QUANTO LOOKBACK OPTIONS

Min Dai

Laboratory of Mathematics and Applied Mathematics, and
Department of Financial Mathematics, Peking University, China

Hoi Ying Wong

Department of Statistics, Chinese University of Hong Kong, Hong Kong

Yue Kuen Kwok

Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong

The lookback feature in a quanto option refers to the payoff structure where the terminal payoff of the quanto option depends on the realized extreme value of either the stock price or the exchange rate. In this paper, we study the pricing models of European and American lookback options with the quanto feature. The analytic price formulas for two types of European style quanto lookback options are derived. The success of the analytic tractability of these quanto lookback options depends on the availability of a succinct analytic representation of the joint density function of the extreme value and terminal value of the stock price and exchange rate. We also analyze the early exercise policies and pricing behaviors of the quanto lookback options with the American feature. The early exercise boundaries of these American quanto lookback options exhibit properties that are distinctive from other two-state American option models.

Key Words: lookback options, quanto feature, early exercise policies

1. INTRODUCTION

Lookback options are contingent claims whose payoff depends on the extremum value of the underlying asset price process realized over a specified period of time within the life of the option. The term "quanto" is an abbreviation for "quantity adjusted", and it refers to the feature where

Address correspondence to Yue Kuen Kwok, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong; e-mail: maykwok@ust.hk.

the payoff of an option is determined by the financial price or index in one currency but the actual payout is realized in another currency.

We examine the pricing models of European and American quanto lookback options whose payoff depends on the joint processes of the stock price and exchange rate. In our option valuation framework, we assume lognormal process for the underlying stock price and exchange rate, and continuous monitoring of the realized maximum of these two stochastic state variables. For the *joint quanto lookback options*, the lookback feature is applied on the stock price process and the exchange rate in the payoff is chosen to be the maximum of a pre-determined floor value and the terminal value at expiry. When the lookback feature is applied on the exchange rate process, this leads to the *maximum rate quanto lookback option*. Here, the exchange rate in the payoff is given by the realized maximum value over some monitoring period.

The pricing of lookback options poses interesting mathematical challenges. The analytic price formulas for European one-asset lookback options have been systematically derived by Goldman *et al.* (1979), and Conze and Viswanathan (1991). For two-state European lookback options, He *et al.* (1998) and Babsiri and Noel (1998) have obtained analytic expressions of the joint probability density functions of the extremum and terminal values of the prices of the underlying assets. However, due to the analytic complexity in their analytic expressions for the density functions, they did not proceed further in evaluating the discounted expectation integrals. Instead, they computed the lookback option prices via numerical integration of the discounted expectation integrals or Monte Carlo simulation.

In this paper, we derive the analytic price formulas for the above two types of European quanto lookback options under the lognormal assumption of the exchange rate and stock price processes. The success of the analytic tractability of these quanto lookback options relies on our derivation of a succinct representation of the joint density function of the extreme value and terminal value of the stock price and the exchange rate. In the derivation procedure, the standard quanto pre-washing techniques for dealing with quanto option models are used. With the availability of the closed form price formulas, we are able to comprehend various contributing factors to the value of these quanto lookback options.

The characteristics of the early exercise regions and optimal early exercise policies of American options on several risky assets are known to depend sensibly on the payoff structures of the options. Broadie and Detemple (1996) and Villeneuve (1999) provided some interesting results on the characterization of the early exercise regions of American extremum options and spread options.

Except for the perpetual American options with very simple payoff structures, like the perpetual Margrabe option and perpetual zero-strike maximum option [see Gerber and Shiu (1996)], it is not feasible to obtain analytic price formulas for multi-state American options. At best, we may obtain the analytic representation of the early exercise premium in terms of an integral that involves the exercise boundary function. The early exercise boundary is then solved via the solution of an integral equation.

It would be interesting to examine how the lookback feature interacts with the American early exercise feature. For example, one would expect early exercise to be delayed when the current asset value is close to the current realized extremum value. One example of an American option with lookback feature is the Russian option (perpetual American lookback option). Closed form price formulas of Russian options have been derived in several papers [Duffie and Harrison (1993); Shepp and Shiryaev (1993)]. Lai and Lim (2003) and Yu *et al.* (2001) examined the exercise boundaries of one-asset American lookback options. In this paper, we examine the optimal early exercise policy of quanto lookback options whose payoff depends on the stock price and exchange rate and the realized extremum value of one of the state variables.

In the next section, we summarize the quanto pre-washing techniques for dealing with the quanto feature in the pricing models, and present the probability density functions that involve the joint processes for the maximum value and the terminal value of the stock price and exchange rate. We then derive the analytic price formulas of the European style joint quanto lookback option and maximum rate quanto lookback option. In Section 3, we analyze the early exercise policies and pricing behaviors of these two types of quanto lookback options with the American feature. The properties of the optimal exercise boundaries are verified through numerical experiments. The paper is ended with conclusive remarks in the last section.

2. EUROPEAN QUANTO LOOKBACK OPTIONS

In this section, we derive the analytic price formulas of two types of European quanto lookback options, where the lookback feature is applied on the exchange rate or the stock price. The usual assumptions of the Black-Scholes option pricing framework are adopted in this paper. Let F_t denote the exchange rate at time t, which is defined as the domestic currency price of one unit of foreign currency. Let r_d and r_f denote the constant domestic and foreign riskless interest rates, respectively. Under the risk neutral measure, the stochastic process of F_t is assumed to be governed by

$$\frac{dF_t}{F_t} = (r_d - r_f)\ dt + \sigma_F\ dZ_F, \tag{2.1}$$

where σ_F is the volatility of F and dZ_F is the standard Wiener process. In the foreign currency world, the stochastic process for the risk neutralized stock price process S_t is assumed to follow

$$\frac{dS_t}{S_t} = (r_f - q)\ dt + \sigma_S\ dZ_S, \tag{2.2}$$

where σ_S and q are the volatility and dividend yield of S, respectively, and dZ_S is the standard Wiener process. By applying the standard quanto prewashing technique [see Dravid *et al.* for a thorough discussion of the technique], the risk neutralized drift rate of S_t in the domestic currency world is given by

$$\delta_S^d = r_f - q - \rho\sigma_S\sigma_F, \tag{2.3}$$

where ρ is the correlation coefficient between dZ_S and dZ_F, with $\rho\ dt = dZ_S\ dZ_F$.

We consider the pricing models of two types of European quanto lookback options whose terminal payoff functions in the domestic currency world are given by

(i) Quanto call option with maximum exchange rate

$$V_{max}(S, F, T) = F_{max}^{[T_0,T]}(S_T - K)^+, \tag{2.4a}$$

where $F_{max}^{[T_0,T]}$ is the realized maximum of the exchange rate F over the time period $[T_0, T]$, and K is the strike price in foreign currency. Here, $x^+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$.

(ii) Joint quanto fixed strike lookback call option

$$V_{joint}(S, F, T; F_c) = \max(F_c, F_T)(S_{max}^{[T_0,T]} - K)^+, \tag{2.4b}$$

where $S_{max}^{[T_0,T]}$ is the realized maximum of the stock price over the time period $[T_0, T]$, and F_c is some pre-specified constant exchange rate.

2.1 Quanto call option with maximum exchange rate

We assume that the current time lies within the period $[T_0, T]$, where the maximum value of the exchange rate is monitored continuously. For convenience, we take the current time to be the zeroth time so that $T_0 < 0 < T$. We define the following unit variance stochastic normal variables

$$X_t = \frac{1}{\sigma_S}\ln\frac{S_t}{S} \quad \text{and} \quad Y_t = \frac{1}{\sigma_F}\ln\frac{F_t}{F}, \quad t > 0, \tag{2.5}$$

where S and F are the current stock price and exchange rate, respectively. In the domestic currency world, the risk neutralized drift rates of X_t and Y_t are given by

$$\mu_X = \frac{r_f - q - \rho\sigma_S\sigma_F - \frac{\sigma_S^2}{2}}{\sigma_S} \quad \text{and} \quad \mu_Y = \frac{r_d - r_f - \frac{\sigma_F^2}{2}}{\sigma_F}, \tag{2.6}$$

respectively. In addition, we define the stochastic random variable M_t to be the logarithm of the normalized maximum value of the exchange rate over the future period $[0, t]$

$$M_t = \frac{1}{\sigma_F} \ln \frac{F_{max}^{[0,t]}}{F}. \tag{2.7}$$

Also, we denote the corresponding quantity for the realized maximum value over the earlier period $[T_0, 0]$ by $M_0 = \dfrac{1}{\sigma_F} \ln \dfrac{F_{max}^{[T_0,0]}}{F}$. In terms of M_t and X_t defined above, the terminal payoff of the quanto call option with maximum exchange rate can be expressed as

$$F_{max}^{[T_0,T]}(S_T - K)^+ = Fe^{\sigma_F \max(M_0, M_T)}(Se^{\sigma_S X_T} - K)^+. \tag{2.8}$$

The value of this European maximum rate quanto call at the current time is given by

$$V_{max} = e^{-r_d T} \int_{-\infty}^{\infty} \int_0^{\infty} Fe^{\sigma_F \max(M_0, m)}(Se^{\sigma_S x} - K)^+ f_{max}(x, m, T) \; dmdx. \tag{2.9}$$

Here, the joint density function of X_T and M_T, $f_{max}(x, m, T)$, is given by

$$f_{max}(x, m, T) = \frac{\partial G_{max}}{\partial m}(x, m, T), \tag{2.10a}$$

where $G_{max}(x, m, T)$ is the distribution function defined by

$$G_{max}(x, m, T) \; dx = P(X \in dx, M \leq m). \tag{2.10b}$$

We write $g_{max}(x, y, T; m)$ as the joint density function of X_T and Y_T with an absorbing barrier m that is greater than y^+, that is,

$$g_{max}(x, y, T; m) \; dxdy = P(X \in dx, Y \in dy, M \leq m). \tag{2.11a}$$

Note that $G_{max}(x, y, T)$ and $g_{max}(x, y, T; m)$ are related by

$$G_{max}(x, m, T) = \int_{-\infty}^{m} g_{max}(x, y, T; m) \; dy. \tag{2.11b}$$

By applying the reflection principle for Brownian processes together with effecting change of measure via the Girsanov Theorem, one can obtain [Heynen and Kat (1994)]

$$g_{max}(x, y, T; m) = \left\{\phi_2(\widetilde{x}, \widetilde{y}, T; \rho) - e^{2\mu_Y m}\phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, T; \rho)\right\} \mathbf{1}_{\{m > y^+\}}, \tag{2.12a}$$

where $\mathbf{1}_A$ is the indicator function with respect to A. Further, we let $\widetilde{x} = x - \mu_X T$ and $\widetilde{y} = y - \mu_Y T$, where μ_X and μ_Y are defined in Eq. (2.6), ρ is the correlation coefficient between dZ_S and dZ_F, and $\phi_2(\widetilde{x}, \widetilde{y}, T; \rho)$ is the bivariate normal density function with zero means and unit variance rates as defined by

$$\phi_2(\widetilde{x}, \widetilde{y}, T; \rho) = \frac{1}{2\pi T\sqrt{1-\rho^2}} \exp\left(-\frac{\widetilde{x}^2 - 2\rho\widetilde{x}\widetilde{y} + \widetilde{y}^2}{2(1-\rho^2)T}\right). \tag{2.12b}$$

As a remark, He *et al.* (1998) obtained an alternative analytic representation for $f_{max}(x, m, T)$ through a tedious procedure (see Theorem 2.3 in their paper). Using their analytic result, it is almost insurmountable to obtain an analytic price formula for any two-state semi-lookback option.

