the relevant density function of the first passage time to the triggering threshold of the capital ratio. When the conversion intensity is state dependent, the CoCo bond price can be obtained by solving a two-dimensional partial differential equation numerically using the finite difference method. Though it is possible to extend to more sophisticated dynamics of stock price and capital ratio, like the inclusion of stochastic volatility or stochastic intensity, the infrequent observation and the lack of historical data of the capital ratio may not well justify the pursue of these more complex dynamic processes.

Our proposed framework is an extension of the equity-credit hybrid modeling framework that models the interaction between equity risk and credit risk (Carr and Linetsky, 2006; Carr and Wu, 2009; Cheridito and Wugalter, 2012). The traditional equity-credit modeling typically adopts a pure reduced form setting in which the jump-to-default of a financial derivative is modeled by a random jump with a stochastic intensity process. These studies mainly concentrate on pricing of defaultable derivatives and credit default swaps (Chung and Kwok, 2014). In this paper, we integrate the equity-credit modeling with the structural approach of capturing the accounting trigger of a CoCo bond via the first passage time feature of the capital ratio process. In addition to capturing the JtNV feature, our reduced form approach is seen to be more flexible since it allows for modeling various embedded options such as the coupon cancellation and callable features. In summary, our enhanced equity-credit model illustrates the versatible approach of capturing the multiple risk exposures in CoCo bonds.

This paper is structured as follows. Section 2 presents the model setup of the enhanced hybrid modeling approach. We present various structural features of the CoCo bond and explain why the determination of the joint distribution of the random time of equity conversion and the stock price at conversion time is crucial in the pricing procedure. In particular, we discuss the characterization of the jump in stock price upon PONV trigger. The arrival of the JtNV is characterized by the intensity of a Poisson jump. Both cases of constant intensity and state dependent intensity are considered. We illustrate the versatibility of our pricing framework by showing how it can be used to cope with more complex structural features, like the floored payoff, coupon cancellation and perpetuality with callable feature. Section 3 presents the numerical examples that perform risk sensitivities analysis of a CoCo bond. We analyze the impact of various model parameters, like correlation, stock price volatility and conversion intensity on the price functions of the CoCo bonds with various contractual features. We also examine the conditional distribution of the stock price at an accounting trigger and analyze the decomposition of the CoCo bonds into their bond and equity components. Section 4 presents summary of results and conclusive remarks.

2 An Enhanced Hybrid Modeling

2.1 The structure

A typical structure of a CoCo bond consists of:

1. Bond component: coupon payments $(c_i)_{i=1,2,\ldots n}$ paid at time points t_i, $i = 1, 2, \ldots, n$, and principal payment F at the maturity T, where $t_n = T$.

2. Equity component: at a trigger event, the bond is converted into G shares of stock of the issuing bank.

The equity exposure to the CoCo bond is revealed through the forced equity conversion. In general, pricing of a CoCo bond is related to interest rate risk (discounting on the coupons and principal), equity risk (equity conversion at a trigger event) and conversion risk (loss absorption mechanism). The conversion risk is the risk of an unfavorable conversion to a declined stock price that wipes off a significant portion of the value of the bond. We emphasize that pricing of CoCos is not directly related to default risk since equity conversion always happens before a bank's default and CoCo bonds are structured to help banks from insolvency. As a part of the challenge in pricing CoCos, it is necessary to model the stock price right after the PONV trigger.

2.2 Model setup

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$ in which Q is a risk neutral measure. All the underlying processes in our equity-credit model are assumed to be observable and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We take the stock price process to be $S_t = \exp(x_t)$ and capital ratio process to be $H_t = \exp(y_t)$, where $X_t = (x_t, y_t)$ follows a bivariate process under the risk neutral measure Q as follows:

$$\begin{aligned} dx_t &= \left[r - q - \frac{\sigma^2}{2} - \gamma\lambda(x_t, y_t)\right] dt + \sigma\, dW_t^1 + \ln(1+\gamma)\, dN_t, && x_0 = x, \\ dy_t &= \kappa\,(\theta - y_t)\, dt + \eta\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\right), && y_0 = y. \end{aligned} \tag{2.1}$$

Here, W_t^1 and W_t^2 are uncorrelated Brownian motions. The log capital ratio y_t is specified as an Ornstein-Unlenbeck (OU) process with the mean reversion feature. This is because banks usually actively manage the amount of regulatory capital in response to changing market values of asset and liability. They have the incentives to maintain a healthy level of capital ratio in order to stay away from any regulatory bleach. The interest rate $r \geq 0$, dividend yield $q \geq 0$, stock price volatility

$\sigma > 0$, capital ratio volatility $\eta > 0$, correlation coefficient $-1 \leq \rho \leq 1$, mean reversion level θ and speed of reversion $\kappa > 0$ of the capital ratio are assumed to be constant. The stock price process is modeled as a Geometric Brownian motion with jump upon equity conversion due to PONV trigger that arises from a sudden deterioration of the bank's financial health or other distressed event. We let N_t denote the Poisson process that models the arrival of the PONV trigger and $\lambda(x_t, y_t) \geq 0$ be the state dependent intensity of N_t. The constant jump magnitude of the stock price upon PONV trigger is denoted by $-1 < \gamma < \infty$.

Let H_B be the contractual threshold of the capital ratio for an accounting trigger and we define $y_B = \ln H_B$. The accounting trigger is modeled by the first passage time of the log capital ratio y_t to a predetermined lower threshold y_B as

$$\tau_B = \inf \{t \geq 0; \, y_t = y_B\}.$$

In addition, the random time of JtNV is modeled by the first jump of the Poisson process N_t, where

$$\tau_R = \inf \{t \geq 0; \, N_t = 1\}.$$

The random time of equity conversion is taken to be the earlier of the first passage time τ_B and the JtNV time τ_R, where

$$\tau = \tau_B \wedge \tau_R.$$

It is well acceptable to assume that $\Pr\{\tau_B = \tau_R\} = 0$ for the sake of simplicity. That is, the two random times do not occur at the same time almost surely. Note that the conversion takes place when at least one of the τ_B and τ_R does not exceed the maturity T.

The no-arbitrage price of a CoCo can be decomposed into three components:

$$P_{CoCo} = P_C + P_F + P_E.$$

1. Value of the coupon payments, P_C:

$$P_C = \sum_{i=1}^{n} \mathbb{E}^Q \left[c_i e^{-rt_i} \mathbf{1}_{\{\tau > t_i\}}\right] = \sum_{i=1}^{n} c_i e^{-rt_i} \left[1 - Q\left(\tau \leq t_i\right)\right],$$

   which is the sum of discount coupon payments received prior to the conversion time τ.[1]

2. Value of the principal payment, P_F:

$$P_F = \mathbb{E}^Q \left[F e^{-rT} \mathbf{1}_{\{\tau > T\}}\right] = F e^{-rT} \left[1 - Q\left(\tau \leq T\right)\right],$$

   in which F is the principal payment when there is no conversion until maturity.

3. Conversion value, P_E:

$$P_E = \mathbb{E}^Q \left[e^{-r\tau} G S_\tau \mathbf{1}_{\{\tau \leq T\}}\right],$$

   where the CoCo is converted into G units of shares of the underlying equity at the conversion time τ.

[1] We ignore the small amount of interest accrual when a conversion occurs between two consecutive coupon payment dates.

It suffices to compute the conversion probability for evaluation of the bond component (sum of P_C and P_F). The key step in computing the conversion value P_E is the determination of the joint modeling of the conversion time τ and stock price S_τ. For conversion into cash, we only need to replace the term GS_τ by a constant cash payment and this reduces to an easier pricing problem. There are several other interesting contractual features, like the floored payoff, coupon cancellation and issuer's call. All these features and their effects on the pricing procedures will be discussed in later sections.

The bivariate first passage time models with a one-sided threshold have been commonly adopted in modeling defaultable bonds and barrier options under stochastic interest rates (Longstaff and Schwartz, 1995; Collin-Dufresne and Goldstein, 2001; Coculescu *et al.*, 2008; Bernard *et al.*, 2008). Our proposed bivariate equity-credit models extend these bivariate first passage time models in several aspects. The accounting trigger is modeled by the first passage time of the mean reversion capital ratio hitting a threshold, which is typical in structural risky bond models. In addition, we adopt the reduced form approach of incorporating a random PONV trigger that is modeled by the first jump of a Poisson process and a jump in stock price upon PONV trigger. When the intensity of PONV trigger is constant, it is possible to simplify the pricing problem to become one-dimensional model. One can then employ an efficient numerical algorithm (Fortet method) to compute the CoCo bond price. When the intensity is chosen to be state dependent, the stock price and capital ratio processes interact through the correlation of the Brownian motions together with the state dependent jump intensity $\lambda(x_t, y_t)$. This form of bivariate dependence of the jump intensity provides additional flexibility to model various triggering mechanisms in a CoCo bond. With some appropriate choices of the state dependent intensity functions, we manage to price the CoCo bonds by solving a two-dimensional pricing model numerically using standard finite difference schemes.

It is worthwhile to discuss the jump in stock price upon equity conversion. At the accounting trigger $\tau = \tau_B$, the stock price is assumed to be continuous with no jump. The rationale is that the stock price should have gradually taken into account the possibility of such a conversion. At the regulatory trigger $\tau = \tau_R$, there would exhibit a fixed jump in stock price as modeled by $S_{\tau_R} = (1+\gamma)\, S_{\tau_{R\_}}$, where $\gamma \in (-1, \infty)$. We rule out the case of $\gamma = -1$, which corresponds to jump to default. Recall that regulatory trigger is structured to avoid default of the issuing bank. Apparently, $\gamma \leq 0$ appears to be reasonable due to signaling of deterioration of bank's financial health by the PONV trigger. However, the other case of $\gamma > 0$ may still be possible since market may react positively to recapitalization upon equity conversion and this leads to a positive jump of stock price.

In practice, it is not straightforward to determine the sign of the parameter γ since it is not sure whether the PONV trigger would enhance equity value or otherwise. The equity conversion helps reduce the liability of the bank by canceling the coupon and principal payments of the bond, which in turn boosts the capital adequacy and leads to a stronger balance sheet and stock price. On the other hand, the conversion into shares of stock causes dilution to existing equity holders which would be reflected by a weakened stock price. The actual effect of equity conversion on the stock price would become better known when there are actual conversion events in the future. Lastly, it is not necessary to make any model assumption on the change of the capital ratio after JtNV as this information is irrelevant to pricing issues.

For illustration purpose, we can follow Ballotta and Kyriakou (2015) and express the solution of eq.(2.1) in terms of the Brownian motions (W_t^1, W_t^2), Poisson process N_t and integrals of the

Brownian motions as follows

$$\begin{aligned} x_t &= \widetilde{x}_t + N_t \ln(1+\gamma) - \gamma \int_0^t \lambda(x_s, y_s)\, ds, \\ y_t &= y_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \int_0^t \eta e^{-\kappa(t-s)} \left(\rho\, dW_s^1 + \sqrt{1-\rho^2}\, dW_s^2\right), \end{aligned}$$

where

$$\widetilde{x}_t = x_0 + \left(r - q - \frac{\sigma^2}{2}\right) t + \sigma\, W_t^1.$$

This representation is useful when we construct the stock price measure in the next section.

2.3 Conversion probability

As part of the pricing procedure, we show how to derive the conversion probability $Q(\tau \leq t)$ under our bivariate equity-credit framework. It is necessary to consider the convolution of the two stopping times, τ_B and τ_R. We make use of the following two building blocks:

$$\begin{aligned} \mathbb{E}^Q\left[\mathcal{H}(\tau_B)\mathbf{1}_{\{\tau_R > \tau_B\}}\right] &= \mathbb{E}^Q\left[e^{-\int_0^{\tau_B} \lambda_u\, du}\, \mathcal{H}(\tau_B)\right], \\ \mathbb{E}^Q\left[\mathcal{H}(\tau_R)\mathbf{1}_{\{\tau_R < \tau_B\}}\right] &= \mathbb{E}^Q\left[\int_0^{\tau_B} \lambda_u e^{-\int_0^u \lambda_s\, ds}\, \mathcal{H}(u)\, du\right], \end{aligned}$$

where $\mathcal{H}(\xi)$ is the payoff at the corresponding stopping time $\xi = \tau_R$ or $\xi = \tau_B$. Note that the state dependent intensity $\lambda_t = \lambda(x_t, y_t)$ introduces dependence among τ_R and τ_B, which adds further complexity to the convolution.

