AMERICAN OPTIONS WITH LOOKBACK PAYOFF

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Abstract. We examine the early exercise policies and pricing behaviors of one-asset American options with lookback payoff structures. The classes of option models considered include floating strike lookback options, Russian options, fixed strike lookback options and pricing model of dynamic protection fund. For each class of the American lookback options, we analyze the optimal stopping region, in particular the asymptotic behavior at times close to expiration and at infinite time to expiration. The inter-relations between the price functions of these American lookback options are explored. The mathematical technique of analyzing the exercise boundary curves of lookback options at infinitesimally small asset value is also applied to the American two-asset minimum put option model.

Key words. Lookback options, American feature, free boundary problems, two-asset minimum put option

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1. Introduction. In this paper, we consider the theoretical analysis of the optimal exercise policies of an American option with lookback payoff. An American lookback option involves the combination of two exotic features: early exercise feature and lookback feature. Like other American option models, the analysis of an American lookback option requires the solution of a free boundary value problem. The solution procedure involves the determination of the free exercise boundary that separates the stopping region and continuation region. The analysis is further complicated by the presence of the path dependent lookback state variable. For floating strike lookback options, the analysis is easier since the dimensionality of the pricing model can be reduced through homogeneity of the price function. This is achieved by taking the asset price as the numeraire. However, for American fixed strike lookback options, the exercise boundary is a two-dimensional curve in the state space described by the asset price and the lookback state variable.

Several earlier papers on American lookback options concentrate on the analysis of the Russian option [7, 17, 18], which is essentially a perpetual zero-strike fixed strike lookback call option. There have been only a few papers which analyze the optimal exercise behaviors of finite time American lookback options. Yu *et al.* [22] develop finite difference algorithms to compute the exercise boundaries of both American fixed strike and floating strike lookback options. In a sequel of two papers [15, 16], Lai and Lim propose the Bernoulli walk approach to compute the price functions and optimal exercise boundaries of American fixed strike and floating strike lookback options. They also obtain analytic price formulas for American lookback options using a decomposition, which expresses the price as the sum of the corresponding European value and an early exercise premium. Dai *et al.* [4] analyze the exercise policies of American floating strike lookback options with quanto payoff. These quanto options involve an underlying foreign currency asset but the payoffs are denominated in domestic currency.

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We would like to provide a more comprehensive and thorough analysis of the exercise behaviors of the commonly traded American lookback options. Our analysis framework relies more on the partial differential equation approach, as opposed to the usual stochastic approach in most earlier works (say, [14, 15]). For the sake of completeness, we attempt to provide a comprehensive list of analytic properties of the exercise boundaries and stopping regions of the lookback option models. The classes of American lookback option models considered in this paper include the floating strike and fixed strike lookback call and put options, Russian options and pricing model of dynamic protection fund. We analyze the exercise boundary of each class of lookback options, in particular the asymptotic behavior at times close to expiration and at infinite time to expiration. The inter-relations between the price functions of these American lookback options are explored. We observe that our mathematical technique developed for analyzing the exercise boundary at infinitesimally small asset value for lookback options can be extended to American two-asset minimum put option model. For all types of American lookback options considered in this paper, we performed numerical calculations to compute the corresponding exercise boundaries. These plots of exercise boundaries serve as the verification to all results derived from the theoretical studies of the optimal exercise policies.

2. Floating strike lookback options. In this section, we explore some analytic properties of the price functions and optimal exercise policies of the American floating strike lookback options. The usual assumptions of the Black-Scholes option pricing framework are adopted in this paper. Let S denote the price of the underlying asset of the lookback option, whose stochastic dynamics under the risk neutral measure is governed by

$$\frac{dS}{S} = (r - q)dt + \sigma\ dZ, \tag{2.1}$$

where t is the calendar time, r is the riskless interest rate, σ and q are the volatility and dividend yield of S, respectively, and Z is the standard Wiener process. We write τ as the time to expiry, $0 \leq \tau < \infty$. Let m and M denote the realized minimum value and realized maximum value, respectively, of the asset price over the lookback monitoring period (continuous monitoring is assumed) up to the current time. The payoff functions of the American floating strike lookback call and lookback put are taken to be

$$(\alpha S - m)^+ \quad \text{and} \quad (M - \alpha S)^+$$

respectively, where α is a positive parameter value, $0 < \alpha < \infty$, and $x^+ = \max(x, 0)$. When $\alpha = 1$, we recover the usual lookback payoffs. While lookback options are less attractive to investors due to their high option premium, the parameter α allows flexible adjustment of the resulting option premium. For example, we may take α to be less (greater) than one in the floating strike call (put) payoff so as to achieve option premium reduction. Furthermore, the addition of the parameter α in the pricing model facilitates our asymptotic analysis of the exercise boundary curves at the limit of infinitesimally small asset value.

2.1. American floating strike lookback call. Let $C_{f\ell}(S, m, \tau)$ denote the price function of an American floating strike lookback call with payoff $(\alpha S - m)^+$. The linear complementarity formulation that governs $C_{f\ell}(S, m, \tau)$ is given by (see [12] and [21])

$$\begin{aligned}&\frac{\partial C_{f\ell}}{\partial \tau} - \mathcal{L}C_{f\ell} \geq 0, \quad C_{f\ell} \geq \alpha S - m,\\ &\left(\frac{\partial C_{f\ell}}{\partial \tau} - \mathcal{L}C_{f\ell}\right)[C_{f\ell} - (\alpha S - m)] = 0, \quad S > m > 0, \tau > 0,\end{aligned} \tag{2.2}$$

with auxiliary conditions:

$$\begin{aligned}&\left.\frac{\partial C_{f\ell}}{\partial m}\right|_{S=m} = 0\\ &C_{f\ell}(S, m, 0) = (\alpha S - m)^+.\end{aligned} \tag{2.3}$$

The operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{\sigma^2}{2}S^2\frac{\partial^2}{\partial S^2} + (r-q)S\frac{\partial}{\partial S} - r.$$

Note that the payoff upon early exercise is guaranteed to be positive so that we can replace the payoff function $(\alpha S - m)^+$ by $\alpha S - m$. However, we cannot do so for the terminal payoff at $\tau = 0$. The dimension of the above formulation can be reduced by one if we define the following transformation of variables:

$$\eta = \frac{m}{S} \quad \text{and} \quad \widetilde{C}_{f\ell}(\eta, \tau) = \frac{C_{f\ell}(S, m, \tau)}{S}. \tag{2.4}$$

This is equivalent to take S as the numeraire. The new linear complementarity formulation for $\widetilde{C}_{f\ell}(\eta, \tau)$ is given by

$$\begin{aligned}&\frac{\partial \widetilde{C}_{f\ell}}{\partial \tau} - \widetilde{\mathcal{L}}\widetilde{C}_{f\ell} \geq 0, \quad \widetilde{C}_{f\ell} \geq \alpha - \eta,\\ &\left(\frac{\partial \widetilde{C}_{f\ell}}{\partial \tau} - \widetilde{\mathcal{L}}\widetilde{C}_{f\ell}\right)[\widetilde{C}_{f\ell} - (\alpha - \eta)] = 0, \quad 0 < \eta < 1, \tau > 0,\end{aligned} \tag{2.5}$$

with auxiliary conditions:

$$\begin{aligned}&\left.\frac{\partial \widetilde{C}_{f\ell}}{\partial \eta}\right|_{\eta=1} = 0\\ &\widetilde{C}_{f\ell}(\eta, 0) = (\alpha - \eta)^+,\end{aligned} \tag{2.6}$$

where the operator $\widetilde{\mathcal{L}}$ is given by

$$\widetilde{\mathcal{L}} = \frac{\sigma^2}{2}\eta^2\frac{\partial^2}{\partial \eta^2} + (q-r)\eta\frac{\partial}{\partial \eta} - q.$$

Remark

The normal reflection condition in Eq. (2.6) plays a crucial role in distinguishing the optimal exercise policies of American lookback options from usual American options. The auxiliary condition is derived from the observation that the lookback option value is insensitive to the running extremum value when the current asset value equals the extremum value. This is because the probability that the current extremum value remains to be the realized extremum value at maturity is essentially zero when the current asset value and running extremum value are equal (see [10]). In a more recent work, Peskir [17] presents a proof on the normal reflection condition for the finite time Russian option. A similar proof can be mimicked for an American lookback option with more general lookback payoff.

The holder optimally exercises the lookback call whenever S reaches sufficiently high level. In terms of η, the holder chooses to exercise when $\eta \leq \eta^*$, where the threshold η^* has dependence on τ. The domain of the pricing model can be divided into two regions: the stopping region $\mathcal{S} = \{(\eta, \tau) : 0 < \eta \leq \eta^*(\tau), 0 < \tau < \infty\}$ inside which it is optimal to exercise the option and the continuation region $\mathcal{S}^C = \{(\eta, \tau) : \eta^*(\tau) < \eta \leq 1, 0 \leq \tau < \infty\}$ inside which it is optimal to continue to hold the option. Upon exercise, we have $\widetilde{C}_{f\ell} = \alpha - \eta$ so that the stopping region is defined by

$$\mathcal{S} = \{(\eta, \tau) : 0 < \eta \leq 1, 0 \leq \tau < \infty \text{ and } \widetilde{C}_{f\ell}(\eta, \tau) = \alpha - \eta\}.$$

The analysis of the optimal exercise policies amounts to the analysis of the analytic properties of $\eta^*(\tau)$ that separates the continuation and stopping regions. Some of the analytic properties of $\eta^*(\tau)$ are summarized in Proposition 2.1.