Theorem 1

The analytic price formula for the European maximum exchange rate quanto lookback call option is given by

$$\begin{aligned} V_{max} = {} & F_{max}^{[T_0,0]} \left[e^{-(r_d - \delta_S^d)T} S N_2(d_1, -e_1; -\rho) - e^{-r_d T} K N_2(d_2, -e_2; -\rho)\right] \\ & + F \left[e^{-qT} S N_2(\widehat{d}_1, \widehat{e}_1; \rho) - e^{-r_f T} K N_2(\widehat{d}_2, \widehat{e}_2; \rho)\right] \\ & + F\sigma_F \int_{M_0}^{\infty} e^{(\sigma_F + 2\mu_Y)m} \left[e^{-(r_d - \delta_S^d)T} e^{2\rho\sigma_S m} S N_2(\widetilde{d}_1, -\widetilde{e}_1; -\rho) \right. \\ & \qquad \left. - e^{-r_d T} K N_2(\widetilde{d}_2, -\widetilde{e}_2; -\rho)\right] dm, \end{aligned} \tag{2.13a}$$

where $N_2(x, y; \rho)$ is the standard bivariate distribution function with zero means and unit variances, and

$$\begin{aligned} d_2 &= \frac{\ln\frac{S}{K} + \mu_X \sigma_S T}{\sigma_S\sqrt{T}}, & e_2 &= \frac{\ln\frac{F}{F_{max}^{[T_0,0]}} + \mu_Y \sigma_F T}{\sigma_F\sqrt{T}}, \\ d_1 &= d_2 + \sigma_S\sqrt{T}, & e_1 &= e_2 + \rho\sigma_S\sqrt{T}, \\ \widehat{d}_2 &= d_2 + \rho\sigma_F\sqrt{T}, & \widehat{e}_2 &= e_2 + \sigma_F\sqrt{T}, \\ \widehat{d}_1 &= \widehat{d}_2 + \sigma_S\sqrt{T}, & \widehat{e}_1 &= \widehat{e}_2 + \rho\sigma_S\sqrt{T}, \\ \widetilde{d}_2 &= d_2 + \frac{2\rho m}{\sqrt{T}}, & \widetilde{e}_2 &= \frac{m + \mu_Y T}{\sqrt{T}}, \\ \widetilde{d}_1 &= \widetilde{d}_2 + \sigma_S\sqrt{T}, & \widetilde{e}_1 &= \widetilde{e}_2 + \rho\sigma_S\sqrt{T}. \end{aligned} \tag{2.13b}$$

The proof of Theorem 1 is presented in Appendix A. The price formula V_{max} consists of three terms. The first term gives the contribution to the option value that is conditional on

$F_{max}^{[T_0,T]} = F_{max}^{[T_0,0]}$ (that is, no updated realized maximum value on F over the future period $[0,T]$) and $S_T \geq K$. The second term corresponds to the scenario where $F_{max}^{[T_0,0]} < F_{max}^{[T_0,T]}$ and $S_T \geq K$. The last term gives the rollover bonus value of potential upward adjustment on the realized value of the exchange rate whenever a new maximum value of exchange rate is realized.

Zero derivative condition at $F = F_{max}^{[T_0,0]}$

When the current value of exchange rate F happens to be at the realized maximum value $F_{max}^{[T_0,0]}$, should the option price be insensitive to infinitesimal changes in $F_{max}^{[T_0,0]}$? Mathematically, this is equivalent to ask whether $\left.\frac{\partial V_{max}}{\partial M_0}\right|_{M_0=0} = 0$. This result can be shown to be true by computing $\frac{\partial V_{max}}{\partial M_0}$ directly using the integral representation of V_{max} in Eq. (2.9) (see Appendix B).

The zero derivative condition at $F = F_{max}^{[T_0,0]}$ is important in the design of the finite difference algorithm for the numerical solution of the quanto lookback option. This is because the full prescription of the boundary conditions of the option model is required in the construction of the finite difference scheme.

2.2 Joint quanto fixed strike lookback call option

For the joint quanto fixed strike lookback call option, the maximum value is monitored continuously on the stock price process S_t. Accordingly, we define the stochastic random variable U_t to be the logarithm of the normalized maximum value over the period $[0,t]$ of the stock price, that is,

$$U_t = \frac{1}{\sigma_S} \ln \frac{S_{max}^{[0,t]}}{S}, \tag{2.14}$$

and denote the corresponding quantity for the realized maximum value over the earlier period $[T_0, 0]$ by $U_0 = \frac{1}{\sigma_S} \ln \frac{S_{max}^{[T_0,0]}}{S}$. In terms of U_T, U_0 and Y_T, the terminal payoff of the joint quanto lookback call can be expressed as

$$\begin{aligned} &\max(F_c, F_T)(S_{max}^{[T_0,T]} - K)^+ \\ &= \begin{cases} F_c(Se^{\sigma_S U_0} - K)^+ & \text{if } F_c \geq F_T \text{ and } S_{max}^{[T_0,0]} \geq S_{max}^{[0,T]} \\ F_c(Se^{\sigma_S U_T} - K)^+ & \text{if } F_c \geq F_T \text{ and } S_{max}^{[0,T]} > S_{max}^{[T_0,0]} \\ Fe^{\sigma_F Y_T}(Se^{\sigma_S U_0} - K)^+ & \text{if } F_T > F_c \text{ and } S_{max}^{[T_0,0]} \geq S_{max}^{[0,T]} \\ Fe^{\sigma_F Y_T}(Se^{\sigma_S U_T} - K)^+ & \text{if } F_T > F_c \text{ and } S_{max}^{[0,T]} > S_{max}^{[T_0,0]} \end{cases}. \end{aligned} \tag{2.15}$$

By following similar derivation procedure as that for $G_{max}(x, m, T)$ in Eq. (2.11b), the density function of the joint processes of Y_T and U_T is given by $\frac{\partial G_{joint}}{\partial u}(y, u, T)$, where

$$G_{joint}(y, u, T) = \int_{-\infty}^{u} g_{joint}(x, y, T; u)\; dx \tag{2.16}$$

and

$$g_{joint}(x, y, T; u) = \{\phi_2(\widetilde{x}, \widetilde{y}, T; \rho) - e^{2\mu_X u}\phi_2(\widetilde{x} - 2u, \widetilde{y} - 2\rho u, T; \rho)\}\mathbf{1}_{\{u > x^+\}}. \tag{2.17}$$

The form of the analytic price formula of the joint quanto lookback option depends on the sign of $S_{max}^{[T_0,0]} - K$. When the option is currently in-the-money or at-the-money (corresponding to $S_{max}^{[T_0,0]} - K \geq 0$), it is guaranteed to expire in-the-money. On the other hand, when $S_{max}^{[T_0,0]} - K < 0$, the option will expire out-of-the-money when $S_{max}^{[T_0,0]} > S_{max}^{[0,T]}$. We derive the price formula of the joint quanto lookback call under the following two cases:

1. $S_{max}^{[T_0,0]} < K$ (currently out-of-the-money)

$$\begin{aligned} V_{joint} = e^{-r_d T} \Bigg\{ & F_c \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} (Se^{\sigma_S u} - K)\frac{\partial G_{joint}}{\partial u}(y, u, T)\; dudy \\ & + F \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} e^{\sigma_F y}(Se^{\sigma_S u} - K)\frac{\partial G_{joint}}{\partial u}(y, u, T)\; dudy \Bigg\}. \end{aligned} \tag{2.18}$$

2. $S_{max}^{[T_0,0]} \geq K$ (currently in-the-money or at-the-money)

$$\begin{aligned} V_{joint} = e^{-r_d T} \Bigg\{ & F_c(S_{max}^{[T_0,0]} - K) \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{0}^{U_0} \frac{\partial G_{joint}}{\partial u}(y, u, T)\; dudy \\ & + \; F_c \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{U_0}^{\infty} (Se^{\sigma_S u} - K)\frac{\partial G_{joint}}{\partial u}(y, u, T)\; dudy \\ & + \; F(S_{max}^{[T_0,0]} - K) \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F_0}}^{\infty} \int_{0}^{U_0} e^{\sigma_F y}\frac{\partial G_{joint}}{\partial u}(y, u, T)\; dudy \\ & + F \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F_0}}^{\infty} \int_{U_0}^{\infty} e^{\sigma_F y}(Se^{\sigma_S u} - K)\frac{\partial G_{joint}}{\partial u}(y, u, T)\; dudy \Bigg\}. \end{aligned} \tag{2.19}$$

By performing the tedious integration precedures (some technical details are presented in Appendix C), we obtain the analytic expressions for V_{joint} as depicted in Theorem 2.