Lemma 1. *For a fixed $t > 0$, the probability of equity conversion is given by*

$$Q(\tau \leq t) = \int_0^t \mathbb{E}^Q\left[\lambda_u e^{-\int_0^u \lambda_s\, ds} \mathbf{1}_{\{\tau_B > u\}}\right] du + \mathbb{E}^Q\left[e^{-\int_0^{\tau_B} \lambda_u\, du} \mathbf{1}_{\{\tau_B \leq t\}}\right]. \tag{2.2}$$

The proof of Lemma 1 is presented in Appendix A.1.

It is interesting to examine the conversion probability and the density function of the conversion time under the hybrid equity-credit framework. Firstly, the conversion density function is non-zero even as $t \to 0^+$ due to the JtNV feature (as revealed by the term $\lambda e^{-\lambda t}$). This property is consistent with exhibiting a finite value of conversion probability even at time close to maturity. Secondly, when the capital ratio is far away from the triggering threshold such that $Q(\tau_B \leq t) \approx 0$, we observe

$$Q(\tau \leq t) \approx \mathbb{E}^Q\left[\mathbf{1}_{\{\tau_R \leq t\}}\right] = Q(\tau_R \leq t),$$

in which equity conversion arises almost solely from JtNV. In this case, we can treat the CoCo bond as a straight bond and one can incorporate a bond yield spread that is related to the jump intensity. This demonstrates the flexibility of the hybrid modeling framework to capture both the equity and fixed income nature of the CoCo bond.

In the next two subsections, we consider the pricing procedures of CoCo bonds under the assumption of constant intensity and state dependent intensity.

2.4 Constant intensity

Suppose the bond is converted into a predetermined G units of shares at the conversion time, the conversion value is given by

$$P_E = G\mathbb{E}^Q\left[e^{-r\tau}S_\tau \mathbf{1}_{\{\tau\le T\}}\right], \qquad \tau = \tau_B \wedge \tau_R.$$

When the intensity of the PONV trigger λ is constant, eq.(2.2) can be simplified as

$$Q(\tau \le t) = \int_0^t \lambda e^{-\lambda u}\left[1 - Q(\tau_B \le u)\right] du + \mathbb{E}^Q\left[e^{-\lambda\tau_B}\mathbf{1}_{\{\tau_B\le t\}}\right].$$

By direct differentiation, the density function of the random conversion time can be obtained as (Campi *et al.*, 2009)

$$Q(\tau \in dt) = \lambda e^{-\lambda t}\left[1 - Q(\tau_B \le t)\right] + e^{-\lambda t} Q(\tau_B \in dt).$$

As expected, under constant intensity, the stopping times τ_B and τ_R are uncorrelated.

When the conversion payoff is a constant multiple of the stock price S, it is possible to adopt the stock price measure Q^* to reduce the dimensionality of the pricing problem by one. We define the adjusted stock price process as follows:

$$\begin{aligned}
\widetilde{S}_t &= S_t e^{-(r-q)t} \\
&= S_0 \exp\left(\sigma W_t^1 - \frac{\sigma^2}{2}t - \lambda\gamma t\right)(1+\gamma)^{N_t}, \quad t \ge 0.
\end{aligned}$$

It is readily to observe that the process $\widetilde{S}_t$ is a positive martingale under the risk neutral measure Q. Hence, we can take $\widetilde{S}_t$ to construct a change-of-measure as

$$Z_t = \left.\frac{dQ^*}{dQ}\right|_{\mathcal{F}_t} = \frac{\widetilde{S}_t}{\widetilde{S}_0} = \frac{e^{-(r-q)t}S_t}{S_0},$$

where Q^* can be interpreted as the *stock price measure*.

Lemma 2. *Suppose that the bivariate dynamics of x_t and y_t are specified by eq.(2.1). Under the stock price measure Q^*, the bivariate process of (x_t, y_t) evolves as*

$$\begin{aligned}
dx_t &= \left(r - q + \frac{\sigma^2}{2} - \gamma\lambda\right)dt + \sigma\, dB_t^1 + \ln(1+\gamma)\, dN_t, & x_0 &= x, \\
dy_t &= \left[\kappa(\theta - y_t) + \rho\sigma\eta\right]dt + \eta\, dB_t^2, & y_0 &= y, \qquad (2.3)
\end{aligned}$$

where $B_t^1 = W_t^1 - \sigma t$, $B_t^2 = \sqrt{1-\rho^2}\, W_t^2 + \rho\, B_t^1$ and $\langle dB_t^1, dB_t^2\rangle = \rho\, dt$. The intensity of the Poisson process N_t becomes

$$\lambda^* = (1+\gamma)\lambda.$$

The proof of Lemma 2 is presented in Appendix A.2.

Now, we can apply the change-of-measure formula to compute the conversion value P_E as follows:

$$P_E = G\mathbb{E}^Q\left[e^{-r\tau}S_\tau\mathbf{1}_{\{\tau\le T\}}\right] = GS_0\mathbb{E}^Q\left[e^{-q\tau}\frac{e^{-(r-q)\tau}S_\tau}{S_0}\mathbf{1}_{\{\tau\le T\}}\right] = GS_0\mathbb{E}^{Q^*}\left[e^{-q\tau}\mathbf{1}_{\{\tau\le T\}}\right],$$

where $\tau = \tau_B \wedge \tau_R$. As a result, we only need to compute a discounted conversion probability under the stock price measure Q^*. The next two propositions provide the formulas to compute the conversion value.

Proposition 3. *Denote $\lambda^* = (1+\gamma)\lambda$. The conversion value P_E under constant intensity λ is given by*

$$P_E = GS_0 \left\{ \int_0^T \lambda^* e^{-(\lambda^*+q)u} \left[1 - Q^*\left(\tau_B \le u\right)\right] du + \mathbb{E}^{Q^*}\left[e^{-(\lambda^*+q)\tau_B} \mathbf{1}_{\{\tau_B \le T\}}\right] \right\}.$$

Proof. By Lemma 2, it suffices to replace λ by $\lambda^* = (1+\gamma)\lambda$ and take the expectation under the stock price measure Q^*. The remaining procedure is similar to that of Lemma 1. □

By following a similar procedure of performing integration by parts (Campi *et al.*, 2009) on the equity-credit hybrid modeling using a CEV process, we can obtain an alternative representation of the conversion value as shown in Proposition 4.

Proposition 4. *When $\lambda^* + q > 0$, the conversion value can be expressed as*

$$P_E = GS_0 \left\{ \frac{\lambda^*}{\lambda^* + q} \left[1 - e^{-(\lambda^*+q)T} Q^*\left(\tau_B > T\right)\right] + \frac{q}{\lambda^* + q} \mathbb{E}^{Q^*}\left[e^{-(\lambda^*+q)\tau_B} \mathbf{1}_{\{\tau_B \le T\}}\right] \right\}.$$

Proof. Applying integration by parts and noting that $\dfrac{\partial}{\partial t} Q^*\left(\tau_B \le t\right)$ gives the density of τ_B, we obtain the result. □

By Propositions 3 and 4, we have effectively reduced a two-dimensional problem into a one-dimensional problem using the change-of-measure formula. As a result, we only need to compute the distribution function of the first passage time $Q^*\left(\tau_B \le T\right)$ and the associated truncated Laplace transform

$$\mathbb{E}^{Q^*}\left[e^{-(\lambda^*+q)\tau_B} \mathbf{1}_{\{\tau_B \le T\}}\right].$$

These two quantities can be readily computed by applying the one-dimensional Fortet method which is essentially a recursive algorithm that solves an integral equation for the conversion density function. The details of the Fortet method are presented in Appendix B. The key steps in the recursive Fortet algorithm are summarized as follows. We discretize the time interval $[0, T]$ into m equal intervals with $t_j = j\Delta t$ for $j = 0, 1, 2, ..., m$, where $m\Delta t = T$. The relevant quantities for the pricing of CoCo can be obtained by the following formulas:

$$Q^*\left(\tau_B \le t_m\right) = \sum_{j=1}^m q_j, \qquad \mathbb{E}^{Q^*}\left[e^{-(\lambda^*+q)\tau_B} \mathbf{1}_{\{\tau_B \le t_m\}}\right] = \sum_{j=1}^m e^{-(\lambda^*+q)t_j} q_j, \tag{2.4}$$

where

$$q_j \approx Q^*\left(\tau_B \in (t_{j-1}, t_j]\right), \quad j = 1, 2, ..., m,$$

can be obtained by the following recursive scheme

$$\begin{aligned} q_1 &= N\left(a\left(t_1\right)\right), \\ q_j &= N\left(a\left(t_j\right)\right) - \sum_{i=1}^{j-1} q_i N\left(b\left(t_j, t_i\right)\right), \quad j = 2, 3, ..., m, \end{aligned}$$

with

$$a(t) = \frac{y_B - \mu(t, 0)}{\Sigma(t, 0)}, \qquad b(t, s) = \left.\frac{y_B - \mu(t, s)}{\Sigma(t, s)}\right|_{y_s = y_B},$$

and

$$\mu(t,s) = y_s e^{-\kappa(t-s)} + \left(\theta + \frac{\rho\sigma\eta}{\kappa}\right)\left[1 - e^{-\kappa(t-s)}\right], \quad \Sigma^2(t,s) = \frac{\eta^2}{2\kappa}\left[1 - e^{-2\kappa(t-s)}\right],$$

in which $N(\cdot)$ here is the cumulative normal distribution function. The same procedure can be applied to the calculation of the conversion probability under the risk neutral measure Q for the coupon and principal payments. Since the Fortet method solves an integral equation by discretization in the time domain, the different choices of the numerical integration quadrature may lead to different forms of the recursive formula. Here, we present the right-point scheme as in Coculescu *et al.* (2008), though one may choose alternative discretization schemes in order to achieve better rate of convergence.

Interest rate risk

The major market risk factors in CoCo bonds are the credit risk and equity risk due to the conversion mechanism, while the interest rate risk plays a less significant role. Unlike fixed income derivatives, the interest rate in pricing CoCo bonds only plays the role as the discounting factor but not in the determination of the payoff. The bond part of a CoCo bond can be treated as a fixed rate coupon bond and its interest rate risk can be quantified by appropriate discounting factors. On the other hand, the conversion component is rather insensitive to interest rate risk due to the following counter-balancing effects. A hike in interest rate leads to a higher expected stock price (due to an increase in risk neutral drift) while the present value of an equity conversion is reduced due to a smaller discounting factor. Hence, the interest rate risk of a CoCo bond is of a less concern than the joint interaction of credit risk and equity risk.