Proposition 2.1
The exercise boundary $\eta^*(\tau; \alpha)$ of the American floating strike lookback call option observes the following properties:

(i) Suppose $(\eta, \tau) \in \mathcal{S}^C$, then $(\lambda_1 \eta, \lambda_2 \tau) \in \mathcal{S}^C$ for all $\lambda_1 \geq 1, \lambda_2 \geq 1$.
(ii) The line $\eta = 1$ always lies inside $\mathcal{S}^C$ for finite value of α.
(iii) The behavior of $\eta^*(\tau; \alpha)$ near expiry, $\tau \to 0^+$, is given by

$$\eta^*(0^+; \alpha) = \min\left(1, \alpha, \frac{q}{r}\alpha\right).$$

When $q > 0, \eta^*(0^+; \alpha)$ is guaranteed to be positive so that there exists at least a line segment: $\tau = 0$, where $0 < \eta < \eta^*(0^+; \alpha)$, in the stopping region. Property (ii) reveals that the line $\eta = 1$ lies in the continuation region. Hence, we can conclude that both the continuation and stopping regions exist in the η-τ plane. Further, by virtue of (i), the free boundary $\eta^*(\tau; \alpha)$ that separates the stopping and continuation regions can be deduced to be monotonically decreasing with respect to τ. In conclusion, for $q > 0$, there exists the monotonic free boundary $\eta^*(\tau; \alpha)$ such that $\widetilde{C}_{f\ell} = \alpha - \eta$ for $\eta \leq \eta^*(\tau; \alpha), \tau > 0$. The details of the proof of Proposition 2.1 is presented in Appendix A. Further asymptotic properties of $\eta^*(\tau; \alpha)$ with respect to $\tau \to \infty$ and $\alpha \to \infty$ are stated in Proposition 2.2

Proposition 2.2
When $q > 0$, the asymptotic behaviors at $\tau \to \infty$ and $\alpha \to \infty$ of the exercise boundary $\eta^*(\tau; \alpha)$ of the American floating strike lookback call option are summarized as follows.

(i) Write $\eta^*_\infty(\alpha)$ as $\lim_{\tau \to \infty} \eta^*(\tau; \alpha)$; $\eta^*_\infty(\alpha)$ is given by the solution of the root inside the interval $(0, 1)$ of the following algebraic equation

$$(\eta^*_\infty)^{\lambda_+ - \lambda_-} = \frac{\lambda_+}{\lambda_-} \frac{(1 - \lambda_-)\eta^*_\infty + \lambda_- \alpha}{(1 - \lambda_+)\eta^*_\infty + \lambda_+ \alpha}$$

where

$$\lambda_\pm = \frac{r-q}{\sigma^2} + \frac{1}{2} \pm \sqrt{\left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2}}.$$

(ii) $\lim_{\alpha\to\infty} \eta^*(\tau;\alpha) = 1$ for all τ.

The proof of Proposition 2.2 is presented in Appendix B. From the monotonic decreasing property of $\eta^*(\tau;\alpha)$ with respect to τ and the finiteness property of $\eta^*_\infty(\alpha)$ for $q > 0$, we infer that $\eta^*(\tau;\alpha) > 0$ exists for all τ when $q > 0$. When $\alpha \to \infty$, the continuation region vanishes.

When the underlying asset is non-dividend paying, $q = 0$, we have $\eta^*(0^+;\alpha) = 0$. Furthermore, since $\eta^*(\tau;\alpha)$ is monotonically decreasing with respect to τ, we deduce that $\eta^*(\tau;\alpha) = 0$ for $\tau > 0$. That is, the stopping region does not exist when $q = 0$. Interpreted in financial sense, it is never optimal to exercise the American floating strike lookback call at any asset price level if the underlying asset is non-dividend paying. Such result agrees intuitively with a similar result of the usual American call.

Figure 1 shows the plot of $\eta^*(\tau;\alpha)$ against τ at varying values of α. The parameter values used in the calculations are: $r = 0.04, q = 0.02$ and $\sigma = 0.3$. The monotonicity properties of $\eta^*(\tau;\alpha)$ with respect to τ and α and the asymptotic behaviors at $\tau \to 0^+$ and $\tau \to \infty$ as shown in the plots do agree with the results stated in Propositions 2.1 and 2.2. Our calculations give the following asymptotic values for $\eta^*(\tau;\alpha)$:

$\eta^*(0^+;0.5) = 0.25,$ $\quad$ $\eta^*(\infty;0.5) = 0.1023,$
$\eta^*(0^+;1) = 0.5,$ $\quad$ $\eta^*(\infty;1) = 0.1988,$
$\eta^*(0^+;2) = 1,$ $\quad$ $\eta^*(\infty;2) = 0.3617,$
$\eta^*(0^+;10) = 1,$ $\quad$ $\eta^*(\infty;10) = 0.7947.$

2.2. American floating strike lookback put. Let $P_{f\ell}(S, M, \tau)$ denote the price function of an American floating strike lookback put with payoff $(M - \alpha S)^+$. The Russian option is the perpetual version of the American floating strike lookback put with $\alpha = 0$. In a similar manner, we use S as the numeraire and define

$$\xi = \frac{M}{S} \quad \text{and} \quad \widetilde{P}_{f\ell}(\xi,\tau) = \frac{P_{f\ell}(S,M,\tau)}{S}. \tag{2.7}$$

The linear complementarity formulation for $\widetilde{P}_{f\ell}(\xi,\tau)$ is given by

$$\begin{aligned} &\frac{\partial \widetilde{P}_{f\ell}}{\partial \tau} - \widetilde{\mathcal{L}}\widetilde{P}_{f\ell} \ge 0, \quad \widetilde{P}_{f\ell} \ge \xi - \alpha, \\ &\left(\frac{\partial \widetilde{P}_{f\ell}}{\partial \tau} - \widetilde{\mathcal{L}}\widetilde{P}_{f\ell}\right)[\widetilde{P}_{f\ell} - (\xi - \alpha)] = 0, \quad 1 < \xi < \infty, \tau > 0, \end{aligned} \tag{2.8}$$

with auxiliary conditions:

$$\begin{aligned} &\left.\frac{\partial \widetilde{P}_{f\ell}}{\partial \xi}\right|_{\xi=1} = 0 \\ &\widetilde{P}_{f\ell}(\xi,0) = (\xi-\alpha)^+. \end{aligned} \tag{2.9}$$

Similarly, we have the free boundary $\xi^*(\tau)$ that separates the stopping region $\{(\xi,\tau) : \xi \ge \xi^*(\tau), 0 \le \tau < \infty\}$ and the continuation region $\{(\xi,\tau) : 1 \le \xi <$

$\xi^*(\tau), 0 \le \tau < \infty\}$. The analytic properties of $\xi^*(\tau)$ are summarized in Proposition 2.3.

Proposition 2.3
The free boundary $\xi^*(\tau;\alpha)$ observes the following properties:

(i) $\xi^*(\tau;\alpha)$ is monotonically increasing with respect to τ and α.

(ii) The behavior of $\xi^*(\tau;\alpha)$ near expiry, $\tau \to 0^+$, is given by

$$\xi^*(0^+;\alpha) = \max\left(1, \alpha, \frac{q}{r}\alpha\right).$$

(iii) Write $\xi^*_\infty(\alpha)$ as $\lim_{\tau\to\infty} \xi^*(\tau;\alpha); \xi^*_\infty(\alpha)$ is given by the solution of the root inside the interval $(1,\infty)$ of the following algebraic equation:

$$(\xi^*_\infty)^{\lambda_+ - \lambda_-} = \frac{\lambda_+}{\lambda_-}\frac{(1-\lambda_-)\xi^*_\infty + \lambda_-\alpha}{(1-\lambda_+)\xi^*_\infty + \lambda_+\alpha}.$$

In particular, when $q = 0$, we have

$$\xi^*_\infty(\alpha) = \infty.$$

As a remark, it is well known that it is never optimal to exercise a Russian option when the underlying asset is non-dividend paying [18]. The above result shows that such optimal exercise policy holds even for non-zero value of α (Russian option is the special case of $\alpha = 0$).

The ideas behind the proof of Proposition 2.3 are similar to those used in proving Propositions 2.1 and 2.2. In Figure 2, we show the plot of $\xi^*(\tau;\alpha)$ against τ with different values of α. The parameter values used in the calculations are: $r = 0.02, q = 0.04$ and $\sigma = 0.3$. We obtained the following asymptotic values for $\xi^*(\tau;\alpha)$:

$\xi^*(0^+;0) = 1,$ $\xi^*(\infty;0) = 3.4939,$
$\xi^*(0^+;0.5) = 1,$ $\xi^*(\infty;0.5) = 4.8536,$
$\xi^*(0^+;1) = 2,$ $\xi^*(\infty;1) = 6.6068,$
$\xi^*(0^+;2) = 4,$ $\xi^*(\infty;2) = 10.7613.$

The monotonic behaviors of $\xi^*(\tau;\alpha)$ as exhibited by the plots in Figure 2 are consistent with the results stated in Proposition 2.3.