Theorem 2

The analytic price formula of the joint quanto fixed strike lookback call option is given by

1. $S_{max}^{[T_0,0]} < K$

$$\begin{aligned} V_{joint} &= e^{-r_d T} F_c \left[S e^{\delta_S^d T} N_2(d_1, -f_1; -\rho) - K N_2(d_2, -f_2; -\rho) \right] \\ &+ \ e^{-r_f T} F \left[S e^{(r_f - q)T} N_2(\widehat{d}_1, \widehat{f}_1; \rho) - K N_2(\widehat{d}_2, \widehat{f}_2; \rho) \right] \\ &+ \ e^{-r_d T} \int_{\frac{1}{\sigma_S} \ln \frac{K}{S}}^{\infty} \sigma_S e^{(2\mu_X + \sigma_S)u} \\ &\qquad \left[F_c S N_2(-\overline{e}_2, -\widetilde{f}_2; \rho) + e^{(r_d - r_f)T} e^{2\sigma_F \rho u} F S N_2(-\overline{e}_1, \widetilde{f}_1; -\rho) \right] du, \end{aligned} \tag{2.20}$$

where $d_1, d_2, \widehat{d}_1$ and $\widehat{d}_2$ are defined in Eq. (2.13), and

$$\begin{aligned} f_2 &= \frac{\ln \frac{F}{F_c} + \mu_Y \sigma_F T}{\sigma_F \sqrt{T}}, & f_1 &= f_2 + \rho \sigma_S \sqrt{T}, \\ \widehat{f}_2 &= f_2 + \sigma_F \sqrt{T}, & \widehat{f}_1 &= \widehat{f}_2 + \rho \sigma_S \sqrt{T}, \\ \widetilde{f}_2 &= f_2 + \frac{2\rho u}{\sqrt{T}}, & \widetilde{f}_1 &= \widetilde{f}_2 + \sigma_F \sqrt{T}, \\ \overline{e}_2 &= \frac{u + \mu_X T}{\sqrt{T}}, & \overline{e}_1 &= \overline{e}_2 + \rho \sigma_F \sqrt{T}. \end{aligned} \tag{2.21}$$

The first two terms in Eq. (2.20) resemble closely to the price formula for the joint quanto European call option (Kwok and Wong, 2000), while the last term can be interpreted as the rollover bonus value for potential upward adjustment on the realized maximum value of the stock price.

2. $S_{max}^{[T_0,0]} \geq K$

$$\begin{aligned} V_{joint} &= (S_{max}^{[T_0,0]} - K) \left[e^{-r_d T} F_c N_2(-d_2^M, -f_2; \rho) + F e^{-r_f T} N_2(-\widehat{d}_2^M, \widehat{f}_2; -\rho) \right] \\ &+ \ e^{-r_d T} F_c \left[S e^{\delta_S^d T} N_2(d_1^M, -f_1; -\rho) - K N_2(d_2^M, -f_2; -\rho) \right] \\ &+ \ e^{-r_f T} F \left[S e^{(r_f - q)T} N_2(\widehat{d}_1^M, \widehat{f}_1; \rho) - K N_2(\widehat{d}_2^M, \widehat{f}_2; \rho) \right] \\ &+ \ e^{-r_d T} \int_{U_0}^{\infty} \sigma_S e^{(2\mu_X + \sigma_S)u} \\ &\qquad \left[F_c S N_2(-\overline{e}_2, -\widetilde{f}_2; \rho) + e^{(r_d - r_f)T} e^{2\sigma_F \rho u} F S N_2(-\overline{e}_1, \widetilde{f}_1; -\rho) \right] du, \end{aligned} \tag{2.22}$$

where $f_1, f_2, \widehat{f}_1, \widehat{f}_2, \widetilde{f}_1, \widetilde{f}_2, \overline{e}_1$ and $\overline{e}_2$ are defined in Eq. (2.21), and

$$\begin{aligned} d_2^M &= \frac{\ln \frac{S}{S_{max}^{[T_0,0]}} + \mu_X \sigma_S T}{\sigma_S \sqrt{T}}, & d_1^M &= d_2^M + \sigma_S \sqrt{T}, \\ \widehat{d}_2^M &= d_2^M + \rho \sigma_F \sqrt{T}, & \widehat{d}_1^M &= \widehat{d}_2^M + \sigma_S \sqrt{T}. \end{aligned} \tag{2.23}$$

The first term corresponds to $S_{max}^{[T_0,0]} > K$ and conditional on no updated realized maximum value on S over the future period $[0, T]$. The second, third and fourth terms are similar to those in Eq. (2.20) except that the strike price K is replaced by $S_{max}^{[T_0,0]}$.

3. AMERICAN QUANTO LOOKBACK OPTIONS

In this section, we would like to analyze the behaviors of the early exercise policies of two types of American quanto lookback options, whose exercise payoffs are defined in Eqs. (2.4a,b). To proceed with the analysis, we first present the linear complementarity formulation of the pricing models, then examine some monotonicity properties of the price functions and the exercise boundaries.

3.1 American maximum exchange rate quanto call

Let $V_M(S, F, \tau; F_{max})$ denote the value of an American maximum rate quanto call option in domestic currency, where τ is the time to expiry and F_{max} is the realized maximum exchange rate up to the current time. By following the variational inequality approach of deriving pricing models for American style path dependent options [see Wilmott *et al.* (1993) for reference], the price function $V_M(S, F, \tau; F_{max})$, if exists, solves the following linear complementarity formulation:

$$
\begin{aligned}
&\frac{\partial V_M}{\partial \tau} - LV_M \geq 0, \quad V_M \geq F_{max} \max(S-K, 0), \\
&\left(\frac{\partial V_M}{\partial \tau} - LV_M\right)[V_M - F_{max}\max(S-K,0)] = 0, \quad S > 0, 0 < F < F_{max}, \tau \in [0, T], \\
&\left.\frac{\partial V_M}{\partial F_{max}}\right|_{F=F_{max}} = 0 \quad \text{and} \quad V_M(S, F, 0; F_{max}) = F_{max}\max(S-K, 0),
\end{aligned} \tag{3.1}
$$

where L is the differential operator defined by

$$
\begin{aligned}
L = {} & \frac{\sigma_S^2}{2} S^2 \frac{\partial^2}{\partial S^2} + \rho \sigma_S \sigma_F S F \frac{\partial^2}{\partial S \partial F} + \frac{\sigma_F^2}{2} F^2 \frac{\partial^2}{\partial F^2} \\
& + \delta_S^d S \frac{\partial}{\partial S} + (r_d - r_f) F \frac{\partial}{\partial F} - r_d.
\end{aligned} \tag{3.2}
$$

In the continuation region, V_M satisfies

$$
\frac{\partial V_M}{\partial \tau} - LV_M = 0 \qquad \text{and} \qquad V_M > F_{max}\max(S-K, 0); \tag{3.3a}
$$

while in the exercise region, V_M satisfies

$$
\frac{\partial V_M}{\partial \tau} - LV_M > 0 \qquad \text{and} \qquad V_M = F_{max}\max(S-K, 0). \tag{3.3b}
$$

The Neumann boundary condition $\left.\frac{\partial V_M}{\partial F_{max}}\right|_{F=F_{max}} = 0$ arises due to the property that if the current exchange rate F happens to be at its maximum value so far, the probability that the current exchange rate remains to be the maximum at later times is essentially zero [see Goldman *et al.* (1979)]. Hence, the option value V_M should be insensitive to infinitesimal change in F_{max}.

The existence of smooth solution (degree of smoothness as required in our subsequent analysis) to the above linear complementarity formulation may not be too easy to establish. Barles (1997) showed that continuous viscosity solution exists for the one-asset lookback options. Similar technique can be used to establish the existence of continuous viscosity solution for the above pricing model of the American maximum rate quanto call. The rigorous proof of the existence of sufficiently smooth solution to Eq. (3.1) is beyond the scope of our paper. Interested readers may consult Friedman's book (1982) about the regularity analysis of solution to linear complementarity formulation. By assuming that the sufficiently smooth price function $V_M(S, F, \tau; F_{max})$ exists, we then *formally* analyze the analytic behaviors of the critical stock price at which it is optimal to exercise the American maximum rate quanto lookback call option.

In the above linear complementarity formulation, F_{max} appears apparently as a parameter. In the subsequent analysis, it is more convenient to use F_{max} as the numeraire and consider the monotonicity properties on the normalized price function

$$U_M(S, \xi, \tau) = V_M(S, F, \tau; F_{max})/F_{max}, \quad \text{where} \quad \xi = F/F_{max}. \tag{3.4}$$

We write the critical stock price as $S_M^*(\xi, \tau)$ with its dependence on ξ and τ.

Proposition 3

The normalized price function $U_M(S, \xi, \tau)$ satisfies the following monotonicity properties with respect to τ and ξ.

(a) $\dfrac{\partial U_M}{\partial \tau} \geq 0$

(b) $\dfrac{\partial U_M}{\partial \xi} \geq 0$.

The proof of Proposition 3 is presented in Appendix D. Similar to other American call options, the continuation region and the exercise region of V_M correspond to $S < S_M^*(\xi, \tau)$ and $S \geq S_M^*(\xi, \tau)$, respectively. Using the above monotonicity properties on U_M, we are able to obtain the following analytic properties on $S_M^*(\xi, \tau)$.

Theorem 4

Consider the optimal exercise boundary $S_M^*(\xi, \tau)$.

(a) At time close to expiry, $\tau \to 0^+$, we have

$$S_M^*(\xi, 0^+) = \begin{cases} \max\left(1, \dfrac{r_d}{r_d - \delta_S^d}\right) K & \text{if } r_d > \delta_S^d \\ \infty & \text{if } r_d \leq \delta_S^d \end{cases}. \tag{3.5}$$

(b) $S_M^*(\xi, \tau)$ is monotonically increasing with respect to τ and ξ.

The proof of Theorem 4 is presented in Appendix E. From Theorem 4, we conclude that when $r_d \leq \delta_S^d$, it is never optimal to exercise the American maximum exchange rate quanto call prematurely. By virtue of the monotonicity property of the critical stock price on ξ, $S_M^*(F, \tau; F_{max})$ would increase with increasing F for fixed F_{max} and decrease with increasing F_{max} for fixed F.

