

Efficient risk measures calculations for generalized CreditRisk⁺ models

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Abstract

Numerical calculations of risk measures and risk contributions in credit risk models amount to the evaluation of various forms of quantiles, tail probabilities and tail expectations of the portfolio loss distribution. Though the moment generating function of the loss distribution in the CreditRisk⁺ model is available in analytic closed form, efficient, accurate and reliable computation of risk measures (Value-at-Risk and Expected Shortfall) and risk contributions for the CreditRisk⁺ model pose technical challenges. We propose various numerical algorithms for risk measures and risk contributions calculations of the enhanced CreditRisk⁺ model under the common background vector framework using the Johnson curve fitting method, saddlepoint approximation method, importance sampling in Monte Carlo simulation and check function formulation. Our numerical studies on stylized credit portfolios and benchmark industrial credit portfolios reveal that the Johnson curve fitting approach works very well for credit portfolios with a large number of obligors, demonstrating high level of numerical reliability and computational efficiency. Once we implement the systematic procedure of finding the saddlepoint within an approximate domain, the saddlepoint approximation schemes provide efficient calculation and accurate numerical results. The importance sampling in Monte Carlo simulation methods are easy to implement, but they compete less favorably in accuracy and reliability with other numerical algorithms. The less commonly used check function formulation is limited to risk measures calculations. It competes favorably in accuracy and reliability, but an extra optimization algorithm is required.

Keywords: Value-at-Risk, Expected Shortfall, CreditRisk⁺ common background vector models, Johnson curve fitting, saddlepoint approximation, importance sampling, check function

1 Introduction

Based on the actuarial science framework of default risks, the CreditRisk⁺ model was first formulated by Credit Suisse First Boston (1997). A good source of research articles on various theoretical extensions and implementation issues on the CreditRisk⁺ framework can be found in the book edited by Matthias and Lehrbass (2004). In order to capture the correlations and severity variations among risky obligors in a portfolio, Akkaya *et al.* (2004) model the dependence structures of sector risk and stochastic severity variations of default losses. Giese (2003) introduces a compound gamma distribution to represent more complex default correlation structures among risky obligors. However, these attempts to incorporate sector correlation are limited to a narrow range of covariance structures. Later extensions of the CreditRisk⁺ framework consider more complex dependence structures among the random sector variables, like the hidden Gamma model of Han and Kang (2008), extreme copula model of Wang *et al.* (2015), and among others.

In this paper, we focus on the common background vector model (CR⁺-CBV) developed by Fischer and Dietz (2011), in which dependence between the sector variables is linked to multiple background variables that may be related to the state of the economy and macroeconomics conditions. Later, Fischer and Mertel (2012)

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extend the sector variables in the CR⁺-CBV model from the usual gamma distribution to the more generalized tempered α -stable distribution. The CR⁺-CBV model can be further extended to include stochastic recovery rates (Fischer *et al.*, 2016) and the mixed vectors with positive and negative correlated common background variables (Zhang *et al.*, 2018). Papalamprou and Antoniou (2019) perform empirical studies on the performance among various CreditRisk⁺ models in the Greek banking sector and show that the CR⁺-CBV model can provide better realistic estimations in the downturn economic conditions when compared with other credit risk models.

Capital allocation for credit portfolios is a crucial risk management procedure in financial institutions. To determine the economic capital of an aggregate credit portfolio, Value-at-Risk (VaR) and Expected Shortfall (ES) are the most common risk measures for the evaluation of capital requirement. Besides, at an individual obligor or sub-portfolio level, the VaR contribution and ES contribution of an obligor measure how much each obligor contributes to the VaR and ES in a portfolio, respectively. The development of efficient, accurate and reliable computational algorithms for calculating risk measures and risk contributions for the CreditRisk⁺ model poses various forms of technical challenges. The common methods include the Panjer recursion schemes (Haaf *et al.*, 2004), Fourier inversion method (Reiß, 2004) and Monte Carlo simulation. The Panjer recursion calculations become computationally demanding when the number of obligors and risk factors are large. The Fourier inversion method typically requires judicious choices of parameter values in the algorithm for its successful implementation and may not be reliable for credit portfolios with complex composition. The Monte Carlo simulation method shows a deterioration of accuracy to an unacceptable level when conditional evaluation of losses is performed in risk contribution calculations at high confidence level. This is because only a small portion of simulation paths can satisfy the condition that the portfolio loss assumes or exceeds the VaR threshold since these cases are typically rare events. Importance sampling is a viable technique to circumvent such difficulties by performing more sampling in the tail of the loss distribution.

In this paper, we consider three analytic approximation methods and their numerical implementation procedures to compute risk measures and risk contributions of a credit portfolio under the CR⁺-CBV model. These include the Johnson curve fitting method, saddlepoint approximation method and check function formulation. Simonato (2011) proposes the use of the family of Johnson distributions (Johnson, 1949) in the fitting curve to compute VaR and ES of a credit portfolio. The curve fitting method employs matching of the first four order moments of the loss distribution with an approximate normal curve via the Johnson curve transform. The family of Johnson distributions can generate genuine distributions for all possible skewness and kurtosis. We then use the approximate distribution to compute the quantile and tail expectation. Gordy (2002) initiates the use of the saddlepoint approximation method in calculating VaR for the CreditRisk⁺ model. The saddlepoint approximation method is based on the asymptotic expansion of a complex integral that represents the tail expectation of the loss distribution. Kwok and Zheng (2018) summarize the use of the saddlepoint approximation methods in calculating risk measures and risk contributions for various industrial risk management models, including the CreditRisk⁺ model. The check function formulation (Koenker and Basset, 1978) relates the distribution quantile (VaR) and tail expectation (ES) of a loss distribution to the minimizer of a check function. Interestingly, VaR is seen to be the argument that gives ES as the minimum value of the check function. Our numerical tests on these three methods reveal that they provide efficient calculations and highly accurate results for computing risk measures and risk contributions of typical credit portfolios under the CR⁺-CBV model and require much less computing time when compared with the Monte Carlo simulation methods.

The remaining sections of this paper are organized as follows. In Section 2, the main features of the CreditRisk⁺ framework under the common background vector model are reviewed. We derive the moment generating functions and cumulant generating functions of the random portfolio loss under the CR⁺-CBV model that are required for the later numerical implementation of the three analytic approximation methods. In Section 3, we show how to relate the two risk measures, VaR and ES, via a check function. We also derive expectation formulas for computing risk contributions under the CR⁺-CBV model. Section 4 discusses the Johnson curve transform of calibrating the four parameters of the approximate normal curve via matching of the first four order moments of the loss distribution. Once the portfolio loss distribution has been fitted by an approximate Johnson S_B distribution, we explain how to find the portfolio VaR and ES together with risk contributions for individual obligors from the approximate Johnson curve. In Section 5, we review the saddlepoint approximation approach and present various saddlepoint approximation formulas for computing risk measures and risk contributions in credit portfolios. Practical implementation issues for finding the saddlepoint

in an appropriate domain are discussed. In Section 6, we review the two-step importance sampling technique in the Monte Carlo simulation of computing risk measures and risk contributions of credit portfolios under the CR⁺-CBV model. In Section 7, comprehensive numerical tests on the performance of calculating risk measures and contributions using various credit portfolios under the CR⁺-CBV model are presented to demonstrate the efficiency, accuracy and reliability of our proposed algorithms. The last section contains conclusive remarks and a summary of the paper.

2 CreditRisk⁺ Model Framework

Some of the essential features of the CreditRisk⁺ framework are summarized as follows:

1. No financial modeling for the default event
The default of an obligor is described as a purely random event, which is characterized by the probability of default. For analytic tractability, the CreditRisk⁺ model takes the Poisson approximation to the Bernoulli type default events.
2. Use of exposure bands
Instead of working with a large number of obligors, the CreditRisk⁺ model groups obligors into a small number of exposure bands to reduce computational costs. Each exposure amount of an obligor is replaced by an integer multiple (rounded to the ceiling) of the basic unit of exposure. To compensate for the error due to rounding, the expected probability of default is adjusted so that the expected loss is maintained to remain the same.
3. Randomization of the probability of default
The probability of default of an obligor is taken to be a random variable that is driven by several sector variables through a linear relationship. This feature is used to generate default correlation among the obligors, which arise only implicitly due to the same sector variables. Conditional independence of default is also assumed where the defaults of obligors are independent given the sector variables.

In the standard CreditRisk⁺ framework, the sector variables are assumed to be independent. In the more recent common background vector model (Fischer and Dietz, 2011) considered in this paper, the sector variables are assumed to be dependent on a set of common background variables and one's own idiosyncratic sector variables. All these variables are assumed to follow the Gamma distribution, which is chosen due to analytic tractability.

2.1 Common Background Vector Model

Let \mathcal{A} be a collection of risky obligors in a credit portfolio. Let $\hat{\epsilon}_A$ and \hat{p}_A denote the respective exposure and expected probability of default of obligor A inside \mathcal{A} . Let l denote the normalized loss unit in dollar amount. We take an adjusted exposure ϵ_A (in units of l) of obligor A to be the ceiling of $\hat{\epsilon}_A/l$. The expected probability of default of obligor A is adjusted to be p_A such that the expected loss of obligor A remains the same; that is,

$$p_A = \frac{\hat{\epsilon}_A \hat{p}_A}{\epsilon_A l}.$$

Let D_A denote the default indicator of obligor A as a Bernoulli random variable: default with probability p_A and no default with probability $1 - p_A$. The normalized random loss of obligor A is given by $\epsilon_A D_A$. The normalized random portfolio loss L is the summation of all normalized random losses from all obligors in \mathcal{A} , where

$$L = \sum_{A \in \mathcal{A}} \epsilon_A D_A. \quad (2.1)$$

For analytic tractability, the CreditRisk⁺ framework takes the default indicator D_A to be a mixture Poisson random variable whose randomized intensity is λ_A with mean p_A . This assumption would implicitly assume

multiple defaults. However, for small value of default probability, this may be considered as an acceptable assumption.

The earlier version of the CreditRisk⁺ framework assumes the probability of default of an obligor is driven by several sector variables. Each sector variable may be associated with an industry type or geographical location. An obligor can participate in various sectors with different weights. We assume the randomized intensity λ_A of D_A (as a mixture Poisson random variable) to be dependent on a number of sector variables $S_k, k = 1, 2, \dots, K$, with weights $0 \leq w_{Ak} \leq 1, k = 1, 2, \dots, K$, where K is the total number of sectors. It is common to standardize the sector variables so that they all have unit mean, where $E[S_k] = 1, k = 1, 2, \dots, K$. There exists the idiosyncratic variable S_0 with weight w_{A0} that has no volatility. For notational convenience, we take $S_0 = 1$. Since all weights must be summed to one, we have

$$w_{A0} = 1 - \sum_{k=1}^K w_{Ak}.$$

The randomized intensity λ_A of D_A assumes the following linear relationship:

$$\lambda_A = p_A(w_{A0} + w_{A1}S_1 + \dots + w_{AK}S_K). \quad (2.2a)$$

Since the mean of $S_k, k = 1, 2, \dots, K$, is set to be one and the sum of all weights is one, it is consistent to observe that the mean of λ_A is the expected probability of default p_A .

In the standard CreditRisk⁺ framework, the sector variables $S_k, k = 1, 2, \dots, K$, are assumed to follow the Gamma distribution for analytic tractability; that is,

$$S_k \sim \Gamma(\theta_k, \delta_k), \quad k = 1, 2, \dots, K.$$

Here, θ_k and δ_k are the parameters of the Gamma distribution Γ with

$$E[S_k] = \theta_k \delta_k = 1 \quad \text{and} \quad \text{var}(S_k) = \theta_k \delta_k^2 = \delta_k, \quad k = 1, 2, \dots, K.$$

Other assumptions on the underlying distribution of the sector variables have been made in the literature. For example, Fischer and Mertel (2012) propose the tempered α -stable distribution for the sector variables. This assumed distribution retains analytic tractability and nests the Gamma distribution asymptotically when the specific parameter α tends to zero.

Next, we present the CR⁺-CBV model, where each sector variable is assumed to be dependent on M common background variables T_1, T_2, \dots, T_M modeled as independent Gamma random variables. We modify the randomized intensity λ_A of the mixture Poisson variable D_A to take the following form:

$$\lambda_A = p_A(w_{A0} + w_{A1}\hat{S}_1 + \dots + w_{AK}\hat{S}_K), \quad (2.2b)$$

where \hat{S}_k is a weighted sum of an idiosyncratic sector variable S_k and independent common background variables T_1, T_2, \dots, T_M . More specifically, we define

$$\hat{S}_k = \delta_k S_k + \sum_{m=1}^M \gamma_{mk} T_m, \quad k = 1, 2, \dots, K, \quad (2.3)$$

where δ_k and γ_{mk} are the respective weights. These variables all follow the Gamma distribution, where

$$S_k \sim \Gamma(\theta_k, 1), \quad k = 1, 2, \dots, K, \quad \text{and} \quad T_m \sim \Gamma(\hat{\theta}_m, 1), \quad m = 1, 2, \dots, M.$$

The scaling property of the Gamma distribution dictates that given $S_k \sim \Gamma(\theta_k, 1)$ and the proportional constant δ_k , we have $\delta_k S_k \sim \Gamma(\theta_k, \delta_k)$. We reformulate λ_A in the CR⁺-CBV model as the standard CreditRisk⁺ model with $K + M$ sector variables, where

$$\lambda_A = p_A(w_{A0} + w_{A1}\tilde{S}_1 + \dots + w_{AK}\tilde{S}_K + w_{A,K+1}\tilde{S}_{K+1} + \dots + w_{A,K+M}\tilde{S}_{K+M}). \quad (2.4)$$

The new variables $\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_{K+M}$ are given by

$$\tilde{S}_i = \begin{cases} \delta_i S_i, & i = 1, 2, \dots, K, \\ T_{i-K}, & i = K + 1, K + 2, \dots, K + M, \end{cases} \quad (2.5)$$

and the reformulated weights $w_{A,K+1}, w_{A,K+2}, \dots, w_{A,K+M}$ can be expressed as

$$w_{A,K+m} = \sum_{k=1}^K w_{Ak} \gamma_{mk}, \quad m = 1, \dots, M. \quad (2.6)$$

Under the reformulation of the weights, the sum of weights $w_{A0} + \sum_{i=1}^{K+M} w_{A,i}$ is no longer equal to one.

Under the CR⁺-CBV model, the mean, variance and covariance for the set of variables $\hat{S}_k, k = 1, 2, \dots, K$, are found to be

$$\begin{aligned} E[\hat{S}_k] &= \delta_k \theta_k + \sum_{m=1}^M \gamma_{mk} \hat{\theta}_m, \\ \text{var}(\hat{S}_k) &= \delta_k^2 \theta_k + \sum_{m=1}^M \gamma_{mk}^2 \hat{\theta}_m, \\ \text{cov}(\hat{S}_i, \hat{S}_j) &= \sum_{m=1}^M \gamma_{mi} \gamma_{mj} \hat{\theta}_m, \quad i \neq j. \end{aligned} \quad (2.7)$$

2.2 Moment Generating Functions and Cumulant Generating Functions

We take the Poisson approximation for the default indicator variable D_A for all $A \in \mathcal{A}$. Conditional on realization of the random variables, $\mathbf{S} = (S_1, \dots, S_K, T_1, \dots, T_M)$, the Poisson default indicator variables are independent. The conditional probability generating function $\Psi_A(t|\mathbf{S})$ for obligor A with exposure ϵ_A is given by $\Psi_A(t|\mathbf{S}) = \exp[\lambda_A(t^{\epsilon_A} - 1)]$. By virtue of conditional independence, the conditional moment generating function (mgf) $M_L(t|\mathbf{S})$ of the portfolio loss L is given by the product of the individual conditional moment generating function $M_A(t|\mathbf{S})$ of all obligors. This gives

$$\begin{aligned} M_L(t|\mathbf{S}) &= \prod_A M_A(t|\mathbf{S}) = \prod_A \Psi_A(e^t|\mathbf{S}) = \exp\left(\sum_A \lambda_A(e^{\epsilon_A t} - 1)\right) \\ &= \exp\left(\sum_A p_A w_{A0}(e^{\epsilon_A t} - 1) + \sum_{k=1}^K \delta_k S_k \sum_A p_A w_{Ak}(e^{\epsilon_A t} - 1) + \sum_{m=1}^M T_m \sum_A p_A w_{A,K+m}(e^{\epsilon_A t} - 1)\right) \\ &= \exp\left(P_0(t) + \sum_{k=1}^K \delta_k S_k P_k(t) + \sum_{m=1}^M T_m P_{K+m}(t)\right), \end{aligned} \quad (2.8)$$

where

$$P_k(t) = \sum_A p_A w_{Ak}(e^{\epsilon_A t} - 1), \quad k = 0, 1, \dots, K, \quad (2.9a)$$

$$P_{K+m}(t) = \sum_A p_A w_{A,K+m}(e^{\epsilon_A t} - 1), \quad m = 1, 2, \dots, M. \quad (2.9b)$$