Having said that, we note that our valuation procedures can be readily extended to deterministic or independent stochastic interest rates. Suppose that the risk free interest rate is stochastic as $(r_t)_{t\geq 0}$ and independent of the conversion time τ. We denote the corresponding zero coupon bond price as $D(0,t) = \mathbb{E}^Q\left[e^{-\int_0^t r_s\,ds}\right]$. The coupon and principal components can be expressed as

$$\begin{aligned} P_C &= \sum_{i=1}^{n} \mathbb{E}^Q\left[c_i e^{-\int_0^{t_i} r_s\,ds}\mathbf{1}_{\{\tau > t_i\}}\right] = \sum_{i=1}^{n} c_i D(0,t_i)\left[1 - Q(\tau \leq t_i)\right], \\ P_F &= \mathbb{E}^Q\left[F e^{-\int_0^T r_s\,ds}\mathbf{1}_{\{\tau > T\}}\right] = F D(0,T)\left[1 - Q(\tau \leq T)\right]. \end{aligned}$$

For the conversion value, we can take $\widetilde{S}_t = S_t e^{\int_0^t r_s\,ds} e^{-qt}$ and modify the change-of-measure as

$$Z_t = \left.\frac{dQ^*}{dQ}\right|_{\mathcal{F}_t} = \frac{\widetilde{S}_t}{\widetilde{S}_0} = \frac{e^{\int_0^t r_s\,ds} e^{-qt} S_t}{S_0},$$

such that

$$P_E = G S_0 \mathbb{E}^Q\left[e^{-q\tau}\frac{e^{-\int_0^\tau r_s\,ds} e^{q\tau} S_\tau}{S_0}\mathbf{1}_{\{\tau \leq T\}}\right] = G S_0 \mathbb{E}^{Q^*}\left[e^{-q\tau}\mathbf{1}_{\{\tau \leq T\}}\right].$$

Therefore, the above discussion with constant JtNV intensity can be extended to independent stochastic interest rate without much complexity.

When the stochastic interest rate is correlated with the capital ratio, the dimensionality of the pricing problem increases by one. The evaluation becomes more cumbersome and analytical tractability of the pricing model depends on the choice of the interest rate model. The evaluation of P_C and P_F can be performed by using the forward measure, while the evaluation of $Q(\tau \leq t)$ and P_E becomes two-dimensional problems (capital ratio and interest rate). On the other hand, the incorporation of stochastic interest rate makes the numerical partial differential equation method suffer from the curse of dimensionality (see Sect. 2.5). Given that interest rate risk is secondary compared to the credit and equity risks, it is not quite worthwhile to introduce correlated stochastic interest rates in the pricing models of CoCo bonds.

2.5 State-dependent intensity

The nice analytical tractability in the direct computation of the conversion probability cannot be retained when we consider more general payoff structure upon equity conversion and/or the intensity of the JtNV trigger becomes state dependent. We show how to formulate the pricing problem into a partial differential equation (PDE) formulation. Once the full characterization of the auxiliary conditions in the PDE formulation is known, one may compute the numerical solution of the pricing problem using standard finite difference schemes.

Since equity conversion occurs either at an accounting trigger or JtNV trigger, we may decompose the stock price at the conversion time τ into the following form:

$$S_\tau = S_{\tau_B}\mathbf{1}_{\{\tau_R>\tau_B\}} + (1+\gamma)S_{\tau_{R\_}}\mathbf{1}_{\{\tau_B>\tau_R\}}$$

where there is a stock price jump at a JtNV trigger ($\tau = \tau_R$). By making use of the following identities:

$$\begin{aligned}
\mathbb{E}^Q\left[e^{-r\tau_B}GS_{\tau_B}\mathbf{1}_{\{\tau_B\le T\}}\mathbf{1}_{\{\tau_B<\tau_R\}}\right] &= \mathbb{E}^Q\left[e^{-\int_0^{\tau_B}(r+\lambda_u)\,du}GS_{\tau_B}\mathbf{1}_{\{\tau_B\le T\}}\right], \\
\mathbb{E}^Q\left[e^{-r\tau_{R_}}G(1+\gamma)S_{\tau_{R_}}\mathbf{1}_{\left\{\tau_{R_}\le T\right\}}\mathbf{1}_{\left\{\tau_{R_}<\tau_B\right\}}\right] &= \mathbb{E}^Q\left[\int_0^{\tau_B\wedge T} e^{-\int_0^u(r+\lambda_s)\,ds}\lambda_u(1+\gamma)GS_u\,du\right],
\end{aligned}$$

where $\mathbf{1}_{\left\{\tau_{R\_}\le T\right\}}\mathbf{1}_{\left\{\tau_{R\_}<\tau_B\right\}} = \mathbf{1}_{\left\{\tau_{R\_}\le \tau_B\wedge T\right\}}$ is used in the second identity, the conversion value can be expressed as

$$P_E = \mathbb{E}^Q\left[e^{-\int_0^{\tau_B}(r+\lambda_u)\,du}GS_{\tau_B}\mathbf{1}_{\{\tau_B\le T\}} + \int_0^{\tau_B\wedge T} e^{-\int_0^u(r+\lambda_s)\,ds}\lambda_u(1+\gamma)GS_u\,du\right]. \tag{2.5}$$

Following a similar interpretation in the employee stock option model of Leung and Sircar (2009), the two terms in eq.(2.5) can be interpreted as follows.

1. When y_t hits the barrier at τ_B, the investor receives the payoff GS_{τ_B};
2. When $t < \tau_B \wedge T$, the investor receives the continuous cash-flow $\lambda(1+\gamma)GS_t$.

In our CoCo bond model, the barrier variable is the log capital ratio y_t while jump only happens in the log stock price process x_t at a JtNV trigger. Before the jump time, the joint process (x_t, y_t) is a bivariate diffusion. Therefore, we can compute the expectation by solving a PDE instead of a PIDE. A similar solution technique has been used in solving an optimal stopping problem for jump-diffusion process with a fixed jump size (Egami and Dayanik, 2012).

Next, we present the PDE formulation of the pricing function $P(x, y, t)$ which solves the conversion value of the CoCo bond, where $P_E = P(x_0, y_0, 0)$. Given the joint dynamics (pre-conversion) of (x_t, y_t) in eq.(2.1), the corresponding generator is given by

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \alpha(x,y)\frac{\partial}{\partial x} + \rho\sigma\eta\frac{\partial^2}{\partial x\partial y} + \frac{\eta^2}{2}\frac{\partial^2}{\partial y^2} + \kappa\,(\theta - y)\,\frac{\partial}{\partial y},$$

where

$$\alpha(x,y) = r - q - \sigma^2/2 - \gamma\lambda(x,y).$$

Under certain mild technical conditions such as (i) $P \in C^{1,2}([0,T), \mathbb{R}^2)$ and (ii) the terminal payoff $P(x,y,T)$ is integrable (Karatzas and Shreve, 1991), the pricing function P is the solution to the following Dirichlet problem

$$\frac{\partial P}{\partial t} + \mathcal{L}P + \lambda(x,y)(1+\gamma)Ge^x = [r + \lambda(x,y)]P, \tag{2.6}$$

for $(x,y,t) \in (-\infty,\infty) \times [y_B,\infty) \times [0,T]$. The boundary condition and terminal condition are

$$\begin{aligned} P(x,y_B,t) &= Ge^x, \\ P(x,y,T) &= 0, \end{aligned}$$

respectively. We assume the usual natural far field boundary conditions for $x \to \pm\infty$ and $y \to \infty$. The formulation in eq.(2.6) can be considered as an inhomogeneous PDE due to the non-linear inhomogeneous term $\lambda(x,y)(1+\gamma)Ge^x$, which represents the payoff due to the random termination at a JtNV trigger. On the other hand, the boundary condition at $y = y_B$ is dictated by the equity conversion payoff at an accounting trigger.

We may model the different forms of dependence of the stock price and/or capital ratio on the financial state of the issuing bank through an appropriate choice of the state dependent intensity function. Some of these choices are discussed below.

1. Stock price dependent intensity:

$$\lambda(x) = \exp(a_0 - a_1 x), \quad a_1 > 0,$$

   which prescribes an inverse relation between the intensity and stock price, with $\lambda(x,y) \to \infty$ as $S = e^x \to 0$. Das and Sundaram (2007) use a similar specification for a hybrid equity-credit modeling of convertible bonds. This specification takes the stock price as a measure of the financial health of the bank in which a low level of stock price indicates a high probability of PONV trigger.

2. Capital ratio dependent intensity:

$$\lambda(y) = b_0 \mathbf{1}_{\{y \le y_{RT}\}}, \quad b_0 > 0,$$

   where y_{RT} is a predetermined level that specifies the warning region in which the capital ratio is close to the contractual threshold such that the supervisory authority initiates the monitoring procedure of potential activation of the PONV trigger.

3. We can combine the above two specifications as a sum of two terms:

$$\lambda(x,y) = \exp(a_0 - a_1 x) + b_0 \mathbf{1}_{\{y \le y_{RT}\}}, \quad a_1 > 0,\, b_0 > 0.$$

   This is seen to provide more flexible modeling of the PONV trigger.

2.6 Other contractual features

We would like to illustrate the versatility of our pricing approach that it can accommodate more complex structural features of the CoCo bonds. We consider three examples of traded CoCo bond contracts with various payoff structures, like the floored payoff, coupon cancellation and perpetuality with callable right.

2.6.1 Floored payoff

The issuer might impose a floor on the conversion value. For example, the Buffer Capital Note (BCN) issued by Credit Suisse has the conversion price limited at a floor of 20 USD per share. The floor feature is attractive from an investor's perspective since this limits the downside risk of a CoCo bond that equity conversion occurs at stock price that may be far too low.

The conversion value with a floored payoff can be formulated as

$$P_E^{Floor} = G\mathbb{E}^Q\left[e^{-r\tau}\max\left(S_\tau, K\right)\mathbf{1}_{\{\tau\leq T\}}\right], \quad \tau = \tau_B \wedge \tau_R,$$

where K is the preset floor. As usual, by performing the following decomposition

$$\max\left(S_\tau, K\right) = \max\left(S_{\tau_B}, K\right)\mathbf{1}_{\{\tau_R>\tau_B\}} + \max\Big((1+\gamma)S_{\tau_{R\_}}, K\Big)\mathbf{1}_{\{\tau_B>\tau_R\}},$$

the conversion value with a floored payoff can be expressed as

$$\begin{aligned} P_E^{Floor} &= G\mathbb{E}^Q\Big[e^{-\int_0^{\tau_B}(r+\lambda_u)\,du}\max\left(S_{\tau_B}, K\right)\mathbf{1}_{\{\tau_B\leq T\}} \\ &\quad + \int_0^{\tau_B\wedge T} e^{-\int_0^u(r+\lambda_s)\,ds}\lambda_u\max\left((1+\gamma)S_u, K\right)du\Big]. \end{aligned}$$

The corresponding PDE formulation is seen to be

$$\frac{\partial P}{\partial t} + \mathcal{L}P + \lambda(x,y)(1+\gamma)G\max\left((1+\gamma)e^x, F\right) = \left[r+\lambda(x,y)\right]P,$$

for $(x,y,t)\in(-\infty,\infty)\times[y_B,\infty)\times[0,T]$. The auxiliary conditions are given by

$$\begin{aligned} P(x,y_B,t) &= G\max\left(e^x, K\right), \\ P(x,y,T) &= 0. \end{aligned}$$

Constant intensity
Under the assumption that the intensity λ is constant and $(1+\gamma)S_{\tau_{R\_}} < K$, we can evaluate the conversion value with a floored payoff using the extended Fortet method (see Appendix B.2 for full details). The key step involves the determination of the numerical approximation to the joint density of (x_{τ_B}, τ_B) defined as follow:

$$Q\left(x_{\tau_B}\in dx, \tau_B\in dt\right) = q\left(x,t\right)dxdt.$$

In the numerical procedure, we discretize the domain for x and t using a set of rectangular grids with the right-point scheme as:

$$\begin{aligned} x_i &= x_{lb} + i\Delta x, \qquad & i &= 1,2,...,n, \\ t_j &= j\Delta t, & j &= 1,2,...,m. \end{aligned}$$

The numerical approximation for the joint density can be computed by the following recursive scheme

$$\begin{aligned} q\left(x_i,t_1\right) &= \Delta x\ \Phi\left(x_i,t_1\right), \quad j=1, \\ q\left(x_i,t_j\right) &= \Delta x\left[\Phi\left(x_i,t_j\right) - \sum_{h=1}^{j-1}\sum_{k=1}^{n} q\left(x_k,t_h\right)\psi\left(x_i,t_j;x_k,t_h\right)\right], \quad j=2,3,...,m, \end{aligned}$$

in which analytic closed form formulas for $\Phi(\cdot)$ and $\psi(\cdot)$ are available for the bivariate Gaussian process (see Appendix B.2).