We performed numerical calculations to compute the exercise boundaries so as to the verify the results obtained in Theorem 4. Figure 1 shows the exercise boundaries of an American maximum exchange rate quanto call option at different times to expiry τ. The parameter values of the option model are $r_d = 0.05, r_f = 0.05, q = 0.02, \sigma_S = 0.2, \sigma_F = 0.2, \rho = 0.5, K = 1$, with $\dfrac{r_d}{r_d - \delta_S^d} K = 1.25$. The monotonicity properties on $S_M^*(\xi, \tau)$ with respect to ξ and τ are clearly revealed in Figure 1. The exercise region and the continuation region are on the right side and the left side of the exercise boundary, respectively. It is interesting to observe that $S_M^*(\xi, \tau)$ changes abruptly at some threshold level of ξ. When ξ increases beyond this τ-dependent threshold level, $S_M^*(\xi, \tau)$ increases quite substantially implying that the holder will wait for much significant increase in stock price in order to exercise the maximum rate quanto lookback call option. In particular, when F becomes close to F_{max}, $S_M^*(\xi, \tau)$ becomes exceedingly large. This is reasonable since it is much likely that a higher value of F_{max} will be realized later so the option holder should restrain from exercising the option prematurely.

The theoretical analysis of the monotonicity property of $V_M(S, F, \tau; \rho)$ with respect to the correlation coefficient ρ is not straightforward, due to the presence of ρ in both the covariance term $\rho\sigma_S\sigma_F SF \dfrac{\partial^2 V_M}{\partial S \partial F}$ and the drift term $\delta_S^d S \dfrac{\partial V_M}{\partial S}$. Since the drift term is expected to predominate over the covariance term and δ_S^d is a decreasing function of ρ, the option value $V_M(S, F, \tau; \rho)$ would be expected to be a decreasing function of ρ. Actually, similar monotonicity behavior on ρ is observed in other quanto call options [Kwok and Wong (2000)]. In all our wide range of numerical experiments that were performed to testify this monotonicity property, we observed that $V_M(S, F, \tau; \rho)$ always appears to be a monotonically decreasing function of ρ. In Figure 2, we show the result of a typical calculation where $V_M(S, F, \tau; \rho)$ decreases monotonically with increasing ρ. The parameter values used in the calculation are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, T = 0.1$ and $K = S = F = F_{max} = 1$.

3.2 American joint quanto fixed strike lookback call option

Let $V_J(S, F, \tau; S_{max})$ denote the value of an American joint quanto fixed strike lookback call option in domestic currency, where S_{max} is the realized maximum value of the stock price up to the current time. The price function $V_J(S, F, \tau; S_{max})$, if exists, solves the following linear complementarity formulation:

$$\frac{\partial V_J}{\partial \tau} - LV_J \geq 0, \qquad V_J \geq \max(F, F_c)\max(S_{max} - K, 0),$$
$$\left(\frac{\partial V_J}{\partial \tau} - LV_J\right)[V_J - \max(F, F_c)\max(S_{max} - K, 0)] = 0, \quad F > 0, 0 < S < S_{max}, \tau \in [0, T],$$
$$\left.\frac{\partial V_J}{\partial S_{max}}\right|_{S=S_{max}} = 0 \quad \text{and} \quad V_J(S, F, 0; S_{max}) = \max(F, F_c)\max(S_{max} - K, 0), \tag{3.6}$$

where L is the differential operator defined in Eq. (3.2). In the continuation region, V_J satisfies

$$\frac{\partial V_J}{\partial \tau} - LV_J = 0 \quad \text{and} \quad V_J > \max(F, F_c)\max(S_{max} - K, 0); \tag{3.7a}$$

while in the exercise region, V_J satisfies

$$\frac{\partial V_J}{\partial \tau} - LV_J > 0 \quad \text{and} \quad V_J = \max(F, F_c)\max(S_{max} - K, 0). \tag{3.7b}$$

Gerber and Shiu (1996) showed that the exercise boundary of an American option on the maximum of two stock prices with zero strike consists of two branches. When the two stock prices are close in value, the holder of this American option should delay premature exercise. This is because the advantage of choosing the maximum of the two stock prices is not distinctive when the stock prices are about the same value. Only when either one of the stock prices is significantly higher than the other should the American option holder chooses to exercise. Under such scenario, the chance of regret of premature exercise would be low.

Due to the presence of the factor $\max(F, F_c)$ in the payoff function, the exercise boundary of an American joint quanto lookback call would be expected to consist of two branches: $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$. Obviously, early exercise is advantageous only when $S_{max} > K$, that is, the option is currently in-the-money. When $S_{max} > K$ but the value of F is close to the predetermined constant F_c, the holder should delay premature exercise since the advantage of taking the maximum of F and F_c is not significant. The chance of regret of early exercise is low only when F is sufficiently above F_c or below F_c. In the F-τ plane, conditional on $S_{max} > K$, the continuation region is bounded by the two branches of the exercise boundary: $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$. When $F \geq F^*_{up}$ or $F \leq F^*_{low}$, it becomes optimal to exercise the American joint quanto lookback call. Therefore, one part of the exercise region is to the right side of the branch $F^*_{up}(S, \tau; S_{max})$ and the other part is to the left of $F^*_{low}(S, \tau; S_{max})$.

We performed numerical calculations to compute the early exercise boundary of the American joint quanto lookback call. In Figure 3, we show the plots of the two branches of the exercise boundary corresponding to different pairs of values of S and S_{max}. The parameter values used in the calculations are $r_d = 0.05, r_f = 0.05, q = 0.02, \sigma_S = 0.2, \sigma_F = 0.2, \rho = 0.5, F_c = 1$ and $K = 1$. The two branches $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ both originate from $F = F_c$ at $\tau \to 0^+$. We observe that $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ are, respectively, monotonically increasing and decreasing with respect to τ. For a fixed value of τ, F^*_{up} is monotonically decreasing with respect to S_{max} (S is fixed) but monotonically increasing with respect to S (S_{max} is fixed). The corresponding monotonicity properties on F^*_{low} are reverse to those on F^*_{up}. These monotonicity properties can be explained by intuitive arguments relating to the chance of regret of premature exercise. The chance of regret decreases with increasing value of S_{max} (option being deeper in-the-money) and decreasing value of S (less chance to realize a new maximum value of the stock price in the future).

Furthermore, the exercise boundaries exhibit properties that depend sensibly on the ratio of S_{max}/S. When S_{max}/S is quite close to 1, it may occur that the exercise boundaries tend to level horizontally at sufficiently large or small value of F, indicating that it is never optimal to exercise at any exchange rate F when the time to expiry τ is beyond certain threshold value. In Figure 3, such phenomena are revealed by the behaviors of the solid boundary curves and dashed boundary curves that correspond to $S_{max}/S = 1.35$ and $S_{max}/S = 1.30$, respectively. However, when S_{max}/S is sufficiently large, $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ are defined for all τ, like that shown by the dotted boundary curves corresponding to $S_{max}/S = 5.0$. In this case, it is always optimal for the holder to exercise the option at any τ when sufficiently high or low value of F is reached. From financial intuition, when S_{max}/S is quite close to 1, the option holder may choose not to exercise at any value of F when the expiration date is sufficiently distant from the current time with the view that a new maximum value on S may be realized. However, when S_{max}/S is relatively large, the probability that a new S_{max} being realized is small. Accordingly, there does not exist some threshold value of τ beyond which the option holder never exercises optimally. In Appendix F, we present the theoretical arguments explaining the above phenomena on the exercise policies for the American joint quanto lookback call.

In Figure 4, we plot the option value of the American joint quanto lookback call at different times to expiry τ. We choose $S = 1$ and $S_{max} = 1.35$, and other parameters of the option model are identical to those used in Figure 3. The intrinsic value, $\max(F, F_c) \max(S_{max} - K, 0)$, of the lookback call is represented by the dotted horizontal line and inclined line. It is observed that the option value curves corresponding to $\tau = 0.25$ and $\tau = 0.5$ intersect tangentially the intrinsic value lines at F^*_{up} above F_c and at F^*_{low} below F_c, but not so for the option value curve corresponding to $\tau = 1.0$. Hence, when $\tau = 1$, it is never optimal to exercise at any level of F. This is consistent with the observation in Figure 3 that the horizontal line $\tau = 1$ always lies in the continuation region corresponding to the case $S = 1.0$ and $S_{max} = 1.35$. Also, the option value is seen to be monotonically increasing with respect to τ.

Some of the properties of the exercise policy and exercise boundary of the American joint quanto lookback call option are stated in Theorem 5, the proof of which is presented in Appendix G.

Theorem 5

The exercise boundary of the American joint quanto fixed strike lookback call option in the F-τ plane consists of two branches: $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$, where

$$F^*_{up}(S, \tau; S_{max}) = \inf\{F > F_c : (S, F, \tau; S_{max}) \in \mathcal{E}\}$$
$$F^*_{low}(S, \tau; S_{max}) = \sup\{F < F_c : (S, F, \tau; S_{max}) \in \mathcal{E}\}.$$

Here, we use $\mathcal{E}$ to denote the exercise region and follow the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. For fixed values of τ, S and S_{max}, conditional on $S_{max} > K$, the option should be optimally exercised when $F \geq F^*_{up}$ or $F \leq F^*_{low}$. The continuation region lies within $F^*_{low}(S, \tau; S_{max})$ and $F^*_{up}(S, \tau; S_{max})$. The two branches of the exercise boundary intersect at $F = F_c$ at $\tau \to 0^+$. At time close to expiry, conditional on $S_{max} > K$, the option should be optimally exercised for any exchange rate F other than F_c.

4. CONCLUSION

The analytic price formulas of two types of European quanto lookback options have been derived. The analytic tractability of these two-state lookback option models has been extended via the use of a succinct analytic representation of the density function of the joint process of the extremal value and terminal value of the exchange rate and stock price. The price formulas help provide the financial interpretation of the contributing factors to the value of the European quanto lookback option.

We have also analyzed the characterization of the exercise boundaries and pricing behaviors of these two types of quanto lookback options with the early exercise privilege. For the American maximum exchange rate quanto call, the critical stock price $S_M^*(F, \tau; F_{max})$ at which it is optimal to exercise the option is seen to be monotonically increasing with respect to time to expiry τ and exchange rate F (for fixed realized maximum exchange rate F_{max}). We show that it is never optimal to exercise the maximum exchange rate quanto call if the effective dividend yield of the foreign stock in domestic currency world is non-positive. Also, when F comes close to F_{max}, it becomes much less likely to exercise the option prematurely. For the American joint quanto fixed strike lookback call, the exercise boundary consists of two branches. Conditional on the option being in-the-money (current realized maximum stock price S_{max} is higher than the strike price K), it is optimal to exercise the option only when the exchange rate F is either sufficiently above or below the predetermined constant exchange rate F_c. At time right before expiry, it is optimal to exercise the American joint quanto lookback call at any level of exchange rate F other than F_c. These results add new insights into the understanding of the characterization of the early exercise policies of the general class of multi-asset American options.