By virtue of (2.6), we have

$$P_{K+m}(t) = \sum_{k=1}^K \gamma_{mk} P_k(t), \quad m = 1, 2, \dots, M.$$

Since $S_k \sim \Gamma(\theta_k, 1)$, $k = 1, 2, \dots, K$, and $T_m \sim \Gamma(\hat{\theta}_m, 1)$, $m = 1, 2, \dots, M$, are all independent, the unconditional moment generating function $M_L(t)$ of the portfolio loss L under the CR⁺-CBV model is obtained by integrating with respect to the product of the Gamma density functions. This gives

$$M_L(t) = \exp(P_0(t)) \prod_{k=1}^K \frac{1}{\hat{\Gamma}(\theta_k)} \int_0^\infty s_k^{\theta_k-1} e^{s_k[\delta_k P_k(t)-1]} ds_k \prod_{m=1}^M \frac{1}{\hat{\Gamma}(\hat{\theta}_m)} \int_0^\infty s_m^{\hat{\theta}_m-1} e^{s_m[P_{K+m}(t)-1]} ds_m,$$

where $\hat{\Gamma}(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the Gamma function. Let $y_k = s_k[1 - \delta_k P_k(t)]$, $k = 1, 2, \dots, K$, and $y_m = s_m[1 - P_{K+m}(t)]$, $m = 1, 2, \dots, M$, we then have

$$\begin{aligned} M_L(t) &= \exp(P_0(t)) \prod_{k=1}^K [1 - \delta_k P_k(t)]^{-\theta_k} \frac{1}{\hat{\Gamma}(\theta_k)} \int_0^\infty y_k^{\theta_k-1} e^{-y_k} dy_k \prod_{m=1}^M [1 - P_{K+m}(t)]^{-\hat{\theta}_m} \frac{1}{\hat{\Gamma}(\hat{\theta}_m)} \int_0^\infty y_m^{\hat{\theta}_m-1} e^{-y_m} dy_m \\ &= \exp(P_0(t)) \prod_{k=1}^K [1 - \delta_k P_k(t)]^{-\theta_k} \prod_{m=1}^M [1 - P_{K+m}(t)]^{-\hat{\theta}_m}. \end{aligned} \quad (2.10)$$

In the later numerical schemes for risk measures calculations derived based on the Johnson curve fitting method and saddlepoint approximation method, the higher order derivatives of the cumulant generating function (cgf) of the random portfolio loss L are involved. Let $\kappa_L(t)$ denote the cgf of L , which is defined as the logarithm of the mgf $M_L(t)$ of L , so that

$$\begin{aligned} \kappa_L(t) &= \ln(M_L(t)) = P_0(t) - \sum_{k=1}^K \theta_k \ln(1 - \delta_k P_k(t)) - \sum_{m=1}^M \hat{\theta}_m \ln(1 - P_{K+m}(t)) \\ &= P_0(t) + \sum_{k=1}^K \theta_k \phi_k(t) + \sum_{m=1}^M \hat{\theta}_m \hat{\phi}_m(t), \end{aligned} \quad (2.11)$$

where

$$\begin{aligned} \phi_k(t) &= -\ln(1 - \delta_k P_k(t)), & k &= 1, 2, \dots, K, \\ \hat{\phi}_m(t) &= -\ln(1 - P_{K+m}(t)), & m &= 1, 2, \dots, M. \end{aligned}$$

The higher order derivatives of $\kappa_L(t)$ can be expressed in terms of the higher order derivatives of $\phi_k(t)$, $k = 1, 2, \dots, K$, and $\hat{\phi}_m(t)$, $m = 1, 2, \dots, M$. For notational convenience, we define

$$\begin{aligned} N_{k,j}(t) &= \frac{P_k^{(j)}(t)}{1 - \delta_k P_k(t)}, & k &= 1, 2, \dots, K, \\ \hat{N}_{m,j}(t) &= \frac{P_{K+m}^{(j)}(t)}{1 - P_{K+m}(t)}, & m &= 1, 2, \dots, M, \end{aligned}$$

where $P_k^{(j)}(t)$ and $P_{K+m}^{(j)}(t)$ denote the j^{th} order derivative of $P_k(t)$ and $P_{K+m}(t)$, respectively. By virtue of (2.9a,b), we have

$$\begin{aligned} P_k^{(j)}(t) &= \sum_A \epsilon_A^j p_A w_{Ak} e^{\epsilon_A t}, & k &= 1, 2, \dots, K, \\ P_{K+m}^{(j)}(t) &= \sum_{k=1}^K \gamma_{mk} P_k^{(j)}(t), & m &= 1, 2, \dots, M. \end{aligned}$$

The derivatives of $N_{k,j}(t)$ and $\hat{N}_{m,j}(t)$ observe the following recurrence relations:

$$\begin{aligned} \frac{d}{dt} N_{k,j}(t) &= N_{k,j+1}(t) + \delta_k N_{k,j}(t) N_{k,1}(t), & k &= 1, 2, \dots, K, \\ \frac{d}{dt} \hat{N}_{m,j}(t) &= \hat{N}_{m,j+1}(t) + \hat{N}_{m,j}(t) \hat{N}_{m,1}(t), & m &= 1, 2, \dots, M. \end{aligned}$$

Based on the above recurrence relations, the higher order derivatives of $\phi_k(t)$, $k = 1, 2, \dots, K$, and $\hat{\phi}_m(t)$, $m = 1, 2, \dots, M$, can be expressed in terms of $N_{k,j}(t)$ and $\hat{N}_{m,j}(t)$. The derivatives of $\phi_k(t)$ and $\hat{\phi}_m(t)$ (up to the fourth order) are found to be

$$\begin{aligned}
\phi'_k(t) &= \delta_k N_{k,1}(t), & \hat{\phi}'_m(t) &= \hat{N}_{m,1}(t), \\
\phi''_k(t) &= \delta_k N_{k,2}(t) + \delta_k^2 N_{k,1}^2(t), & \hat{\phi}''_m(t) &= \hat{N}_{m,2}(t) + \hat{N}_{m,1}^2(t), \\
\phi'''_k(t) &= \delta_k N_{k,3}(t) + 3\delta_k^2 N_{k,1}(t)N_{k,2}(t) + 2\delta_k^3 N_{k,1}^3(t), \\
\hat{\phi}'''_m(t) &= \hat{N}_{m,3}(t) + 3\hat{N}_{m,1}(t)\hat{N}_{m,2}(t) + 2\hat{N}_{m,1}^3(t), \\
\phi''''_k(t) &= \delta_k N_{k,4}(t) + 3\delta_k^2 N_{k,2}^2(t) + 4\delta_k^2 N_{k,1}(t)N_{k,3}(t) + 12\delta_k^3 N_{k,1}^2(t)N_{k,2}(t) + 6\delta_k^4 N_{k,1}^4(t), \\
\hat{\phi}''''_m(t) &= \hat{N}_{m,4}(t) + 3\hat{N}_{m,2}^2(t) + 4\hat{N}_{m,1}(t)\hat{N}_{m,3}(t) + 12\hat{N}_{m,1}^2(t)\hat{N}_{m,2}(t) + 6\hat{N}_{m,1}^4(t).
\end{aligned} \tag{2.12}$$

The skewness and kurtosis of the loss variables L can be expressed in terms of the higher order derivatives of $\kappa_L(t)$ as follows:

$$\text{skewness} = \frac{\kappa_L'''(0)}{\kappa_L''(0)^{3/2}}, \quad \text{kurtosis} = \frac{\kappa_L''''(0)}{\kappa_L''(0)^2} + 3. \tag{2.13}$$

3 Risk Measures and Risk Contributions

At the portfolio level, the bank's risk management procedure requires the determination of the economic capital as a buffer against unexpected catastrophic losses in a credit portfolio. The economic capital of a credit portfolio is commonly set to be the difference between a high quantile (VaR at a chosen confidence level) of the portfolio loss distribution and the expected loss. The risk measure VaR is widely used in theory and practice in the financial industry as the basis in economic capital allocation. Since VaR does not satisfy subadditivity, ES is used as an alternative risk measure in setting the economic capital (Tasche, 2002). Once the economic capital allocation step has been completed, the risk managers would be interested to break down the unexpected loss of the portfolio to individual obligor level in order to identify the risk drivers within the credit portfolio. The detailed risk analysis requires an understanding of the risk contributions from individual obligors to the overall portfolio risk.

In this section, we review the definitions of VaR and ES, and their mathematical relation in the formulation of minimization of a check function (Bertsimas *et al.*, 2004). More precisely, we show that VaR is the argument that gives ES as the minimum value of the check function. The minimization in the check function formulation can be solved via an optimization software in a single optimization calculation under a credit risk model. We also present the expectation formulas for the computation of risk contributions to VaR and ES for individual obligor under the CR⁺-CBV model.

3.1 Relation of VaR and Expected Shortfall in Check Function

At a given confidence level $\alpha \in (0, 1)$, VaR_α of the random portfolio loss L is defined as the left α -quantile $q_L^{-1}(\alpha)$, where

$$\text{VaR}_\alpha = q_L^{-1}(\alpha) = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}. \tag{3.1}$$

Here, $F_L(l)$ is the cumulative distribution function (cdf) of L and the confidence level α is usually chosen to be sufficiently close to 1.0, says, $\alpha = 0.99$. Assuming L to be a continuous distribution, ES_α of L at the confidence level α is defined by

$$\text{ES}_\alpha = \mathbb{E}(L|L \geq \text{VaR}_\alpha) = \frac{1}{1-\alpha} \int_{\text{VaR}_\alpha}^{\infty} l f_L(l) dl = \frac{1}{1-\alpha} \int_{\alpha}^1 \text{VaR}_u du, \tag{3.2}$$

where \mathbb{E} and $f_L(l)$ are the expectation operator and the probability density function (pdf) of L , respectively. After some algebraic manipulation, we establish that

$$\text{ES}_\alpha = F_L^{-1}(\alpha) + \frac{1}{1-\alpha} \int_{\alpha}^1 [F_L^{-1}(u) - F_L^{-1}(\alpha)] du = \text{VaR}_\alpha + \frac{1}{1-\alpha} \mathbb{E}[(L - \text{VaR}_\alpha)^+].$$

This motivates the following definition of the check function $f(x)$ associated with the loss variable L :

$$f(x) = x + \frac{1}{1-\alpha} \mathbb{E}[(L-x)^+] = x + \frac{1}{1-\alpha} \int_x^\infty [1 - F_L(l)] dl. \quad (3.3)$$

By taking $x_0 = \text{VaR}_\alpha$, one can show that

1. $x_1 > x_0$: For $l \in (x_0, x_1)$, we observe $1 - F_L(l) \leq 1 - \alpha$ so that

$$f(x_1) - f(x_0) = x_1 - x_0 - \frac{1}{1-\alpha} \int_{x_0}^{x_1} [1 - F_L(l)] dl \geq x_1 - x_0 - \frac{1}{1-\alpha} (1-\alpha)(x_1 - x_0) = 0;$$

2. $x_2 < x_0$: For $l \in (x_2, x_0)$, we observe $1 - F_L(l) > 1 - \alpha$ so that

$$f(x_2) - f(x_0) = \frac{1}{1-\alpha} \int_{x_2}^{x_0} [1 - F_L(l)] dl - (x_0 - x_2) > \frac{1}{1-\alpha} (1-\alpha)(x_0 - x_2) - (x_0 - x_2) = 0.$$

We establish that $f(x)$ attains its global minimum value at $x_0 = \text{VaR}_\alpha$ since x_1 is any value greater than x_0 and x_2 is any value smaller than x_0 . In conclusion, we obtain the following mathematical relation between VaR_α and ES_α with regard to the check function:

$$\text{VaR}_\alpha = \underset{x \in \mathbb{R}}{\text{argmin}} f(x) \quad \text{and} \quad \text{ES}_\alpha = \min_{x \in \mathbb{R}} f(x). \quad (3.4)$$

3.2 Risk Contributions

From risk management perspective, it is important to examine the risk contributions of individual exposure to the overall portfolio risk. The marginal risk contribution (commonly called the Euler allocation) examines the impact of investing an additional small amount in an obligor on the overall portfolio risk (Rosen and Saunders, 2010). Let $\rho(L)$ denote a chosen risk measure of a portfolio loss distribution. Suppose each obligor A with exposure ϵ_A and default indicator D_A is in the collection \mathcal{A} for a credit portfolio. The marginal risk contribution of the obligor A with respect to the risk measure $\rho(L)$ is given by

$$C_A^\rho(L) = \lim_{\delta \rightarrow 0} \frac{\rho(L + \delta \epsilon_A D_A) - \rho(L)}{\delta} = \epsilon_A \frac{\partial \rho(L)}{\partial \epsilon_A}. \quad (3.5)$$

Assuming that $\rho(L)$ is positive homogeneous in $\epsilon_A, A \in \mathcal{A}$, where

$$\rho(\lambda L) = \lambda \rho(L), \quad \lambda > 0,$$

one can establish

$$\sum_{A \in \mathcal{A}} C_A^\rho(L) = \sum_{A \in \mathcal{A}} \epsilon_A \frac{\partial \rho(L)}{\partial \epsilon_A} = \rho(L), \quad (3.6)$$

where the sum of marginal risk contributions from all obligors equals the total portfolio risk.

The two risk measures, VaR_α and ES_α , both satisfy the positive homogeneous property. Assuming VaR_α and ES_α to be differentiable with respect to the exposures of obligors, the corresponding marginal risk contribution of VaR_α and ES_α of obligor A with exposure ϵ_A are defined by

$$\text{VaRC}_\alpha^A = \epsilon_A \frac{\partial \text{VaR}_\alpha}{\partial \epsilon_A} \quad \text{and} \quad \text{ESC}_\alpha^A = \epsilon_A \frac{\partial \text{ES}_\alpha}{\partial \epsilon_A}. \quad (3.7)$$

Tasche (2004) shows that the above two marginal risk contributions can be represented by the following pair of conditional expectation formulas:

$$\text{VaRC}_\alpha^A = \epsilon_A \mathbb{E}[D_A | L = \text{VaR}_\alpha] = \epsilon_A \frac{\mathbb{E}[D_A \mathbf{1}_{\{L = \text{VaR}_\alpha\}}]}{\mathbb{E}[\mathbf{1}_{\{L = \text{VaR}_\alpha\}}]} = \epsilon_A \frac{\mathbb{E}[\lambda_A \mathbf{1}_{\{L = \text{VaR}_\alpha - \epsilon_A\}}]}{\mathbb{E}[\mathbf{1}_{\{L = \text{VaR}_\alpha\}}]}, \quad (3.8a)$$

$$\text{ESC}_\alpha^A = \epsilon_A \mathbb{E}[D_A | L \geq \text{VaR}_\alpha] = \epsilon_A \frac{\mathbb{E}[D_A \mathbf{1}_{\{L \geq \text{VaR}_\alpha\}}]}{\mathbb{P}[L \geq \text{VaR}_\alpha]} = \epsilon_A \frac{\mathbb{E}[\lambda_A \mathbf{1}_{\{L \geq \text{VaR}_\alpha - \epsilon_A\}}]}{\mathbb{P}[L \geq \text{VaR}_\alpha]}, \quad (3.8b)$$

where \mathbb{E} and \mathbb{P} are the expectation operator and probability measure under the loss distribution L , respectively. By following similar procedures [proof of Theorem 4 in Tasche (2004)] in the evaluation of the conditional expectation formulas, we obtain the following formulas for the marginal risk contributions under the CR^+ -CBV model:

$$\begin{aligned} & \text{VaRC}_\alpha^A \\ &= \epsilon_A p_A \frac{w_{A0} \mathbb{E}[\mathbf{1}_{\{L = \text{VaR}_\alpha - \epsilon_A\}}] + \sum_{k=1}^K w_{Ak} \delta_k \theta_k \mathbb{E}_{L_1(k)}[\mathbf{1}_{\{L = \text{VaR}_\alpha - \epsilon_A\}}] + \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m \mathbb{E}_{L_2(m)}[\mathbf{1}_{\{L = \text{VaR}_\alpha - \epsilon_A\}}]}{\mathbb{E}[\mathbf{1}_{\{L = \text{VaR}_\alpha\}}]}, \end{aligned} \quad (3.9a)$$

$$\begin{aligned} & \text{ESC}_\alpha^A \\ &= \epsilon_A p_A \frac{w_{A0} \mathbb{P}[L \geq \text{VaR}_\alpha - \epsilon_A] + \sum_{k=1}^K w_{Ak} \delta_k \theta_k \mathbb{P}_{L_1(k)}[L \geq \text{VaR}_\alpha - \epsilon_A] + \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m \mathbb{P}_{L_2(m)}[L \geq \text{VaR}_\alpha - \epsilon_A]}{\mathbb{P}[L \geq \text{VaR}_\alpha]}. \end{aligned} \quad (3.9b)$$

For the operators $\mathbb{E}_{L_1(k)}$ and $\mathbb{P}_{L_1(k)}$, $k = 1, 2, \dots, K$, and $\mathbb{E}_{L_2(m)}$ and $\mathbb{P}_{L_2(m)}$, $m = 1, 2, \dots, M$, they have similar interpretation as \mathbb{E} and \mathbb{P} under the loss distribution L , except that $L_1(k)$ and $L_2(m)$ correspond to some other loss distribution modified from L . By virtue of (2.5), to obtain $L_1(k)$, $k = 1, 2, \dots, K$, we require the Gamma distribution for the k^{th} sector variable $\delta_k S_k$ to be changed from $\delta_k \Gamma(\theta_k, 1)$ to $\delta_k \Gamma(\theta_k + 1, 1)$ while those for all other variables \tilde{S}_i , $i \neq k$, remain unchanged. Similarly, to obtain $L_2(m)$, $m = 1, 2, \dots, M$, we require the Gamma distribution for the m^{th} common background variable T_m to be changed from $\Gamma(\hat{\theta}_m, 1)$ to $\Gamma(\hat{\theta}_m + 1, 1)$ while those for all other variables \tilde{S}_i , $i \neq K + m$, stay the same. The proof of the above marginal risk contribution formulas (3.9a,b) is presented in Appendix A.

