Options with combined reset rights on strike and maturity

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Abstract

Reset clauses on the strike price and maturity date are commonly found in derivative contracts, like insurance segregated funds, bonds and executive warrants. We analyze the optimal reset policy adopted by the holder of an option that possesses the reset rights on the strike price and date of maturity. The optimal reset policy relates closely to the temporal rate of change of the value of the new option received by the holder at the reset moment. The characterization of the optimal reset policy requires the solution of a free boundary value problem. As part of the solution procedure, we determine the critical asset price for a given time to expiry at which the holder chooses to activate the reset clause optimally. Depending on the specific nature of the reset clauses, the reset policies exhibit a wide variety of behaviors. We also manage to obtain analytic price formulas for several specific types of reset options.

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1 Introduction

Reset clauses on both the strike price and date of maturity are commonly found in derivative contracts. The reset rights can be exercised at any time

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during the life of the contract or only limited to some pre-determined dates. Also, the allowable number of resets may be more than one. Early examples of financial contracts with reset rights are extendible bonds issued by the Canadian Government in 1950's. In these bonds, the holder possesses the right to extend the bond's maturity on or before a fixed date. Besides the federal government, provincial governments and private firms in Canada also issued extendible bonds (Ananthanarayanan and Schwartz, 1980). Longstaff (1990) considers a wide variety of extendible options with applications including American options with stochastic dividends, shared equity mortgages and debt negotiation under financial distress of the issuer. Windcliff *et al.* (2001) perform a comprehensive analysis of the Canadian segregated funds. These funds allow the holder to reset the guarantee level and the maturity date a preset number of times during the life of the fund contract.

In this paper, we consider the pricing issues of put options with reset rights on the strike price and maturity date, and in particular, we examine the optimal reset policy to be adopted by the holder. We assume that the reset can be exercised at any time but only once during the life of the contract. Upon reset, the new strike becomes a multiple of the prevailing asset price. This preset multiplicative factor can be greater than, equal to or less than one. For maturity extension, we consider extension of a fixed time period beyond the original maturity date. The present models appear to be somewhat simplified version of the real life derivative contracts cited above. However, they represent extensions of the reset option models analyzed in our earlier paper (Dai *et al.*, 2004), in which we consider simpler situation where there is no maturity extension and the reset strike price is constrained to be equal to the prevailing price of the underlying asset. The simplified assumptions adopted in our models allow us to perform thorough analysis of the optimal reset policies. The analytic results would provide some insight on the understanding of the reset features embedded in various derivative contracts traded in the market.

Since the reset can occur at any time, the option pricing model leads to a free boundary value problem. As part of the solution of the pricing model, we have to determine the optimal reset policy. This amounts to the determination of the critical asset price at which the holder should choose to activate the reset optimally. The optimal reset policy would depend on the relative magnitude of the parameter values in the pricing model. These parameters include the riskless interest rate, dividend yield, volatility of the asset price process, original strike price, multiplicative factor in determining the new strike price, time to expiry and the duration of maturity extension. Depending on the values of these parameters, our analysis shows that the critical asset price may exist for all times or only over certain time interval or does not exist at all (that is, the holder should never activate the reset).

The paper is organized as follows. In the next section, we establish the linear complementarity formulation of the pricing model with reset rights on both strike and maturity. We then examine the time dependent behaviors of the price function of the new option received upon reset. The temporal monotonicity properties of the price function of the new option and its time derivative play important roles in the determination of the optimal reset policies. In Section 3, we consider the pricing behaviors of put options with only strike reset. The optimal reset policies depend on whether the multiplicative factor in strike reset is greater than one or otherwise. The most interesting phenomena in the optimal reset policies occur in those cases where the riskless interest rate is greater than the dividend yield and the mutiplicative factor is greater than one. In some special cases, we manage to obtain closed form solution of the price functions. In Section 4, we consider put options with reset rights on both the strike price and maturity date. A wide variety of optimal reset policies are exhibited, depending on the specific nature of the reset clause and parameter values in the reset option model. The paper is concluded with a summary of the main results.