Once the joint density values at successive time points are known, the conversion value with a floored payoff can be evaluated by performing numerical integration of the following integral formula:

$$P_E^{Floor} = G\int_0^T \int_{-\infty}^{\infty} q\left(x,u\right) e^{-(r+\lambda)u} \max\left(e^x, K\right) dx du + GK\int_0^T \lambda e^{-(r+\lambda)u} Q\left(\tau_B > u\right) du.$$

Note that the optionality in $\max((1+\gamma)S_{\tau_{R\_}}, K)$ is avoided by assuming sufficiently strong downward jump in stock price upon PONV trigger such that $(1+\gamma)S_{\tau_{R\_}} < K$.

2.6.2 Coupon cancellation

It is straightforward to modify our equity-credit framework to deal with other interesting features that appear in some issued CoCo bonds. One of these features is the coupon cancellation mechanism that entitles the issuer to the discretionary right to cancel the coupon payments before the conversion event. This can be considered as a partial conversion in which the issuing bank restructures her balance sheet by reducing the liability on future cash outflows. Corcuera *et al.* (2014) discuss the coupon cancellation feature of the CoCo bond issued by the Spanish bank BBVA, in which a coupon payment can be cancelled upon the sole discretion of the bank or the supervisory authority.

We can model the coupon cancellation (CC) event in a reduced form manner by introducing an independent exponential random variable $\tau_C \sim \mathrm{Exp}(\lambda_C)$. The CoCo bond price can be expressed as

$$P_E^{CC} = \sum_{i=1}^{n} \mathbb{E}^Q\left[c_i e^{-rt_i}\mathbf{1}_{\{\tau_C > t_i\}}\right] + \mathbb{E}^Q\left[F e^{-rT}\mathbf{1}_{\{\tau > T\}}\right] + \mathbb{E}^Q\left[e^{-r\tau} G S_\tau \mathbf{1}_{\{\tau \le T\}}\right], \quad \tau = \tau_B \wedge \tau_R,$$

such that the coupon payments are related to the stopping time τ_C only, while the principal payment and conversion value remain the same. The exogenous feature of τ_C is also consistent with the discretionary nature of coupon deferral or coupon cancellation in reality. Assuming that the random time τ_C is independent of the other random times, it is straightforward to extend to time-dependent intensity like those with piecewise constant intensity so that the term structure of coupon cancellation probability can be modeled.

2.6.3 Perpetual CoCo bond with callable feature

One innovative design of CoCos involves perpetual maturity along with a callable feature. For example, the Perpetual Subordinated Contingent Convertible Securities issued by the HSBC bank can be redeemed at par by the issuer on any coupon reset date. The major challenge in pricing these CoCo bonds lies in the modeling of the issuer's call policy. As noted in Jarrow *et al.* (2010), one may approximate the issuing bank's call policy using a reduced form approach from market perspective. The reduced form approach of modeling the call policy can be found in other applications, like mortgage prepayment and exercise of employee's stock options.

Suppose that the CoCo bond is perpetual with a constant coupon stream c and call price K. We introduce an independent exponential random variable $\tau_C \sim \mathrm{Exp}(\lambda_C)$ as the random time of issuer's call. This is seen to be equivalent to our equity-credit framework by setting $\gamma = 0$ and $\lambda = \lambda_C$. In view of this observation, we can formulate the CoCo bond price as

$$P_E^{Callable} = \sum_{i=1}^{\infty} \mathbb{E}^Q\left[c e^{-rt_i}\mathbf{1}_{\{\tau_C \wedge \tau_B > t_i\}}\right] + \mathbb{E}^Q\left[e^{-r\tau_C} G K \mathbf{1}_{\{\tau_B > \tau_C\}}\right] + \mathbb{E}^Q\left[e^{-r\tau_B} G S_{\tau_B}\mathbf{1}_{\{\tau_C > \tau_B\}}\right],$$

where K is the call price as specified in the contract. The first term is the coupon payment stream up to issuer's call or accounting trigger, the second term is the discounted expected value when the issuer call before conversion while the third term is the discounted expected value when the accounting trigger occurs prior to a call.

3 Numerical Examples

We consider a CoCo bond with the following contractual specification: 5-year maturity, 10% coupon rate with the face value F of \$100, and the number of conversion shares G is set to be 200 (which implies the conversion price to be \$0.5). The accounting triggering level is set to be 5%, which gives $y_B = \ln 0.05$. Unless otherwise stated, the current stock price is set to be $S_0 = 0.5$.

3.1 Model parameters

Given the contractual specification of a CoCo bond (c_i, F and G), we need to perform calibration on the model parameters $(\sigma, \rho, \eta, \kappa, \theta)$ for the bivariate process and (λ, γ) for the embedded JtNV feature. While our proposed model is versatile to capture various CoCo bond features, the parameter calibration is not easy due to the lack of market data. For instance, the difficulty to infer the time-series of capital ratio makes the estimation of the capital ratio volatility η and correlation coefficient ρ not an easy task. Moreover, the consistent estimation of the mean reversion parameters requires not only a large number of observations but also a long time span of historical data (Yu, 2012). On the other hand, since an actual equity conversion is yet to occur, it is hard to disentangle the Poisson jump part and pin down the intensity and corresponding jump size in the stock price.

In our sample calculations, we set some economically reasonable parameter values in the pricing model of CoCos as follows.

r	q	σ	ρ	η	κ	θ	λ	γ
0.02	0.01	0.40	0.50	0.30	0.20	ln(0.10)	0.05	-0.70

We set the correlation coefficient to be 0.5 to reflect the moderate but imperfect comovement of the capital ratio and the stock price. We take the stock price volatility to be 40% to reflect a volatile market environment such that the equity risk is important. For the capital ratio dynamics, we assume a long-term level of 10% with a mean reversion speed of 0.2, indicating that the bank takes around 5 years to adjust its capital ratio to its long-term mean level. This follows from Collin-Dufresne and Goldstein (2001) who take the value of 0.18 for the mean reversion speed of their dynamic leverage ratio model. The annualized volatility of the log capital ratio taken at the level of 30% reflects a non-negligible probability of accounting trigger for a time horizon of 5 years.

For the jump part, we assume that the JtNV intensity is $\lambda = 0.05$ with the jump size $\gamma = -0.70$. The intensity of 5% can be translated to a credit spread of roughly 350 bps using the rule-of-thumb: $s_{CDS} = \lambda(1 - R)$ with the recovery rate R of 30%.[2] As discussed, the jump size upon conversion is rather arbitrary and it requires the trader's own judgment. We set a high level of 70% write-down in stock price to reflect the arrival of a negative shock (bad news) leading to a PONV conversion. In the case of state dependent intensity, we choose $a_1 = 0, 0.5$ or 1.0 in the functional form of the stock price dependence and a_0 is set by matching $\lambda(x_0) = 0.05$. This allows us to have a benchmark comparison with the constant intensity case. Figure 1 illustrates the inverse relationship between the conversion intensity and stock price with varying values of a_1.

[2] We apply the CDS analogy here because a conversion always happens before a default. Hence, the CDS implied default intensity can be justified as a lower bound of the conversion intensity.

3.2 Check for numerical accuracy among various numerical schemes

In our sample calculations for checking numerical accuracy among various numerical schemes for pricing the CoCo bonds, we used the parameter values: $S_0 = 100$ and $G = 1$ for convenience. We set the time horizon to be $t = 1, 2, 3, 4, 5$. We calculated the conversion value P_E using the Fortet method, explicit finite difference (FD) scheme and Monte Carlo simulation. For the FD scheme, we adopted 500 grid points in both the x- and y-direction and chose the natural boundary points to be sufficiently far such that the numerical results converge. In order to ensure that the explicit FD scheme is stable, we chose a small time step $\Delta t = 0.0001$. For the Monte Carlo simulation, we also took a small time step $\Delta t = 0.0001$ in order to capture the barrier hitting event effectively. The number of simulation paths was taken to be $100,000$ in order to achieve sufficiently good accuracy. We implemented the Fortet method using the right point scheme and chose $\Delta t = 0.0001$ for consistency with other numerical schemes. Comparison of the numerical results using these three numerical methods are presented in Tables 1a, 1b and 1c. Good agreement of the numerical results for P_E using various numerical schemes is confirmed.

3.3 Sensitivity analysis: constant intensity

We performed sensitivity analysis to examine how various model parameters may affect the CoCo bond price. First, we focus on the simple case with constant intensity and conversion into shares with no other complex contractual provision. In this case, the pricing formula can be expressed succinctly as

$$P = \sum_{i=1}^{n} c_i e^{-rt_i} \left[1 - Q\left(\tau \le t_i\right)\right] + Fe^{-rT}\left[1 - Q\left(\tau \le T\right)\right] + GS_0\mathbb{E}^{Q^*}\left[e^{-q\tau}\mathbf{1}_{\{\tau \le T\}}\right]. \tag{3.1}$$

Here, Q is the risk neutral measure and Q^* is the stock price measure. The correlation coefficient ρ and stock price volatility σ enter into the pricing formula implicitly through the adjustment term added to the capital ratio dynamics under the stock price measure Q^* in the conversion value P_E term [the last term in eq.(3.1)]. In other words, the correlation coefficient and stock price volatility have no direct impact on the bond components P_C and P_F [the first two terms in eq.(3.1)].[3] From Lemma 2, we observe that the conversion value P_E is related to the conversion probability under the stock price measure Q^*. When the correlation coefficient is positive, $\rho > 0$, the capital ratio dynamics has a higher mean reversion level due to the adjustment term $\rho\sigma\eta > 0$. This implies a smaller conversion probability under Q^* and hence a smaller conversion value P_E. These theoretical observations are essential to understand the numerical plots of the pricing properties of the CoCo bonds presented below.

Impact of correlation coefficient, ρ

The joint density of the conversion time and stock price at an accounting trigger is highly dependent on the correlation coefficient ρ. When the correlation coefficient ρ is more positive, the capital ratio exhibits a stronger correlated movement with the stock price. As the capital ratio declines in value and hits the accounting trigger threshold, the stock price tends to stay at a lower level, thus resulting a smaller conversion value P_E. This explains why the CoCo price is a decreasing function of ρ. By comparing Figures 2a and 2b, we observe that the impact of ρ on the CoCo price is more significant

[3] This differs from the equity derivative approach in which the conversion time is proxied by the first passage time of the stock price to an implied threshold, in which the stock price volatility affects directly the CoCo bond price through the conversion probability.

when the current capital ratio is further from the trigger threshold and the stock price volatility is higher. In Section 3.6, we present more detailed discussion of ρ on the conditional distribution of the stock price at an accounting trigger and the resulting CoCo price.

Impact of stock price volatility, σ

Figure 3 shows that the CoCo bond price is decreasing with stock price volatility σ when the correlation coefficient ρ is positive. This pricing property is somewhat similar to that of a reverse convertible. When ρ is zero, the CoCo bond price is insensitive to the stock price volatility (vega value is zero). This is expected since the conversion value is not affected by σ due to the martingale property of the discounted stock price and the bond component is not affected by the stock price when the intensity is constant. The sensitivity of the CoCo price with respect to stock price volatility is stronger when the correlation coefficient is closer to one. Similar to a reverse convertible, there is no upside gain on the CoCo price when the stock price increases. However, the downside move of the stock price correlated with downside move in the capital ratio at positive correlation leads to a higher probability of activation of the loss absorption feature in the CoCo bond. This negative effect on the CoCo bond becomes more prominent when the stock price volatility is higher.