4 Johnson Curve Fitting Method

The Johnson curve transform (Johnson, 1949) is commonly used to transform an arbitrary continuous random variable X with an unknown distribution into a standard normal random variable Z . There are three classic Johnson distributions: the lognormal distribution (S_L), unbounded distribution (S_U) and bounded distribution (S_B), the details of which have been summarized in Elderton and Johnson (1969). For a given set of data generated from the random variable X with the observed mean, variance, skewness and kurtosis, one can derive a unique four-parameter Johnson distribution by matching the first four order moments of the given set of data using the Johnson curve fitting method.

Hill *et al.* (1976) provide a comprehensive algorithm to determine which Johnson distribution is required and how to evaluate the four parameters of the Johnson distribution based on the first four order moments of the random variable X . Other methods that estimate the four parameters of the Johnson distribution include the percentile approach (Slifker and Shapiro, 1980) and quantile method (Wheeler, 1980). The Johnson curve toolbox developed by Jones (2014) provides a set of Matlab functions for estimating the four parameters of the Johnson distribution using moments or quantiles. Once an appropriate Johnson distribution is found, the percentiles of the random variable X can be estimated efficiently from the percentiles of the standard normal variable Z .

For the CR^+ -CBV model specifically considered in this paper, we are interested in evaluating the four-parameter Johnson S_B distribution that is bounded on both ends based on the first four order moments. This is because the loss variable L is bounded by the minimal value $L_{\min} = 0$ and the maximal value $L_{\max} = \sum_A \epsilon_A$, where ϵ_A is the exposure for obligor $A \in \mathcal{A}$. The first four order moments of the loss variable L can be effectively computed using the derivatives in (2.12). The risk measures and risk contributions of the portfolio loss distribution are then calculated approximately from the approximate Johnson S_B distribution. Details of the algorithm to find the four parameters of the Johnson S_B distribution by using a Newton-Raphson iterative scheme is presented in Appendix B.

Consider the following transform of the random variable X as

$$Y = \frac{X - c}{d}, \quad (4.1)$$

where c is the minimum value of X and $c + d$ is the maximum value of X . The transform that changes the random variable X into the standard normal variable Z for the Johnson S_B distribution is defined by

$$Z = a + b * \ln \left(\frac{Y}{1 - Y} \right) = a + b \ln \left(\frac{X - c}{c + d - X} \right), \quad c < X < c + d, \quad (4.2)$$

where a and b are the shape parameters, c is the location parameter and d is the scale parameter. The last two parameters, c and d , do not change the shape of the curve but translate and scale the density curve, respectively. The inverse transformation to change the standard normal variable Z back into the random variable X is given by

$$X = c + \frac{d}{1 + \exp\left(\frac{a-Z}{b}\right)}. \quad (4.3)$$

The pdf of X for the Johnson S_B curve can be derived from the standard normal variable Z , which is found to be

$$f_B(x) = \frac{bd}{\sqrt{2\pi}(x-c)(c+d-x)} \exp \left(-\frac{1}{2} \left[a + b * \ln \left(\frac{x-c}{c+d-x} \right) \right]^2 \right), \quad c < x < c + d. \quad (4.4)$$

The cdf of X can be calculated through the cdf of the standard normal variable Z , that is

$$F_B(x) = \Phi \left(a + b \ln \left(\frac{x-c}{c+d-x} \right) \right), \quad (4.5)$$

where $\Phi(\cdot)$ is the standard normal distribution function.

After determining the four parameters by using the algorithm in Appendix B, one can obtain the value of VaR_α in (3.1) through the quantile function of the cdf of X . Let $\phi(\cdot)$ be the standard normal density function and $\phi^{-1}(\alpha)$ be the quantile function of the standard normal variable Z evaluated at confidence level $\alpha \in (0, 1)$. The corresponding VaR_α under the Johnson S_B distribution is then given by

$$\text{VaR}_\alpha = F_B^{-1}(\alpha) = c + \frac{d}{1 + \exp\left(\frac{a-\phi^{-1}(\alpha)}{b}\right)}. \quad (4.6)$$

Using (3.2), ES_α under the Johnson S_B distribution is given by

$$\text{ES}_\alpha = \mathbb{E}(L|L \geq \text{VaR}_\alpha) = \frac{1}{1-\alpha} \int_{z_\alpha}^{\infty} \left[c + \frac{d}{1 + \exp\left(\frac{a-z}{b}\right)} \right] \phi(z) \, dz, \quad (4.7)$$

where

$$z_\alpha = a + b * \ln \frac{\text{VaR}_\alpha - c}{c + d - \text{VaR}_\alpha}.$$

Also, the VaR contribution and ES contribution defined in (3.9a,b) can be easily obtained using the approximate Johnson S_B distribution. To compute the VaR contribution from each obligor, the expectation calculations required in (3.9a) can be preformed using the pdf of the Johnson S_B distribution in (4.4). Similarly, to compute the ES contribution from each obligor, the tail probabilities required in (3.9b) can be estimated using the cdf of the Johnson S_B distribution in (4.5). When we calculate $\mathbb{E}_{L_1(k)}$ and $\mathbb{P}_{L_1(k)}$, $k = 1, 2, \dots, K$, and $\mathbb{E}_{L_2(m)}$ and $P_{L_2(m)}$, $m = 1, 2, \dots, M$, it is necessary to modify the four parameters of the Johnson S_B distribution to fit the first four order moments of the modified loss distribution $L_1(k)$, $k = 1, 2, \dots, K$, and $L_2(m)$, $m = 1, 2, \dots, M$.

5 Saddlepoint Approximation Methods

The calculations of risk measures and risk contributions of a credit portfolio involve numerical evaluation of quantiles, tail probabilities and tail expectations of the portfolio loss L . The saddlepoint approximation approach has been known to be a versatile tool in statistics that is based on the steepest descent method to derive analytic approximation formulas of the Laplace inversion integrals for the density function, tail probability

and tail expectation of a random variable. In the derivation procedure, we deform a contour in the complex plane to pass through a saddlepoint along the steepest descent path. The dominant contribution to the value of the complex inversion integral can be obtained by considering a local expansion around the saddlepoint. The saddlepoint is obtained via the solution of an algebraic equation involving the derivative of the cgf of the random variable. The saddlepoint approximation approach works well when the analytic form of the higher order derivatives of the cgf are available. Fortunately, such analytic tractability requirements of the cgf are observed for the portfolio loss variable under the CR⁺-CBV model (see Sec. 2.2).

There have been numerous research works on the use of the saddlepoint approximation methods in risk measures calculations in the CreditRisk⁺ model and other industrial credit risk models, like the Vasicek model (Huang *et al.*, 2007) and Gaussian copula models (Glasserman, 2006). Gordy (2002) pioneers the use of the Lugannani-Rice saddlepoint formula in VaR calculation for the standard CreditRisk⁺ model. He also proposes a systematic procedure to find the upper bound of the solution of the saddlepoint equation so as to ensure the existence of saddlepoint for VaR calculation. This is an important procedure for the success of the saddlepoint approximation method, the details of which are presented in Appendix C. Studer (2001) proposes the use of the measure change in the saddlepoint approximation procedure to compute ES. More recently, Fischer *et al.* (2016) develop the saddlepoint approximation formula for computing VaR under the CR⁺-CBV model with stochastic recovery rates. Besides the CreditRisk⁺ model, Huang *et al.* (2007) derive various saddlepoint approximation formulas for computing risk measures and risk contribution for the Vasicek portfolio credit loss model. There are several sources of comprehensive review of the use of the saddlepoint approximation methods in credit risk models, like the two book chapters by Broda and Paoletta (2012) and Martin (2013) as well as the monograph by Kwok and Zheng (2018).

We reveal the classical saddlepoint approximation formulas for the numerical approximation of the pdf and cdf of a continuous random variable. For any continuous random variable L with cgf $\kappa_L(t)$, Daniels (1954) devises the following first order saddlepoint approximation to the pdf $f_L(x)$:

$$f_L(x) \approx \frac{1}{\sqrt{2\pi\kappa_L''(\hat{t})}} \exp(\kappa_L(\hat{t}) - \hat{t}x), \quad (5.1)$$

where \hat{t} denotes the unique solution to the saddlepoint equation:

$$\kappa_L'(\hat{t}) = x. \quad (5.2)$$

In addition, the second order saddlepoint approximation to the pdf $f_L(x)$ is given by

$$f_L(x) \approx \frac{\exp(\kappa_L(\hat{t}) - \hat{t}x)}{\sqrt{2\pi\kappa_L''(\hat{t})}} \left\{ 1 + \frac{1}{8} \left[\lambda_4(\hat{t}) - \frac{5}{3} \lambda_3^2(\hat{t}) \right] \right\}, \quad (5.3)$$

where

$$\lambda_3(\hat{t}) = \frac{\kappa_L'''(\hat{t})}{\kappa_L''(\hat{t})^{3/2}} \quad \text{and} \quad \lambda_4(\hat{t}) = \frac{\kappa_L''''(\hat{t})}{\kappa_L''(\hat{t})^2}. \quad (5.4)$$

Note that the third order and fourth order cumulants are involved in the second order saddlepoint approximation formula, which may lead to cumbersome calculations.

Lugannani and Rice (1980) devise the following renowned saddlepoint approximation to the cdf $F_L(x)$:

$$F_L(x) \approx \begin{cases} \Phi(w) - \phi(w) \left(\frac{1}{u} - \frac{1}{w} \right), & x \neq \mu_L, \\ \frac{1}{2} + \frac{\kappa_L''(0)}{6\sqrt{2\pi\kappa_L''(0)^{3/2}}}, & x = \mu_L, \end{cases} \quad (5.5)$$

where $\mu_L = E[L]$ is the mean of L and

$$w = \text{sgn}(\hat{t}) \sqrt{2[\hat{t}x - \kappa_L(\hat{t})]} \quad \text{and} \quad u = \hat{t} \sqrt{\kappa_L''(\hat{t})}. \quad (5.6)$$

The higher order approximation formulas for the cdf $F_L(x)$ is given by

$$F_L(x) \approx \Phi(w) - \phi(w) \left\{ \frac{1}{u} - \frac{1}{w} + \frac{1}{w^3} - \frac{1}{u^3} - \frac{\lambda_3(\hat{t})}{2u^2} + \frac{1}{u} \left[\frac{\lambda_4(\hat{t})}{8} - \frac{5\lambda_3^2(\hat{t})}{24} \right] \right\}, \quad (5.7)$$

where $\lambda_3(\hat{t})$ and $\lambda_4(\hat{t})$ are defined in (5.4).

5.1 Saddlepoint Approximation to VaR and ES

Gordy (2002) uses the Lugannani-Rice saddlepoint approximation for efficient and accurate computation of tail quantiles of the loss distribution in the standard CreditRisk⁺ model. Even though there is no direct analytical formula for VaR in saddlepoint approximation, the calculation of VaR_α is simply the inverse problem of computing the tail probabilities. Given a specified confidence level α , we set the tail probability $1 - F_L(x) = 1 - \alpha$, where $F_L(x)$ is the cdf of the portfolio loss L . Using a root-finding procedure, we solve for the above equation, whose solution is VaR_α .

For $x > E[L]$, the solution \hat{t} of the saddlepoint equation $\kappa'_L(\hat{t}) = x$ [see (5.2)] can be shown to fall within the interval $(0, t^*)$, where $t^* = \min\{t_1^*, \dots, t_K^*, t_{K+1}^*, \dots, t_{K+M}^*\}$. Here, t_k^* , $k = 1, 2, \dots, K$, and t_{K+m}^* , $m = 1, 2, \dots, M$, are the roots of the individual logarithmic terms in $\kappa_L(t)$ of the CR⁺-CBV model [see (2.11)]. That is, $1 - \delta_k P_k(t_k^*) = 0$, $k = 1, 2, \dots, K$, and $1 - P_{K+m}(t_{K+m}^*) = 0$, $m = 1, 2, \dots, M$. They are seen to observe the following bounds:

$$\begin{aligned} 0 < t_k^* &\leq \ln\left(\frac{1}{\delta_k} + \mu_k\right) / \sum_A p_A w_{Ak} \epsilon_A, & k = 1, 2, \dots, K, \\ 0 < t_{K+m}^* &\leq \ln\left(1 + \sum_{k=1}^K \gamma_{mk} \mu_k\right) / \sum_{k=1}^K \gamma_{mk} \sum_A p_A w_{Ak} \epsilon_A, & m = 1, 2, \dots, M. \end{aligned} \quad (5.8)$$

The proof of these analytic properties on the saddlepoint \hat{t} is presented in Appendix C. Therefore, for any confidence level $\alpha \in (F_L(E[L]), 1)$, one can establish the existence of saddlepoint \hat{t} . An effective root finding procedure to estimate VaR_α using the Lugannani-Rice formula for the cgf $F_L(x)$ are outlined below:

1. Determine t_k^* , $k = 1, 2, \dots, K$, and t_{K+m}^* , $m = 1, 2, \dots, M$, as the roots to the equations obtained by setting individual logarithmic terms in $\kappa_L(t)$ in (2.11) to be zero. Let $t^* = \min\{t_1^*, \dots, t_K^*, t_{K+1}^*, \dots, t_{K+M}^*\}$;
2. Establish a set of finite points t_1, t_2, \dots, t_N between 0 and t^* , where N is the total number of points;
3. For each point t_i , $i = 1, 2, \dots, N$, x_i is set to be $\kappa'_L(t_i)$. Use the Lugannani-Rice formula to obtain the tail probability $1 - F_L(x_i)$. Record the pairs $\{1 - F_L(x_i), x_i\}$, $i = 1, 2, \dots, N$.
4. At the confidence level α , interpolate to find \hat{x} such that $1 - F_L(\hat{x}) = 1 - \alpha$. Here, \hat{x} gives an estimate of VaR_α .

For a given confidence level α , Martin (2006) obtains the first order saddlepoint approximation formula for ES_α as given by

$$\text{ES}_\alpha \approx \frac{1}{1 - \alpha} \left[\mu_L [1 - \Phi(w)] + \phi(w) \left(\frac{x}{u} - \frac{\mu_L}{w} \right) \right], \quad x = \text{VaR}_\alpha, \quad (5.9)$$

where $\mu_L = E[L]$ and \hat{t}, u, w are given in (5.2) and (5.6).

To consider higher order saddlepoint approximation and avoid the third and fourth order derivatives of $\kappa_L(t)$, Bulter and Wood (2004) obtain the following second order saddlepoint approximation for ES_α

$$\text{ES}_\alpha \approx \frac{1}{1 - \alpha} \left[\mu_L [1 - \Phi(w)] + \phi(w) \left(\frac{x}{u} - \frac{\mu_L}{w} + \frac{\mu_L - x}{w^3} + \frac{1}{u\hat{t}} \right) \right]. \quad (5.10)$$

Studer (2001) introduces the measure change in the saddlepoint approximation procedure to compute ES. The procedure changes the original loss variable L to a new loss variable \hat{L} , which is similar to the change of measure in the saddlepoint approximation. We introduce a new probability density function $f_{\hat{L}}(x) = x f_L(x) / \mu_L$ for a new random variable \hat{L} . For a given confidence level α , ES_α can be written as:

$$\text{ES}_\alpha = \frac{\mu_L}{1 - \alpha} \int_{\text{VaR}_\alpha}^{\infty} \frac{x f_L(x)}{\mu_L} dx = \frac{\mu_L}{1 - \alpha} \int_{\text{VaR}_\alpha}^{\infty} f_{\hat{L}}(x) dx = \frac{\mu_L}{1 - \alpha} \mathbb{P}[\hat{L} \geq \text{VaR}_\alpha]. \quad (5.11)$$

The saddlepoint approximation for $\mathbb{P}[\hat{L} > \text{VaR}_\alpha]$ is found by using the Lugannani-Rice formula (5.5) or (5.7), which requires the cgf $\kappa_{\hat{L}}(t)$ of this new random variable \hat{L} . Note that the relationship between $\kappa_{\hat{L}}(t)$ and $\kappa_L(t)$ is given by

$$\kappa_{\hat{L}}(t) = \kappa_L(t) + \ln(\kappa'_L(t)) - \ln(\kappa'_L(0)).$$

Accordingly, the higher order derivatives of $\kappa_{\hat{L}}(t)$ are found to be

$$\begin{aligned}\kappa'_{\hat{L}}(t) &= \kappa'_L(t) + \frac{\kappa''_L(t)}{\kappa'_L(t)}, \\ \kappa''_{\hat{L}}(t) &= \kappa''_L(t) + \frac{\kappa'''_L(t)}{\kappa'_L(t)} - \frac{\kappa''_L(t)^2}{\kappa'_L(t)^2}.\end{aligned}$$

Furthermore, we can obtain the corresponding VaR_α and ES_α simultaneously by using the check function defined in (3.3). The tail expectation $\mathbb{E}[(L-x)^+]$ is involved in the check function, which can be estimated by using the corresponding saddlepoint approximation formula as given by [see (2.28) in Kwok and Zheng (2018)]

$$\mathbb{E}[(L-x)^+] \approx \Delta[1 - \Phi(w)] - \phi(w) \left(\frac{\Delta}{w} - \frac{\Delta}{w^3} - \frac{1}{tu} \right), \quad (5.12)$$

where $\Delta = \kappa'_L(0) - x$. Substituting the formula into (3.3) and choosing an appropriate solver for the optimization problem, we obtain ES_α as the minimum value of the check function and VaR_α as the argument that gives ES_α .