3. Fixed strike lookback options.

We now consider the pricing behaviors and optimal exercise policies of American fixed strike lookback options, where the payoff involves the strike price K and either realized maximum value M or realized minimum value m. The payoff functions of the American fixed strike lookback call and lookback put are given by

$$(M-K)^+ \quad \text{and} \quad (K-m)^+,$$

respectively. We also consider American option model with lookback payoff of the form

$$\max(M, K),$$

which is related to the pricing model of dynamic protection fund with early withdrawal right [7, 9]. According to the guarantee clause, the fund holder acquires more units of the fund from the fund sponsor whenever the fund value falls below the guaranteed protection floor. The early withdrawal right embedded in the protection fund resembles the early exercise right of an American option. When we set $K = 0$ in the payoff $\max(M, K)$, the option model becomes the finite-time Russian option.

It is tempting to seek possible fixed-floating symmetry relations between American lookback call and put options, similar to those obtained by Detemple [6] for usual American options. While it is possible to obtain symmetry relations between the grant-date price functions of European lookback options (with no dependence on the running extremum value), such relations do not hold for the in-progress counterparts. We do not expect to have nice fixed-floating symmetry relations between the price functions of in-progress American lookback options.

3.1. American fixed strike lookback call. Let $C_{fix}(S, M, \tau; K)$ denote the price function of an American fixed strike lookback call with payoff $(M - K)^+$. The linear complementarity formulation that governs $C_{fix}(S, M, \tau; K)$ is given by

$$\begin{aligned} &\frac{\partial C_{fix}}{\partial \tau} - \mathcal{L}C_{fix} \geq 0, \quad C_{fix} \geq (M - K), \\ &\left(\frac{\partial C_{fix}}{\partial \tau} - \mathcal{L}C_{fix}\right)[C_{fix} - (M - K)] = 0, \quad 0 < S < M, \tau > 0, \end{aligned} \tag{3.1}$$

with auxiliary conditions:

$$\begin{aligned} &\left.\frac{\partial C_{fix}}{\partial M}\right|_{S=M} = 0, \\ &C_{fix}(S, M, 0) = (M - K)^+. \end{aligned} \tag{3.2}$$

Let $\mathcal{S}(K)$ denote the stopping region of the American fixed strike lookback call with strike price K. Inside $\mathcal{S}(K)$, the price function equals the exercise payoff, that is,

$$\mathcal{S}(K) = \{(S, M, \tau) \in \{0 < S \leq M\} \times (0, \infty) : C_{fix}(S, M, \tau) = (M - K)^+\}.$$

Propositions 3.1–3.2 summarize the characterization of the optimal exercise policy of the American fixed strike lookback call and the analytic properties of the stopping region.

Proposition 3.1
The stopping region $\mathcal{S}(K)$ and the price function $C_{fix}(S, M, \tau; K)$ of the American fixed strike lookback call observe the following properties:

(i) $C_{fix}(S, M, \tau; K_2) - C_{fix}(S, M, \tau; K_1) \leq K_1 - K_2$ if $K_1 > K_2$,

(ii) $\mathcal{S}(K_1) \subset \mathcal{S}(K_2)$ if $K_1 > K_2$,

(iii) Suppose $(S, M, \tau) \in \mathcal{S}(K)$ and $0 < \lambda_1 \leq 1, \lambda_2 \geq 1, 0 < \lambda_3 \leq 1$, we have

$$(\lambda_1 S, \lambda_2 M, \lambda_3 \tau) \in \mathcal{S}(K).$$

The proof of Proposition 3.1 is presented in Appendix C. In Figure 3, we plot the exercise boundary that separates the stopping region and continuation region in the S-M plane, and use $M^*(S, \tau; K)$ to denote the exercise boundary. Such representation reveals the dependence of the critical realized maximum value M^* on S, τ and K. By virtue of (iii) in Proposition 3.1, we deduce that the stopping region lies to the upper left side of the exercise boundary in the S-M plane. Hence, we may rewrite $\mathcal{S}(K)$ in the following alternative form:

$$\mathcal{S}(K) = \{(S, M, \tau) \in \{0 < S \leq M\} \times (0, \infty) : \quad M > M^*(S, \tau)\}.$$

Further properties on $M^*(S,\tau;K)$ are summarized in Proposition 3.2.

Proposition 3.2
Let $M^*(S,\tau;K)$ denote the exercise boundary of the American fixed strike lookback call in the S-M plane, then $M^*(S,\tau;K)$ observes the following properties:

(i) $\lim_{\tau\to 0^+} M^*(S,\tau;K) = K$ for all S,

(ii) $M^*(S,\tau;K)$ is a monotonically increasing with respect to S and τ,

(iii) $\lim_{S\to 0^+} M^*(S,\tau;K) = K$ for all τ,

(iv) When $K=0$, $M^*(S,\tau;0)$ is a linear function of S. Furthermore, $\dfrac{M^*(S,\tau;0)}{S}$ is a monotonically increasing function of τ and

$$\lim_{S\to\infty} \frac{M^*(S,\tau;K)}{S} = \frac{M^*(S,\tau;0)}{S} \quad \text{for} \quad K>0. \tag{3.3}$$

Part (i) gives the zeroth order asymptotic expansion of $M^*(S,\tau;K)$ as $\tau\to 0^+$ (see [15] for a higher order asymptotic expansion of $M^*(S,\tau;K)$ as $\tau\to 0^+$). One can prove Part (i) by following a similar approach as that of (iii) in Proposition 2.1. Part (ii) is a corollary of part (iii) in Proposition 3.1. The proof of parts (iii) and (iv) in Proposition 3.2 is presented in Appendix D.

In Figure 3, we show the plot of the exercise boundaries of the American fixed strike lookback call option with varying values of maturity τ in the S-M plane. The parameter values used in the calculations are: $K=1, r=0.02, q=0.04$ and $\sigma=0.3$. The exercise boundary corresponding to the zero-strike lookback call is a straight line, the slope of which depends on τ. By virtue of Eq. (3.3), the exercise boundaries for the non-zero strike lookback call options tend to those of their zero-strike counterparts as $S\to\infty$. Note that $M^*(S,\tau;0)/S = \xi^*(\tau;0)$, where $\xi^*(\tau;\alpha)$ denotes the exercise boundary in the pricing model for $\widetilde{P}_{f\ell}(\xi,\tau)$ [see Eqs. (2.8, 2.9)]. Our calculations give the following numerical values for $\xi^*(\tau;0)$:

$$\begin{aligned} \xi^*(\infty;0) &= 3.4939 \\ \xi^*(2;0) &= 2.0300 \\ \xi^*(0.5;0) &= 1.5450. \end{aligned}$$

The finite-time Russian option is seen to be identical to the zero-strike American fixed strike lookback call. Let $V_{Rus}(S,M,\tau)$ denote the price function of the finite-time Russian option so that

$$V_{Rus}(S,M,\tau) = C_{fix}(S,M,\tau;0). \tag{3.4}$$

Since K does not appear in the price function $V_{Rus}(S,M,\tau)$, the asset value S can be used as a numeraire. We may write

$$\widetilde{V}_{Rus}(\xi,\tau) = \frac{V_{Rus}(S,M,\tau)}{S} \quad \text{where} \quad \xi = \frac{M}{S}. \tag{3.5}$$

This explains why $M^*(S,\tau;0)/S$ becomes independent of S. More detailed theoretical analysis of the price function $V_{Rus}(S,M,\tau)$ can be found in Peskir's paper [16].

The exercise boundaries plotted in Figure 3 do agree with our financial intuition about the optimal early exercise policies of the American fixed strike lookback call options. Either $S \to 0^+$ or $\tau \to 0^+$, the chance of achieving a higher realized maximum value M becomes vanishingly small, so it becomes optimal to exercise even when M reaches the level K. When the asset price is very high, $M^*(S, \tau; K)$ becomes almost insensitive to the strike price K since the value K has only small effect on the exercise payoff. Hence, when $S \to \infty$, the asymptotic behavior of $M^*(S, \tau; K)$ as stated in Eq. (3.3) is observed.

3.2. American fixed strike lookback put. Consider an American fixed strike lookback put with payoff $(K - m)^+$, the linear complementarity formulation that governs its price function $P_{fix}(S, m, \tau)$ is given by

$$
\begin{aligned}
&\frac{\partial P_{fix}}{\partial \tau} - \mathcal{L}P_{fix} \geq 0, \qquad P_{fix} \geq (K - m), \\
&\left(\frac{\partial P_{fix}}{\partial \tau} - \mathcal{L}P_{fix}\right)[P_{fix} - (K - m)] = 0, \qquad 0 < m < S, \tau > 0,
\end{aligned}
\tag{3.6}
$$

with auxiliary conditions:

$$
\begin{aligned}
&\left.\frac{\partial P_{fix}}{\partial m}\right|_{S=m} = 0 \\
&P_{fix}(S, m, 0) = (K - m)^+.
\end{aligned}
\tag{3.7}
$$

In a similar manner, we let $m^*(S, \tau; K)$ denote the exercise boundary that separates the stopping region and continuation region in the S-m plane. The analytic properties of $m^*(S, \tau; K)$ are summarized in Proposition 3.3.