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APPENDIX A: PROOF OF THEOREM 1

From Eqs. (2.9), (2.10a,b), (2.11a,b) and (2.12a,b), we obtain

$$V_{max} = e^{-r_d T} F e^{\sigma_F M_0} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_0^{M_0} (Se^{\sigma_S x} - K) \frac{\partial G_{max}}{\partial m}(x, m, T)\ dmdx$$
$$+\ e^{-r_d T} F \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{M_0}^{\infty} e^{\sigma_F m}(Se^{\sigma_S x} - K) \frac{\partial G_{max}}{\partial m}(x, m, T)\ dmdx. \qquad (A1)$$

By performing the inner integration with respect to m, the first integral can be expressed as

$$I_1 = e^{-r_d T} F e^{\sigma_F M_0} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{-\infty}^{M_0} (Se^{\sigma_S x} - K)$$
$$[\phi_2(\widetilde{x}, \widetilde{y}, T; \rho) - e^{2\mu_Y M_0} \phi_2(\widetilde{x} - 2\rho M_0, \widetilde{y} - 2M_0, T; \rho)]\ dydx. \qquad (A2)$$

For the second term in Eq. (A1), we consider the second term in $\dfrac{\partial G_{max}}{\partial m}$ and apply parts integration to obtain

$$-\int_{M_0}^{\infty} e^{\sigma_F m} \frac{\partial}{\partial m} \left[\int_{-\infty}^{m} e^{2\mu_Y m} \phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, T; \rho)\ dy \right] dm$$
$$= e^{\sigma_F M_0} \int_{-\infty}^{M_0} e^{2\mu_Y M_0} \phi_2(\widetilde{x} - 2\rho M_0, \widetilde{y} - 2M_0, T; \rho)\ dy$$
$$-\lim_{m\to\infty} e^{\sigma_F m} \int_{-\infty}^{m} e^{2\mu_Y m} \phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, T; \rho)\ dy$$
$$+\ \sigma_F \int_{M_0}^{\infty} e^{(2\mu_Y + \sigma_F)m} \int_{-\infty}^{m} \phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, T; \rho)\ dydm. \qquad (A3)$$

Note that the second term in I_1 [see Eq. (A2)] cancels with the double integral arising from the first term in Eq. (A3). The second term in Eq. (A3) can be shown to be zero even with the presence of the two exponential factors $e^{\sigma_F m}$ and $e^{2\mu_Y m}$. One can show that the negative quadratic terms in m in the exponent of the exponential function associated with ϕ_2 causes the term tending to zero as $m \to \infty$ at a rate faster than the growth of the two exponential factors. By observing this cancellation, Eq. (A1) can be expressed as

$$V_{max} = e^{-r_d T} F \left[e^{\sigma_F M_0} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{-\infty}^{M_0} (Se^{\sigma_S x} - K)\phi_2(\widetilde{x}, \widetilde{y}, T; \rho)\ dydx \right.$$
$$+ \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{M_0}^{\infty} e^{\sigma_F m}(Se^{\sigma_S x} - K)\phi_2(\widetilde{x}, \widetilde{m}, T; \rho)\ dmdx$$
$$+ \sigma_F \int_{M_0}^{\infty} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_{-\infty}^{m} e^{(\sigma_F + 2\mu_Y)m}(Se^{\sigma_S x} - K)$$
$$\left. \phi_2(\widetilde{x} - 2\rho m, \widetilde{y} - 2m, T; \rho)\ dydxdm \right], \qquad (A4)$$

where $\widetilde{m} = m - \mu_Y T$. Here, the first two integrals are expressible in terms of $N_2(\cdot\,,\cdot\,;\rho)$ while the last integral can be simplified to become a single integral with the integrand involving $N_2(\cdot\,,\cdot\,;\rho)$.

APPENDIX B: PROOF OF THE ZERO DERIVATIVE CONDITION AT $F = F_{max}^{[T_0,0]}$

By differentiating V_{max} in Eq. (A1) with respect to M_0, we obtain

$$\begin{aligned}\frac{\partial V_{max}}{\partial M_0} = e^{-r_d T} F \Bigg[& \sigma_F e^{\sigma_F M_0} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \int_0^{M_0} (S e^{\sigma_S x} - K) \frac{\partial G_{max}}{\partial m}(x, m, T)\ dm dx \\ & + \ e^{\sigma_F M_0} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} (S e^{\sigma_S x} - K) \frac{\partial G_{max}}{\partial m}(x, M_0, T)\ dx \\ & - \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} e^{\sigma_F M_0} (S e^{\sigma_S x} - K) \frac{\partial G_{max}}{\partial m}(x, M_0, T)\ dx \Bigg].\end{aligned}$$

The second and the third terms cancel with each other, and the first term becomes zero when M_0 is set equal to zero. Hence, we obtain

$$\left.\frac{\partial V_{max}}{\partial M_0}\right|_{M_0=0} = 0.$$

APPENDIX C: PROOF OF THEOREM 2

We consider the following two separate cases:

1. $S_{max}^{[T_0,0]} < K$

   By observing that $U_0 < \dfrac{1}{\sigma_S} \ln \dfrac{K}{S}$ and the first term in $g_{joint}(x, y, T; u)$ is independent of u, we transform V_{joint} in Eq. (2.18) into the following form

$$\begin{aligned}V_{joint} = e^{-r_d T} \Bigg\{ & F_c \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} (S e^{\sigma_S u} - K) \phi_2(\widetilde{u}, \widetilde{y}, T; \rho)\ du dy \\ & - F_c \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} (S e^{\sigma_S u} - K) \\ & \quad \frac{\partial}{\partial u} \left[\int_{-\infty}^{u} e^{2\mu_X u} \phi_2(\widetilde{x} - 2u, \widetilde{y} - 2\rho u, T; \rho)\ dx \right]\ du dy \\ & + \ F \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} e^{\sigma_F y} (S e^{\sigma_S u} - K) \phi_2(\widetilde{u}, \widetilde{y}, T; \rho)\ du dy \\ & - \ F \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} e^{\sigma_F y} (S e^{\sigma_S u} - K) \\ & \quad \frac{\partial}{\partial u} \left[\int_{-\infty}^{u} e^{2\mu_X u} \phi_2(\widetilde{x} - 2u, \widetilde{y} - 2\rho u, T; \rho)\ dx \right]\ du dy \Bigg\},\end{aligned}$$

where $\widetilde{u} = u - \mu_X T$. The first and third integrals can be expressed in terms of $N_2(\cdot\,,\cdot\,;\rho)$ in a straightforward manner. By applying parts integration like that in Eq. (A3) and observing similar occurrences of vanishing boundary terms arising from parts integration, the second and the fourth integrals can be expressed as

$$\begin{aligned}
\text{second integral } &= e^{-r_d T} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \sigma_S e^{(2\mu_X+\sigma_S)u} \\
&\left[F_c S \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{-\infty}^{u} \phi_2(\widetilde{x}-2u, \widetilde{y}-2\rho u, T;\rho)\, dxdy \right] du, \\
\text{fourth integral} &= e^{-r_d T} \int_{\frac{1}{\sigma_S}\ln\frac{K}{S}}^{\infty} \sigma_S e^{(2\mu_X+\sigma_S)u} \\
&\left[FS \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{-\infty}^{u} e^{\sigma_F y} \phi_2(\widetilde{x}-2u, \widetilde{y}-2\rho u, T;\rho)\, dxdy \right] du.
\end{aligned}$$

Both of the above two integrals can be expressed as a single integral with integrand involving $N_2(\cdot\,,\cdot\,;\rho)$.

2. $S_{max}^{[T_0,0]} \geq K$

   The first and third integrals in Eq. (2.19) can be expressed as

$$e^{-r_d T} F_c(S_{max}^{[T_0,0]} - K) \int_{-\infty}^{\frac{1}{\sigma_F}\ln\frac{F_c}{F}} \int_{-\infty}^{U_0} \phi_2(\widetilde{x}, \widetilde{y}, T;\rho)\, dxdy$$

   and

$$e^{-r_d T} F(S_{max}^{[T_0,0]} - K) \int_{\frac{1}{\sigma_F}\ln\frac{F_c}{F}}^{\infty} \int_{-\infty}^{U_0} e^{\sigma_F y} \phi_2(\widetilde{x}, \widetilde{y}, T;\rho)\, dxdy,$$

   respectively. These two integrals can be simplified to become the first term in price formula (2.22) [with the common factor $(S_{max}^{[T_0,0]} - K)$]. The second and the fourth integrals in Eq. (2.19) are similar to the two integrals in Eq. (2.18) except that the lower integration limit becomes U_0 instead of $\dfrac{1}{\sigma_S}\ln\dfrac{K}{S}$. By applying similar parts integration procedures and again observing the occurrences of vanishing boundary terms arising from parts integration, the sum of these two integrals give the remaining terms in price formula (2.22).

APPENDIX D: PROOF OF PROPOSITION 3

(a) For any American options, the value of the longer-lived one is always worth at least that of its shorter-lived counterpart, so $\dfrac{\partial U_M}{\partial \tau} = \dfrac{1}{F_{max}} \dfrac{\partial V_M}{\partial \tau} \geq 0$.

(b) For a given value of F_{max}, $V_M(S, F, \tau)$ is a non-decreasing function of F since a higher value of F would mean at least the same or a higher value of $F_{max}^{[T_0,T]}$ to be realized at expiry compared to the counterpart with a lower value of F. We then have $\dfrac{\partial U_M}{\partial \xi} = \dfrac{\partial V_M}{\partial F} \geq 0$.