5.2 Saddlepoint Approximation for Risk Contribution

Martin *et al.* (2001) pioneer the one-term saddlepoint approximation to find a simplified saddlepoint approximation formula for VaR contribution. We start with the following Bromwich integral for the tail probability of the loss variable L in terms of the cgf $\kappa_L(t)$, where

$$\mathbb{P}[L > \text{VaR}_\alpha] = \frac{1}{2\pi i} \int_{\zeta-i\infty}^{\zeta+i\infty} \frac{e^{\kappa_L(t) - t\text{VaR}_\alpha}}{t} dt. \quad (5.13)$$

The vertical Bromwich contour $\text{Re } t = \zeta$ is chosen to lie within the domain of analyticity of $\kappa_L(t)$ in the complex plane. To find the risk contribution VaRC_α^A of obligor A with exposure ϵ_A at confidence level α , we differentiate both sides of the above equation with respect to ϵ_A and obtain

$$\text{VaRC}_\alpha^A = \epsilon_A \frac{\int_{\zeta-i\infty}^{\zeta+i\infty} \frac{\partial \kappa_L(t)}{\partial \epsilon_A} \frac{1}{t} \exp(\kappa_L(t) - t\text{VaR}_\alpha) dt}{\int_{\zeta-i\infty}^{\zeta+i\infty} \exp(\kappa_L(t) - t\text{VaR}_\alpha) dt}. \quad (5.14)$$

Martin *et al.* (2001) derive the saddlepoint approximation for VaRC_α^A by freezing the term $\frac{1}{t} \frac{\partial \kappa_L(t)}{\partial \epsilon_A}$ at the saddlepoint \hat{t} , where $\kappa'_L(\hat{t}) = \text{VaR}_\alpha$. This leads to the following one-term saddlepoint approximation formula:

$$\text{VaRC}_\alpha^A \approx \frac{\epsilon_A}{\hat{t}} \frac{\partial \kappa_L(t)}{\partial \epsilon_A} \Big|_{t=\hat{t}}. \quad (5.15)$$

In the framework of $\text{CR}^+ - \text{CBV}$ model, the corresponding one-term saddlepoint approximation formula for VaRC_α^A is given by

$$\text{VaRC}_\alpha^A \approx \epsilon_A p_A e^{\epsilon_A t} \left[w_{A0} + \sum_{k=1}^K \theta_k \delta_k \frac{w_{Ak}}{1 - \delta_k P_k(t)} + \sum_{m=1}^M \hat{\theta}_m \frac{w_{A,K+m}}{1 - P_{K+m}(t)} \right] \Big|_{t=\hat{t}}, \quad (5.16)$$

where $P_k(t)$, $k = 1, 2, \dots, K$, and $P_{K+m}(t)$, $m = 1, 2, \dots, M$, are defined in (2.9a,b).

Another method for risk contributions calculation makes use of the two marginal risk contribution formulas (3.9a,b). We substitute the saddlepoint approximation formula for the pdf (4.3) or formula for the cdf (4.5) into the expectation terms in (3.9a) for VaR contribution and probability terms in (3.9b) for ES contribution, respectively. Be aware that it is necessary to modify the corresponding cgf and its derivatives when we calculate the expectation and tail probability for different loss variables, namely, $L_1(k)$, $k = 1, 2, \dots, K$, and $L_2(m)$, $m = 1, 2, \dots, M$.

6 Two-step Importance Sampling in Monte Carlo Simulation

In the calculations of risk measures and risk contributions for credit portfolios, Monte Carlo simulation methods are much computationally demanding since defaults of obligors are rare events. For example, given one million simulation paths, there may be only 0.1% or even less of the simulation paths that satisfy the VaR threshold. Estimation of risk measures and risk contributions can only be made on limited number of simulation paths that end in the tail of the loss distribution.

Importance sampling is an efficient algorithm to reduce variance since it allows for more sampling in the tail of the loss distribution L . Glasserman and Li (2003) propose a two-step importance sampling (IS) method on the mixed Poisson model, which solves the problem of estimating the tail probabilities. Han and Kang (2008) summarize the two-step IS algorithm for the hidden gamma model to compute risk measures. As for risk contributions, both Merino and Nyfeler (2004) and Glasserman (2006) develop the IS estimators for marginal risk contributions. A good review of Monte Carlo methods for computing VaR and VaRC for credit portfolios can be found in Hong *et al.* (2014).

Glasserman and Li (2003) propose the two-step IS algorithm that employs the exponential twisting to change the original portfolio loss distribution into a new portfolio loss distribution, where these two distributions are linked through the likelihood ratio. Here, we adopt similar exponential twisting procedures under the CR^+ -CBV model. The two-step IS algorithm consists of the following two steps:

- For each sector variables and common background variables, modify the parameters of the underlying Gamma distribution by the exponential twisting procedure.
- Conditional on the realized sector variables and common background variables, for each obligor $A \in \mathcal{A}$, change the intensity of the Poisson distribution of the default indicator by the exponential twisting procedure.

In the first step, we perform exponential twisting on each Gamma distribution in (2.5), namely, $\delta_k S_k, k = 1, 2, \dots, K$, and $T_m, m = 1, 2, \dots, M$, into a new set of Gamma distributions by some twisting parameters $p_k, k = 1, 2, \dots, K$, and $\hat{p}_m, m = 1, 2, \dots, M$, respectively. The new set of Gamma distributions have the same shape parameters but different scale parameters as the original set of Gamma distributions, which are defined by

$$\begin{aligned}\bar{S}_k &\sim \Gamma(\theta_k, \frac{\delta_k}{1 - \delta_k p_k}), & k = 1, 2, \dots, K, \\ \bar{T}_m &\sim \Gamma(\hat{\theta}_m, \frac{1}{1 - \hat{p}_m}), & m = 1, 2, \dots, M.\end{aligned}\tag{6.1}$$

The choices of p_k and \hat{p}_m will be discussed later [see (6.4)]. Conditional on the realized sector variables and common background variables from the new set of Gamma distributions, $\bar{\mathbf{S}} = (\bar{S}_1, \dots, \bar{S}_K, \bar{T}_1, \dots, \bar{T}_M)$, each Poisson distribution D_A for obligor $A \in \mathcal{A}$ is independent with each other and has the modified intensity as given by

$$\lambda_A^{\bar{\mathbf{S}}} = p_A(w_{A0} + \sum_{k=1}^K w_{Ak} \bar{S}_k + \sum_{m=1}^M w_{A,K+m} \bar{T}_m).\tag{6.2}$$

The second step is to apply exponential twisting on the conditional Poisson distribution into a new conditional Poisson distribution by the twisting parameter \hat{t} . The new Poisson distribution conditional on $\bar{\mathbf{S}}$ becomes

$$\hat{D}_A \sim \text{Poisson}(\lambda_A^{\bar{\mathbf{S}}} e^{\epsilon_A \hat{t}}).\tag{6.3}$$

When $\hat{t} > 0$, the conditional intensity of the new Poisson distribution for each obligor A increases. The obligor A then becomes more likely to default, thus we can generate more samples in the tail area of the loss distribution.

According to Glasserman and Li (2003), the exponential twisting parameter \hat{t} for the Poisson distribution is chosen to be the saddlepoint that satisfies $\kappa'_L(\hat{t}) = x$ [see (5.2)], where x is some predetermined portfolio loss which lies within the tail area of the loss distribution L , like $\text{VaR}_{\alpha=0.9}$. This works on the principle that both the two-step IS algorithm and saddlepoint approximation method estimate the loss distribution around

the saddlepoint. Accordingly, the exponential twisting parameters for the new set of Gamma distributions are chosen to be

$$\begin{aligned} p_k &= \sum_{A \in \mathcal{A}} p_A w_{Ak} (e^{\epsilon_A \hat{t}} - 1) = P_k(\hat{t}), & k &= 1, 2, \dots, K, \\ \hat{p}_m &= \sum_{A \in \mathcal{A}} p_A w_{A, K+m} (e^{\epsilon_A \hat{t}} - 1) = P_{K+m}(\hat{t}), & m &= 1, 2, \dots, M. \end{aligned} \quad (6.4)$$

To guarantee that the scale parameters for the new set of Gamma distributions (6.1) are positive, we require $p_k < 1/\delta_k$, $k = 1, 2, \dots, K$, and $\hat{p}_m < 1$, $m = 1, 2, \dots, M$.

The likelihood ratio for the realized portfolio loss $L_i = \sum_A \epsilon_A \hat{D}_A$ in i^{th} simulation path after these two-step importance sampling procedures becomes

$$\xi(L_i) = \exp(-\hat{t}L_i + \kappa_L(\hat{t})). \quad (6.5)$$

The two-step IS estimator of the tail probability is then given by

$$\mathbb{P}[L > x] = \xi(L) \mathbf{1}_{\{L > x\}}. \quad (6.6)$$

The implementation procedures of the two-step IS algorithm are summarized in Algorithm 1 as follows:

Algorithm 1: Pseudo code for the two-step IS algorithm

For any pre-determined x , solve the saddlepoint \hat{t} that satisfies $\kappa'_L(\hat{t}) = x$. The exponential twisting parameters for the Gamma distributions are

$$p_k = P_k(\hat{t}), \quad k = 1, 2, \dots, K, \quad \text{and} \quad \hat{p}_m = P_{K+m}(\hat{t}), \quad m = 1, 2, \dots, M.$$

for $i = 1 : N$ (number of the simulation paths) **do**

1. For each sector variable and common background variable, draw samples from

$$\bar{S}_k \sim \Gamma(\theta_k, \frac{\delta_k}{1 - \delta_k p_k}), \quad k = 1, 2, \dots, K, \quad \text{and} \quad \bar{T}_m \sim \Gamma(\hat{\theta}_m, \frac{1}{1 - \hat{p}_m}), \quad m = 1, 2, \dots, M.$$

2. Compute the conditional default probabilities $\lambda_A^{\bar{S}}$ as defined in (6.2).

3. For each obligor $A \in \mathcal{A}$, draw samples from

$$\hat{D}_A \sim \text{Poisson}(\lambda_A^{\bar{S}} e^{\epsilon_A \hat{t}}).$$

4. Record the portfolio loss and the likelihood ratio as

$$L_i = \sum_A \epsilon_A \hat{D}_A \quad \text{and} \quad \xi(L_i) = \exp(-\hat{t}L_i + \kappa_L(\hat{t})).$$

(5). Record \hat{D}_A for each obligor $A \in \mathcal{A}$ when $L_i = \text{VaR}_\alpha$ or $L_i \geq \text{VaR}_\alpha$. (This step is used for risk contribution calculations.)

return the two-step IS estimator (6.6).

Given a confidence level α , VaR_α is the α -quantile of the equation $P(L > x) = 1 - \alpha$. The estimator of ES_α is the mean value for all realized $L_i \geq \text{VaR}_\alpha$, which is given by

$$\text{ES}_\alpha = \mathbb{E}(L | L \geq \text{VaR}_\alpha) = \frac{\mathbb{E}[L \mathbf{1}_{\{L \geq \text{VaR}_\alpha\}}]}{\mathbb{P}[L \geq \text{VaR}_\alpha]} = \frac{\sum_{i=1}^N L_i \xi(L_i) \mathbf{1}_{\{L_i \geq \text{VaR}_\alpha\}}}{\sum_{i=1}^N \xi(L_i) \mathbf{1}_{\{L_i \geq \text{VaR}_\alpha\}}}. \quad (6.7)$$

where N is the number of simulation paths. From the formulas of VaRC and ESC in (3.8a,b), we obtain the estimators of VaRC and ESC as given by

$$\text{VaRC}_\alpha^A = \epsilon_A \frac{\mathbb{E}[D_A \mathbf{1}_{\{L = \text{VaR}_\alpha\}}]}{\mathbb{E}[\mathbf{1}_{\{L = \text{VaR}_\alpha\}}]} = \epsilon_A \frac{\sum_{i=1}^N D_A \xi(L_i) \mathbf{1}_{\{L_i = \text{VaR}_\alpha\}}}{\sum_{i=1}^N \xi(L_i) \mathbf{1}_{\{L_i = \text{VaR}_\alpha\}}}, \quad (6.8a)$$

$$\text{ESC}_\alpha^A = \epsilon_A \frac{\mathbb{E}[D_A \mathbf{1}_{\{L \geq \text{VaR}_\alpha\}}]}{\mathbb{P}[L \geq \text{VaR}_\alpha]} = \epsilon_A \frac{\sum_{i=1}^N D_A \xi(L_i) \mathbf{1}_{\{L_i \geq \text{VaR}_\alpha\}}}{\sum_{i=1}^N \xi(L_i) \mathbf{1}_{\{L_i \geq \text{VaR}_\alpha\}}}. \quad (6.8b)$$

7 Numerical Results

We performed extensive numerical experiments to assess the performance in terms of accuracy and efficiency of the Johnson curve fitting method, saddlepoint approximation method and check function formulation for calculating risk measures and risk contributions for various CreditRisk⁺ models on the stylized credit portfolios and benchmark industrial credit portfolios. The Monte Carlo simulation calculations are set as the benchmark for comparison of the performance for these numerical methods.

7.1 Stylized credit portfolios

The stylized credit portfolio shown in Table 1 is slightly modified from the portfolio adopted in Kurth and Tasche (2003). There are 31,615 obligors in the portfolio, which are grouped separately in two sectors. The obligors with exposures that are equal to or less than 0.01 are put in sector 1 and the other obligors with larger exposures are put in sector 2. In sector 1, the probability of default of an obligor can be either 0.5% or 1% while the probability of default ranges from 0.1% to 1.75% for obligors in sector 2. The expected loss for the whole stylized portfolio is $\mu_L = E[L] = \sum_A p_A \epsilon_A = 3.3994$.

	sector 1			sector 2						
exposure per obligor	0.01	0.01	0.005	0.05	0.08	0.1	0.15	0.2	0.3	1
number of obligors	10,000	10,000	10,000	1,000	500	100	10	2	2	1
probability of default	0.5%	1%	1%	1%	1.75%	1.75%	1.25%	0.7%	0.3%	0.1%

Table 1: Exposures and probabilities of default for obligors in different sectors for the stylized credit portfolio.

In the first set of risk calculations, we consider the standard CreditRisk⁺ model with two independent sector variables $\hat{S}_k, k = 1, 2$, which follow the Gamma distribution $\delta_k \Gamma(\theta_k, 1), k = 1, 2$. The parameters of the two Gamma distributions are chosen to be $\delta_1 = 0.0256, \delta_2 = 0.1296$ and $\theta_k = 1/\delta_k, k = 1, 2$, respectively. In the second set of risk calculations, we aim to capture the correlation between different sectors by adding two common background variables for each sector. This leads to the CR⁺-CBV(2) model. Each sector variable $\hat{S}_k, k = 1, 2$, is the weighted sum of an idiosyncratic sector variable $S_k, k = 1, 2$, and two independent common background variables $T_m, m = 1, 2$, based on the following linear relationship:

$$\hat{S}_k = \delta_k S_k + \gamma_{1k} T_1 + \gamma_{2k} T_2, \quad k = 1, 2. \quad (7.1)$$

Here, $S_k \sim \Gamma(\theta_k, 1), k = 1, 2$, and $T_m \sim \Gamma(\hat{\theta}_m, 1), m = 1, 2$. The parameters for the CR⁺-CBV(2) model are summarized in Table 2.

k	δ_k	θ_k	$\hat{\theta}_k$	γ_{1k}	γ_{2k}
1	0.0256	23.4375	4.8000	0.0625	0.0016
2	0.1296	4.6296	62.5000	0.0625	0.0016

Table 2: Parameter values of the CR⁺-CBV(2) model for the stylized credit portfolio.

Based on the portfolio information and model parameter values given in Tables 1 and 2, respectively, the Plain Monte Carlo simulation [MC (Plain)] and the two-step IS simulation [MC (IS)] were performed to compute the risk measures and risk contributions. The implementation of the Plain Monte Carlo simulation uses the intensity λ_A of the Poisson default indicator for each obligor in (2.4) directly. In the simulation calculations, the Plain Monte Carlo simulation is achieved by using 10 subsamples with 10,000 simulation paths each. On the other hand, the two-step IS simulation is obtained through 10 subsamples with 8,000 simulation paths each. For each simulation calculations, risk measures are the sample means of the 10 corresponding subsample estimates. The standard error for each risk measure is taken to be the sample standard deviation of the 10 corresponding

subsample estimates. The numerical results from these two simulation calculations are used to assess accuracy and efficiency of other numerical methods.

7.1.1 Numerical calculations of risk measures

Risk measures calculations of VaR_α and ES_α by using the Johnson curve transform were implemented on the standard CR^+ model and CR^+ -CBV model for the stylized credit portfolio. The Johnson curve transform provides an approximate fitting curve for the loss distribution by matching the first four order moments of the stylized credit portfolio with those of the Johnson S_B distribution. After finding the four parameters of the Johnson S_B distribution through the calibration procedure in Appendix B, VaR_α can be estimated using the analytic formula (4.6). The evaluation of ES_α given in (4.7) requires numerical integration, which can be implemented with ease.