Proposition 3.3
The exercise boundary $m^*(S, \tau; K)$ of the American fixed strike lookback put satisfies the following properties:

(i) $\lim_{\tau \to 0^+} m^*(S, \tau; K) = K \qquad$ for all S,

(ii) $m^*(S, \tau; K)$ is monotonically increasing with respect to S,

(iii) $\lim_{S \to \infty} m^*(0, \tau; K) = K \qquad$ for all τ,

(iv) $\lim_{S \to 0^+} \dfrac{m^*(S, \tau; K)}{S} = 1 \qquad$ for all τ.

Parts (i) - (iii) in Proposition 3.3 can be proven by using similar arguments as those used in proving parts (i) - (iii) in Proposition 3.2. The proof of (iv) in Proposition 3.3 is interesting and challenging. It relies on the asymptotic result on $\eta^*(\tau; \alpha)$ as stated in (ii) in Proposition 2.2 (see Appendix E for details).

Figure 4 shows the plot of the exercise boundaries $m^*(S, \tau; K)$ of the American fixed strike lookback put with varying values of maturity τ in the S-m plane. The parameter values used in the calculations are: $K = 1, r = 0.04, q = 0.02$ and $\sigma = 0.3$. According to (iii) and (iv) in Proposition 3.3, the exercise boundaries are seen to tend asymptotically to $m = K$ as $S \to \infty$ and $m = S$ as $S \to 0^+$.

3.3. American lookback option with payoff $\max(M, K)$**.** Let $V_M(S, M, \tau)$ denote the price function of the American option with lookback payoff $\max(M, K)$.

First, we argue from financial intuition that $V_M(S, M, \tau)$ should be insensitive to the *current* realized maximum value of asset price M when $M < K$, that is,

$$\frac{\partial V_M}{\partial M} = 0 \quad \text{for} \quad M < K. \tag{3.8}$$

The option payoff is given by K if the *future* realized maximum value of asset price is less than or equal to K; otherwise, the payoff equals the future realized maximum value. In either case, the current realized maximum value M does not enter into the payoff function. Hence, $V_M(S, M, \tau)$ does have dependence on M when $M < K$. On the other hand, when $M \geq K$, the future realized maximum value is always greater than or equal to K, so the payoff is simply given by M. This is the same payoff as that of the finite-time Russian option. Hence, we have

$$V_M(S, M, \tau) = V_{Rus}(S, M, \tau) \quad \text{for} \quad M \geq K. \tag{3.9}$$

By virtue of the continuity property of the price function $V_M(S, M, \tau)$ with respect to M, we then have

$$V_M(S, M, \tau) = \begin{cases} V_{Rus}(S, M, \tau) & \text{for } M \geq K \\ V_{Rus}(S, K, \tau) & \text{for } M < K \end{cases}. \tag{3.10}$$

For $M \geq K, V_M$ and V_{Rus} should share the same optimal exercise policy. At $M = K$, the exercise boundary of the finite-time Russian option is given by $S = K/\xi^*(\tau; 0)$. Hence, for $M < K$, the American option with payoff $\max(M, K)$ will be exercised optimally when $S \leq K/\xi^*(\tau; 0)$ and unexercised if otherwise.

In Figure 5, we plot the stopping region and continuation region in the S-M plane of the American option with payoff $\max(M, K)$. The set of parameter values used in the calculations are: $K = 1, r = 0.02, q = 0.04$ and $\sigma = 0.3$. When $M \geq K$, the stopping region and continuation region for fixed value of τ are separated by the oblique line: $M = S\xi^*(\tau; 0)$. On the other hand, when $M < K$, the exercise boundary becomes the vertical line: $S = K/\xi^*(\tau; 0)$.

3.4. A related two-asset American option model. As a slight departure from the option models with lookback payoff structures, we consider the optimal exercise policies of a two-asset American option with a put payoff on the minimum of two asset values. There have been several comprehensive papers that analyze the early exercise policies of two-asset American options [2, 5, 9, 13, 14, 19, 20]. We would like to demonstrate that the mathematical technique of analyzing the exercise boundaries of the American fixed strike lookback put option at $S \to 0^+$ can be adopted to resolve the mystery on the asymptotic behaviors of the exercise boundaries of the two-asset American minimum put option at infinitesimally small asset values.

Let S_1 and S_2 denote the prices of the two underlying assets, whose dynamics under the risk neutral measure are governed by

$$\frac{dS_i}{S_i} = (r - q_i)dt + \sigma_i \, dZ_i \qquad i = 1, 2, \tag{3.11}$$

where $dZ_1 \, dZ_2 = \rho \, dt$, ρ is the correlation coefficient between the two Wiener processes dZ_1 and dZ_2. The exercise payoff is given by $(K - \min(S_1, S_2))^+$, where K is the strike price. Let $P_{min}(S_1, S_2, \tau; K)$ denote the price function of this two-asset American minimum put option. Let $\mathcal{S}_2(K)$ denote the continuation region in the S_1-S_2 plane, with dependence on K. The linear complementarity formulation for $P_{min}(S_1, S_2, \tau; K)$ is given by

$$\begin{aligned} &\frac{\partial P_{min}}{\partial \tau} - \mathcal{L}_2 P_{min} \geq 0, \qquad P_{min} \geq (K - \min(S_1, S_2))^+, \\ &\left[\frac{\partial P_{min}}{\partial \tau} - \mathcal{L}_2 P_{min}\right][P_{min} - (K - \min(S_1, S_2))^+] = 0, \\ &\qquad 0 < S_1 < \infty, 0 < S_2 < \infty, \tau > 0. \end{aligned} \tag{3.12}$$

The operator $\mathcal{L}_2$ is defined by

$$\begin{aligned} \mathcal{L}_2 = &\frac{\sigma_1^2}{2} S_1^2 \frac{\partial^2}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{\sigma_2^2}{2} S_2^2 \frac{\partial^2}{\partial S_2^2} \\ &+ (r - q_1) S_1 \frac{\partial}{\partial S_1} + (r - q_2) S_2 \frac{\partial}{\partial S_2} - r. \end{aligned} \tag{3.13}$$

In Figure 6, we show the plot of the exercise boundaries of the two-asset American minimum put option in the S_1-S_2 plane. The following set of parameter values are used in the calculations: $K = 1, r = 0.02, q_1 = 0, q_2 = 0.03, \sigma_1 = \sigma_2 = 0.3$ and $\rho = 0.5$. The whole line $S_1 = S_2$ always lie in the continuation region. The continuation region is bounded by the two branches of the exercise boundaries. In the region $S_1 > S_2$, we let $S_2^*(S_1, \tau)$ denote the exercise boundary at time to expiry τ. We observe that the curve $S_2^*(S_1, \tau)$ tends to the line $S_1 = S_2$ as $S_1 \to 0^+$ and tends to some asymptotic limit as $S_1 \to \infty$. Similar phenomena occur in the region $S_2 > S_1$, where the exercise boundary at time to expiry τ is represented by $S_1^*(S_2, \tau)$. For the above set of parameter values chosen for the option model, we obtain

$$\begin{aligned} &\lim_{S_1 \to \infty} S_2^*(S_1, 0.1) = 0.6277, \quad \lim_{S_1 \to \infty} S_2^*(S_1, 1) = 0.4855, \quad \lim_{S_1 \to \infty} S_2^*(S_1, \infty) = 0.2268, \\ &\lim_{S_2 \to \infty} S_1^*(S_2, 0.1) = 0.8118, \quad \lim_{S_2 \to \infty} S_1^*(S_2, 1) = 0.6100, \quad \lim_{S_2 \to \infty} S_1^*(S_2, \infty) = 0.3077. \end{aligned}$$

Some of the analytic properties of the exercise boundaries $S_1^*(S_2, \tau)$ and $S_2^*(S_1, \tau)$ are summarized in Proposition 3.4.

Proposition 3.4
Let $S_1^*(S_2, \tau)$ and $S_2^*(S_1, \tau)$ denote the exercise boundaries at time to expiry τ in the two respective regions, $S_2 > S_1$ and $S_1 > S_2$, in the S_1-S_2 plane of the two-asset American minimum put option. The exercise boundaries and the continuation region observe the following properties:

(i) Let $S_{1,P}^*(\tau)$ and $S_{2,P}^*(\tau)$ denote the exercise boundary of the one-asset American put option with the underlying asset S_1 and S_2, respectively. We have

$$\lim_{S_2 \to \infty} S_1^*(S_2, \tau) = S_{1,P}^*(\tau) \quad \text{and} \quad \lim_{S_1 \to \infty} S_2^*(S_1, \tau) = S_{2,P}^*(\tau).$$

(ii) Both $S_1^*(S_2, \tau)$ and $S_2^*(S_1, \tau)$ are monotonically decreasing with respect to time to expiry and monotonically increasing with respect to the asset price level.

(iii) The whole line $S_1 = S_2$ is contained completely inside the continuation region.