2 Mathematical formulation of the pricing models

In our pricing models for the put option with reset rights on strike or maturity or both, we follow the usual Black-Scholes risk neutral valuation framework. The asset value S is assumed to follow the risk neutral lognormal diffusion process

$$\frac{dS}{S} = (r-q)dt + \sigma\, dZ, \tag{1}$$

where r and q are the constant riskless interest rate and dividend yield, respectively, Z is the standard Wiener process and σ is the volatility. We let X and T denote the original strike price and expiration date of the put option, and let $\widehat{X}$ and $\widehat{T}$ denote the new strike price and expiration date upon reset. Let τ and $\widehat{\tau}$ denote the original and new time to expiry, respectively, so that $\tau = T - t$ and $\widehat{\tau} = \widehat{T} - t$, where t denotes the current time.

In this paper, we limit the new strike after reset to be of the form $\widehat{X} = \alpha S_\xi$, where S_ξ is the asset value at the reset moment ξ. Here, the constant multiplicative factor α can be greater than, equal to or less than one. This form of reset strike is commonly found in the majority of derivatives with reset feature. Besides, with such choice of the reset strike price, the price function of the new option received becomes linearly homogeneous in S and it takes the form $SP_\alpha(\widehat{\tau})$, where

$$P_\alpha(\widehat{\tau}) = \alpha e^{-r\tau} N(-\widehat{d}_-) - e^{-q\tau} N(-\widehat{d}_+), \tag{2a}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi,$$

$$\widehat{d}_- = \frac{\ln \frac{1}{\alpha} + \left(r - q - \frac{\sigma^2}{2}\right) \widehat{\tau}}{\sigma\sqrt{\widehat{\tau}}} \quad \text{and} \quad \widehat{d}_+ = \widehat{d}_- + \sigma\sqrt{\widehat{\tau}}. \tag{2b}$$

For maturity reset, we assume the new maturity date to be δ time periods beyond the original maturity date, that is, $\widehat{T} = T + \delta$ or $\widehat{\tau} = \tau + \delta, \delta > 0$.

2.1 Linear complementarity formulation

The critical asset value at which the holder should reset optimally is not known aprior but has to be determined as part of the solution. This leads to a free boundary value problem. Let $V(S, \tau; X, r, q)$ denote the value of the reset put option and write $\delta = \widehat{\tau} - \tau$. The linear complementarity formulation of the price function $V(S, \tau)$ is given by

$$\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV \geq 0, \quad V(S,\tau) \geq SP_\alpha(\tau+\delta), \tag{3a}$$

$$\left[\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV\right] [V(S,\tau) - SP_\alpha(\tau+\delta)] = 0; \tag{3b}$$

with the terminal payoff condition:

$$V(S, 0) = \max(\max(X - S, 0), SP_\alpha(\delta)). \tag{3c}$$

Unlike usual American options, the exercise payoff and terminal payoff in our reset options are different. More specifically, the exercise payoff is time dependent and the exercise payoff right before expiration is different from the terminal payoff. In our subsequent analysis, we derive two necessary conditions for the commencement of optimal reset based on the specific nature of the exercise payoff in the reset option [see Eqs.(5-6)]. Most of the interesting phenomena on the optimal reset policies are dictated by these two necessary conditions and the functional form of the terminal payoff.

When $\delta = 0$, there is no maturity extension. Correspondingly, we observe the property: $SP_\alpha(\tau; r, q) = e^{-q\tau} SP_\alpha(\tau; r-q, 0)$. It can be shown that $V(S, \tau; X, r, q) = e^{-q\tau} V(S, \tau; X, r-q, 0)$. Hence, without loss of generality, it suffices to consider the price function of $V(S, \tau;\ X, r, 0)$ where the dividend yield q is taken to be zero and the interest rate r can be positive or negative. When $\delta > 0$, the obstacle function $SP_\alpha(\tau+\delta; r, q)$ is equal to $e^{-q\delta}[e^{-q\tau} SP_\alpha(\tau+\delta; r-q, 0)]$. Due to the additional factor $e^{-q\delta}$, we also have to rescale the asset price and the strike price in order to obtain a simple relation between the price functions corresponding to non-zero and zero dividend yield. To simplify the analytic procedures, we take $q = 0$ in all subsequent analysis. We should be alerted that when the value of the "redefined" interest rate is negative, this corresponds to the scenario where the "actual" riskless interest rate is less than the dividend yield.