Another method for computing risk measures is the saddlepoint approximation method. The monotonicity of $\kappa'_L(t)$ guarantees the existence of unique solution \hat{t} of the saddlepoint equation: $\kappa'_L(\hat{t}) = x$, for any $x \geq \mu_L$ (see Appendix C). The root-finding procedure is required to find VaR_α as discussed in Section 5.1, while ES_α is calculated by either the first order or second order saddlepoint approximation formulas in (5.9) and (5.10), respectively. The measure change saddlepoint approximation formula in (5.11) is also an efficient method to obtain ES_α , which requires the change of the loss variable L into a new variable \hat{L} with a new pdf $f_{\hat{L}} = x f_L(x) / \mu_L$. We also apply the check function formulation to compute VaR_α and ES_α by solving an optimization problem (3.4) via an optimization solver. The tail expectation calculation as part of the procedure in the check function is estimated by using (5.12).

The calculation results for VaR_α and ES_α on the standard CreditRisk^+ model and CR^+ -CBV(2) model under four confidence levels $\alpha = 0.9, 0.95, 0.99, 0.999$ are presented in Tables 3 and 4, respectively.

	$\text{VaR}_{\alpha=0.90}$	$\text{VaR}_{\alpha=0.95}$	$\text{VaR}_{\alpha=0.99}$	$\text{VaR}_{\alpha=0.999}$	CPU
MC (Plain)	4.3121 (0.0124)	4.6286 (0.0156)	5.2636 (0.0486)	6.0605 (0.0879)	2201
MC (IS)	4.3101 (0.0031)	4.6252 (0.0026)	5.2685 (0.0041)	6.0750 (0.0089)	1375
Johnson curve (4.6)	4.3103	4.6252	5.2688	6.0796	1.025
VaR-1 st order (5.5)	4.3101	4.6253	5.2693	6.0779	0.0253
VaR-2 nd order (5.7)	4.3103	4.6255	5.2694	6.0778	0.0497
check function (3.4)	4.3103	4.6254	5.2694	6.0778	0.0434
	$\text{ES}_{\alpha=0.90}$	$\text{ES}_{\alpha=0.95}$	$\text{ES}_{\alpha=0.99}$	$\text{ES}_{\alpha=0.999}$	CPU
MC (Plain)	4.7379 (0.0198)	5.0216 (0.0298)	5.6124 (0.0646)	6.3572 (0.0938)	2201
MC (IS)	4.7386 (0.0017)	5.0228 (0.0012)	5.6180 (0.0026)	6.4045 (0.0064)	1375
Johnson curve (4.7)	4.7373	5.0223	5.6246	6.4047	1.025
ES-1 st order (5.9)	4.7495	5.0373	5.6448	6.4281	0.0019
ES-2 nd order (5.10)	4.7375	5.0226	5.6243	6.4003	0.0020
measure change (5.11)	4.7383	5.0234	5.6252	6.4011	0.0039
check function (3.4)	4.7375	5.0226	5.6243	6.4003	0.0434

Table 3: For comparison of accuracy and efficiency, we list the numerical values of VaR_α and ES_α for the standard CreditRisk^+ model on the stylized credit portfolio at different confidence levels α obtained via various numerical algorithms. The equations used in the calculations are shown in brackets besides the method types. The numbers inside the brackets besides the Monte Carlo simulation results are the standard errors. The last column shows the average CPU time used for each numerical method.

From Tables 3 and 4, the CPU time required by both the Plain Monte Carlo simulation and two-step IS algorithm is of order of thousand seconds. The Johnson curve fitting algorithm typically requires one second, while only a few percent of one second or less is required for the saddlepoint approximation calculations. The two-step IS algorithm outperforms the Plain Monte Carlo simulation with less CPU time and smaller standard errors. The results from the Johnson curve fitting method, saddlepoint approximation method and check formula agree very well with those obtained from the two-step IS simulation. In some cases, the numerical

results from different numerical methods match with each other up to 4 significant figures accuracy. The higher CPU time required for the Johnson curve fitting method compared with the saddlepoint approximation calculations mainly comes from the iteration process to find the four parameters of the Johnson S_B distribution. The computation efforts required by the saddlepoint approximation calculations arise mainly from finding the saddlepoint \hat{t} , which involves simple root finding procedure of a scalar equation. The second order saddlepoint approximation does provide better accuracy than the first order saddlepoint approximation on the estimation of VaR_α and ES_α , revealing the merit of applying a higher-order approximation at slightly higher computational costs. Meanwhile, the check function formulation gives a good verification on accuracy of the second order saddlepoint approximation. Though these two numerical methods are based on different derivation principles, they do produce almost the same value of ES_α . An iteration optimization solver is required to solve the check function (3.4). A nice feature is that the solution from the optimization procedure provides the value of VaR_α and ES_α simultaneously [see (3.4)]. The measure change saddlepoint approximation formula (5.11) also provides highly accurate results on ES_α for both the standard CreditRisk^+ model and $\text{CR}^+\text{-CBV}(2)$ model.

	$\text{VaR}_{\alpha=0.90}$	$\text{VaR}_{\alpha=0.95}$	$\text{VaR}_{\alpha=0.99}$	$\text{VaR}_{\alpha=0.999}$	CPU
MC (Plain)	4.3921(0.0200)	4.7260 (0.0245)	5.4310 (0.0460)	6.2770 (0.0898)	2243
MC (IS)	4.3821 (0.0017)	4.7231 (0.0019)	5.4263 (0.0016)	6.2966(0.0038)	1737
Johnson curve (4.6)	4.3825	4.7239	5.4214	6.2952	1.2430
VaR-1 st order (5.5)	4.3824	4.7240	5.4214	6.2947	0.0349
VaR-2 nd order (5.7)	4.3825	4.7241	5.4213	6.2946	0.0493
check function (3.4)	4.3825	4.7240	5.4213	6.2945	0.0582
	$\text{ES}_{\alpha=0.90}$	$\text{ES}_{\alpha=0.95}$	$\text{ES}_{\alpha=0.99}$	$\text{ES}_{\alpha=0.999}$	CPU
MC (Plain)	4.8514 (0.0209)	5.1611 (0.0282)	5.8201 (0.0548)	6.6519 (0.0957)	2243
MC (IS)	4.8476 (0.0011)	5.1594 (0.0023)	5.8089 (0.0014)	6.6415 (0.0042)	1737
Johnson curve (4.7)	4.8453	5.1540	5.8050	6.6426	1.2431
ES-1 st order (5.9)	4.8570	5.1682	5.8243	6.6682	0.0023
ES-2 nd order (5.10)	4.8453	5.1540	5.8047	6.6419	0.0025
measure change (5.11)	4.8456	5.1544	5.8051	6.6421	0.0064
check function (3.4)	4.8453	5.1540	5.8047	6.6419	0.0582

Table 4: For comparison of accuracy and efficiency, we list the numerical values of VaR_α and ES_α for the $\text{CR}^+\text{-CBV}(2)$ model on the stylized credit portfolio at different confidence levels α obtained via various numerical algorithms. The equations used in the calculations are shown in brackets besides the method types. The numbers inside the brackets besides the Monte Carlo simulation results are the standard errors. The last column shows the average CPU time used for each numerical method.

7.1.2 Numerical calculations of risk contributions

The risk contributions from individual obligors to the whole stylized credit portfolio are computed by using the marginal risk contribution formulas (3.9a,b). The pdf and cdf of the loss distribution L from the Johnson curve fitting method given in (4.4) and (4.5) can be used directly to calculate the expectations and tail probabilities in formulas (3.9a,b), respectively. For the loss distributions $L_1(k), k = 1, 2, \dots, K$, and $L_2(m), m = 1, 2, \dots, M$, modified from L , it is necessary to change the corresponding k^{th} sector variable or m^{th} common background variable into a new variable while all other variables stay the same. Due to these modifications, the four parameters of the Johnson S_B distribution need to be recomputed to fit the first four order moments of the new set of model parameters for each $L_1(k)$ or $L_2(m)$. The expectation and tail probability of $L_1(k)$ or $L_2(m)$ are estimated through the pdf and cdf under the new Johnson S_B distribution. Specifically, since (4.6) is the quantile function of (4.5), here we adopted (4.6) to evaluate the tail probability via the root finding procedure in our numerical calculations for the ES contributions. Similarly, the pdf and cdf of the loss distribution L by the saddlepoint approximation method presented in (5.1) or (5.3) and (5.5) or (5.7) are used to calculate the expectation and tail probability, respectively. It is also required to modify the corresponding cgf and its

derivatives for different loss distributions $L_1(k)$ and $L_2(m)$ when employing the saddlepoint approximation method to calculate the expectation and tail probability. Besides, the one-term saddlepoint approximation formula (5.16) for VaR_α (Martin *et al.*, 2001) is also used to compute VaRC_α for both the standard CreditRisk^+ model and CR^+ -CBV(2) model.

The calculation results for risk contributions on the standard CreditRisk^+ model and CR^+ -CBV(2) model under three confidence levels $\alpha = 0.95, 0.99, 0.999$ are presented in Tables 5 and 6, respectively.

	sector 1			sector 2						sum of contributions	
$\text{VaRC}_{\alpha=0.95}$											(4.6254)
MC (Plain)	0.5716	1.1394	0.5611	0.7961	1.2021	0.3046	0.0331	0.0046	0.0048	0.0084	4.6258
MC (IS)	0.5667	1.1362	0.5606	0.8020	1.1982	0.3111	0.0370	0.0052	0.0041	0.0057	4.6268
Johnson curve (4.4)	0.5677	1.1354	0.5622	0.8102	1.1926	0.3081	0.0358	0.0058	0.0043	0.0047	4.6267
pdf-1 st order (5.1)	0.5674	1.1348	0.5619	0.8098	1.1919	0.3079	0.0358	0.0058	0.0043	0.0047	4.6242
pdf-2 nd order (5.3)	0.5676	1.1352	0.5621	0.8100	1.1922	0.3080	0.0358	0.0058	0.0043	0.0047	4.6255
Martin (5.16)	0.5585	1.1171	0.5537	0.8194	1.2083	0.3127	0.0365	0.0059	0.0045	0.0085	4.6253
$\text{VaRC}_{\alpha=0.99}$											(5.2694)
MC (Plain)	0.5868	1.1530	0.5738	1.0003	1.4947	0.3891	0.0468	0.0084	0.0068	0.0081	5.2678
MC (IS)	0.5836	1.1698	0.5776	0.9958	1.4771	0.3912	0.0463	0.0097	0.0079	0.0197	5.2787
Johnson curve (4.4)	0.5869	1.1738	0.5798	0.9917	1.4829	0.3872	0.0462	0.0077	0.0061	0.0112	5.2734
pdf-1 st order (5.1)	0.5866	1.1732	0.5795	0.9901	1.4802	0.3865	0.0461	0.0077	0.0060	0.0111	5.2670
pdf-2 nd order (5.3)	0.5868	1.1736	0.5797	0.9906	1.4811	0.3867	0.0461	0.0077	0.0061	0.0112	5.2670
Martin (5.16)	0.5780	1.1559	0.5716	0.9993	1.4959	0.3911	0.0468	0.0078	0.0063	0.0167	5.2693
$\text{VaRC}_{\alpha=0.999}$											(6.0778)
MC (Plain)	0.6084	1.2196	0.6030	1.2177	1.8505	0.4711	0.0670	0.0120	0.0068	0.0125	6.0686
MC (IS)	0.6042	1.2041	0.5970	1.2255	1.8430	0.4874	0.0605	0.0113	0.0079	0.0303	6.0712
Johnson curve (4.4)	0.6038	1.2077	0.5953	1.2231	1.8537	0.4885	0.0596	0.0101	0.0084	0.0239	6.0742
pdf-1 st order (5.1)	0.6040	1.2080	0.5954	1.2229	1.8534	0.4884	0.0596	0.0101	0.0084	0.0238	6.0742
pdf-2 nd order (5.3)	0.6042	1.2084	0.5956	1.2239	1.8550	0.4888	0.0597	0.0101	0.0084	0.0239	6.0780
Martin (5.16)	0.5959	1.1918	0.588	1.2316	1.8679	0.4926	0.0603	0.0103	0.0086	0.0311	6.0779
$\text{ESC}_{\alpha=0.95}$											(5.0226)
MC (Plain)	0.5806	1.1593	0.5739	0.9157	1.3723	0.3550	0.0426	0.0067	0.0054	0.0086	5.0201
MC (IS)	0.5782	1.1576	0.5726	0.9219	1.3710	0.3567	0.0433	0.0071	0.0053	0.0091	5.0228
Johnson curve (4.6)	0.5794	1.1588	0.5729	0.9220	1.3715	0.3569	0.0422	0.0069	0.0054	0.0088	5.0248
cdf-1 st order (5.5)	0.5793	1.1585	0.5728	0.9215	1.3706	0.3566	0.0422	0.0069	0.0054	0.0088	5.0226
cdf-2 nd order (5.7)	0.5793	1.1586	0.5728	0.9214	1.3705	0.3566	0.0422	0.0069	0.0054	0.0088	5.0225
$\text{ESC}_{\alpha=0.99}$											(5.6243)
MC (Plain)	0.5908	1.1881	0.5842	1.0925	1.6571	0.4237	0.0495	0.0087	0.0094	0.0140	5.6180
MC (IS)	0.5964	1.1932	0.5879	1.0901	1.6465	0.4288	0.0530	0.0093	0.0076	0.0153	5.6281
Johnson curve (4.6)	0.5946	1.1892	0.5868	1.0931	1.6453	0.4316	0.0521	0.0088	0.0071	0.0166	5.6252
cdf-1 st order (5.5)	0.5947	1.1893	0.5869	1.0926	1.6446	0.4314	0.0521	0.0087	0.0071	0.0166	5.6239
cdf-2 nd order (5.7)	0.5947	1.1894	0.5869	1.0927	1.6447	0.4314	0.0521	0.0087	0.0071	0.0166	5.6243
$\text{ESC}_{\alpha=0.999}$											(6.4003)
MC (Plain)	0.6093	1.2209	0.6124	1.3068	2.0026	0.5365	0.0687	0.0133	0.0123	0.0320	6.4148
MC (IS)	0.6080	1.2178	0.5985	1.3140	2.0110	0.5414	0.0633	0.0119	0.0087	0.0338	6.4084
Johnson curve (4.6)	0.6088	1.2175	0.5998	1.3133	1.9983	0.5280	0.0649	0.0111	0.0094	0.0305	6.3816
cdf-1 st order (5.5)	0.6093	1.2187	0.6003	1.3183	2.0063	0.5302	0.0652	0.0112	0.0094	0.0306	6.3995
cdf-2 nd order (5.7)	0.6094	1.2187	0.6004	1.3185	2.0067	0.5303	0.0652	0.0112	0.0094	0.0306	6.4004

Table 5: Risk contributions for VaRC_α and ESC_α are computed for the standard CR^+ model on the stylized credit portfolio. The numerical values of VaRC_α and ESC_α at different confidence levels α are obtained via various numerical algorithms. The equations used in the calculations are shown in brackets besides the method types. The numbers shown in bracket in the last column are the best estimates of VaR_α and ES_α (sum of contributions) from Table 3.