Impact of conversion intensity, λ

Figure 4 reveals the sensitivity of the CoCo bond price to the level of JtNV intensity λ at different sizes of negative stock price jump γ at conversion. We vary the jump size as γ = -0.3, -0.5, -0.7, which means that the proportional drop in stock price due to a JtNV is assumed to be 30%, 50% and 70%, respectively. Since the capital ratio of 8% is quite far away from the triggering threshold, most of the conversion risk of the CoCo bond lies at the possibility of a JtNV with the associated drop of the stock price and conversion value. The CoCo bond price can be highly sensitive to the intensity under the assumption of a large downward jump size. In other words, the added JtNV feature provides the flexibility to incorporate a risk premium that lowers the value of a CoCo bond. This may explain lowering of the market price of the CoCo bond even when the conversion probability generated by the capital ratio diffusion is relatively small.

3.4 Floored payoff

We would like to examine how the floored payoff may affect the conversion risk. In Figure 5, we show the CoCo bond price with different levels of stock price (delta risk) and stock price volatility (vega risk) with $K = 0.2$ and $K = 0.4$, corresponding to setting the floor to be 40% and 80% of the current stock price level $S_0 = 0.5$, respectively. For a high floor value set at $K = 0.4$, the CoCo bond price approaches the bond floor as the stock price falls close to the floor level of $K = 0.4$. For the vega risk, Figure 5(b) shows that the CoCo bond with a floored payoff is less sensitive to the stock price volatility and hence exhibits lower vega sensitivity. When we have the floor level set at a lower level $K = 0.2$, the floored payoff serves to protect the investor only at times with extremely low stock price. When the stock price level and volatility are at normal levels, the CoCo bond behaves like a standard CoCo bond without the floor and exhibits similar vega sensitivity. To summarize, the floored payoff provides protection for the investor from conversion risk and set a lower bound on the CoCo bond price. This generates a positive convexity of the CoCo bond price versus the stock price as well as a reduction on the vega sensitivity.

3.5 State dependent intensity

We explore the impact of stock price dependent intensity as exhibited by the following function

$$\lambda(x) = \exp(a_0 - a_1 x), \quad a_1 > 0;$$

in particular, the impact of the coefficient a_1 on the delta risk (see Figure 6). When $a_1 = 0$, the intensity function is reduced to the constant case and the CoCo bond price has a linear dependence to the stock price (due to the martingale property). When we introduce the stock price dependent intensity with $a_1 = 0.5$ and $a_1 = 1.0$, a negative convexity effect appears when the stock price is low. This is due to a higher probability of a JtNV trigger at low stock price such that the CoCo bond price becomes lower. On the other hand, the JtNV intensity decreases gradually as the stock price increases due to the inverse relationship. The CoCo bond prices converge to those under constant intensity ($a_1 = 0$) at high stock price. The incorporation of an inverse relation of the state dependent intensity on the stock price is important since the pricing model can generate negative convexity that models the *death spiral* effect in which equity sensitivity of a CoCo bond increases as the stock price is under distressed.

We would like to examine how the stock price dependent intensity induces additional vega sensitivity of the CoCo bond price. Figure 7(a) shows that when the correlation coefficient is zero, the CoCo bond price decreases with increasing stock price volatility. Recall from Figure 3 that the vega sensitivity is zero under constant intensity when $\rho = 0$. When the intensity has an inverse stock price dependence, a higher stock price volatility implies a higher chance of JtNV trigger since the stock price is more likely to diffuse to a lower level. As expected, the vega sensitivity is higher when we set a larger value of a_1. In Figure 7(b), we observe at $\rho = 0.5$ that a stronger stock price dependence on the intensity leads to higher vega sensitivity of the CoCo bond price, though the impact is less when compared with the case where the correlation is zero.

3.6 Conditional distribution of the stock price at an accounting trigger

We would like to examine the conditional distribution of the stock price at an accounting trigger. This can be seen as an extension to the notion of implied trigger in Spiegeleer and Schoutens (2012) when there is an imperfect correlation between the stock price and capital ratio.

By the law of conditional probability, we deduce that the conditional distribution of x_t at the accounting trigger τ_B is given by

$$\Pr\left(x_{\tau_B} \in dx \middle| \tau_B \in dt\right) = \frac{\Pr\left(x_{\tau_B} \in dx, \tau_B \in dt\right)}{\Pr\left(\tau_B \in dt\right)}. \tag{3.2}$$

In order to examine the impact of varying levels of interaction between the stock price and capital ratio, we chose $\rho = 0.3$ and $\rho = 0.9$ in our sample calculations. The dispersion of the stock price distribution can be significant when $\rho = 0.3$. In Table 2, we observe that the mean of the implied stock price distribution can be much higher than the initial stock price of 0.5, indicating the possibility of significant divergence between the capital ratio and stock price at an accounting trigger. When ρ is 0.9, the mean of the implied stock price distribution concentrates around and lies below 0.5. Figures 8(a) and 8(b) show that the implied stock price distribution becomes more concentrated when ρ increases. This indicates that under a sufficiently high value of ρ, it becomes reasonable to infer the chance of an accounting trigger by monitoring the stock price level. This is consistent with the equity derivative approach that models the accounting trigger by the hitting time of the stock price to a lower barrier (down-and-out).

3.7 Decomposition into the bond and equity components

We examine the decomposition of the CoCo bond value into its bond and equity components with varying values of the capital ratio. This serves to illustrate the change in equity exposure of the CoCo bond as the capital ratio increases.

As revealed in Figure 9, the proportion of the equity component can be almost half of the CoCo bond value when the capital ratio is close to the accounting trigger threshold of 5%. This highlights the importance of modeling the interaction of the bivariate stock price and capital ratio dynamics. When there is no JtNV ($\lambda = 0$), the equity part of a CoCo bond becomes small as the capital ratio increases to 12%, indicating diminishing equity exposure. When there exists more significant JtNV risk ($\lambda = 0.05$), we observe a residual equity component of the CoCo bond even when the capital ratio increases to 12%. This is consistent with the market consensus that the CoCo bond remains to face with some equity risk when the possibility of JtNV persists.

4 Conclusion

We propose an enhanced equity-credit model with underlying bivariate process of the stock price and capital ratio, together with equity conversion feature that is subject to accounting trigger and point-of-non-viability (PONV) trigger. The accounting trigger is modeled using the structural approach as the event of the capital ratio hitting a preset contractual threshold value from above. On the other hand, we model the PONV trigger by the reduced form approach and consider it as a Poisson arrival with specified conversion intensity. We consider various specifications of the contractual payoff upon equity conversion from actual CoCo bond contracts traded in the market. We also discuss different functional forms of state dependence of the conversion intensity on the stock price and capital ratio. For general payoff structure upon equity conversion and state dependent conversion intensity under PONV trigger, we manage to formulate the pricing problem as a two-dimensional partial differential equation (PDE). The numerical solution of the governing PDE can be computed using standard finite difference schemes. Under certain simplifying assumption on the conversion payoff and constant conversion intensity, we reduce the pricing problem to evaluation of the conversion probability. We show how to construct efficient numerical algorithms using the Fortet method to compute prices of the CoCo bonds. We also discuss various issues in the calibration of the model parameters from market prices of traded instruments. The pricing properties of the CoCo bonds under various market conditions and the sensitivity analysis of the price functions under varying values of the model parameters are examined. Our bivariate equity-credit model provides a flexible pricing framework that can incorporate various modeling features and contractual specifications of CoCo bonds traded in the market.

Acknowledgements

We appreciate the constructive comments and suggestions from two anonymous referee which significantly contributed to improving the paper. The views and opinions expressed here are those of the authors and do not reflect the views of the Mitsubishi UFJ Trust Investment Technology Institute.

References

[1] Albul, B., Jaffee, D.M. and Tchistyi, A. (2010). Contingent convertible bonds and capital structure decisions. Working paper, Coleman Fung Risk Management Research Center 2010-01, University of California Berkeley.

[2] Bates, D.S. (1991). The crash of '87: was it expected? the evidence from options markets. *The Journal of Finance*, **46(3)**, 1009-1044.

[3] Avdjiev, S., Kartasheva, A. and Bogdanova, B. (2013). CoCos: a Primer. *BIS Quarterly Review.*

[4] Ballotta, L. and Kyriakou, L. (2015) Convertible bonds valuation in a jump diffusion setting with stochastic interest rates. *Quantitative Finance*, **15(1)**, 115-129.

[5] Bernard, C., Le Courtois, O. and Quittard-Pinon, F. (2008). Pricing derivatives with barriers in a stochastic interest rate environment. *Journal of Economic Dynamics and Control*, **32(9)**, 2903-2938.

[6] Brigo, D., Garcia, J. and Pede N. (2015). CoCo bonds pricing with credit and equity calibrated first-passage firm value models. *International Journal of Theoretical and Applied Finance*, **18(3)**, 31 pages.

[7] Campi, L., Polbennikov, S. and Sbuelz, A. (2009). Systematic equity-based credit risk: A CEV model with jump to default. *Journal of Economic Dynamics and Control*, **33(1)**, 93-108.

[8] Carr, P. and Linetsky, V. (2006). A jump to default extended CEV model: An application of Bessel processes. *Finance and Stochastics*, **10(3)**, 303-330.

[9] Carr, P. and Wu, L. (2009). Stock options and credit default swaps: A joint framework for valuation and estimation. *Journal of Financial Econometrics*, **8(4)**, 409-449.

[10] Chen N. and Kou, S. (2009). Credit spread, implied volatility, and optimal capital structures with jump risk and endogenous defaults. *Mathematical Finance*, **19**, 343-378.

[11] Cheridito, P. and Wugalter, A. (2012). Pricing and hedging in affine models with possibility of default. *SIAM Journal of Financial Mathematics*, **3(1)**, 328-350.

[12] Cheridito, P. and Xu., Z. (2015). A reduced form CoCo model with deterministic conversion intensity. *Journal of Risk*, **17(3)**, 1-18.

[13] Chung, T.K. and Kwok, Y.K. (2014). Equity-credit modeling under affine jump-diffusion models with jump-to-default. *Journal of Financial Engineering*, **1(2)**, 1-25.

[14] Collin-Dufresne P. and Goldstein, R.S. (2001). Do credit spreads reflect stationary leverage ratios? *Journal of Finance*, **56(5)**, 1929-1957.

[15] Coculescu, D., Geman, H. and Jeanblanc, M. (2008). Valuation of default-sensitive claims under imperfect information. *Finance and Stochastics*, **12(1)**, 195-218.

[16] Corcuera, J.M., Spiegeleer J.D., Ferreiro-Castilla, A., Kyprianou, A.E., Madan, D.B. and Schoutens, W. (2013). Pricing of contingent convertibles under smile conform models. *Journal of Credit Risk*, **9(3)**, 121-140.

[17] Corcuera, J.M., Spiegeleer, J.D., Fajardo, J., Jonsson, H., Schoutens, W. and Valdivia, A. (2014). Close form pricing formulas for Coupon Cancellable CoCos. *Journal of Banking and Finance*, **42**, 339-351.

[18] Dai, M., Kwok, Y.K. and You, H. (2007). Intensity-based framework and penalty formulation of optimal stopping problems. *Journal of Economic Dynamics and Control*, **31**, 3680-3880.

[19] Das, S.R., Sundaram, R.K. (2007). An integrated model for hybrid securities. *Management Science*, **53(9)**, 1439-1451.

[20] Duffie, D and Singleton, K. (1999). Modeling term structures of defaultable bonds. *Review of Financial Studies*, **12**, 687-720.

[21] Egami, M. and Dayanik, S. (2012). Optimal stopping problems for asset management. *Advances in Applied Probability*, **44(3)**, 655-677.

[22] Eom, Y.H., Helwege, J. and Huang, J.Z. (2004). Structural models of corporate bond pricing: an empirical analysis. *Review of Financial Studies*, **17**, 499-544.

[23] Glasserman, P. and Nouri, B. (2012). Contingent capital with a capital-ratio trigger. *Management Science*, **58(10)**, 1816-1833.