In the remaining part of this section and Section 3, we confine our discussion to put option with reset right on the strike price only. When there is no maturity reset right, we have $\delta = 0$. Next, we would like to deduce two necessary conditions on the properties of $P_\alpha(\tau)$ in order that the decision to reset may be activated optimally.

Consider the function $D(S,\tau) = V(S,\tau) - SP_\alpha(0)$. By substituting $D(S,\tau)$ into Eq. (3a-c) (with $\delta = 0$ and $q = 0$), we obtain

$$\frac{\partial D}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 D}{\partial S^2} - rS\frac{\partial D}{\partial S} + rD \geq 0, \quad D(S,\tau) \geq S[P_\alpha(\tau) - P_\alpha(0)], \tag{4a}$$

$$\left[\frac{\partial D}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 D}{\partial S^2} - rS\frac{\partial D}{\partial S} + rD\right]\{D(S,\tau) - S[P_\alpha(\tau) - P_\alpha(0)]\} = 0, \tag{4b}$$

with the terminal payoff condition:

$$D(S,0) = \max(X - (1 + P_\alpha(0))S, 0). \tag{4c}$$

The value of $D(S,\tau)$ must stay positive for $\tau > 0$. This is because $D(S,\tau)$ must be greater than the corresponding European option with the same terminal payoff and this European option value is always positive for $\tau > 0$. Since $D(S,\tau) = S[P_\alpha(\tau) - P_\alpha(0)]$ upon reset and $D(S,\tau) > 0$ for all $\tau > 0$, we then obtain

$$P_\alpha(\tau) > P_\alpha(0) = \begin{cases} \alpha - 1 & \text{when } \alpha > 1 \\ 0 & \text{when } \alpha \leq 1 \end{cases} \tag{5}$$

to be one of the necessary condition for the commencement of optimal reset.

In the stopping region (the reset right has been activated), the option value $V(S,\tau) = SP_\alpha(\tau)$ satisfies the first inequalities in Eq. (3a). To be more precise, we have $q = 0$ and the inequality is strict. This leads to

$$\left(\frac{\partial}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2}{\partial S^2} - rS\frac{\partial}{\partial S} + r\right)SP_\alpha(\tau) = SP_\alpha'(\tau) > 0. \tag{6}$$

The positivity of $SP_\alpha'(\tau)$ is the other necessary condition on $P_\alpha(\tau)$ for the activation of strike reset.

2.2 Properties of $P'_\alpha(\tau)$ and $P_\alpha(\tau) - (\alpha - 1)$

As deduced in the last subsection, the function $P_\alpha(\tau)$ must satisfy both conditions: $P'_\alpha(\tau) > 0$ and $P_\alpha(\tau) > P_\alpha(0)$ for the commencement of optimal reset. The properties of sign change of the two functions, $P'_\alpha(\tau)$ and $P_\alpha(\tau) - P_\alpha(0)$, have crucial importance on the determination of the optimal reset policy. When $\alpha \leq 1$, $P_\alpha(\tau) > P_\alpha(0)$ is automatically satisfied for $\tau > 0$ since the option price function $SP_\alpha(\tau)$ is always positive and $P_\alpha(0) = 0$. Hence, it is only necessary to consider the properties of $P_\alpha(\tau) - (\alpha - 1)$, where $\alpha > 1$. In this subsection, we try to analyze in full details all of its interesting phenomena and search for the appropriate set of conditions that determine (i) the sign of $P'_\alpha(\tau)$ for $\alpha > 0$ and (ii) $P_\alpha(\tau) - (\alpha - 1)$ for $\alpha > 1$.

From Eqs. (2a,b), by taking $q = 0$ and substituting $\widehat{\tau}$ by τ, we can express $P_\alpha(\tau)$ as

$$P_\alpha(\tau) = \alpha e^{-r\tau} N(-d_-) - N(-d_+) \tag{7a}$$

where

$$d_- = \gamma_- \sqrt{\tau} + \frac{\beta}{\sqrt{\tau}}, \quad \gamma_- = \frac{r - \frac{\sigma^2}{2}}{\sigma}, \quad \beta = \frac{\ln \frac{1}{\alpha}}{\sigma},$$

$$d_+ = \gamma_+ \sqrt{\tau} + \frac{\beta}{\sqrt{\tau}}, \quad \gamma_+ = \frac{r + \frac{\sigma^2}{2}}{\sigma}. \tag{7b}$$