	sector 1				sector 2						sum of contributions
VaRC $_{\alpha=0.95}$											(4.7240)
MC (Plain)	0.6131	1.2339	0.6088	0.7796	1.1492	0.2958	0.0292	0.0059	0.0037	0.0031	4.7223
MC (IS)	0.6183	1.2383	0.6142	0.7712	1.1314	0.3020	0.0327	0.0062	0.0047	0.0041	4.7231
Johnson curve (4.4)	0.6210	1.2420	0.6156	0.7722	1.1335	0.2924	0.0338	0.0054	0.0040	0.0045	4.7244
pdf-1 st order (5.1)	0.6208	1.2415	0.6154	0.7718	1.1329	0.2922	0.0338	0.0054	0.0040	0.0045	4.7224
pdf-2 nd order (5.3)	0.6209	1.2419	0.6156	0.7721	1.1334	0.2923	0.0338	0.0054	0.0040	0.0045	4.7241
Martin (5.16)	0.6181	1.2361	0.6131	0.7745	1.1377	0.2937	0.0341	0.0055	0.0042	0.0071	4.7240
VaRC $_{\alpha=0.99}$											(5.4213)
MC (Plain)	0.6723	1.3315	0.6598	0.9272	1.4114	0.3514	0.0478	0.0055	0.0044	0.0074	5.4187
MC (IS)	0.6812	1.3559	0.6729	0.9211	1.3692	0.3657	0.0428	0.0053	0.0057	0.0082	5.4280
Johnson curve (4.4)	0.6788	1.3576	0.6715	0.9210	1.3715	0.3572	0.0423	0.0070	0.0055	0.0097	5.4221
pdf-1 st order (5.1)	0.6786	1.3571	0.6712	0.9203	1.3705	0.3569	0.0423	0.0070	0.0055	0.0097	5.4190
pdf-2 nd order (5.3)	0.6788	1.3576	0.6714	0.9208	1.3712	0.3571	0.0423	0.0070	0.0055	0.0097	5.4214
Martin (5.16)	0.6765	1.3529	0.6695	0.9223	1.3739	0.3580	0.0425	0.0070	0.0056	0.0132	5.4214
VaRC $_{\alpha=0.999}$											(6.2945)
MC (Plain)	0.7517	1.5108	0.7402	1.1528	1.6424	0.4443	0.0292	0.0089	0.0050	0.0120	6.2973
MC (IS)	0.7543	1.4939	0.7371	1.0979	1.6677	0.4509	0.0530	0.0111	0.0070	0.0182	6.2911
Johnson curve (4.4)	0.7494	1.4987	0.7398	1.1078	1.6705	0.4387	0.0531	0.0090	0.0073	0.0191	6.2934
pdf-1 st order (5.1)	0.7492	1.4984	0.7397	1.1073	1.6698	0.4385	0.0531	0.0090	0.0073	0.0191	6.2915
pdf-2 nd order (5.3)	0.7495	1.4990	0.7399	1.1080	1.6709	0.4388	0.0532	0.0090	0.0073	0.0191	6.2947
Martin (5.16)	0.7480	1.4960	0.7388	1.1083	1.6715	0.4391	0.0533	0.0090	0.0074	0.0234	6.2947
ESC $_{\alpha=0.95}$											(5.1540)
MC (Plain)	0.6569	1.3099	0.6501	0.8668	1.2817	0.3331	0.0396	0.0059	0.0044	0.0095	5.1579
MC (IS)	0.6566	1.3137	0.6503	0.8650	1.2777	0.3332	0.0390	0.0066	0.0051	0.0075	5.1547
Johnson curve (4.6)	0.6566	1.3132	0.6500	0.8639	1.2803	0.3323	0.0391	0.0064	0.0049	0.0078	5.1546
cdf-1 st order (5.5)	0.6526	1.3053	0.6480	0.8424	1.2289	0.3155	0.0360	0.0057	0.0040	0.0072	5.0458
cdf-2 nd order (5.7)	0.6566	1.3132	0.6500	0.8638	1.2801	0.3323	0.0391	0.0064	0.0049	0.0078	5.1540
ESC $_{\alpha=0.99}$											(5.8047)
MC (Plain)	0.7013	1.4095	0.6985	1.0044	1.5024	0.3894	0.0459	0.0076	0.0072	0.0244	5.7906
MC (IS)	0.7096	1.4224	0.7014	1.0040	1.5016	0.3914	0.0466	0.0077	0.0063	0.0139	5.8049
Johnson curve (4.6)	0.7099	1.4197	0.7015	1.0030	1.5028	0.3930	0.0471	0.0079	0.0063	0.0137	5.8048
cdf-1 st order (5.5)	0.7074	1.4149	0.7003	0.9898	1.4713	0.3827	0.0453	0.0074	0.0058	0.0133	5.7382
cdf-2 nd order (5.7)	0.7099	1.4198	0.7016	1.0030	1.5027	0.3930	0.0471	0.0079	0.0063	0.0137	5.8048
ESC $_{\alpha=0.999}$											(6.6419)
MC (Plain)	0.7723	1.5815	0.7661	1.1667	1.7353	0.4566	0.0495	0.0056	0.0033	0.0403	6.5772
MC (IS)	0.7671	1.5421	0.7596	1.1839	1.7983	0.4747	0.0607	0.0098	0.0067	0.0258	6.6287
Johnson curve (4.6)	0.7770	1.5540	0.7666	1.1815	1.7886	0.4710	0.0574	0.0097	0.0081	0.0238	6.6378
cdf-1 st order (5.5)	0.7755	1.5509	0.766	1.1727	1.7666	0.4637	0.0561	0.0095	0.0077	0.0203	6.5889
cdf-2 nd order (5.7)	0.7773	1.5546	0.7669	1.1825	1.7901	0.4714	0.0575	0.0098	0.0081	0.0238	6.6420

Table 6: Risk contributions for VaRC $_{\alpha}$ and ESC $_{\alpha}$ are computed for the CR⁺-CBV(2) model on the stylized credit portfolio. The numerical values of VaRC $_{\alpha}$ and ESC $_{\alpha}$ at different confidence levels α are obtained via various numerical algorithms. The equations used in the calculations are shown in brackets besides the method types. The numbers shown in bracket in the last column are the best estimates of VaR $_{\alpha}$ and ES $_{\alpha}$ (sum of contributions) from Table 4.

The performance of Monte Carlo simulation on risk contributions deteriorates as the confidence level α increases, because the number of simulation paths ending in the tail area of loss distribution becomes more limited. The two-step IS algorithm can generate more simulation paths ending in the tail area, so the risk contributions calculations are more accurate. In the VaRC calculation, the simulation path with realized loss amount L_i that exactly hits the value of VaR_α is a rare event, so we preset a small threshold d_α to proxy the event $\{L_i = \text{VaR}_\alpha\}$ by $\{L_i \in (\text{VaR}_\alpha - d_\alpha, \text{VaR}_\alpha + d_\alpha)\}$. Usually, there are less number of simulation paths ending beyond VaR_α when α is larger. The chosen threshold d_α has to be larger for larger α in order to capture more eligible simulation paths.

In Tables 5 and 6, we check the sum of risk contributions from all obligors, for both VaRC and ESC, using different numerical methods. For most cases, the sum of risk contributions is close to the best estimate of the corresponding risk measure from Table 3 or 4. The two-step IS algorithm provides more accurate results on VaRC_α and ESC_α comparing with the Plain Monte Carlo simulation. This is reasonable since the estimates from the two-step IS simulation are based on more simulation paths in the tail area of the loss distribution. Though the one-term saddlepoint approximation formula (5.16) is derived based on a crude analytic approximation, the sum of all individual VaR contributions estimated by this saddlepoint approximation formula agrees with the best estimate of VaR very well. The implementation procedures of the Johnson curve fitting method and saddlepoint approximation method on risk contributions become more tedious, but they are reliable methods on the estimation of both VaRC and ESC. The calculation results for VaRC and ESC from these two numerical methods are very close to each other, up to 3 significant figures even at $\alpha = 0.999$.

7.2 Benchmark industrial credit portfolio

We examine the performance of various numerical methods for calculating risk measures and risk contributions on a benchmark industrial credit portfolio with a large number of obligors and industry sectors. We choose the benchmark industrial credit portfolio from the International Association of Credit Portfolio Management and International Swaps and Derivatives Association (see the websites www.isda.org and www.iapcm.org). The credit portfolio has two term loans with 3,000 obligors each. All obligors are classified into 10 industry sectors. The information on the risk parameters for each industry sector of the benchmark industrial credit portfolio is presented in Table 7. The sector variances and correlation matrix are estimated from the real market data, which are specified by using actual cases of US corporate defaults from 1995 to 2009. More detailed information on the benchmark industrial credit portfolio can be found in Fischer and Dietz (2011). In our numerical tests, we investigate the three CR^+ -CBV models with 1, 2 and 3 common background variables, namely, CBV(1), CBV(2) and CBV(3). We restrict the case where each obligor is given full weight to the sector it belongs. The risk measures and risk contributions are given in unit of billion US dollars.

Sector	Sector name	Number of obligors	Exposure at default	Portfolio potential loss	Expected loss
1	Energy	239	12 382 000 000	5 021 260 000	57 988 047
2	Materials	472	17 422 000 000	7 104 600 000	134 098 406
3	Industrials	614	14 186 000 000	5 927 880 000	80 364 896
4	Consumer discretionary	634	20 180 000 000	7 803 100 000	148 957 421
5	Consumer staples	222	3 835 000 000	1 578 860 000	17 808 856
6	Health care	103	1 049 000 000	406 480 000	5 358 676
7	Financials	312	14 277 000 000	5 853 480 000	47 012 264
8	Information technology	150	5 012 000 000	2 071 460 000	68 278 467
9	Telecommunication services	84	7 497 000 000	3 066 040 000	11 312 357
10	Utilities	170	4 160 000 000	1 719 760 000	34 078 735
Total		3000	100 000 000 000	40 552 920 000	605 258 124

Table 7: Risk parameters of the benchmark industrial credit portfolio by industry sectors.

The statistics information for these three CR^+ -CBV models including the mean, variance, skewness and kurtosis are shown in Table 8. Since the mean, variance and covariance are used to calibrate the different CR^+ -CBV models, the variations of mean and variance are insignificant among the three CR^+ -CBV models.

When the number of common background variables increases, both the skewness and kurtosis increase since there are more weights on the tail area under larger number of common background variables.

Model	mean	variance	skewness	kurtosis
CBV(1)	0.6053	0.3301	2.2745	11.4060
CBV(2)	0.6040	0.3353	2.3483	11.8148
CBV(3)	0.6060	0.3362	2.6634	14.8248

Table 8: Statistics information of the CR⁺-CBV models with varying number of common background variables.

Based on the portfolio information and model parameters, the Plain Monte Carlo simulation calculations were performed using 10 subsamples with 100,000 simulation paths in each subsample. The two-step IS simulation calculations were performed using 10 subsamples with 80,000 simulation paths in each subsample. For each simulation calculations, risk measures and standard errors are taken to be the sample means and sample standard deviations of the 10 corresponding subsample estimates, respectively.

7.2.1 Numerical results of risk measures

Tables 9 and 10 present the different numerical values of VaR_α and ES_α at three confidence levels α = 0.90, 0.95, 0.99 obtained via various numerical algorithms for the CBV(1), CBV(2) and CBV(3), respectively. We adopt the same Johnson curve fitting algorithm, saddlepoint approximation formulas and check function formulation as what we did in the stylized credit portfolio, only with different model parameters.

	VaR _{α=0.90}	VaR _{α=0.95}	VaR _{α=0.99}	CPU
CBV(1)				
MC (Plain)	1.3144 (0.0054)	1.7297 (0.0108)	2.7586 (0.0223)	1414
MC (IS)	1.3119 (0.0066)	1.7221 (0.0064)	2.7573 (0.0088)	1144
Johnson curve (4.6)	1.3115	1.7229	2.7553	0.8032
VaR-1 st order (5.5)	1.3030	1.7267	2.7643	9.727
VaR-2 nd order (5.7)	1.2925	1.7079	2.7322	12.26
check function (3.4)	1.2913	1.7092	2.7414	34.28
CBV(2)				
MC (Plain)	1.3153 (0.0097)	1.7438 (0.0133)	2.7864 (0.0196)	2258
MC (IS)	1.3147 (0.0085)	1.7417 (0.0081)	2.7942 (0.0061)	1264
Johnson curve (4.6)	1.3112	1.7333	2.7969	1.267
VaR-1 st order (5.5)	1.3092	1.7399	2.7873	11.97
VaR-2 nd order (5.7)	1.3064	1.7366	2.7892	14.93
check function (3.4)	1.3092	1.7296	2.7713	58.17
CBV(3)				
MC (Plain)	1.2864 (0.0080)	1.7253 (0.0117)	2.8487 (0.0180)	1618
MC (IS)	1.2862 (0.0037)	1.7270 (0.0072)	2.8515 (0.0047)	1069
Johnson curve (4.6)	1.2885	1.7191	2.8521	1.843
VaR-1 st order (5.5)	1.2345	1.6678	2.7898	11.64
VaR-2 nd order (5.7)	1.2761	1.7098	2.7694	15.27
check function (3.4)	1.2255	1.6524	2.7446	30.84

Table 9: The numerical values of VaR_α at different confidence levels α for the three CR⁺-CBV models on the benchmark industrial credit portfolio are obtained via various numerical algorithms. The equations used in the calculations are shown in brackets besides the method types. The numbers inside the brackets besides the Monte Carlo simulation results are the standard errors. The last column shows the average CPU time used for each numerical method.

The small amount of CPU time required for the Johnson curve fitting method indicates high computational

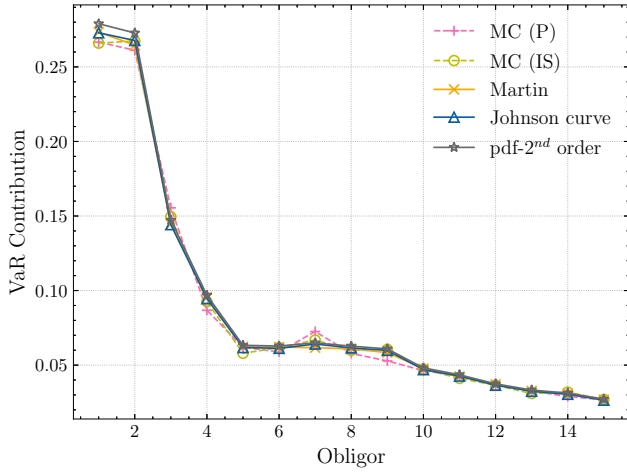
efficiency in finding the four parameters of the Johnson S_B distribution by the two-dimensional Newton-Raphson iterative scheme. In the saddlepoint approximation method, it requires more significant computational efforts to find the saddlepoint \hat{t} under the more complex industrial credit portfolio. The same increase in computational complexity appears when we employ the check function formula. In terms of accuracy, the two-step IS algorithm again outperforms the Plain Monte Carlo simulation with smaller standard errors in most cases. The results from other numerical methods agree with those obtained from the two-step IS simulation very well, especially the Johnson curve fitting approach [see the good agreement of numerical results under CBV(1) and CBV(2)]. The check function formula and the second order saddlepoint approximation formula serve to cross check each other in calculating ES_α . The slightly larger difference in VaR_α between these two methods may be caused by the fact that the optimization process for the check function formula tends to terminate at the close neighborhood of the correct value of VaR_α . Furthermore, the number of common background variables is a crucial factor that affects accuracy of these numerical methods. The agreement of numerical results among different numerical methods worsens with an increase of common background variables [see CBV(3)]. The second order saddlepoint approximation method is likely to underestimate VaR_α and ES_α when the benchmark industrial credit portfolio has a larger number of common background variables. The Johnson curve fitting approach gives better estimates on VaR_α but overestimates ES_α . This may be caused by the fat tail of the estimated Johnson S_B distribution.

	$ES_{\alpha=0.90}$	$ES_{\alpha=0.95}$	$ES_{\alpha=0.99}$	CPU
<hr/>				
CBV(1)				
MC (Plain)	1.9333 (0.0119)	2.3704 (0.0192)	3.4264 (0.0524)	1414
MC (IS)	1.9373 (0.0026)	2.3564 (0.0024)	3.4113 (0.0030)	1144
Johnson curve (4.7)	1.9481	2.3548	3.3960	0.8536
ES-2 nd order (5.10)	1.9134	2.3524	3.4123	9.643
measure change (5.11)	1.9620	2.3905	3.4084	12.13
check function (3.4)	1.9133	2.3522	3.4119	34.28
<hr/>				
CBV(2)				
MC (Plain)	1.9487 (0.0119)	2.3942 (0.0157)	3.4727 (0.0361)	2258
MC (IS)	1.9487 (0.0014)	2.3962 (0.0032)	3.4636 (0.0028)	1264
Johnson curve (4.7)	1.9494	2.3757	3.5281	1.269
ES-2 nd order (5.10)	1.9356	2.3785	3.4454	17.32
measure change (5.11)	1.9798	2.4261	3.5066	19.91
check function (3.4)	1.9355	2.3784	3.4452	58.17
<hr/>				
CBV(3)				
MC (Plain)	1.9547 (0.0083)	2.4317 (0.0102)	3.6281 (0.0306)	1618
MC (IS)	1.9554 (0.0025)	2.4345 (0.0034)	3.6249 (0.0033)	1069
Johnson curve (4.7)	2.0733	2.5687	3.8812	1.947
ES-2 nd order (5.10)	1.8754	2.3382	3.5041	11.76
measure change (5.11)	2.0633	2.5935	3.9201	16.32
check function (3.4)	1.8753	2.3381	3.5027	30.84

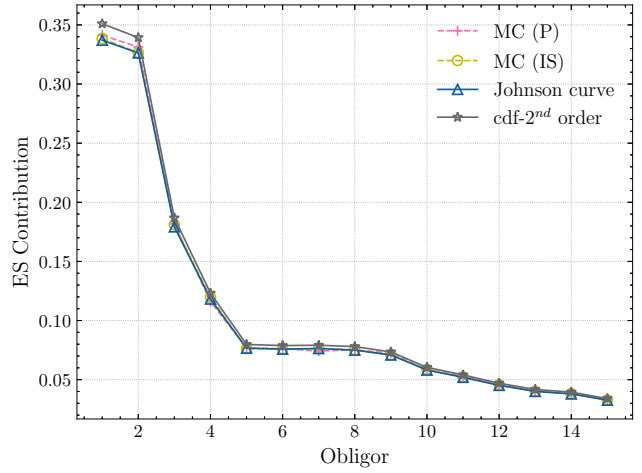
Table 10: The numerical values of ES_α at different confidence levels α for the three CR^+ -CBV models on the benchmark industrial credit portfolio are obtained via various numerical algorithms. The equations used in the calculations are shown in brackets besides the method types. The numbers inside the brackets besides the Monte Carlo simulation results are the standard errors. The last column shows the average CPU time used for each numerical method.