(iv) At infinitesimally small asset values, we have

$$\lim_{S_1\to 0^+} \frac{S_2^*(S_1,\tau)}{S_1} = 1 \quad \text{and} \quad \lim_{S_2\to 0^+} \frac{S_1^*(S_2,\tau)}{S_2} = 1 \quad \text{for all } \tau. \tag{3.14}$$

All exercise boundaries tend asymptotically to the line $S_1 = S_2$ as S_1 and S_2 both tend to zero.

The intuition behind the asymptotic properties stated in part (i) of Proposition 3.4 is quite obvious. When $S_1 \to \infty, P_{min}(S_1, S_2, \tau; K) \longrightarrow P(S_2, \tau; K)$, where $P(S_2, \tau; K)$ denotes the price function of the one-asset American put option with underlying asset S_2. We would expect that both option models follow the same optimal exercise strategy, thus leading to the asymptotic properties stated in (i). The proof of these asymptotic properties can be pursued by following similar arguments used in the proof of Proposition 4.8 in Villeneuve's paper [20]. Also, the monotonicity properties of $S_1^*(S_2, \tau)$ and $S_2^*(S_1, \tau)$ have been discussed in other papers (say [2] and [20]). Property (iii) states that when $S_1 = S_2$, it is never optimal to exercise the two-asset American minimum put option. This optimal exercise policy is similar to that of the two-asset American maximum call option. The proof of (iii) can follow a similar argument presented by Detemple *et al*. [5] on the American maximum call option. The proof of the asymptotic behavior of the exercise boundaries at $S_1 \to 0$ and $S_2 \to 0$ requires specifically the technique developed in the proof of property (iii) in Proposition 3.3. The proof of part (iv) of Proposition 3.4 is presented in Appendix F.

4. Conclusion. This paper demonstrates the richness of the optimal exercise behaviors adopted by holders of the American options with payoff structures involving lookback state variables. The analysis of the optimal exercise policies of an American lookback option is complicated by the presence of an additional lookback state variable. For fixed strike lookback options, we characterize the exercise behaviors by analyzing the analytic properties of the stopping region and continuation region in the two-dimensional state space (asset price and lookback state variable). For floating strike lookback options, the dimension of the pricing model can be reduced by one if the asset price is used as the numeraire. We reveal the close relationship between the price functions of the finite-time Russian option and the dynamic protection fund with withdrawal right. For the American put option on the minimum value of two assets, the exercise region consists of two branches of exercise surfaces. Compared to earlier works, our analyses provide more comprehensive understanding of the optimal exercise policies of commonly traded American lookback options. In particular, we provide more precise description of the asymptotic behaviors of the exercise boundaries. All the optimal exercise policies of American lookback options derived from our theoretical studies have been verified by plots of the exercise boundaries obtained via numerical calculations.

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APPENDIX A — Proof of Proposition 2.1

(i) First, we show that if $(\eta, \tau) \in \mathcal{S}^C$, then $(\eta, \lambda_2\tau) \in \mathcal{S}^C$ for $\lambda_2 \geq 1$. By applying the comparison principle, one can show that $\dfrac{\partial \widetilde{C}_{f\ell}}{\partial \tau} > 0$. This is consistent with the financial intuition that the price function of any American option is an increasing function of τ. Suppose (η, τ) lies in the continuation region, then $\widetilde{C}_{f\ell}(\eta, \tau) > \alpha - \eta$. By virtue of $\dfrac{\partial \widetilde{C}_{f\ell}}{\partial \tau} > 0$, we deduce that $\widetilde{C}_{f\ell}(\eta, \lambda_2\tau) > \alpha - \eta$ for $\lambda_2 \geq 1$. Hence, $(\eta, \lambda_2\tau)$ also lies in the continuation region.

Next, we show that if $(\eta, \tau) \in \mathcal{S}^C$, then $(\lambda_1\eta, \tau) \in \mathcal{S}^C$ for $\lambda_1 \geq 1$. If suffices to show that

$$\frac{\partial}{\partial \eta}[\widetilde{C}_{f\ell}(\eta, \tau) - (\alpha - \eta)] \geq 0. \tag{A.1}$$

We write $U(\eta, \tau) = \widetilde{C}_{f\ell}(\eta, \tau) - (\alpha - \eta)$, then the linear complementarity formulation for $U(\eta, \tau)$ is given by

$$\frac{\partial U}{\partial \tau} - \widetilde{\mathcal{L}}U \geq r\eta - q\alpha, \quad U \geq 0,$$
$$\left(\frac{\partial U}{\partial \tau} - \widetilde{\mathcal{L}}U\right) U = 0, \quad 0 < \eta < 1, \tau > 0,$$

with auxiliary conditions:

$$\left.\frac{\partial U}{\partial \eta}\right|_{\eta=1} = 1 \quad \text{and} \quad U(\eta, 0) = (\eta - \alpha)^+.$$

Both the initial condition $(\eta - \alpha)^+$ and the non-homogeneous term $r\eta - q\alpha$ are increasing functions of η, and $\left.\dfrac{\partial U}{\partial \eta}\right|_{\eta=1} > 0$. By virtue of the comparison principle, we deduce that $\dfrac{\partial U}{\partial \eta} \geq 0$.

(ii) We prove by contradiction. Suppose there exists $\tau_0 > 0$ such that $(1, \tau_0) \in S$, by applying Eq. (A.1), we can show that $(\eta, \tau_0) \in S$ for $\eta < 1$. We then have

$$\widetilde{C}_{f\ell}(\eta, \tau_0) = \alpha - \eta, \quad \eta < 1.$$

This implies

$$\frac{\partial \widetilde{C}_{f\ell}}{\partial \eta} = -1 \quad \text{at} \quad (1, \tau_0),$$

which contradicts the Neumann boundary condition stated in Eq. (2.6).

(iii) A necessary condition for (η, τ) lying inside $\mathcal{S}$ is given by

$$\left(\frac{\partial}{\partial \tau} - \widetilde{\mathcal{L}}\right)(\alpha - \eta) = \alpha q - r\eta \geq 0,$$

that is, $\eta \leq \dfrac{q}{r}\alpha$. Hence, we should have $\eta^*(0^+) \leq \dfrac{q}{r}\alpha$. Since the exercise payoff must be non-negative, so another necessary condition is given by $\eta \leq \alpha$.

Lastly, the feasible region for η is $\{\eta : \eta \leq 1\}$. Combining all three necessary conditions, we should have

$$\eta^*(0^+) \leq \min\left(1, \alpha, \frac{q}{r}\alpha\right).$$

Suppose $\eta^*(0^+) < \min\left(1, \alpha, \frac{q}{r}\alpha\right)$, then for $\eta \in \left(\eta^*(0^+), \min\left(1, \alpha, \frac{q}{r}\alpha\right)\right)$, we have

$$\left.\frac{\partial \widetilde{C}_{f\ell}}{\partial \tau}\right|_{\tau=0} = \left.\widetilde{\mathcal{L}}\widetilde{C}_{f\ell}\right|_{\tau=0} = \widetilde{\mathcal{L}}(\alpha - \eta) = r\eta - \alpha q < 0.$$

This contradicts with $\frac{\partial \widetilde{C}_{f\ell}}{\partial \tau} \geq 0$ for all τ. Hence, we obtain

$$\eta^*(0^+) = \min\left(1, \alpha, \frac{q}{r}\alpha\right).$$

APPENDIX B — Proof of Proposition 2.2

(i) Write $\eta^*_\infty(\alpha) = \lim_{\tau\to\infty} \eta^*(\tau;\alpha)$ and $\widetilde{C}^\infty_{f\ell}(\eta) = \lim_{\tau\to\infty} \widetilde{C}_{f\ell}(\eta,\tau)$, then $\widetilde{C}^\infty_{f\ell}(\eta)$ satisfies the following differential equation:

$$\widetilde{\mathcal{L}}\ \widetilde{C}^\infty_{f\ell} = 0, \quad \eta^*_\infty < \eta < 1,$$

subject to the auxiliary conditions:

$$\widetilde{C}^\infty_{f\ell}(\eta^*_\infty) = \alpha - \eta^*_\infty, \quad \frac{\partial \widetilde{C}_{f\ell}}{\partial \eta}(\eta^*_\infty) = -1, \quad \frac{\partial \widetilde{C}_{f\ell}}{\partial \eta}(1) = 0.$$

The general solution to $\widetilde{C}^\infty_{f\ell}(\eta)$ is given by

$$\widetilde{C}^\infty_{f\ell}(\eta) = A_1\eta^{\lambda_+} + A_2\eta^{\lambda_-}, \quad \eta^*_\infty < \eta < 1.$$

Applying the auxiliary conditions, we obtain

$$A_1 = \frac{(1-\lambda_-)\eta^*_\infty + \lambda_-\alpha}{(\lambda_- - \lambda_+)(\eta^*_\infty)^{\lambda_+}} \quad \text{and} \quad A_2 = \frac{(1-\lambda_+)\eta^*_\infty + \lambda_+\alpha}{(\lambda_+ - \lambda_-)(\eta^*_\infty)^{\lambda_-}},$$

and η^*_∞ satisfies the non-linear algebraic equation

$$(\eta^*_\infty)^{\lambda_+ - \lambda_-} = \frac{\lambda_+}{\lambda_-}\frac{(1-\lambda_-)\eta^*_\infty + \lambda_-\alpha}{(1-\lambda_+)\eta^*_\infty + \lambda_+\alpha}. \tag{B.1}$$

The above algebraic equation has two roots, one lies in $(0,1)$ and the other lies in $(1,\infty)$ (the proof of these properties can be found in [3]). Here, η^*_∞ corresponds to the root in $(0,1)$. Hence, the results in part (i) are established.