Suppose we set $\alpha = X/S$, then $SP_\alpha(\tau)$ becomes the price function of a European put. One then visualizes that the analysis of the time dependent behaviors of $P'_\alpha(\tau)$ is directly related to that of the theta of a European put. The derivative of $P_\alpha(\tau)$ is found to be

$$P'_\alpha(\tau) = \alpha e^{-r\tau} \left[-rN(-d_-) + n(-d_-) \frac{\sigma}{2\sqrt{\tau}} \right], \tag{8}$$

where $n(x) = N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

First of all, we see that when $r \leq 0, P'_\alpha(\tau) > 0$ for all $\tau \geq 0$. This is not surprising since an American put without dividend payment is equivalent to a European put when the interest rate is non-positive. The positivity of $P'_\alpha(\tau)$ then follows from the property that the τ-derivative of the value of an American put is always positive. However, when $r > 0$, the sign of $P'_\alpha(\tau)$ depends on the sign behaviors of β and γ_-. Some representative graphs of $P'_\alpha(\tau)$ as functions of τ for different values of α are displayed in Figure 1. The results are summarized in Lemmas 1 and 2, corresponding to $\beta \geq 0$ and $\beta < 0$, respectively.

Lemma 1 *When $r > 0$ and $\beta \geq 0$ (equivalent to $\alpha \leq 1$), there exists a* *unique $\tau^* > 0$ at which $P'_\alpha(\tau)$ changes sign, and that $P'_\alpha(\tau) > 0$ for $\tau < \tau^*$ and $P'_\alpha(\tau) < 0$ for $\tau > \tau^*$.*

Before presenting the next lemma, we first state the definitions of several quantities. Consider the quadratic polynomial: $p_2(\tau) = \gamma_-\gamma_+\tau^2 - [\beta(\gamma_- + \gamma_+) + 1]\tau + \beta^2$, where $\gamma_- \neq 0$, and let τ_1 and τ_2 denote its two real roots, where $\tau_1 < \tau_2$. The discriminant of $p_2(\tau)$ is found to be $\Delta = \beta^2\sigma^2 + 1 + 4\dfrac{\beta r}{\sigma}$. In particular, when $\gamma_- = 0$, the quadratic equation degenerates into a linear equation. We let τ_0 denote the corresponding single root. Define $f(\tau)$ by the relation: $P'_\alpha(\tau) = \alpha e^{-r\tau} f(\tau)$. Note that $f(\tau)$ depends implicitly on the parameter α. When $r > 0$ and $\beta < 0$ (that is, $\alpha > 1$), the sign behaviors of $P'_\alpha(\tau)$ depend on the sign behaviors of γ_-, $f(\tau_0)$, $f(\tau_1)$ and $f(\tau_2)$, etc., the details of which are stated in Lemma 2.

Lemma 2 *When $r > 0$ and $\beta < 0$ (that is, $\alpha > 1$), there are two possibilities for the sign behaviors of $P'_\alpha(\tau)$ for $\tau \geq 0$.*

(1) There may exist a time interval (τ_1^*, τ_2^*) such that $P'_\alpha(\tau) > 0$ when $\tau \in (\tau_1^*, \tau_2^*)$ and $P'_\alpha(\tau) \leq 0$ if otherwise. This occurs only when either one of the following cases occurs:
   (i) $\gamma_- < 0$ and $f(\tau_2) > 0$;
   (ii) $\gamma_- > 0, \beta(\gamma_- + \gamma_+) + 1 > 0, \Delta > 0$ and $f(\tau_1) > 0$;
   (iii) $\gamma_- = 0, \beta(\gamma_- + \gamma_+) + 1 > 0$ and $f(\tau_0) > 0$.

   Here, τ_1 and τ_2 are the two real roots of $p_2(\tau)$ and $\tau_0 = \dfrac{\beta^2\sigma}{2r\beta + \sigma}$.

(2) When none of the above conditions (i)-(iii) are satisfied, then $P'_\alpha(\tau) \leq 0$ for all $\tau \geq 0$.

The proofs of the above two lemmas are quite technical and they are presented in the Appendix. Lemma 2 shows explicitly the existence of an interval (τ_1^*, τ_2^*) such that $P'_\alpha(\tau) > 0$ when certain conditions on the parameter values are satisfied, and state explicitly the conditions under which the τ-derivative of a European put price function may stay positive over an interval of time. Hull (2003) notes that the theta of an in-the-money put on a non-dividend paying asset might be positive. To our knowledge, the full analysis of the theta property of the price function of a European put option has not been explored in the literature.