7.2.2 Numerical plots of risk contributions

In Figures 2-4, we plot the VaR and ES contributions for the major 15 obligors at confidence level $\alpha = 0.99$ for the three CR^+ -CBV models. When compared with the Plain Monte Carlo simulation, the numerical risk contribution results obtained from the two-step IS simulation provides better agreement with those obtained

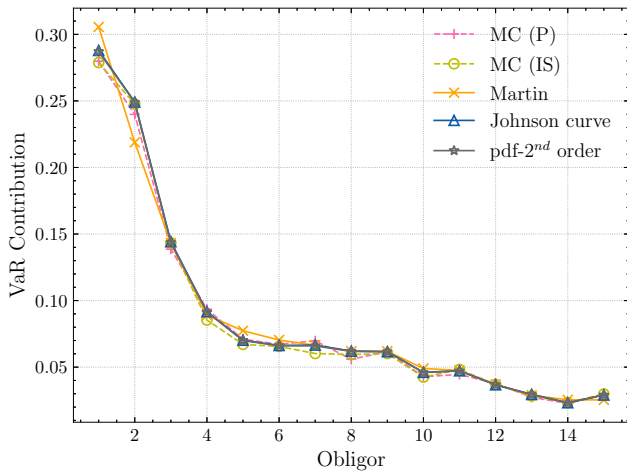


(a) VaR contribution of CBV(1) at $\text{VaR}_{\alpha=0.99}$

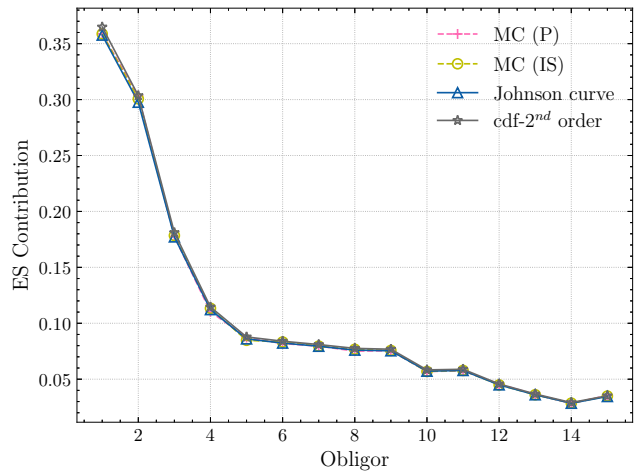


(b) ES contribution of CBV(1) at $\text{ES}_{\alpha=0.99}$

Figure 1: Plots of VaR contribution and ES contribution for the major 15 obligors of the benchmark industrial credit portfolio for the CBV(1) model at confidence level $\alpha = 0.99$ obtained via various numerical algorithms.

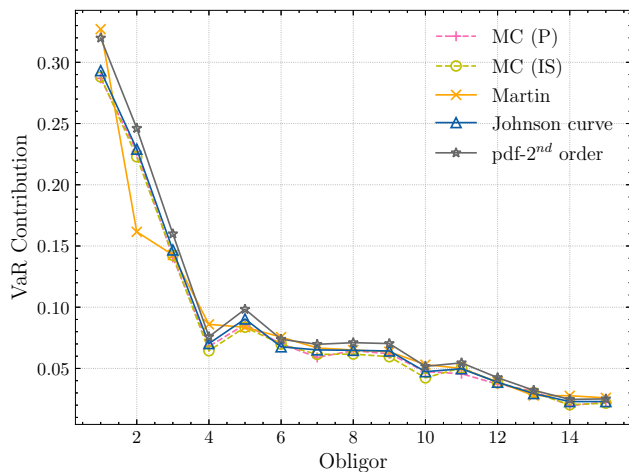


(a) VaR contribution of CBV(2) at $\text{VaR}_{\alpha=0.99}$

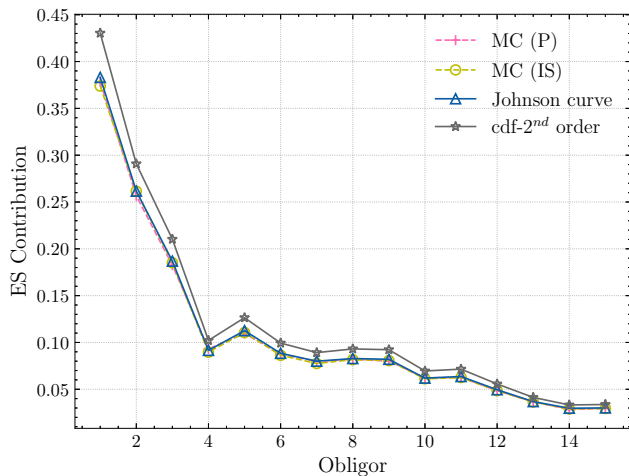


(b) ES contribution of CBV(2) at $\text{ES}_{\alpha=0.99}$

Figure 2: Plots of VaR contribution and ES contribution for the major 15 obligors of the benchmark industrial credit portfolio for the CBV(2) model at confidence level $\alpha = 0.99$ obtained via various numerical algorithms.



(a) VaR contribution of CBV(3) at $\text{VaR}_{\alpha=0.99}$



(b) ES contribution of CBV(3) at $\text{ES}_{\alpha=0.99}$

Figure 3: Plots of VaR contribution and ES contribution for the major 15 obligors of the benchmark industrial credit portfolio for the CBV(3) model at confidence level $\alpha = 0.99$ obtained via various numerical algorithms.

using marginal risk contribution formulas (3.9a,b) and one-term saddlepoint approximation formula (5.16). The calculations by using the Johnson curve fitting method to estimate the expectations and tail probabilities in (3.9a,b) give the most consistent numerical risk contribution results among all numerical algorithms. Also, the CPU time required for the Johnson curve fitting method is much less compared with those of the Monte Carlo simulation and saddlepoint approximation calculations in (3.9a,b) due to the use of an efficient iteration algorithm. The saddlepoint approximation calculations of the expectations and tail probabilities in (3.9a,b) become more time consuming since one has to solve for the saddlepoint \hat{t} for each obligor and each loss distribution. The performance worsens slightly when the number of common background variables becomes larger. The same phenomenon appears when we implement the one-term saddlepoint approximation formula.

8 Conclusion

We have developed and implemented various numerical algorithms for calculating risk measures (VaR and ES) and risk contributions from risky obligors in credit portfolios under the enhanced common background vector framework in the CreditRisk⁺ model. Besides the established saddlepoint approximation approach and two-step importance sampling algorithm, we also presented the detailed implementation of the less popularly known Johnson curve fitting algorithm and check function formulation. We have compared the performance of these numerical algorithms for calculating risk measures and risk contributions using both stylized credit portfolios and complex industrial credit portfolios.

In most of our VaR and ES calculations, the Johnson curve fitting algorithm and saddlepoint approximation methods perform very well in terms of accuracy, efficiency and reliability. In many cases, these two numerical methods show very good agreement of numerical results up to 3 significant figures and reveal no reported cases of numerical failure even for complex industrial credit portfolios. The numerical calculations of risk contributions by using the Monte Carlo simulation may pose challenges due to rare events of extreme losses, in particular at high confidence level in VaR and ES. On the other hand, the Johnson curve fitting algorithm delivers accurate results on risk contributions in most cases. The saddlepoint approach works well in calculating risk contributions for most credit portfolios, given that we implement the systematic procedure of finding the saddlepoint that lies within an appropriate domain. In terms of computational efficiency, the Johnson curve fitting algorithm only requires around one second in CPU time, which is typically less than 0.1% of that required by the Monte Carlo simulation method. The efficient implementation of the Johnson curve fitting algorithm and saddlepoint approximation method relies on the analytic expressions of the first four order moments of the credit loss distribution. Fortunately, these analytic requirements are satisfied in the CR⁺-CBV model.

This paper shows the successes and limitations of various numerical algorithms in calculating risk measures

and risk contributions for the CR^+ -CBV model. Practitioners are advised to use different numerical algorithms to cross check accuracy and reliability of the numerical results on risk measures and risk contributions of credit portfolios.

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Appendix A Proof of marginal risk contribution formulas (3.9a,b)

This proof represents a refinement and nontrivial extension of the proof in Tasche (2004). Equation (3.8a) reveals that the VaR contribution of obligor A at confidence level α can be expressed as

$$\text{VaRC}_\alpha^A = \epsilon_A \frac{\mathbb{E}[D_A \mathbf{1}_{\{L = \text{VaR}_\alpha\}}]}{\mathbb{E}[\mathbf{1}_{\{L = \text{VaR}_\alpha\}}]} = \epsilon_A \frac{\mathbb{E}[\lambda_A \mathbf{1}_{\{L = \text{VaR}_\alpha - \epsilon_A\}}]}{\mathbb{E}[\mathbf{1}_{\{L = \text{VaR}_\alpha\}}]}.$$

To evaluate the expectation in the numerator, we consider the expansion of $\epsilon_A \mathbb{E}[D_A e^{tL}]$, where $\epsilon_A D_A$ is the individual random loss of obligor A and $L = \sum_A \epsilon_A D_A$ is the total random portfolio loss. Here, ϵ_A is the exposure of obligor A and D_A is the default indicator that is assumed to be a mixture Poisson random variable in the CR⁺-CBV model. Note that L is a discrete non-negative random variable, whose value can be written as L_1, L_2, \dots, L_N . The expansion of $\epsilon_A \mathbb{E}[D_A e^{tL}]$ admits the following form

$$\epsilon_A \mathbb{E}[D_A e^{tL}] = \sum_{i=1}^N \epsilon_A \mathbb{E}[D_A e^{tL_i} \mathbf{1}_{\{L=L_i\}}] = \sum_{i=1}^N \epsilon_A e^{tL_i} \mathbb{E}[\lambda_A \mathbf{1}_{\{L=L_i - \epsilon_A\}}]. \quad (\text{A.1})$$

The expansion in (A.1) involves the expectation $\mathbb{E}[\lambda_A \mathbf{1}_{\{L=L_i - \epsilon_A\}}]$, which has the same form as the numerator in the VaR contribution formula when VaR_α is set to be L_i . Alternatively, we consider another representation of $\epsilon_A \mathbb{E}[D_A e^{tL}]$; that is

$$\epsilon_A \mathbb{E}[D_A e^{tL}] = \epsilon_A \mathbb{E}[\lambda_A e^{t(L + \epsilon_A)}] = \epsilon_A e^{t\epsilon_A} \mathbb{E}[\lambda_A e^{tL}]. \quad (\text{A.2})$$

We then compute $\mathbb{E}[\lambda_A e^{tL}]$ with reference to the CR⁺-CBV model, where λ_A is defined in (2.4). Conditional on the realization of the random variables $\mathbf{S} = (S_1, \dots, S_K, T_1, \dots, T_M)$, we deduce from (2.8) that

$$\begin{aligned} \mathbb{E}[\lambda_A e^{tL}] &= \mathbb{E}[\lambda_A E[e^{tL} | \mathbf{S}]] = \mathbb{E}[\lambda_A M_L(t | \mathbf{S})] \\ &= p_A w_{A0} M_L(t) + p_A \sum_{k=1}^K w_{Ak} \mathbb{E}[\delta_k S_k e^{\delta_k S_k P_k(t)}] \mathbb{E} \left[\exp \left(P_0(t) + \sum_{\substack{j=1 \\ j \neq k}}^K \delta_j S_j P_j(t) + \sum_{n=1}^M T_n P_{K+n}(t) \right) \right] \\ &\quad + p_A \sum_{m=1}^M w_{A, K+m} \mathbb{E}[T_m e^{T_m P_{K+m}(t)}] \mathbb{E} \left[\exp \left(P_0(t) + \sum_{j=1}^K \delta_j S_j P_j(t) + \sum_{\substack{n=1 \\ n \neq m}}^M T_n P_{K+n}(t) \right) \right]. \end{aligned}$$

Performing the expectation calculations similar with those in (2.10), we obtain

$$\begin{aligned} \mathbb{E} \left[\exp \left(P_0(t) + \sum_{\substack{j=1 \\ j \neq k}}^K \delta_j S_j P_j(t) + \sum_{n=1}^M T_n P_{K+n}(t) \right) \right] &= \exp(P_0(t)) \prod_{\substack{j=1 \\ j \neq k}}^K [1 - \delta_j P_j(t)]^{-\theta_j} \prod_{n=1}^M [1 - P_{K+n}(t)]^{-\hat{\theta}_n}, \\ \mathbb{E} \left[\exp \left(P_0(t) + \sum_{j=1}^K \delta_j S_j P_j(t) + \sum_{\substack{n=1 \\ n \neq m}}^M T_n P_{K+n}(t) \right) \right] &= \exp(P_0(t)) \prod_{j=1}^K [1 - \delta_j P_j(t)]^{-\theta_j} \prod_{\substack{n=1 \\ n \neq m}}^M [1 - P_{K+n}(t)]^{-\hat{\theta}_n}. \end{aligned}$$

Using the property of the Gamma function: $\hat{\Gamma}(x+1) = x\hat{\Gamma}(x)$, we establish

$$\begin{aligned}\mathbb{E}[\delta_k S_k e^{\delta_k S_k P_k(t)}] &= \frac{1}{\hat{\Gamma}(\theta_k)} \int_0^\infty \delta_k s^{\theta_k} e^{s[\delta_k P_k(t)-1]} ds = \delta_k [1 - \delta_k P_k(t)]^{-(\theta_k+1)} \frac{1}{\hat{\Gamma}(\theta_k)} \int_0^\infty y^{\theta_k+1-1} e^{-y} dy \\ &= \delta_k \theta_k [1 - \delta_k P_k(t)]^{-(\theta_k+1)}, \quad k = 1, 2, \dots, K,\end{aligned}$$

$$\begin{aligned}\mathbb{E}[T_m e^{T_m P_{K+m}(t)}] &= \frac{1}{\hat{\Gamma}(\hat{\theta}_m)} \int_0^\infty s^{\hat{\theta}_m} e^{s[P_{K+m}(t)-1]} ds = [1 - P_{K+m}(t)]^{-(\hat{\theta}_m+1)} \frac{1}{\hat{\Gamma}(\hat{\theta}_m)} \int_0^\infty y^{\hat{\theta}_m+1-1} e^{-y} dy \\ &= \hat{\theta}_m [1 - P_{K+m}(t)]^{-(\hat{\theta}_m+1)}, \quad m = 1, 2, \dots, M.\end{aligned}$$

For notational convenience, we let $M_{L_1(k)}(t)$, $k = 1, 2, \dots, K$, and $M_{L_2(m)}(t)$, $m = 1, 2, \dots, M$, denote the moment generating functions of $L_1(k)$ and $L_2(m)$, respectively. Here, $L_1(k)$ and $L_2(m)$ correspond to loss variables modified from L . By virtue of (2.5), for $L_1(k)$, $k = 1, 2, \dots, K$, the Gamma distribution for the k^{th} sector variable $\delta_k S_k$ is changed from $\delta_k \Gamma(\theta_k, 1)$ to $\delta_k \Gamma(\theta_k + 1, 1)$; while for $L_2(m)$, $m = 1, 2, \dots, M$, the Gamma distribution for the m^{th} common background variable T_m is changed from $\Gamma(\theta_m, 1)$ to $\Gamma(\theta_m + 1, 1)$. All other sector variables and common background variables remain unchanged. Combining all the above results together, we have

$$\begin{aligned}\mathbb{E}[\lambda_A e^{tL}] &= p_A w_{A0} M_L(t) + p_A \sum_{k=1}^K w_{Ak} \delta_k \theta_k [1 - \delta_k P_k(t)]^{-(\theta_k+1)} \exp(P_0(t)) \prod_{\substack{j=1 \\ j \neq k}}^K [1 - \delta_j P_j(t)]^{-\theta_j} \prod_{n=1}^M [1 - P_{K+n}(t)]^{-\hat{\theta}_n} \\ &\quad + p_A \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m [1 - P_{K+m}(t)]^{-(\hat{\theta}_m+1)} \exp(P_0(t)) \prod_{j=1}^K [1 - \delta_j P_j(t)]^{-\theta_j} \prod_{\substack{n=1 \\ n \neq m}}^M [1 - P_{K+n}(t)]^{-\hat{\theta}_n} \\ &= p_A w_{A0} M_L(t) + p_A \sum_{k=1}^K w_{Ak} \delta_k \theta_k M_{L_1(k)}(t) + p_A \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m M_{L_2(m)}(t).\end{aligned}\tag{A.3}$$

Recall that $M_{L_1(k)}(t) = \mathbb{E}_{L_1(k)}[e^{tL_1(k)}]$, $k = 1, 2, \dots, K$, and $M_{L_2(m)}(t) = \mathbb{E}_{L_2(m)}[e^{tL_2(m)}]$, $m = 1, 2, \dots, M$, where $\mathbb{E}_{L_1(k)}$, $k = 1, 2, \dots, K$, and $\mathbb{E}_{L_2(m)}$, $m = 1, 2, \dots, M$, represent the expectation under the modified loss distribution $L_1(k)$ and $L_2(m)$, respectively. Substituting (A.3) into (A.2), we obtain

$$\begin{aligned}\epsilon_A \mathbb{E}[D_A e^{tL}] &= \epsilon_A \mathbb{E}[\lambda_A e^{t(L+\epsilon_A)}] \\ &= \epsilon_A p_A \left[w_{A0} \mathbb{E}[e^{t(L+\epsilon_A)}] + \sum_{k=1}^K w_{Ak} \delta_k \theta_k \mathbb{E}_{L_1(k)}[e^{t[L_1(k)+\epsilon_A]}] + \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m \mathbb{E}_{L_2(m)}[e^{t[L_2(m)+\epsilon_A]}] \right] \\ &= \sum_{i=1}^N \epsilon_A p_A e^{tL_i} \left[w_{A0} \mathbb{E}[\mathbf{1}_{\{L=L_i-\epsilon_A\}}] + \sum_{k=1}^K w_{Ak} \delta_k \theta_k \mathbb{E}_{L_1(k)}[\mathbf{1}_{\{L=L_i-\epsilon_A\}}] + \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m \mathbb{E}_{L_2(m)}[\mathbf{1}_{\{L=L_i-\epsilon_A\}}] \right].\end{aligned}\tag{A.4}$$