[24] Gupta, A., Akuzawa, T. and Nishiyama, Y. (2013). Quantitative evaluation of contingent capital and its applications. *North American Journal of Economics and Finance*, **26**, 457-486.

[25] Jarrow, R., Li, H., Liu, S. and Wu, C. (2010). Reduced-form valuation of callable corporate bonds: Theory and evidence. *Journal of Financial Economics*, **95(2)**, 227-248.

[26] Karatzas, I. and Shreve, S.E. (1991) Brownian Motion and Stochastic Calculus (Second Edition). New York, Springer-Verlag.

[27] Leung, C.M. and Kwok, Y.K. (2015). Numerical pricing of CoCo bonds with Parisian trigger feature using the Fortet method. Working paper, Hong Kong Baptist University.

[28] Leung, T. and Sircar, R. (2009). Accounting for risk aversion, vesting, job termination risk and multiple exercises in valuation of employee stock options. *Mathematical Finance*, **19(1)**, 99-128.

[29] Longstaff, F.A. and Schwartz, E.S. (1995). A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance*, **50(3)**, 789-819.

[30] Pennacchi, G. (2011). A structural model of contingent bank capital. Federal Reserve Bank of Cleveland Working Paper 10-04.

[31] Shreve, S.E. (2004). *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer Finance, New York.

[32] Spiegeleer, J.D. and Schoutens, W. (2012). Pricing contingent convertibles: A derivatives approach. *Journal of Derivatives*, **20(2)**, 27-36.

[33] Wilkens, S. and Bethke, N. (2014). Contingent convertible bonds: A first empirical assessment of selected pricing models. *Financial Analysts Journal*, **70(2)**, 59-77.

[34] Yu, J. (2012). Bias in the estimation of the mean reversion parameter in continuous time models. *Journal of Econometrics*, **169**, 114-122.

A Proof of Lemmas

A.1 Lemma 1

We consider a general stochastic intensity $\lambda_t = \lambda(X_t)$ with Markov process $X_t \in \mathbb{R}^2$. Since $\tau = \tau_B \wedge \tau_R$, the decomposition into the two events $\{\tau_B > \tau_R\}$ and $\{\tau_R > \tau_B\}$ gives

$$Q(\tau \leq t) = \mathbb{E}^Q\left[\mathbf{1}_{\{\tau_B \wedge \tau_R \leq t\}}\right] = \mathbb{E}^Q\left[\mathbf{1}_{\{\tau_R \leq t\}}\mathbf{1}_{\{\tau_B > \tau_R\}}\right] + \mathbb{E}^Q\left[\mathbf{1}_{\{\tau_B \leq t\}}\mathbf{1}_{\{\tau_R > \tau_B\}}\right].$$

The first term corresponds to the scenario where the JtNV trigger occurs prior to the accounting trigger. By virtue of iterated expectation, we obtain

$$\begin{aligned}\mathbb{E}^Q\left[\mathbf{1}_{\{\tau_R \leq t\}}\mathbf{1}_{\{\tau_B > \tau_R\}}\right] &= \mathbb{E}^Q\left[\mathbb{E}^Q\left[\mathbf{1}_{\{\tau_R \leq t\}}\mathbf{1}_{\{\tau_B > \tau_R\}}\middle| \tau_R = u\right]\right] \\ &= \int_0^t \mathbb{E}^Q\left[\lambda(X_u)\, e^{-\int_0^u \lambda(X_s)\,ds}\mathbf{1}_{\{\tau_B > u\}}\right] du.\end{aligned}$$

The second term gives the conversion probability conditional on accounting trigger occurring prior to the JtNV trigger. It is straightforward to obtain

$$\mathbb{E}^Q\left[\mathbf{1}_{\{\tau_B \leq t\}}\mathbf{1}_{\{\tau_R > \tau_B\}}\right] = \mathbb{E}^Q\left[e^{-\int_0^{\tau_B} \lambda(X_u)\,du}\mathbf{1}_{\{\tau_B \leq t\}}\right].$$

A.2 Lemma 2

The proof follows from Lemma 11.6.1 and Theorem 11.6.2 in Shreve (2004). From the adjusted stock price process $\widetilde{S}_t$, we can decompose the Radon-Nikodym density as

$$Z_t = Z_t^c Z_t^J,$$

where

$$Z_t^c = \exp\left(\sigma W_t^1 - \frac{\sigma^2}{2}t\right) \quad \text{and} \quad Z_t^J = \exp(-\gamma\lambda t)(1+\gamma)^{N_t},$$

corresponding to the change-of-measure for the continuous path and jump path, respectively. Since the Brownian motion is only affected by the change-of-measure Z_t^c, by the Girsanov theorem, we have

$$dB_t^1 = dW_t^1 - \sigma\,dt.$$

We have the log stock price under the stock price measure Q^* as

$$dx_t = \left(r - q + \frac{\sigma^2}{2} - \lambda\gamma\right)dt + \sigma\,dB_t^1 + \ln(1+\gamma)\,dN_t,$$

while the dynamic equation for y_t under Q^* is given by

$$dy_t = \kappa(\theta - y_t)\,dt + \eta\left(\sqrt{1-\rho^2}\,dW_t^2 + \rho\,dW_t^1\right) = [\kappa(\theta - y_t) + \rho\sigma\eta]\,dt + \eta\,dB_t^2,$$

where $dB_t^2 = \sqrt{1-\rho^2}\,dW_t^2 + \rho\,dB_t^1$ and $\langle dB_t^1, dB_t^2\rangle = \rho\,dt$. On the other hand, the Poisson process is affected only by the change-of-measure Z_t^J as

$$Z_t^J = \exp(-\gamma\lambda t)(1+\gamma)^{N_t} = e^{(\lambda-\lambda^*)t}\left(\frac{\lambda^*}{\lambda}\right)^{N_t},$$

where the second equality is derived from the change-of-measure for a Poisson process with constant intensity. Hence, the intensity becomes $\lambda^* = (1+\gamma)\lambda$ under the stock price measure Q^*.

B Fortet Method

B.1 One-dimensional Fortet scheme

Consider the following dynamic equation of the capital ratio y_t under the stock price measure Q^*, where

$$dy_t = [\kappa(\theta - y_t) + \rho\sigma\eta]\, dt + \eta\, dB_t.$$

Let τ_B denote the first passage time of y_t hitting the threshold y_B and $q(t)$ be the corresponding density function defined by $\Pr\{\tau_B \in dt\} = q(t)\, dt$. The transition probability density for the process y_t conditional on $s < t$ is defined by

$$\Pr\left\{ y_t \in dy \middle| y_s \in dy' \right\} = f\left(y, t; y', s\right) dy.$$

Suppose the capital ratio process starts at y_0 at $t = 0$ and moves downstream to y at time t, where $y < y_B < y_0$, then the process must cross the threshold y_B at some time prior to time t as it moves from y_0 to y. By the continuity and strong Markov property of the capital ratio process, we observe

$$f(y, t; y_0, 0) = \int_0^t q(s)\, f(y, t; y_B, s)\, ds.$$

Integrating y on both sides over $(-\infty, y_B]$, we obtain the Fortet equation as follows:

$$N\left(\frac{y_B - \mu(t, 0)}{\Sigma(t, 0)}\right) = \int_0^t q(s)\, N\left(\frac{y_B - \mu(t, s)}{\Sigma(t, s)}\right)\Bigg|_{y_s = y_B} ds, \tag{B.1}$$

where

$$\mu(t, s) = y_s e^{-\kappa(t-s)} + \left(\theta + \frac{\rho\sigma\eta}{\kappa}\right)\left[1 - e^{-\kappa(t-s)}\right], \quad \Sigma^2(t, s) = \frac{\eta^2}{2\kappa}\left[1 - e^{-2\kappa(t-s)}\right].$$

We write

$$a(t) = \frac{y_B - \mu(t, 0)}{\Sigma(t, 0)}, \qquad b(t, s) = \frac{y_B - \mu(t, s)}{\Sigma(t, s)}\Bigg|_{y_s = y_B},$$

so that eq.(B.1) can be expressed as

$$N(a(t)) = \int_0^t q(s)\, N(b(t, s))\, ds.$$

This is seen to be a Volterra integral equation of the first kind. To solve the integral equation numerically, we apply the right point scheme of numerical integration. By approximating the integral over the discrete time points, $t_j = j\Delta t$, $j = 0, 1, ..., m$, where Δt is the uniform time step, we obtain

$$N(a(t_j)) \approx \sum_{i=1}^{j} q_i N(b(t_j, t_i)).$$

Here, we have chosen the discrete approximation

$$q_j \approx \Pr\{\tau_B \in (t_{j-1}, t_j]\}, \qquad j = 1, 2, ..., m.$$

We then deduce the following recursive scheme for calculating q_j, $j = 1, 2, \ldots, m$, successively (Longstaff and Schwartz, 1995; Coculescu *et al.*, 2008):

$$\begin{aligned} q_1 &= N(a(t_1)), \\ q_j &= N(a(t_j)) - \sum_{i=1}^{j-1} q_i N(b(t_j, t_i)), \qquad j = 2, 3, ..., m. \end{aligned}$$

B.2 Two-dimensional Fortet scheme

The calculation for floored payoff and the conditional stock price distribution require the joint density of (x_{τ_B}, τ_B) under the risk neutral measure Q. The joint dynamics of (x_t, y_t) conditional on (x_s, y_s), $s < t$, and the event $\{\tau_R > t\}$ can be written as

$$\begin{aligned} x_t &= x_s + \left(r - q - \frac{1}{2}\sigma^2 - \lambda\gamma\right)(t-s) + \int_s^t \sigma \, dB_u^1, \\ y_t &= y_s e^{-\kappa(t-s)} + \theta\left[1 - e^{-\kappa(t-s)}\right] + \int_s^t \eta e^{-\kappa(t-u)} dB_u^2, \end{aligned}$$

where $\langle dB_t^1, dB_t^2 \rangle = \rho \, dt$. Note that an adjustment term $-\lambda\gamma$ is added in the drift term of x_t due to the constant intensity jump upon equity conversion. We would like to apply the Fortet method to find the numerical approximation of the joint density $q(x,t)$ of (x_{τ_B}, τ_B) as defined by

$$\Pr\{x_{\tau_B} \in dx, \tau_B \in dt\} = q(x,t) \, dxdt.$$

Let $f(y,x,t;y',x',s)$ denote the transition probability density for the bivariate process (y_t, x_t) conditional on (x_s, y_s), $s < t$, where

$$\Pr\left\{ y_t \in dy, x_t \in dx \middle| \, y_s \in dy', x_s \in dx' \right\} = f\left(y,x,t;y',x',s\right) dydx.$$

Similar to the one-dimensional case, suppose y_t starts at y_0 at $t=0$ and moves downstream to y at time t, then the capital ratio process crosses y_B at some time prior to time t, where $y < y_B < y_0$. By the continuity and strong Markov property of the joint process (y_t, x_t), we have

$$f(y,x,t;y_0,x_0,0) = \int_0^t \int_{-\infty}^{\infty} q\left(x',s\right) f\left(y,x,t;y_B,x',s\right) dx'ds.$$

Integrating y on both sides over $(-\infty, y_B]$, we obtain the extended Fortet equation as follows:

$$\Phi(x,t) = \int_0^t \int_{-\infty}^{\infty} q\left(x',s\right) \psi\left(x,t;x',s\right) dx'ds, \tag{B.2}$$

where the marginal distribution functions are given by

$$\Phi(x,t) = \int_{-\infty}^{y_B} f(y,x,t;y_0,x_0,0) \, dy, \quad \psi\left(x,t;x',s\right) = \int_{-\infty}^{y_B} f\left(y,x,t;y_B,x',s\right) dy,$$

and whose closed form expressions will be presented later. Since $y_0 > y_B$, it is seen that

$$\lim_{t \to 0} \Phi(x,t) = 0, \qquad \lim_{t \to s} \psi\left(x,t;x',s\right) = \delta\left(x - x'\right), \tag{B.3}$$

where $\delta(\cdot)$ is the Dirac delta function. In order to solve the two-dimensional integral equation (B.2) numerically, we discretize the domain for x and t using a set of rectangular grids as follows:

$$x_i = x_{lb} + i\Delta x, \quad i = 1,2,...,n, \quad \text{and} \quad t_j = j\Delta t, \quad j = 1,2,...,m.$$