(ii) When $\alpha \to \infty$, the non-linear algebraic equation (B.1) reduces to

$$(\eta^*_\infty)^{\lambda_+ - \lambda_-} = 1$$

so that the solution for η^*_∞ becomes 1. Also, $\eta^*(0^+) = 1$ when α becomes sufficiently large. Since $\eta^*(\tau)$ is monotonically decreasing with respect to τ, and $\eta^*(0^+) = \eta^*(\infty) = 1$ as $\alpha \to \infty$, we can deduce that

$$\lim_{\alpha\to\infty} \eta^*(\tau;\alpha) = 1 \quad \text{for all } \tau.$$

APPENDIX C — Proof of Proposition 3.1

(i) Define the function $V(S, M, \tau; K) = C_{fix}(S, M, \tau; K) + K$. Similar to Eqs. (3.1–3.2), the linear complementarity formulation for $V(S, M, \tau; K)$ is given by

$$\frac{\partial V}{\partial \tau} - \mathcal{L}V \geq rK, \quad V \geq \max(M, K)$$
$$\left[\frac{\partial V}{\partial \tau} - \mathcal{L}V - rK\right][V - \max(M, K)] = 0,$$

with auxiliary conditions:

$$\left.\frac{\partial V}{\partial M}\right|_{S=M} = 0 \quad \text{and} \quad V(S, M, 0; K) = \max(M, K).$$

By virtue of the comparison principle, we have

$$V(S, M, \tau; K_1) \geq V(S, M, \tau; K_2) \quad \text{if} \quad K_1 > K_2,$$

and hence the result.

(ii) From (i), for $K_1 > K_2$, we have

$$(C.1) \qquad C_{fix}(S, M, \tau; K_1) - (M - K_1) \geq C_{fix}(S, M, \tau; K_2) - (M - K_2).$$

Suppose $(S, M, \tau) \in \mathcal{S}^C(K_2)$, where $\mathcal{S}^C(K_2)$ denotes the continuation region. In the continuation region, the option value is strictly greater than the exercise payoff so that

$$C_{fix}(S, M, \tau; K_2) > M - K_2.$$

Combining with Inequality (C.1), we can deduce

$$C_{fix}(S, M, \tau; K_1) > M - K_1,$$

so that $(S, M, \tau) \in \mathcal{S}^C(K_1)$. Hence, we establish $\mathcal{S}^C(K_2) \subset \mathcal{S}^C(K_1)$; and so $\mathcal{S}(K_1) \subset \mathcal{S}(K_2)$.

(iii) Since $C_{fix}(S, M, \tau)$ is monotonically increasing with respect to both S and τ, and the exercise payoff is independent of S and τ, we deduce that if $(S, M, \tau) \in \mathcal{S}(K)$, then

$$(\lambda_1 S, M, \lambda_3 \tau) \in \mathcal{S}(K) \quad \text{for all} \quad 0 < \lambda_1 \leq 1 \text{ and } 0 < \lambda_3 \leq 1.$$

Next, we would like to show that $(S, M, \tau) \in \mathcal{S}(K)$ would imply $(S, \lambda_2 M, \tau) \in \mathcal{S}(K)$, for all $\lambda_2 \geq 1$. Suppose $(S, M, \tau) \in \mathcal{S}(K)$, then $(S/\lambda_2, M, \tau) \in \mathcal{S}(K)$ for $\lambda_2 \geq 1$. Furthermore, by virtue of the linear homogeneity property of the price function and the price function and the result in (i), we obtain

$$\begin{aligned} C_{fix}(S, \lambda_2 M, \tau; K) &= \lambda_2 C_{fix}\left(\frac{S}{\lambda_2}, M, \tau; \frac{K}{\lambda_2}\right) \\ &\leq \lambda_2 \left[C_{fix}\left(\frac{S}{\lambda_2}, M, \tau; K\right) + \left(1 - \frac{1}{\lambda_2}\right)K\right] \\ &= \lambda_2 \left[M - K + \left(1 - \frac{1}{\lambda_2}\right)K\right] = \lambda_2 M - K. \end{aligned}$$

On the other hand, the option value $C_{fix}(S, \lambda_2 M, \tau; K)$ cannot fall below the exercise payoff $\lambda_2 M - K$. Combining the results, we then have

$$C_{fix}(S, \lambda_2 M, \tau; K) = \lambda_2 M - K,$$

that is, $(S, \lambda_2 M, \tau) \in \mathcal{S}(K)$. Hence, we obtain the desired result.

APPENDIX D — Proof of Proposition 3.2

(iii) It is clear that $M^*(0^+, \tau; K) \geq K$. From the monotonic increasing property of $M^*(S, \tau; K)$ with respect to S, suppose we can show that the line $M = M_0$ lies in the stopping region in the S-M plane for any $M_0 > K$, then one can deduce that $M^*(S, \tau; K) \to K$ as $S \to 0^+$. This is because the minimum value of $M^*(S, \tau; K)$ is achieved when S is approaching zero from above, and this minimum value is K. We write $U_{fix}(S, \tau) = C_{fix}(S, M_0, \tau) - (M_0 - K)$. The linear complementarity formulation of $U_{fix}(S, \tau)$ is given by

$$\left(\frac{\partial}{\partial \tau} - \mathcal{L}\right) U_{fix} \geq -r(M_0 - K), \quad U_{fix} \geq 0,$$
$$\left[\left(\frac{\partial}{\partial \tau} - \mathcal{L}\right) U_{fix}\right] U_{fix} = 0$$

with initial condition: $U_{fix}(S, 0) = 0$. Since the right-hand term $-r(M_0 - K)$ is always negative and the initial value has compact support, we apply the theorem by Brezis and Friedman [1] that the solution $U_{fix}(S, \tau)$ has compact support too. The stopping region is non-empty, that is, there exists (S, τ) such that $C_{fix}(S, M_0, \tau) = M_0 - K$ for any $M_0 > K$. Hence, the line $M = M_0 \in \mathcal{S}(K)$ for any $M_0 > K$.

(iv) When $K = 0$, the American fixed strike lookback call is the same as the American floating strike lookback put [with $\alpha = 0$ in Eq. (2.8)]. The monotonically increasing property of $\xi^*(\tau) = M^*(S, \tau; 0)/S$ follows directly from Proposition 2.3(i).

For $K > 0$, by virtue of the linear homogeneity property of $M^*(S, \tau; K)$, we obtain

$$\lim_{S \to \infty} \frac{M^*(S, \tau; K)}{S} = \lim_{S \to \infty} \frac{M^*\left(\frac{S}{K}, \tau; 1\right)}{\frac{S}{K}} = \lim_{K \to 0} \frac{M^*\left(\frac{S}{K}, \tau; 1\right)}{\frac{S}{K}}$$
$$= \lim_{K \to 0} \frac{M^*(S, \tau; K)}{S} = \frac{M^*(S, \tau; 0)}{S}.$$

APPENDIX E — Proof of Proposition 3.3

(iv) First, we consider the proof with $q > 0$, whose arguments rely on the existence of $\eta^*(\tau; \alpha)$. Since $\eta^*(\tau; \alpha)$ does not exist when $q = 0$, we will deal with the special case of zero dividend separately later. For $\alpha \geq 1$, we observe that

$$(K - m)^+ \leq (K - \alpha S)^+ + \alpha S - m$$

so that

$$P_{fix}(S, m, \tau; K) \leq \alpha P\left(S, \tau; \frac{K}{\alpha}\right) + C_{f\ell}(S, m, \tau; \alpha), \tag{E.1}$$

where $P\left(S,\tau;\dfrac{K}{\alpha}\right)$ denotes the price function of the American vanilla put option with strike price $\dfrac{K}{\alpha}$. Let $S_P^*\left(\tau;\dfrac{K}{\alpha}\right)$ be the critical asset price of the American vanilla put with payoff $\left(\dfrac{K}{\alpha}-S\right)^+$. Consider the point $(\widehat{S},\widehat{m})$ in the S-m plane which lies inside the region

$$R_\alpha = \left\{(S,m): \quad m \le S\eta^*(\tau;\alpha) \qquad \text{and} \quad S \le S_P^*\left(\tau;\frac{K}{\alpha}\right)\right\},$$

$(\widehat{S},\widehat{m})$ lies in the corresponding stopping region of both the American floating strike call and American vanilla put. We then have

$$(E.2) \qquad P\left(\widehat{S},\tau;\frac{K}{\alpha}\right) = \frac{K}{\alpha} - \widehat{S} \quad \text{and} \quad C_{f\ell}(\widehat{S},\widehat{m},\tau;\alpha) = \alpha\widehat{S} - \widehat{m}.$$

Now, we argue that $(\widehat{S},\widehat{m})$ also lies in the stopping region of the American fixed strike put. To establish the claim, it suffices to show that

$$(E.3) \qquad P_{fix}(\widehat{S},\widehat{m},\tau;K) = K - \widehat{m}.$$

Combining the results in Eqs. (E.1) and (E.2), we obtain $P_{fix}(\widehat{S},\widehat{m},\tau;K) \le K - \widehat{m}$. Since the option value of the American fixed strike put cannot fall below its exercise payoff, the result in Eq. (E.3) is then established.