By comparing the coefficient of $\epsilon_A e^{tL_i}$ in the two expansions in (A.1) and (A.4), we deduce that

$$\begin{aligned}\mathbb{E}[\lambda_A \mathbf{1}_{\{L=L_i-\epsilon_A\}}] \\ &= p_A \left[w_{A0} \mathbb{E}[\mathbf{1}_{\{L=L_i-\epsilon_A\}}] + \sum_{k=1}^K w_{Ak} \delta_k \theta_k \mathbb{E}_{L_1(k)}[\mathbf{1}_{\{L=L_i-\epsilon_A\}}] + \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m \mathbb{E}_{L_2(m)}[\mathbf{1}_{\{L=L_i-\epsilon_A\}}] \right].\end{aligned}\tag{A.5}$$

In terms of VaR_α , we may replace the dummy value L_i in (A.5) by VaR_α . We finally obtain the marginal risk contribution formula for VaRC_α^A [see (3.8a)] as follows:

$$\begin{aligned}\text{VaRC}_\alpha^A \\ &= \epsilon_A p_A \frac{w_{A0} \mathbb{E}[\mathbf{1}_{\{L=\text{VaR}_\alpha-\epsilon_A\}}] + \sum_{k=1}^K w_{Ak} \delta_k \theta_k \mathbb{E}_{L_1(k)}[\mathbf{1}_{\{L=\text{VaR}_\alpha-\epsilon_A\}}] + \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m \mathbb{E}_{L_2(m)}[\mathbf{1}_{\{L=\text{VaR}_\alpha-\epsilon_A\}}]}{\mathbb{E}[\mathbf{1}_{\{L=\text{VaR}_\alpha\}}]}.\end{aligned}$$

By following a similar procedure, we also obtain

$$\begin{aligned} & \text{ESC}_\alpha^A \\ &= \epsilon_{APA} \frac{w_{A0} \mathbb{P}[L \geq \text{VaR}_\alpha - \epsilon_A] + \sum_{k=1}^K w_{Ak} \delta_k \theta_k \mathbb{P}_{L_1(k)}[L \geq \text{VaR}_\alpha - \epsilon_A] + \sum_{m=1}^M w_{A,K+m} \hat{\theta}_m \mathbb{P}_{L_2(m)}[L \geq \text{VaR}_\alpha - \epsilon_A]}{\mathbb{P}[L \geq \text{VaR}_\alpha]}. \end{aligned}$$

Here, \mathbb{P} is the probability measure under the loss distribution L , $\mathbb{P}_{L_1(k)}$, $k = 1, 2, \dots, K$, and $\mathbb{P}_{L_2(m)}$, $m = 1, 2, \dots, M$, are the probability measures under the loss distribution $L_1(k)$ and $L_2(m)$, respectively.

Appendix B Johnson S_B distribution algorithm

To estimate the parameters of the Johnson S_B distribution, the algorithm proposed by Hill *et al.* (1976) matches the first four order moments (mean μ , variance σ^2 , skewness κ_3 and kurtosis κ_4) of the target distribution with those of the Johnson S_B distribution. The moments of the Johnson S_B distribution are evaluated by using Goodwin's integral (Goodwin, 1949). The shape parameters a and b are estimated by a two-dimensional Newton-Raphson iterative scheme. Once these two shape parameters are determined, the location parameter c and scale parameter d are available by using the first two order moments of the Johnson S_B distribution.

Recall that Y is defined as the transform of the random variable X via (4.1) and the random variable X can be changed into the standard normal variable Z via (4.2). By Goodwin's integral, the r^{th} order moment $\mathbb{E}(Y^r)$ of Y about zero can be written in terms of a and b as follows:

$$\begin{aligned} M_r(a, b) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} [1 + e^{(a-z)/b}]^{-r} dz \\ &= \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \left[1 + e^{(a-\sqrt{2}nh)/b} \right]^{-r} e^{-n^2 h^2} + O_r(h), \end{aligned} \quad (\text{B.1})$$

where h is the length of small intervals when using the numerical integration, and $O_r(h)$ is the error term for the r^{th} moment. The error term can be expressed explicitly as

$$O_r(h) \approx -e^{-\pi^2/h^2} \left\{ \left[1 + e^{(ah-\sqrt{2}i\pi)/bh} \right]^{-r} + \left[1 + e^{(ah+\sqrt{2}i\pi)/bh} \right]^{-r} \right\}.$$

Note that the error term tends rapidly to zero with the length h . A sufficiently small value in the length h is chosen in order to achieve the required accuracy. Given the first four order moments of Y obtained via (B.1), the mean, variance, skewness and kurtosis of Y are obtained by:

$$\begin{aligned} f_1(a, b) &= E(a, b) = M_1(a, b), \\ f_2(a, b) &= \text{var}(a, b) = M_2(a, b) - M_1^2(a, b), \\ f_3(a, b) &= \text{skew}(a, b) = \frac{M_3(a, b) - 3M_1(a, b)M_2(a, b) + 2M_1^3(a, b)}{[M_2(a, b) - M_1^2(a, b)]^{3/2}}, \\ f_4(a, b) &= \text{kurt}(a, b) = \frac{M_4(a, b) - 4M_1(a, b)M_3(a, b) + 6M_1^2(a, b)M_2(a, b) - 3M_1^4(a, b)}{[M_2(a, b) - M_1^2(a, b)]^2}. \end{aligned}$$

To estimate the shape parameters a and b , Hill *et al.* (1976) consider the two-dimensional Newton-Raphson iterative scheme using the skewness $f_3(a, b)$ and kurtosis $f_4(a, b)$. Assuming that the estimator of the parameters in the i^{th} iterative step is (a_i, b_i) , the $(i+1)^{\text{th}}$ iterate is obtained by

$$\begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + J(a_i, b_i)^{-1} \begin{pmatrix} f_3(a_{i+1}, b_{i+1}) - f_3(a_i, b_i) \\ f_4(a_{i+1}, b_{i+1}) - f_4(a_i, b_i) \end{pmatrix}, \quad (\text{B.2})$$

where the Jacobian matrix $J(a_i, b_i)$ and its inversion are given by

$$J(a_i, b_i) = \begin{pmatrix} \frac{\partial f_3}{\partial a}(a_i, b_i) & \frac{\partial f_3}{\partial b}(a_i, b_i) \\ \frac{\partial f_4}{\partial a}(a_i, b_i) & \frac{\partial f_4}{\partial b}(a_i, b_i) \end{pmatrix}, \quad J(a_i, b_i)^{-1} = \frac{1}{|J(a_i, b_i)|} \begin{pmatrix} \frac{\partial f_4}{\partial b}(a_i, b_i) & -\frac{\partial f_3}{\partial b}(a_i, b_i) \\ -\frac{\partial f_4}{\partial a}(a_i, b_i) & \frac{\partial f_3}{\partial a}(a_i, b_i) \end{pmatrix}. \quad (\text{B.3})$$

The determinant of the Jacobian matrix is defined as

$$|J(a_i, b_i)| = \frac{\partial f_3}{\partial a}(a_i, b_i) \frac{\partial f_4}{\partial b}(a_i, b_i) - \frac{\partial f_3}{\partial b}(a_i, b_i) \frac{\partial f_4}{\partial a}(a_i, b_i).$$

The iteration is continued until the skewness $f_3(a_{i+1}, b_{i+1})$ and the kurtosis $f_4(a_{i+1}, b_{i+1})$ are sufficiently close to those of the input data κ_3 and κ_4 . The estimated parameters are then given as

$$\begin{aligned} \hat{a} &= \begin{cases} a_{i+1}, & \kappa_3 > 0, \\ -a_{i+1}, & \kappa_3 < 0, \end{cases} \\ \hat{b} &= b_{i+1}, \\ \hat{d} &= \frac{\sigma}{\sqrt{M_2(a_{i+1}, b_{i+1}) - M_1^2(a_{i+1}, b_{i+1})}}, \\ \hat{c} &= \begin{cases} \mu - \hat{d} * M_1(a_{i+1}, b_{i+1}), & \kappa_3 > 0, \\ \mu - \hat{d} * [1 - M_1(a_{i+1}, b_{i+1})], & \kappa_3 < 0. \end{cases} \end{aligned}$$

Additional details on the implementation of the algorithm are presented below:

- Initialization of the shape parameters a and b

To initiate the two-dimensional Newton-Raphson iterative scheme, the initial estimates of the parameters a and b are obtained as follows:

Let $\beta_2 = \omega^4 + 2\omega^3 + 3\omega^2 - 3$ with

$$w = \left(1 + 0.5\kappa_3^2 + \kappa_3\sqrt{1 + 0.25\kappa_3^2}\right)^{1/3} + \left(1 + 0.5\kappa_3^2 + \kappa_3\sqrt{1 + 0.25\kappa_3^2}\right)^{-1/3} - 1.$$

The initial estimate of the shape parameter b is found by interpolating between 0 and $(\ln w)^{-\frac{1}{2}}$ as

$$b_0 = \begin{cases} \frac{0.626\beta_2 - 0.408}{(3.0 - \beta_2)^{0.479}}, & \beta_2 \geq 1.8, \\ 0.8(\beta_2 - 1), & \text{otherwise.} \end{cases} \quad (\text{B.4})$$

As suggested by Draper (1952), the initial estimate of the shape parameter a is given by

$$|a_0| = \frac{\sqrt{1 - 2b_0^2} - 2b_0^2 \tanh^{-1} \sqrt{1 - 2b_0^2}}{b_0}. \quad (\text{B.5})$$

- Partial derivatives of $M_r(a, b)$

As required in the two-dimensional Newton-Raphson iteration, the partial derivatives of $M_r(a, b)$ with respect to a and b are derived as follows:

$$\begin{aligned} \frac{\partial M_r(a, b)}{\partial a} &= -\frac{r}{b} \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-n^2 h^2} e^{(a - \sqrt{2}nh)/b} \left[1 + e^{(a - \sqrt{2}nh)/b}\right]^{-r-1} \\ &= \frac{r}{b} \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-n^2 h^2} \left\{ \left[1 + e^{(a - \sqrt{2}nh)/b}\right]^{-r-1} - \left[1 + e^{(a - \sqrt{2}nh)/b}\right]^{-r} \right\} \\ &= \frac{r}{b} [M_{r+1}(a, b) - M_r(a, b)], \end{aligned}$$

$$\begin{aligned}\frac{\partial M_r(a, b)}{\partial b} &= \frac{r}{b^2} \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-n^2 h^2} e^{(a-\sqrt{2}nh)/b} (a - \sqrt{2}nh) \left[1 + e^{(a-\sqrt{2}nh)/b}\right]^{-r-1} \\ &= \frac{ar}{b^2} [M_r(a, b) - M_{r+1}(a, b)] - \frac{r}{b^2} \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-n^2 h^2} e^{(a-\sqrt{2}nh)/b} \sqrt{2}nh \left[1 + e^{(a-\sqrt{2}nh)/b}\right]^{-r-1}.\end{aligned}$$

Due to the appearance of $\sqrt{2}nh$ in the second term, we cannot express the second term as the difference of two moments as in the first term. The algorithm of Hill *et al.* (1976) adopts the following approximation formula:

$$\frac{\partial M_r(a, b)}{\partial b} \approx \frac{r}{b^3} \{(ab - r)[M_r(a, b) - M_{r+1}(a, b)] + (r + 1)[M_{r+1}(a, b) - M_{r+2}(a, b)]\}. \quad (\text{B.6})$$

Since the partial derivative of $M_r(a, b)$ with respect to b requires the $(r + 2)^{\text{th}}$ moments in (B.6), the Jacobian matrix in (B.3) involves the first six order moments of the Johnson S_B distribution.

Appendix C Existence of saddlepoint

This proof is a refinement of the proof in Gordy (2002) and represents an extension from the standard CreditRisk⁺ model to the CR⁺-CBV model. Let \hat{t} be the solution of the saddlepoint equation $\kappa'_L(\hat{t}) = x$ [see (5.2)]. We would like to show that for $x > E[L]$, the solution \hat{t} lies within $(0, t^*)$, where $t^* = \min\{t_1^*, \dots, t_K^*, t_{K+1}^*, \dots, t_{K+M}^*\}$. Here, $t_k^*, k = 1, 2, \dots, K$, and $t_{K+m}^*, m = 1, 2, \dots, M$, are the roots of the equations obtained by setting the individual logarithmic terms in $\kappa_L(t)$ [see (2.11)] to be zero. That is,

$$\begin{aligned}1 - \delta_k P_k(t_k^*) &= 1 - \delta_k \sum_A p_A w_{Ak} (e^{\epsilon_A t_k^*} - 1) = 0, & k = 1, 2, \dots, K, \\ 1 - P_{K+m}(t_{K+m}^*) &= 1 - \sum_A p_A w_{A, K+m} (e^{\epsilon_A t_{K+m}^*} - 1) = 0, & m = 1, 2, \dots, M.\end{aligned} \quad (\text{C.1})$$

We first establish the upper bounds on $t_k^*, k = 1, 2, \dots, K$, and $t_{K+m}^*, m = 1, 2, \dots, M$, then derive the corresponding upper bound t^* for the saddlepoint \hat{t} . From (C.1), we observe that

$$\begin{aligned}P_k(t_k^*) &= 1/\delta_k, & k = 1, 2, \dots, K, \\ P_{K+m}(t_{K+m}^*) &= 1, & m = 1, 2, \dots, M.\end{aligned} \quad (\text{C.2})$$

We deduce from (2.9a,b) that each of $P_k(t), k = 1, 2, \dots, K$, and $P_{K+m}(t), m = 1, 2, \dots, M$, is an increasing and convex function over t . Since $P_k(0) = 0$ and $P_{K+m}(0) = 0$, we conclude that $t_k^* > 0$ and $t_{K+m}^* > 0$. Let $\mu_k = \sum_A p_A w_{Ak}, k = 1, 2, \dots, K$, by virtue of Jensen's inequality, we then have

$$\begin{aligned}P_k(t_k^*) &= \sum_A p_A w_{Ak} e^{\epsilon_A t_k^*} - \mu_k \geq \exp(t_k^* \sum_A p_A w_{Ak} \epsilon_A) - \mu_k, & k = 1, 2, \dots, K, \\ P_{K+m}(t_{K+m}^*) &= \sum_{k=1}^K \gamma_{mk} \sum_A p_A w_{Ak} e^{\epsilon_A t_{K+m}^*} - \sum_{k=1}^K \gamma_{mk} \mu_k \geq \sum_{k=1}^K \gamma_{mk} \exp(t_{K+m}^* \sum_A p_A w_{Ak} \epsilon_A) - \sum_{k=1}^K \gamma_{mk} \mu_k \\ &\geq \exp(t_{K+m}^* \sum_{k=1}^K \gamma_{mk} \sum_A p_A w_{Ak} \epsilon_A) - \sum_{k=1}^K \gamma_{mk} \mu_k, & m = 1, 2, \dots, M.\end{aligned} \quad (\text{C.3})$$

Substituting (C.2) into (C.3) and observing $t_k^* > 0$ and $t_{K+m}^* > 0$, the bounds of t_k^* and t_{K+m}^* are given by

$$\begin{aligned}0 < t_k^* &\leq \ln\left(\frac{1}{\delta_k} + \mu_k\right) / \sum_A p_A w_{Ak} \epsilon_A, & k = 1, 2, \dots, K, \\ 0 < t_{K+m}^* &\leq \ln\left(1 + \sum_{k=1}^K \gamma_{mk} \mu_k\right) / \sum_{k=1}^K \gamma_{mk} \sum_A p_A w_{Ak} \epsilon_A, & m = 1, 2, \dots, M.\end{aligned} \quad (\text{C.4})$$

Let $t^* = \min\{t_1^*, \dots, t_K^*, t_{K+1}^*, \dots, t_{K+M}^*\}$, which observes $t^* > 0$. The convergence strip of $\kappa_L(t)$ as $t \in (-\infty, t^*)$ contains the origin. Also, $\kappa_L(t)$ is strictly convex which implies $\kappa_L''(t) > 0$. On the other hand, when $t \rightarrow t^*$, one of the terms in the denominator in $\kappa_L'(t)$, either $1 - \delta_k P_k(t)$ or $1 - P_{K+m}(t)$, tends to zero, which dictates that $\lim_{t \rightarrow t^*} \kappa_L'(t) = \infty$. The mapping $\kappa_L'(t) : (-\infty, t^*) \rightarrow (0, \infty)$ is one-to-one and onto (bijection). Since $\kappa_L'(t)$ is increasing for all $t \in (-\infty, t^*)$, the solution \hat{t} of the saddlepoint equation $\kappa_L'(\hat{t}) = x$ should be less than t^* . When $x > E[L]$, the property $\kappa_L'(0) = E[L]$ implies that the solution \hat{t} lies within $(0, t^*)$. Other solutions of the saddlepoint equation may exist outside the range of $(-\infty, t^*)$, but these solutions are erroneous.