We approximate the integral equation (B.2) by

$$\Phi(x_i, t_j) = \sum_{\ell=1}^{j} \sum_{k=1}^{n} q(x_k, t_\ell) \, \psi(x_i, t_j; x_k, t_\ell),$$

where $q(x_i, t_j)$ denotes the discrete approximation

$$q\left(x_i, t_j\right) \approx \Pr\left\{x_{\tau_B} \in (x_{i-1}, x_i], \tau_B \in (t_{j-1}, t_j]\right\}.$$

For $j = 1$, we obtain the approximation formula

$$\Delta x \ \Phi\left(x_i, t_1\right) = \Delta x \sum_{k=1}^{n} q\left(x_k, t_1\right) \psi\left(x_i, t_1; x_k, t_1\right) = q\left(x_i, t_1\right),$$

using the properties in eq.(B.3). For $j > 1$, we obtain the approximation scheme

$$\Delta x \ \Phi\left(x_i, t_j\right) = \Delta x \sum_{k=1}^{n} q\left(x_k, t_j\right) \psi\left(x_i, t_j; x_k, t_j\right) + \Delta x \sum_{\ell=1}^{j-1} \sum_{k=1}^{n} q\left(x_k, t_\ell\right) \psi\left(x_i, t_j; x_k, t_\ell\right).$$

By eq.(B.3), the first term on the right hand side gives $q\left(x_i, t_j\right)$. As a result, we have the following recursive scheme for finding $q(x_i, t_j)$ at successive time points t_j, $j = 1, 2, \ldots, n$, and at varying level of x_i, $i = 1, 2, \ldots, n$:

$$\begin{aligned} q\left(x_i, t_1\right) &= \Delta x \ \Phi\left(x_i, t_1\right), \\ q\left(x_i, t_j\right) &= \Delta x \left[\Phi\left(x_i, t_j\right) - \sum_{\ell=1}^{j-1} \sum_{k=1}^{n} q\left(x_k, t_\ell\right) \psi\left(x_i, t_j; x_k, t_\ell\right)\right], \quad j = 2, 3, ..., m. \end{aligned} \tag{B.4}$$

Lastly, we derive the closed form expressions for Φ and ψ in eq.(B.4). To this end, we observe that $y_t|_{x_t, \mathcal{F}_s} \sim N\left(\mu\left(t; s\right), \Sigma^2\left(t; s\right)\right)$ is Gaussian and whose conditional moments under the risk neutral measure Q can be obtained by the projection theorem as follows:

$$\begin{aligned} \mu\left(t; s\right) &\triangleq \mathbb{E}_s\left[y_t | x_t\right] = \mathbb{E}_s\left[y_t\right] + \frac{\mathrm{cov}_s\left(y_t, x_t\right)}{\mathrm{var}_s\left[x_t\right]}\left[x_t - \mathbb{E}_s\left[x_t\right]\right], \\ \Sigma^2\left(t; s\right) &\triangleq \mathrm{var}_s\left[y_t | x_t\right] = \mathrm{var}_s\left[y_t\right] - \frac{\mathrm{cov}_s\left(y_t, x_t\right)^2}{\mathrm{var}_s\left[x_t\right]}. \end{aligned}$$

The unconditional moments are readily obtained as

$$\begin{aligned} &\mathbb{E}_s\left[x_t\right] = x_s + \left(r - q - \frac{1}{2}\sigma^2 - \lambda\gamma\right)\left(t - s\right), \quad \mathbb{E}_s\left[y_t\right] = y_s e^{-\kappa(t-s)} + \theta\left[1 - e^{-\kappa(t-s)}\right], \\ &\mathrm{var}_s\left[x_t\right] = \sigma^2\left(t - s\right), \quad \mathrm{var}_s\left[y_t\right] = \frac{\eta^2}{2\kappa}\left[1 - e^{-2\kappa(t-s)}\right], \quad \mathrm{cov}_s\left(y_t, x_t\right) = \rho\sigma\eta\left[\frac{1 - e^{-\kappa(t-s)}}{\kappa}\right]. \end{aligned}$$

The conditional probability relation gives

$$f\left(y_t, x_t, t; y_s, x_s, s\right) = f\left(x_t, t; x_s, s\right) f\left(y_t, t; y_s, x_s, s | x_t\right),$$

where

$$\begin{aligned} f\left(x_t, t; x_s, s\right) &= \frac{1}{\sqrt{2\pi\sigma^2\left(t - s\right)}} \exp\left(-\frac{\left[x_t - x_s - \left(r - q - \frac{1}{2}\sigma^2 - \lambda\gamma\right)\left(t - s\right)\right]^2}{2\sigma^2\left(t - s\right)}\right), \\ f\left(y_t, t; y_s, x_s, s | x_t\right) &= \frac{1}{\sqrt{2\pi\Sigma^2\left(t; s\right)}} \exp\left(-\frac{\left[y_t - \mu\left(t; s\right)\right]^2}{2\sigma^2\Sigma^2\left(t; s\right)}\right). \end{aligned}$$

The integration of the above conditional density formula with respect to y over $(-\infty, y_B)$ gives the following closed form expressions for $\Phi(x,t)$ and $\psi(x,t;x',s)$:

$$\begin{aligned}
\Phi(x,t) &= f(x_t,t;x_0,0)\, N\left(\frac{y_B-\mu(t;0)}{\Sigma(t;0)}\right), \\
\psi\left(x,t;x',s\right) &= f(x_t,t;x_s,s)\, N\left(\frac{y_B-\mu(t;s)}{\Sigma(t;s)}\right)\Bigg|_{y_s=y_B}.
\end{aligned}$$

(i) Capital ratio = 6%

Time	Fortet	FD Scheme	Monte Carlo
1	37.762	37.769	37.420 (0.139)
2	46.612	46.617	46.115 (0.141)
3	51.090	51.094	50.670 (0.142)
4	54.080	54.083	53.681 (0.141)
5	56.354	56.355	55.871 (0.143)

(ii) Capital ratio = 8%

Time	Fortet	FD Scheme	Monte Carlo
1	6.627	6.629	6.459 (0.062)
2	14.526	14.527	14.418 (0.091)
3	20.027	20.028	19.877 (0.104)
4	24.191	24.190	24.004 (0.113)
5	27.573	27.565	27.600 (0.121)

(iii) Capital ratio = 10%

Time	Fortet	FD Scheme	Monte Carlo
1	2.241	2.241	2.235 (0.031)
2	6.638	6.639	6.664 (0.057)
3	10.949	10.948	10.917 (0.074)
4	14.733	14.726	14.729 (0.087)
5	18.059	18.038	17.850 (0.097)

Table 1a: Numerical calculations of the conversion value with constant intensity $\lambda = 0.05$ and varying values of the capital ratio using the Fortet method, finite difference (FD) scheme and Monte Carlo simulation method. The bracket quantities are the standard errors of the Monte Carlo simulation results. Good agreement of the numerical results using different numerical methods is observed.

(i) Capital ratio = 6%

Time	FD Scheme	Monte Carlo
1	37.720	37.170 (0.138)
2	46.472	46.029 (0.141)
3	50.827	50.381 (0.141)
4	53.674	53.300 (0.141)
5	55.785	55.400 (0.142)

(ii) Capital ratio = 8%

Time	FD Scheme	Monte Carlo
1	6.591	6.511 (0.063)
2	14.379	14.141 (0.090)
3	19.714	19.663 (0.103)
4	23.666	23.623 (0.112)
5	26.795	26.795 (0.118)

(iii) Capital ratio = 10%

Time	FD Scheme	Monte Carlo
1	2.209	2.172 (0.030)
2	6.503	6.385 (0.056)
3	10.643	10.657 (0.073)
4	14.193	14.152 (0.085)
5	17.230	17.380 (0.094)

Table 1b: Numerical calculations of the conversion value with state dependent intensity specified as $\lambda(x) = \exp(a_0 - a_1 x)$, where $a_1 = 0.5$ and a_0 is set by matching $\lambda(x_0) = 0.05$. Good agreement of the numerical results using the finite difference (FD) scheme and Monte Carlo simulation method is observed. The bracket quantities are the standard errors of the Monte Carlo simulation results.

(i) Capital ratio = 6%

Time	FD Scheme	Monte Carlo
1	38.701	38.364 (0.139)
2	47.540	47.158 (0.143)
3	51.900	51.701 (0.143)
4	54.734	54.335 (0.144)
5	56.828	56.704 (0.146)

(ii) Capital ratio = 8%

Time	FD Scheme	Monte Carlo
1	6.932	6.913 (0.064)
2	14.915	14.846 (0.092)
3	20.342	20.389 (0.106)
4	24.343	24.198 (0.113)
5	27.501	27.506 (0.120)

(iii) Capital ratio = 10%

Time	FD Scheme	Monte Carlo
1	2.279	2.293 (0.031)
2	6.703	6.643 (0.056)
3	10.938	10.873 (0.073)
4	14.554	14.722 (0.087)
5	17.639	17.584 (0.095)

Table 1c: Numerical calculations of the conversion value with state dependent intensity specified as $\lambda(x,y) = \exp(a_0 - a_1 x) + b_0 \mathbf{1}_{\{y \leq y_{RT}\}}$, where $a_1 = 0.5$, a_0 is set by matching $\lambda(x_0) = 0.05$, $b_0 = 0.1$ and $y_{RT} = 0.07$. Good agreement of the numerical results using the finite difference (FD) scheme and Monte Carlo simulation method is observed. The bracket quantities are the standard errors of the Monte Carlo simulation results.

Time	1	2	3	4	5
$\rho = 0.3$	0.48	0.58	0.69	0.82	0.96
$\rho = 0.9$	0.28	0.31	0.34	0.38	0.43

Table 2: Conditional mean of the stock price distribution at an accounting trigger. At $\rho = 0.3$, the mean of the implied stock price distribution can be much larger than the initial stock price of 0.5.

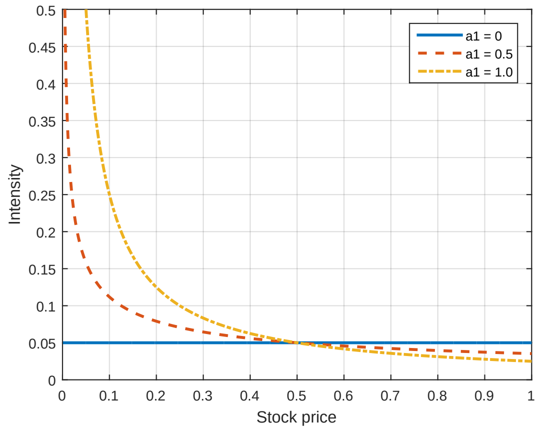
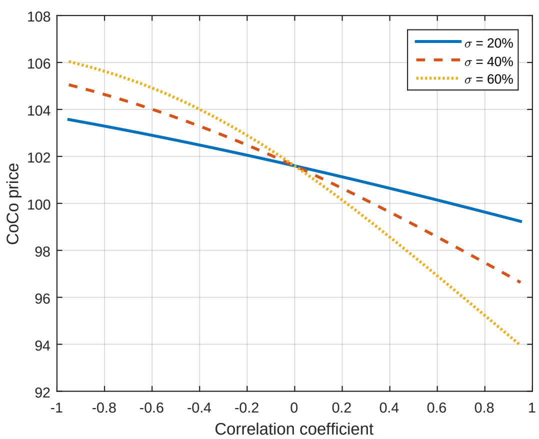
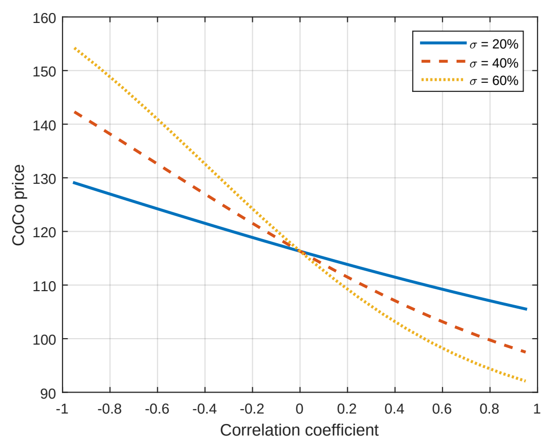


Figure 1: Inverse relationship between conversion intensity and stock price with varying values of a_1 in the state dependent intensity function.