Lastly, we take the limit $\alpha \to \infty$ and observe that

$$\lim_{\alpha\to\infty} \eta^*(\tau;\alpha) = 1 \quad \text{and} \quad \lim_{\alpha\to\infty} S_P^*\left(\tau;\frac{K}{\alpha}\right) = 0$$

for all τ. As $\alpha \to \infty$, R_α shrinks to an infinitesimally small triangular wedge with the oblique side: $S = m$. Hence, we can deduce that as $S \to 0^+$ and for all values of τ, all the exercise boundaries $m^*(S,\tau;K)$ tend to the oblique asymptotic line: $S = m$.

Lastly, we consider the case where $q = 0$. We add the parameter q in the price function $P_{fix}(S,m,\tau;K,q)$ and exercise boundary $m^*(S,\tau;q)$, and write the corresponding stopping region as $\mathcal{S}(q)$ with dependence on q. From the pricing property

$$P_{fix}(S,m,\tau;K,0) \le P_{fix}(S,m,\tau;K,q),$$

we deduce that

$$\mathcal{S}(q) \subset \mathcal{S}(0), \quad q > 0.$$

Hence, we have $m^*(S,\tau;0) \ge m^*(S,\tau;q)$ so that

$$\frac{m^*(S,\tau;q)}{S} \le \frac{m^*(S,\tau;0)}{S} \le 1, \quad q > 0.$$

Since we have established $\dfrac{m^*(S,\tau;q)}{S} \to 1$ as $S \to 0$, so $\displaystyle\lim_{S\to 0^+} \frac{m^*(S,\tau;0)}{S} = 1$.

APPENDIX F — Proof of Proposition 3.4

(iii) We only show the proof of

$$\lim_{S_1 \to 0^+} \frac{S_2^*(S_1, \tau)}{S_1} = 1.$$

The proof of the other limiting property in Eq. (3.14) can be pursued in a similar manner. Following a similar approach in Appendix E, we employ the following inequality

$$(F.1) \qquad (K - \min(S_1, S_2))^+ \leq (K - \alpha S_2)^+ + (\alpha S_2 - \min(S_1, S_2))^+,$$

and examine the stopping region $\widehat{\mathcal{S}}_\alpha$ of the American two-asset option with payoff $(\alpha S_2 - \min(S_1, S_2))^+$. Also, we let $S_{2,P}^*$ be the critical asset price of the American put with payoff $\left(\frac{K}{\alpha} - S_2\right)^+$. By applying inequality (F.1) and following a similar argument presented in Appendix E, one can show that the stopping region of the two-asset American minimum put option is contained inside

$$\overline{R}_\alpha = \left\{(S_1, S_2) : (S_1, S_2) \in \widehat{\mathcal{S}}_\alpha \quad \text{and} \quad S_2 \leq S_{2,P}^*\left(\tau; \frac{K}{\alpha}\right)\right\}.$$

The asymptotic behavior of $S_2^*(S_1, \tau)$ at infinitesimally small value of S_1 is established once we can show that the boundaries of $\overline{R}_\alpha$ are bounded by the line $S_1 = S_2$ as $\alpha \to \infty$.

Let V_α denote the price function of the American two-asset option with payoff $(\alpha S_2 - \min(S_1, S_2))^+, \alpha \geq 1$. We let $x = S_1/S_2$ and define $W_\alpha = V_\alpha/S_2$. The exercise boundary of the American option model $W_\alpha(x, \tau)$ has two branches, and let them be denoted by $x_h^*(\tau)$ and $x_\ell^*(\tau)$. The continuation region is represented by $\{(x, \tau) : x_\ell^*(\tau) < x < x_h^*(\tau), 0 \leq \tau < \infty\}$. The linear complementarity formulation of $W_\alpha(x, \tau)$ is given by

$$\frac{\partial W_\alpha}{\partial \tau} - \frac{1}{2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)x^2 \frac{\partial^2 W_\alpha}{\partial x^2} - (q_2 - q_1)x \frac{\partial W_\alpha}{\partial x} + q_2 W_\alpha = 0,$$
$$x_\ell^*(\tau) < x < x_h^*(\tau), \quad \tau > 0,$$

with auxiliary conditions:

$$W_\alpha(x_\ell^*, \tau) = \alpha - x_\ell^*, \qquad \frac{\partial W_\alpha}{\partial x}(x_\ell^*, \tau) = -1,$$
$$W_\alpha(x_h^*, \tau) = \alpha - 1, \qquad \frac{\partial W_\alpha}{\partial x}(x_h^*, \tau) = 0,$$
$$W_\alpha(x, 0) = \begin{cases} \alpha - x & \text{if } x \leq 1 \\ \alpha - 1 & \text{if } x > 1 \end{cases}.$$

For $q_2 > 0$, one can show that $x_\ell^*(\tau)$ and $x_h^*(\tau)$ are monotonic functions of τ. Also, $x_\ell^*(0^+) = x_h^*(0^+) = 1$ when $\alpha > \frac{q_1}{q_2}$. Similar to Property (ii) in Proposition 2.2, we would like to establish the following asymptotic results

$$(F.2) \qquad \lim_{\alpha \to \infty} x_\ell^*(\tau; \alpha) = 1 \quad \text{and} \quad \lim_{\alpha \to \infty} x_h^*(\tau; \alpha) = 1$$

so that the boundary of $\overline{R}_\alpha$ will be bounded by $S_1 = S_2$ as $\alpha \to \infty$. By virtue of the monotonicity properties of $x_\ell^*(\tau)$ and $x_h^*(\tau)$ with respect to τ, the asymptotic properties in (F.2) are valid if we can show

$$\lim_{\alpha\to\infty} x_\ell^*(\infty;\alpha) = 1 \quad \text{and} \quad \lim_{\alpha\to\infty} x_h^*(\infty;\alpha) = 1. \tag{F.3}$$

When $q_2 = 0, x_\ell^*(\tau)$ does not exist but $\lim_{\alpha\to\infty} x_h^*(\tau;\alpha) = 1$ remains valid. The arguments in the proof presented below have to be modified slightly for this degenerate case.

The proof of Eq. (F.3) requires the solution of $W_\alpha^\infty(x)$, the perpetual limit of $W_\alpha(x,\tau)$. The governing equation for $W_\alpha^\infty(x)$ is given by

$$\frac{1}{2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)x^2\frac{d^2W_\alpha^\infty}{dx^2} + (q_2 - q_1)x\frac{dW_\alpha^\infty}{dx} - q_2W_\alpha^\infty = 0,$$
$$x_\ell^*(\infty) < x < x_h^*(\infty),$$

with auxiliary conditions:

$$W_\alpha^\infty(x_\ell^*(\infty)) = \alpha - x_\ell^*(\infty), \qquad \frac{dW_\alpha^\infty}{dx}(x_\ell^*(\infty)) = -1,$$
$$W_\alpha^\infty(x_h^*(\infty)) = \alpha - 1, \qquad \frac{dW_\alpha^\infty}{dx}(x_h^*(\infty)) = 0.$$

By following a similar approach in Appendix B, we can show that

$$\lim_{\alpha\to\infty} \frac{x_h^*(\infty;\alpha)}{x_\ell^*(\infty;\alpha)} = 1,$$

and hence the relations in Eq. (F.3) are established.

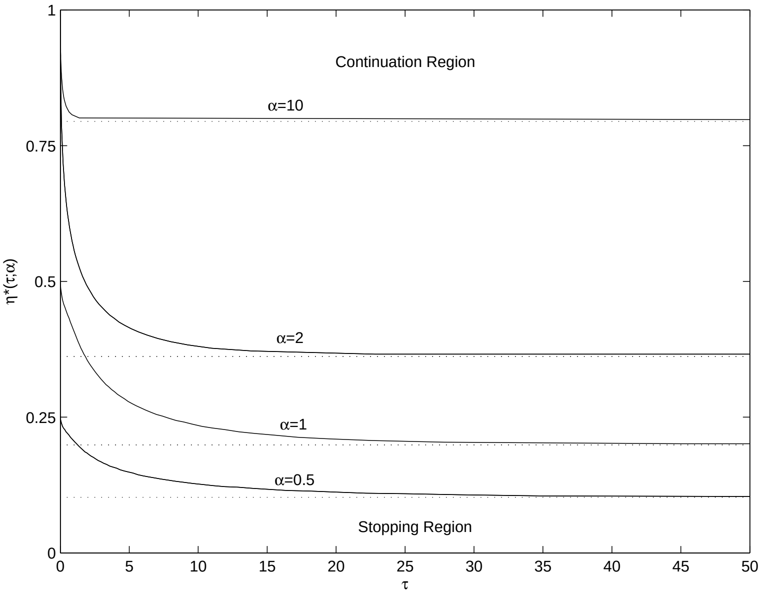

FIG 1. The critical threshold $\eta^*(\tau;\alpha)$ of the American floating strike lookback call option is plotted against τ for different values of α. The parameter values of the pricing model are: $r = 0.04, q = 0.02$ and $\sigma = 0.3$.