APPENDIX E: PROOF OF THEOREM 4

The monotonicity property: $\dfrac{\partial U_M}{\partial \tau} > 0$ is maintained in the continuation region even when $\tau \to 0^+$. First, it is obvious that $S_M^*(\xi, 0^+) \geq K$. For $S \in (K, S_M^*(\xi, 0^+))$, we have $U_M(S, \xi, 0^+) = S - K$. Since $U_M(S, \xi, 0^+)$ should satisfy $\dfrac{\partial U_M}{\partial \tau} = \widehat{L} U_M$, where

$$\begin{aligned}\widehat{L} = &\frac{\sigma_S^2}{2} S^2 \frac{\partial^2}{\partial S^2} + \rho \sigma_S \sigma_F S \xi \frac{\partial^2}{\partial S \partial \xi} + \frac{\sigma_F^2}{2} \xi^2 \frac{\partial^2}{\partial \xi^2} \\ &+ \delta_S^d S \frac{\partial}{\partial S} + (r_d - r_f) \xi \frac{\partial}{\partial \xi} - r_d,\end{aligned}$$

we obtain

$$\left.\frac{\partial U_M}{\partial \tau}\right|_{\tau=0} = \delta_S^d S - r_d(S - K) = r_d K - (r_d - \delta_S^d) S.$$

For $r_d > \delta_S^d$, the condition: $\left.\dfrac{\partial U_M}{\partial \tau}\right|_{\tau=0} > 0$ is satisfied only for $S < \dfrac{r_d}{r_d - \delta_S^d} K$. On the other hand, when $r_d \leq \delta_S^d$, $\left.\dfrac{\partial U_M}{\partial \tau}\right|_{\tau=0} > 0$ always holds true. We then conclude that

$$S_M^*(\xi, 0^+) = \begin{cases} \max\left(1, \dfrac{r_d}{r_d - \delta_S^d}\right) K & \text{if } r_d > \delta_S^d \\ \infty & \text{if } r_d \leq \delta_S^d \end{cases}.$$

The above result agrees with the usual result for critical asset price close to expiry for American call options when we visualize $r_d - \delta_S^d$ as the effective dividend yield of the foreign stock in the domestic currency world.

To show the monotonicity property of $S_M^*(\xi, \tau)$ with respect to τ, we let $\tau_2 > \tau_1$ and consider the evaluation of $U_M(S, \xi, \tau)$ at stock price level $S = S_M^*(\xi, \tau_1)$ and at two times τ_1 and τ_2. By virtue of the monotonicity property of U_M on τ, we have

$$U_M(S_M^*(\xi, \tau_1), \xi_1, \tau_2) > U_M(S_M^*(S, \tau_1), \xi_1, \tau_1) = S_M^*(\xi, \tau_1) - K.$$

This implies that the American option remains in the continuation region when $S = S_M^*(\xi, \tau_1)$ and $\tau = \tau_2$. Since the exercise region is on the right side of the continuation region, we deduce that

$$S_M^*(\xi, \tau_2) > S_M^*(\xi, \tau_1), \qquad \tau_2 > \tau_1.$$

The monotonicity property of $S_M^*(\xi,\tau)$ with respect to ξ can be established by using the monotonicity property of U_M on ξ and following a similar argument as above.

APPENDIX F: SOME PHENOMENA ON THE EXERCISE POLICIES OF AN AMERICAN JOINT QUANTO LOOKBACK CALL

Let $V_{fix}(S,F,\tau;S_{max})$ and $V_{float}(S,F,\tau;S_{max})$ denote the price function of the lookback options with terminal payoffs $F_c(S_{max}-K)^+$ and $F(S_{max}-K)^+$, respectively. Obviously, we have

$$V_J(S,F,\tau;S_{max}) \geq V_{fix}(S,F,\tau;S_{max}) \quad \text{and} \quad V_J(S,F,\tau;S_{max}) \geq V_{float}(S,F,\tau;S_{max}), \quad (F1)$$

since the exchange rate in the payoff of V_J is given by $\max(F,F_c)$. For the one-asset fixed strike lookback call option with payoff $(S_{max}-K)^+$, interest rate r and cost of carry δ, we denote its price function by $W(S,\tau;S_{max},r,\delta)$. It can be shown easily that

$$V_{fix}(S,F,\tau;S_{max}) = F_c W(S,\tau;S_{max},r_d,\delta_S^d) \quad (F2a)$$

$$V_{float}(S,F,\tau;S_{max}) = FW(S,\tau;S_{max},r_f,r_f-q). \quad (F2b)$$

To show that there exists a threshold value $\tau_{joint}^*(S,S_{max})$ such that the American joint quanto lookback call should never be exercised at any F whenever $\tau > \tau_{joint}^*(S,S_{max})$, we rely on a similar result for the one-asset fixed strike lookback call option.

Proposition

When S_{max}/S is sufficiently close to 1, there exists some threshold value $\tau^*(S,S_{max};r,\delta)$ such that it is never optimal to exercise the one-asset fixed strike lookback call option when $\tau > \tau^*(S,S_{max};r,\delta)$.

Remark on the proposition

A comprehensive analysis of the optimal exercise policies of one-asset fixed strike lookback options has been performed in the papers by Lai and Lim (2003) and Dai and Kwok (2003). Here, we reproduce similar plots of the exercise boundaries (see Figure 5) so as to reveal the intuition behind the understanding of the result in the proposition.

Figure 5 shows the plots of the exercise boundaries in the (S,S_{max})-plane of the one-asset fixed strike lookback call option at different times to expiry. The parameter values used in the calculations are: $r=0.05, \delta=0.1, \sigma=0.2$ and $K=1$. For a given time to expiry τ, the exercise region lies above the corresponding exercise boundary. Consider the point B which corresponds to $S = 3.85$ and $S_{max} = 5$, it lies in the continuation region corresponding to $\tau = 2$ and the exercise region corresponding to $\tau = 0.5$. There exists a threshold value $\tau^*(3.85, 5)$, where $0.5 < \tau^*(3.85, 5) < 2$, such that B always stays in the continuation region corresponding to those values of time to expiry τ, where $\tau > \tau^*(3.85, 5)$. The existence of such threshold value $\tau^*(3.85, 5)$ can be justified by the monotonic properties of the exercise boundaries with respect to τ and the analytic properties of the exercise boundaries at $\tau = 0$ and $\tau = \infty$. Suppose S drops to 1 while S_{max} remains at 5 (represented by point A), point A always lies in the exercise region corresponding to all values of time to expiry since it lies above the exercise boundary corresponding to $\tau = \infty$. This agrees with the intuition that when S is far below S_{max}, the chance of realizing a higher S_{max} is low so that it is optimal to exercise the lookback option prematurely at all times. With the same value of S but S_{max} drops to 1.35 (represented by point C), it may become optimal to exercise the lookback option only when the option life is sufficiently close to expiry. Both points B and C have a low value of the ratio S_{max}/S (closer to 1) compared to that of point A.

We now return to the theoretical justification of the exercise policies as depicted in Figure 3. By observing the results in Eqs. (F1) and (F2a,b), and together with the Proposition, one can deduce that when S_{max}/S is sufficently close to 1, there exist $\tau^*_{fix}(S, S_{max})$ and $\tau^*_{float}(S, S_{max})$ such that

$$V_{fix}(S, F, \tau; S_{max}) > F_c(S_{max} - K)^+ \quad \text{for} \quad \tau > \tau^*_{fix}(S, S_{max})$$

and

$$V_{float}(S, F, \tau; S_{max}) > F(S_{max} - K)^+ \quad \text{for} \quad \tau > \tau^*_{float}(S, S_{max}).$$

Taking $\tau^*_{joint}(S, S_{max}) = \max(\tau^*_{fix}(S, S_{max}), \tau^*_{float}(S, S_{max}))$, we then have

$$V_J(S, F, \tau; S_{max}) > \max(F, F_c)(S_{max} - K)^+ \quad \text{for} \quad \tau > \tau^*_{joint}(S, S_{max}).$$

Since the continuation value is higher than the exercise payoff, we can deduce that the American joint quanto lookback call should never be exercised optimally at any value of F when $\tau > \tau^*_{joint}(S, S_{max})$.

APPENDIX G: PROOF OF THEOREM 5

At $F = F_c$ and $\tau > 0$, conditional on $S_{max} > K$, the option should remain alive. If otherwise, the option value is equal to the exercised payoff. Substituting $V_J = \max(F, F_c)(S_{max} - K)$ into Eq. (3.7b), we observe that

$$\frac{\partial V_J}{\partial \tau} - LV_J = -\left[\frac{\sigma_F^2}{2}F^2\delta(F - F_c)(S_{max} - K) + (r_d - r_f)F(S_{max} - K)\mathbf{1}_{\{F>F_c\}} - r_d \max(F, F_c)(S_{max} - K)\right] \longrightarrow -\infty \text{ when } F = F_c,$$

where $\delta(x)$ and $\mathbf{1}_{\{\cdot\}}$ are the delta function and indicator function, respectively. Since the condition: $\dfrac{\partial V_J}{\partial \tau} - LV_J \geq 0$ is not satisfied, the option should not be optimally exercised at $F = F_c$ and $\tau > 0$. The whole vertical line $F = F_c$ in the F-τ plane lies in the continuation region.

Next, we would like to show that the exercise regions contain the two horizontal line segments: $\{\tau = 0, F < F_c\}$ and $\{\tau = 0, F > F_c\}$ in the F-τ plane. Assume the contrary, suppose there exists a finite interval $(F^*_{low}(S, 0^+), F^*_{up}(S, 0^+))$ at $\tau \to 0^+$ that lies completely within the continuation region. Let $F \in (F^*_{low}(S, 0^+), F^*_{up}(S, 0^+))$; by continuity, the option value evaluated at F and $\tau \to 0^+$ is $\max(F, F_c)(S_{max} - K)$. Substituting this option value into Eq. (3.7a), we then have

$$\left.\frac{\partial V_J}{\partial \tau}\right|_{\tau=0} = \begin{cases} -r_f F(S_{max} - K) & \text{for } F > F_c \\ -r_d F_c(S_{max} - K) & \text{for } F < F_c \end{cases}.$$

In both cases, $\left.\dfrac{\partial V_J}{\partial \tau}\right|_{\tau=0} < 0$, which is in contradiction to the property: $\left.\dfrac{\partial V_J}{\partial \tau}\right|_{\tau=0} \geq 0$. This would then imply the non-existence of such finite interval. Hence, at time close to expiry and conditional on $S_{max} > K$, the option should be optimally exercised for any exchange rate F other than F_c.