For $\alpha > 1$, optimal reset can commence only within the time interval where $P'_\alpha(\tau) > 0$ and $P_\alpha(\tau) > \alpha - 1$. Our next goal is to find the conditions under which such time interval may exist. Consider the equation

$$F_\alpha(\tau) = P_\alpha(\tau) - (\alpha - 1) = 0, \quad \alpha > 1 \text{ and } \tau > 0, \tag{9}$$

which may have no root, unique root or multiple roots. Let τ_m^* denote the smallest positive solution to the above equation, if solutions do exist. First, we observe that the value of $F_\alpha(\tau)$ decreases monotonically with α since $\frac{\partial}{\partial\alpha}F_\alpha(\tau) = e^{-r\tau}N(\frac{\ln\alpha - \gamma_+\tau}{\sigma\sqrt{\tau}}) - 1 < 0$ for $\tau > 0$. When $\alpha \to 1^+$, we have $\lim_{\alpha\to 1^+} F_\alpha(\tau) = P_1(\tau) > 0$ for $\tau > 0$. On the other hand, we note that $F_\alpha(0) = 0$ and $F'_\alpha(0) = P'_\alpha(0) < 0$ so that $F_\alpha(0^+) = 0^-$. We then deduce that for $\alpha > 1$, there exists an interval $(0, \tau^*_m)$ such that $F_\alpha(\tau) < 0$, with $F_\alpha(\tau^*_m) = 0$ and $F_\alpha(\tau) > 0$ for some $\tau > \tau^*_m$. Furthermore, if τ^*_m exists for $\alpha = \alpha_2 > 1$, then τ^*_m also exists for α_1 such that $1 < \alpha_1 < \alpha_2$ and τ^*_m decreases as α decreases. In particular, τ^*_m tends to zero as α tends to 1^+. In other words, τ^*_m does exist for α being sufficiently close to but greater than 1.

The existence of τ^*_m for $\alpha > 1$ implies that the curve $y = F_\alpha(\tau)$ starts at zero at $\tau = 0$, falls below the τ-axis over certain time interval, reaches the minima, then increases until the curve eventually crosses the τ-axis and continues to increase for some time beyond τ^*_m. Such prediction of the time dependent behaviors of $F_\alpha(\tau)$ is consistent with the results in Lemma 2 since the curve $y = F_\alpha(\tau)$ can be increasing only over one time interval (τ^*_1, τ^*_2), provided that such interval exists. Suppose τ^*_m exists, the curve $y = F_\alpha(\tau)$ must be increasing for some time interval, otherwise the curve stays below the τ-axis for all $\tau > 0$ and never crosses the τ-axis at $\tau = \tau^*_m$. Hence, we deduce that the existence of τ^*_m implies the existence of the interval (τ^*_1, τ^*_2), where $P'_\alpha(\tau) > 0$ for $\tau \in (\tau^*_1, \tau^*_2)$. Furthermore, we have $\tau^*_1 < \tau^*_m \leq \tau^*_2$. The two conditions: $P'_\alpha(\tau) > 0$ and $F_\alpha(\tau) > 0$ will be satisfied within the interval (τ^*_m, τ^*_2), if such interval exists.

We would like to explore the monotonicity properties of $\tau^*_1(\alpha)$ and $\tau^*_2(\alpha)$ as a function of α. From Lemma 1, we know that there exists a unique $\tau^*(1)$ such that $P'_1(\tau) > 0$ for $\tau \in (0, \tau^*(1))$. Hence, we deduce by the continuity property on α that

$$\lim_{\alpha\to 1^+} \tau^*_2(\alpha) = \tau^*(1) \text{ and } \lim_{\alpha\to 1^+} \tau^*_1(\alpha) = 0. \tag{10}$$