(a) Capital ratio = 5.125%.



(b) Capital ratio = 8%.

Figure 2: Impact of correlation coefficient ρ on the CoCo bond price.

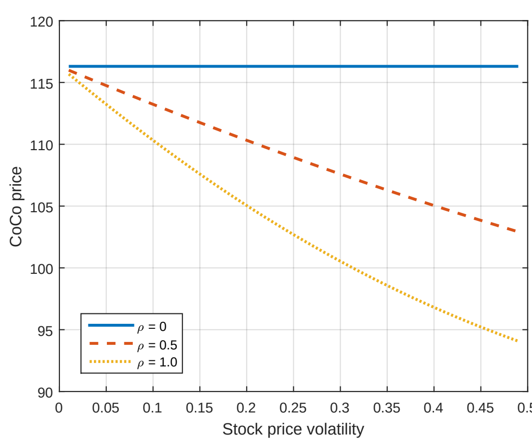


Figure 3: Impact of stock price volatility σ on the CoCo bond price with capital ratio of 8%.

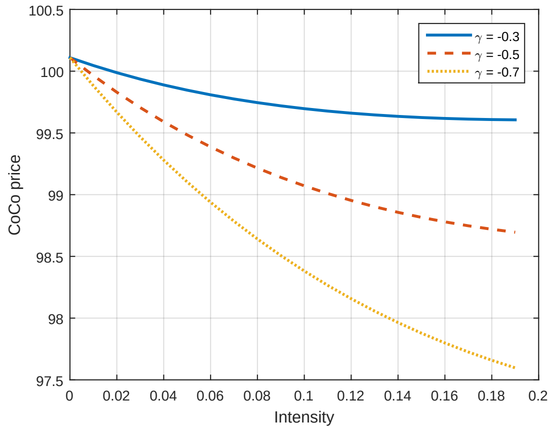
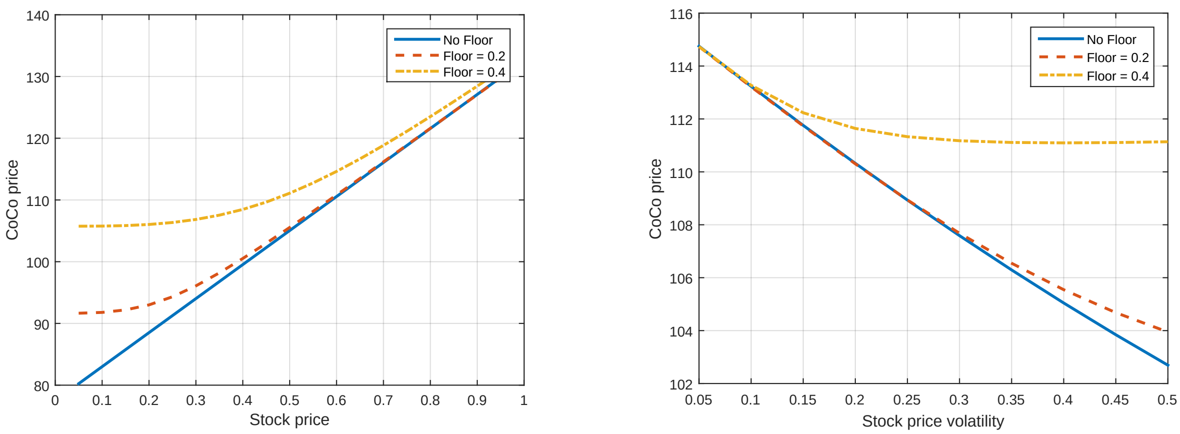


Figure 4: Impact of JtNV intensity λ on the CoCo bond price with capital ratio of 8%.



(a) delta risk

(b) vega risk

Figure 5: Impact of floored payoff on the CoCo bond price.

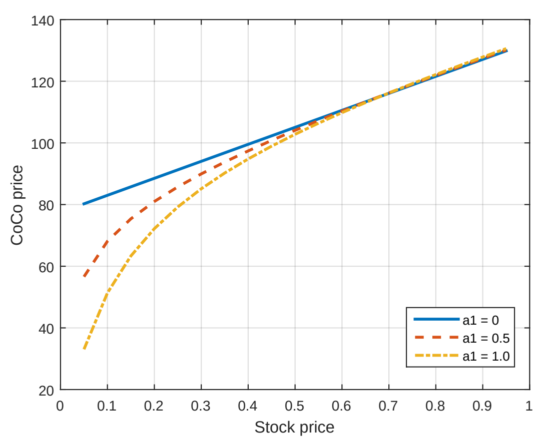
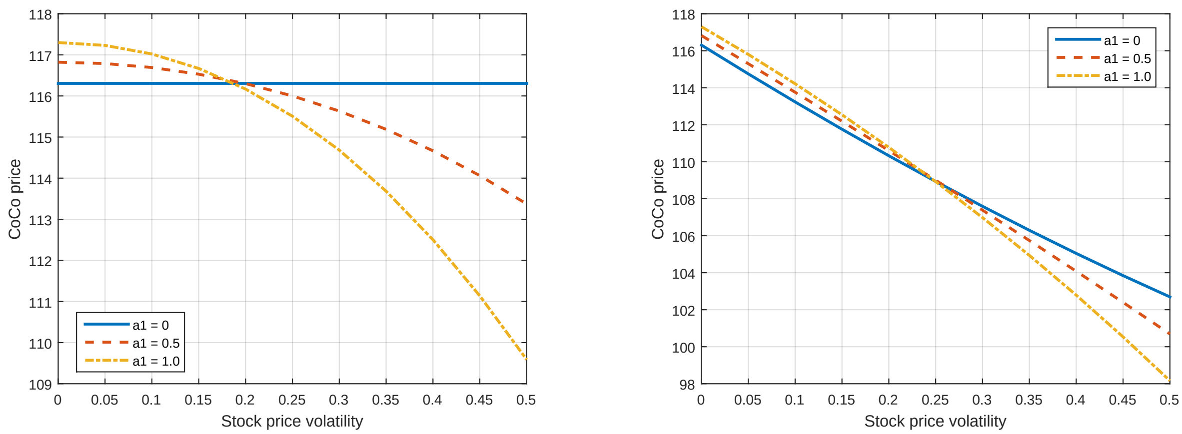


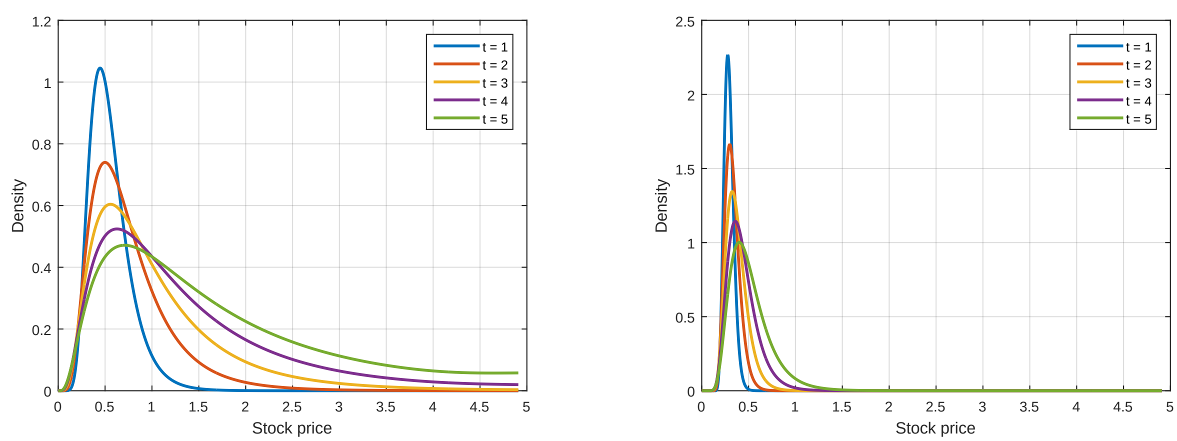
Figure 6: Effect of stock price dependent intensity on the delta risk of the CoCo bond prices.



(a) Zero correlation.

(b) Correlation coefficient = 0.5.

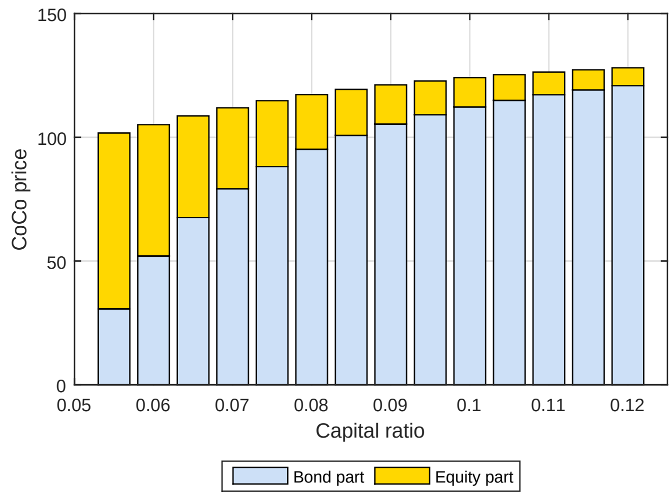
Figure 7: Effect of stock price dependent intensity on the vega risk of the CoCo bond prices.



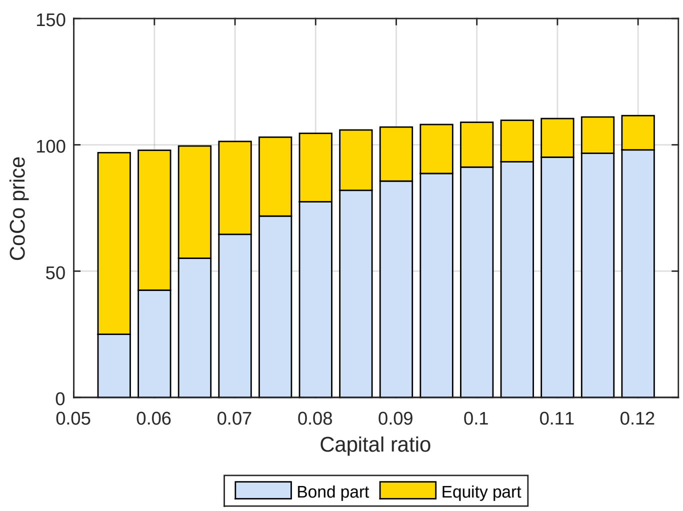
(a) Correlation coefficient, $\rho = 0.3$.

(b) Correlation coefficient, $\rho = 0.9$.

Figure 8: Plots of the conditional distribution of stock price at an accounting trigger with varying values of the correlation coefficient.



(a) The JtNV intensity is set to zero: the equity exposure diminishes when the capital ratio stays around 12%.



(b) The JtNV intensity is set to be 5%: the equity exposure remains to be significant even when the capital ratio stays around 12%.

Figure 9: Decomposition of the CoCo bond value into the bond and equity components.