In the F-τ plane, the vertical line $F = F_c$ is in the continuation region while the two horizontal line segments: $\{\tau = 0, F < F_c\}$ and $\{\tau = 0, F > F_c\}$ are in the exercise regions. We then deduce that for a fixed value of τ, there exist some critical values F^*_{up} and F^*_{low} ($F^*_{up} > F_c$ and $F^*_{low} < F_c$) such that the option should be optionally exercised when $F \geq F^*_{up}$ or $F \leq F^*_{low}$ (see Figure 3). Due to the monotonic increasing property of the option value with respect to τ, it can be shown that $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ are unique. In other words, the exercise boundary consists of exactly one branch $F^*_{up}(S, \tau; S_{max})$ that lies completely to the right of the vertical line $F = F_c$ and another unique branch $F^*_{low}(S, \tau; S_{max})$ to the left of $F = F_c$. The two branches $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ intersect at $F = F_c$ when $\tau \to 0^+$. Further, $F^*_{up}(S, \tau; S_{max})$ and $F^*_{low}(S, \tau; S_{max})$ are, respectively, monotonically increasing and decreasing with respect to τ.

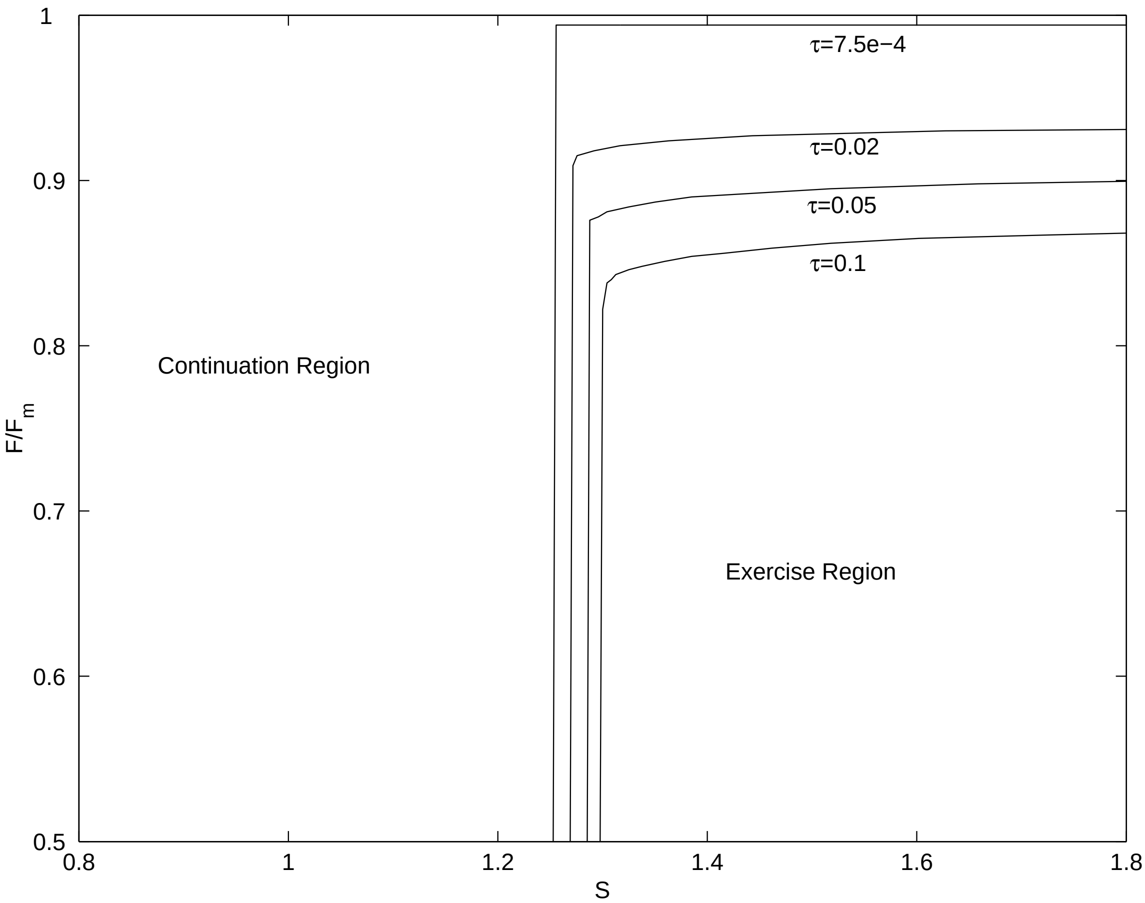

Figure 1 The exercise boundaries of an American maximum exchange rate quanto call option at different times to expiry τ are plotted. The parameters of the option model are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, \rho = 0.5$ and $K = 1$. The exercise region and the continuation region are on the right and left side of the exercise boundary respectively.

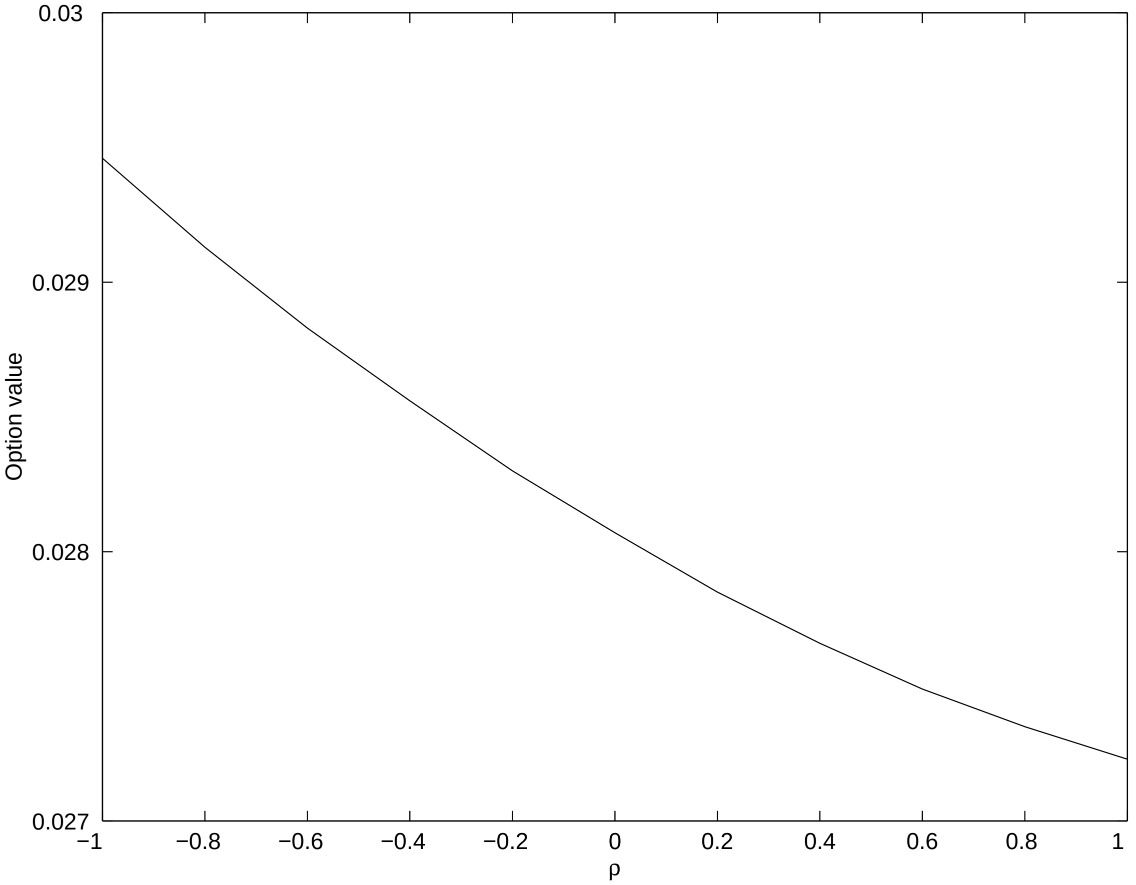

Figure 2 The value of an American maximum exchange rate quanto call option is plotted against the correlation coefficient ρ. The parameters of the option model are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, T = 0.1$ and $K = S = F = F_{max} = 1$. The option value is seen to be monotonically decreasing with respect to increasing correlation coefficient.