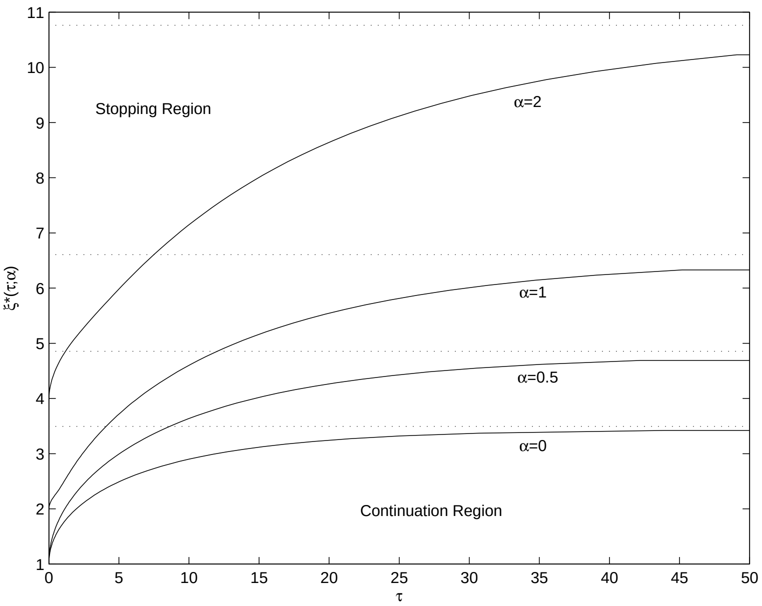

FIG 2. The critical threshold $\xi^*(\tau;\alpha)$ of the American floating strike lookback put option is plotted against τ for different values of α. The parameter values of the pricing model are: $r = 0.02, q = 0.04$ and $\sigma = 0.3$.

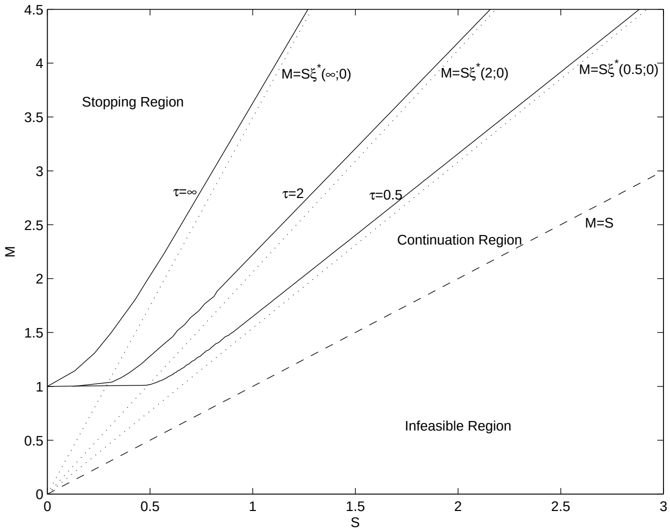

FIG 3. The exercise boundaries (solid curves) of the American fixed strike lookback call option with varying values of maturity τ are plotted in the S-M plane. At a given τ, the stopping region is lying to the left and above of the corresponding exercise boundary. The dotted lines are asymptotic lines of the exercise boundaries, corresponding to the exercise boundaries of the zero-strike counterparts. The stopping region of the Russian option lies to the left of the dotted line: $M = S\xi^*(\infty; 0)$. The parameter values used in the calculations are: $K = 1, r = 0.02, q = 0.04$ and $\sigma = 0.3$.

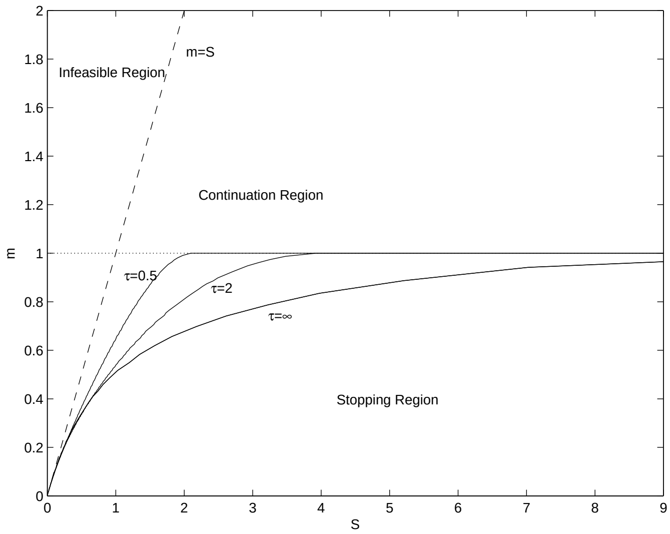

FIG 4. The exercise boundaries of the American fixed strike lookback put option with varying values of maturity τ are plotted in the S-m plane. All exercise boundaries tend to the oblique asymptotic line: $m = S$ as $S \to 0^+$, and the horizontal asymptotic line: $m = K$ as $S \to \infty$. The parameter values used in the calculations are: $K = 1, r = 0.04, q = 0.02$ and $\sigma = 0.3$.

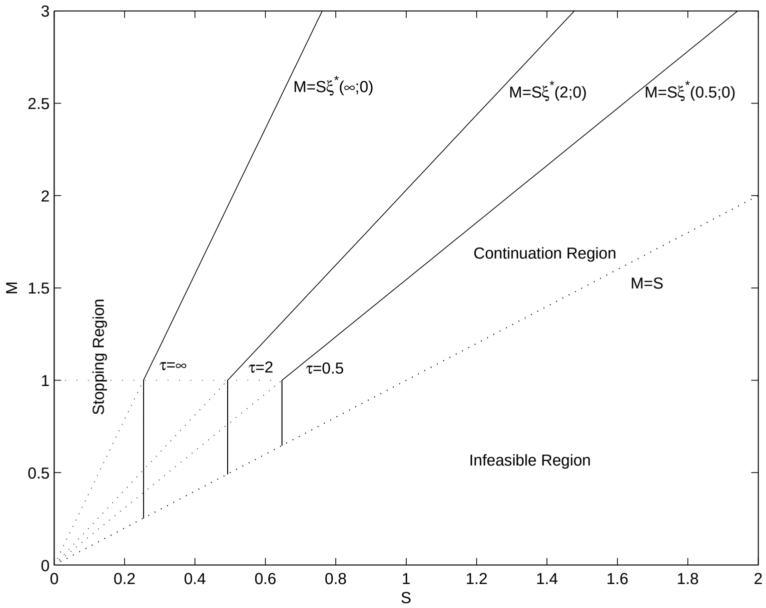

FIG 5. The exercise boundaries of the American option with payoff function $\max(M, K)$ with varying values of maturity τ are plotted in the S-M plane. The parameter values used in the calculations are: $K = 1, r = 0.02, q = 0.04$ and $\sigma = 0.3$.

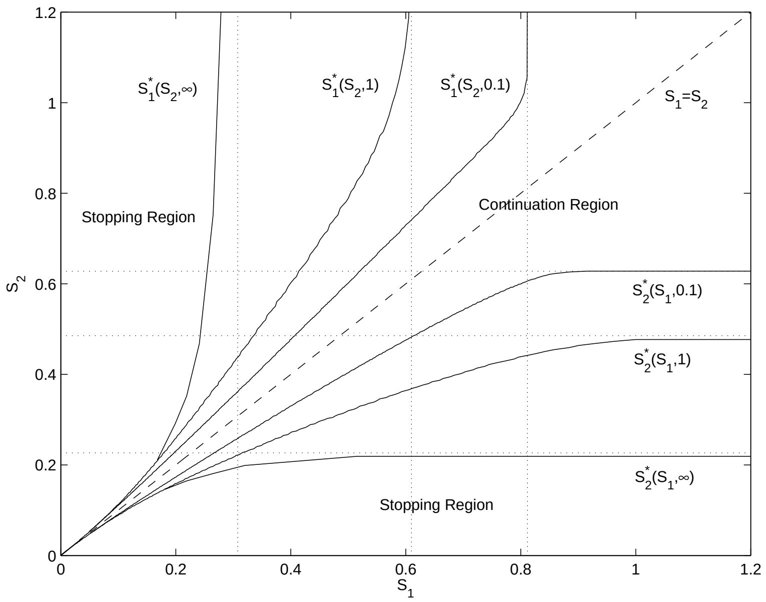

FIG 6. The exercise boundaries of the two-asset American minimum put option with varying values of maturity τ are plotted in the S_1-S_2 plane. The continuation region is bounded between the two branches of the exercise boundaries. The parameter values used in the calculations are: $K = 1, r = 0.02, q_1 = 0, q_2 = 0.03, \sigma_1 = \sigma_2 = 0.3$ and $\rho = 0.5$.