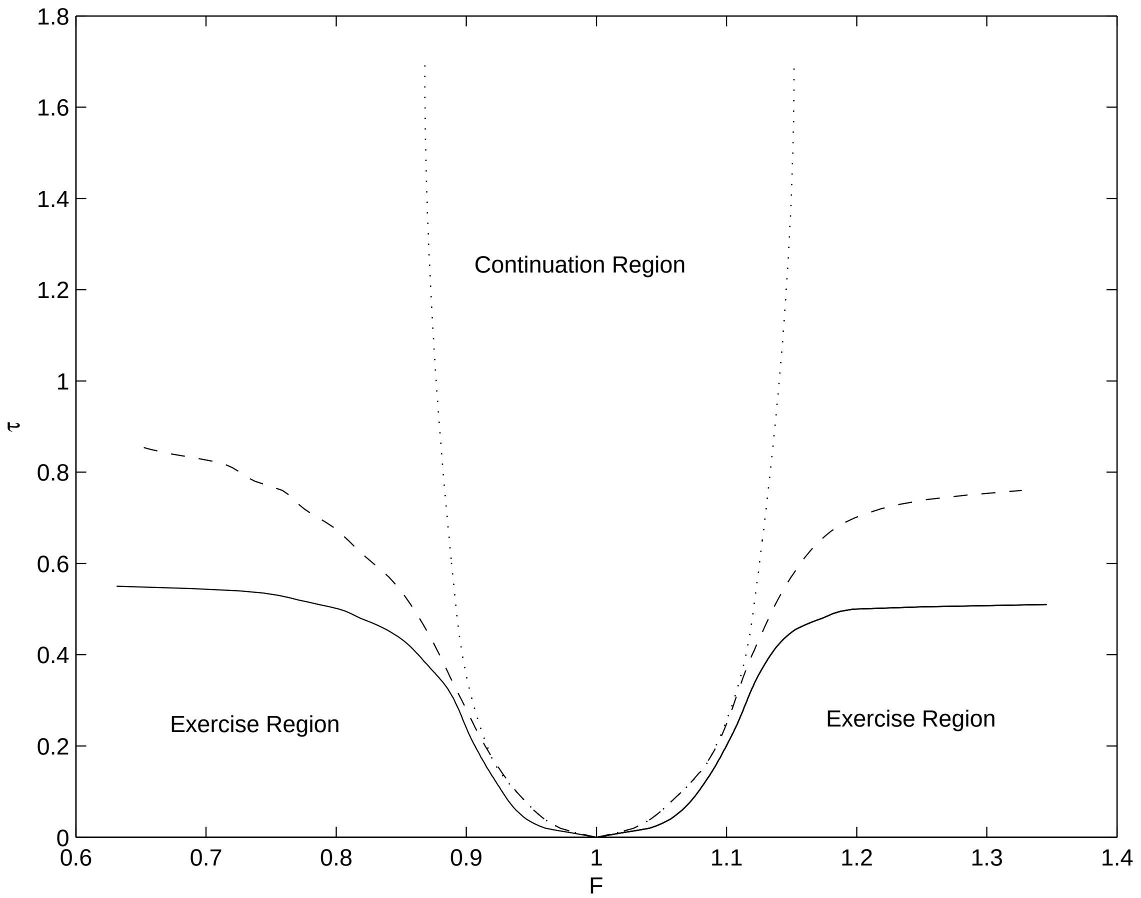

Figure 3 The exercise boundaries of an American joint quanto fixed strike lookback call option at different pairs of values of S and S_{max} are plotted. The parameters in the option model are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, \rho = 0.5$ and $K = F_c = 1$. The solid curve corresponds to $S = 1.0, S_{max} = 1.35$; the dashed curve corresponds to $S = 3.85, S_{max} = 5.0$; and the dotted curve corresponds to $S = 1.0, S_{max} = 5.0$. The exercise boundary consists of two branches with the continuation region lying in between. The solid curve and dashed curve are seen to level horizontally at high and low exchange rate F. Such phenomena reveal that when $S = 1.0$, $S_{max} = 1.35$ or $S = 3.85$, $S_{max} = 5.0$, there exists threshold value for τ such that it is never optimal to exercise at any level of F when τ is larger than the threshold value. The dotted curve is bounded by two vertical asymptotes, indicating that $F^*_{up}(S, \tau, S_{max})$ and $F^*_{low}(S, \tau, S_{max})$ are defined for all τ corresponding to $S = 1.0$, $S_{max} = 5.0$.

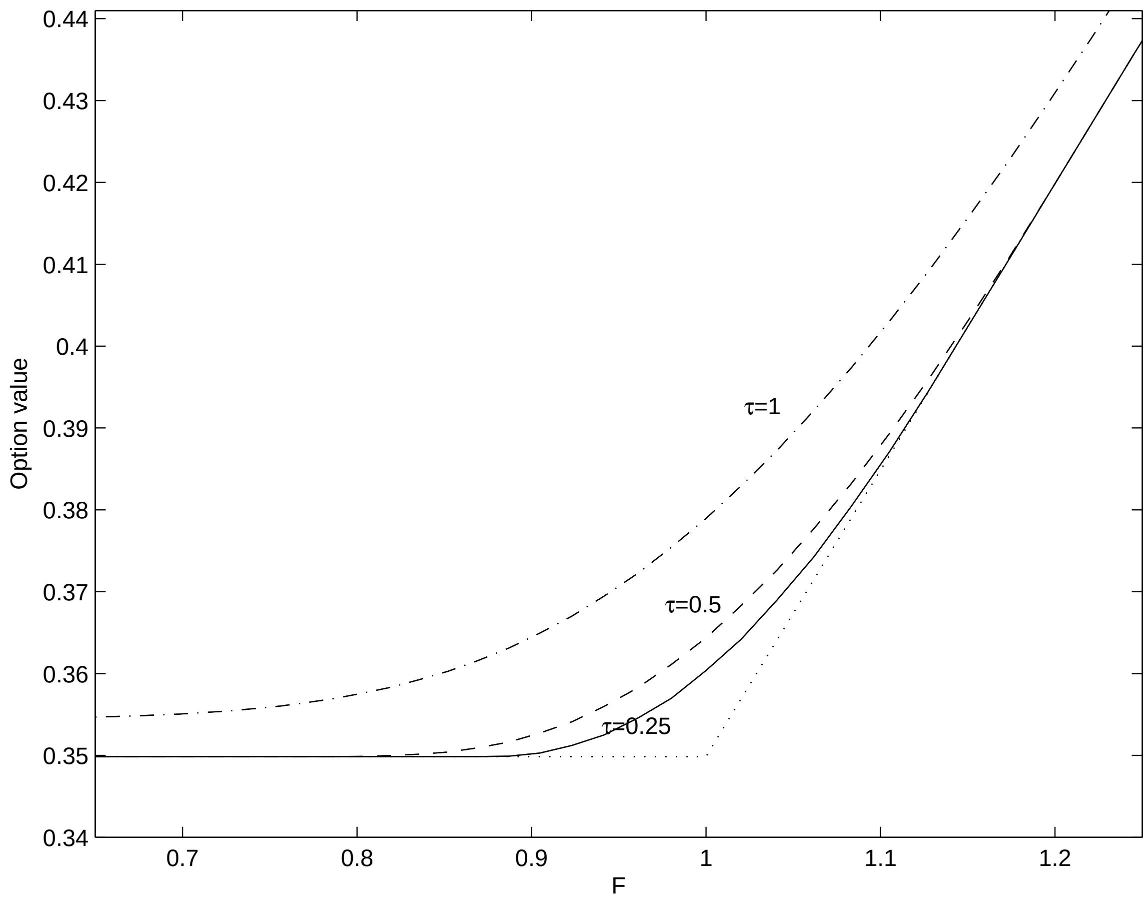

Figure 4 The value of an American joint quanto fixed strike lookback call option is plotted against the exchange rate F at different times to expiry τ. The parameters of the option model are $r_d = r_f = 0.05, q = 0.02, \sigma_S = \sigma_F = 0.2, \rho = 0.5, K = 1, F_c = S = 1$ and $S_{max} = 1.35$. For $\tau = 0.25$ and $\tau = 0.5$, both option value curves cut tangentially the intrinsic value lines at two critical exchange rates $F^*_{low}(S, \tau; S_{max})$ and $F^*_{up}(S, \tau; S_{max})$. However, for $\tau = 1$, the option value curve never intersects with the intrinsic value lines.

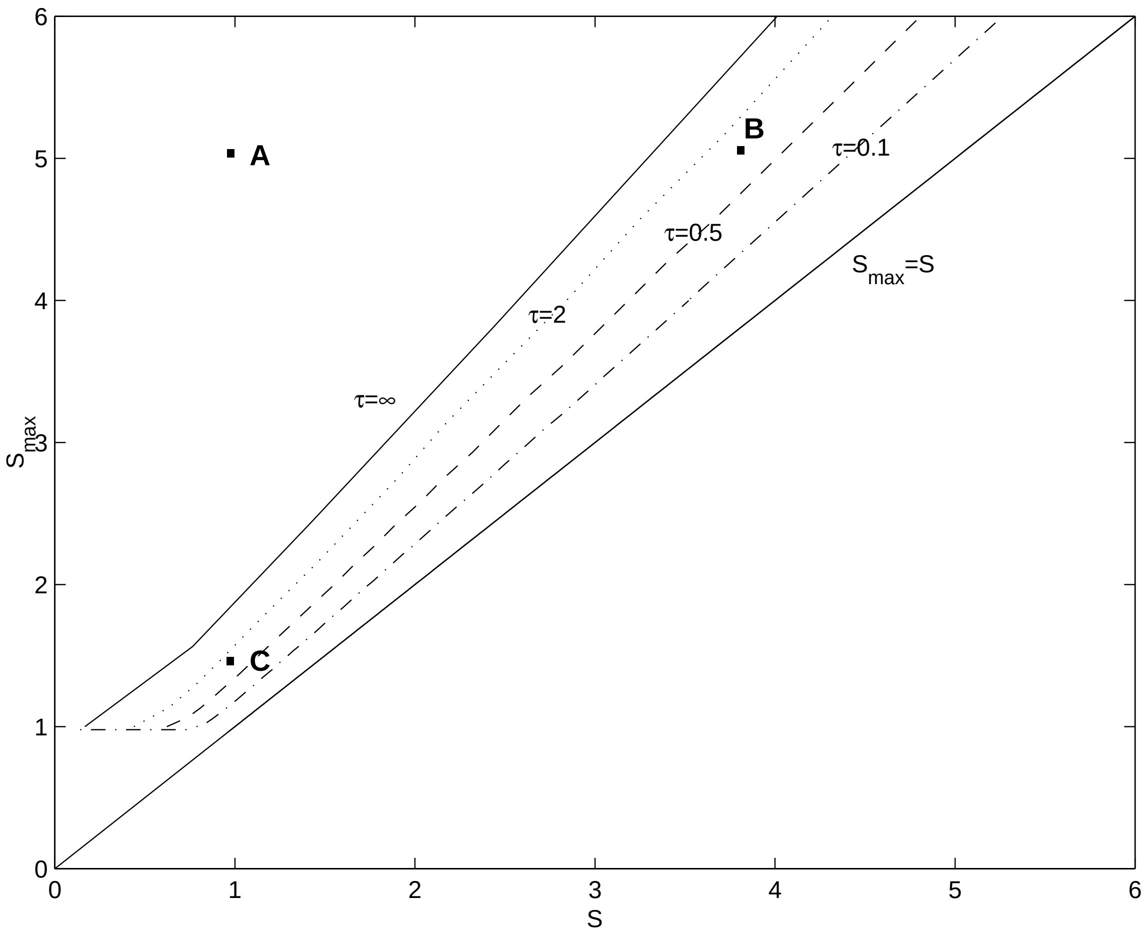

Figure 5 The figure shows the plots of the exercise boundaries in the (S, S_{max})-plane of the one-asset fixed strike lookback call option with varying values of τ. The following parameter values are used in the calculations: $r = 0.05, \delta = 0.1, \sigma = 0.2, K = 1$. The points A, B and C correspond to $(1, 5), (3.85, 5)$ and $(1, 1.35)$, respectively. For any point lying within the exercise boundaries corresponding to $\tau = 0$ and $\tau = \infty$ (like points B and C), there exists a threshold value τ^* such that the point lies in the continuation region corresponding to those values of τ such that $\tau > \tau^*$. Since the point A lies above the exercise boundary corresponding to $\tau = \infty$, this indicates that at $S = 1$ and $S_{max} = 5$, it is optimal to exercise the lookback call at any value of τ.