The monotonicity properties of $\tau^*_1(\alpha)$ and $\tau^*_2(\alpha)$ depend on the monotonicity properties of $P'_\alpha(\tau)$ as a function of α. Since the function $f(\tau) = \frac{P'_\alpha(\tau)}{\alpha e^{-r\tau}}$ has the same sign as that of $P'_\alpha(\tau)$, it suffices to consider the corresponding monotonicity property of $f(\tau)$. Now, we consider

$$\frac{\partial}{\partial\alpha}f(\tau;\alpha) = -\frac{e^{-r\tau}n(-d_-)}{\sigma\sqrt{\tau}}\frac{\ln\alpha + \gamma_+\tau}{2\tau}, \tag{11}$$

which is seen to be always negative for $\alpha > 1$. Therefore, the interval (τ^*_1, τ^*_2) where $P'_\alpha(\tau)$ stays positive would shrink as α increases. The width of the interval achieves its maximum value when α tends to one [see Eq. (10)]. In other words, for $\alpha_2 > \alpha_1 > 1$, suppose $\tau_1^*(\alpha_2)$ and $\tau_2^*(\alpha_2)$ exist, we deduce that $\tau_1^*(\alpha_1)$ and $\tau_2^*(\alpha_1)$ exist, and $\tau_1^*(\alpha_1) < \tau_1^*(\alpha_2)$, $\tau_2^*(\alpha_1) > \tau_2^*(\alpha_2)$.

Since the interval (τ_1^*, τ_2^*) shrinks as α increases, we would ask whether τ_m^*, τ_1^* and τ_2^* exist for all values of α greater than one. The answer is "no". The conditions on α in order that τ_m^*, τ_1^* and τ_2^* exist are stated in Theorem 3.

Theorem 3 *The condition on α for the existence of τ_m^*, τ_1^* and τ_2^* are summarized as follows.*

(1) There exists a threshold value α_1^* such that $\tau_1^*(\alpha)$ and $\tau_2^*(\alpha)$ exist for $1 < \alpha \leq \alpha_1^*$, and do not exist if otherwise. At $\alpha = \alpha_1^*$, the interval (τ_1^*, τ_2^*) shrinks to zero width so that

$$\tau_1^*(\alpha_1^*) = \tau_2^*(\alpha_1^*).$$

(2) There exists a threshold value α_m^* such that $\tau_m^*(\alpha)$ exists for $1 < \alpha \leq \alpha_m^*$, and does not exist if otherwise. At $\alpha = \alpha_m^*$, the interval (τ_m^*, τ_2^*) shrinks to zero so that

$$\tau_m^*(\alpha_m^*) = \tau_2^*(\alpha_m^*).$$

The proof of Theorem 3 is presented in the Appendix. The threshold values α_1^* and α_m^* depend on the parameter values r and σ, and they can be determined by numerical iterative procedures.

We performed numerical calculations to verify the properties on $\tau_1^*(\alpha)$, $\tau_2^*(\alpha)$ and $\tau_m^*(\alpha)$ as stated in Theorem 3, and the numerical results are plotted in Figures 1 and 2. Figure 1 shows the plot of $P_\alpha'(\tau)$ against τ for varying values of α, where $\alpha > 1$. The parameter values used in the calculations are: $r = 0.04$, $q = 0$, $\sigma = 0.2$. We observe that $P_\alpha'(\tau)$ stays positive for some interval (τ_1^*, τ_2^*) when $1 < \alpha < 1.18$. That is, for this set of values for r, q and σ, we obtain $\alpha_1^* = 1.18$. The width of the interval (τ_1^*, τ_2^*) is seen to shrink in size with increasing value of α.

The dependence of τ_1^*, τ_2^* and τ_m^* on α are shown in Figure 2. We used the same set of parameter values as those for plotting Figure 1. The plots verify that the interval (τ_1^*, τ_2^*) shrinks in size monotonically with increasing α, and its width becomes zero at $\alpha_1^* = 1.18$. This is revealed by the intersection of the two curves of $\tau_1^*(\alpha)$ and $\tau_2^*(\alpha)$ at $\alpha = \alpha_1^*$. The curves of $\tau_m^*(\alpha)$ always lie within the two curves of $\tau_1^*(\alpha)$ and $\tau_2^*(\alpha)$, which agrees with the property that τ_m^* always lies within (τ_1^*, τ_2^*). The curve of $\tau_m^*(\alpha)$ is monotonically increasing and it intersects that of $\tau_1^*(\alpha)$ at $\alpha_m^* = 1.15$. The plots clearly reveal the existence of the threshold values α_1^* and α_m^*, where the interval (τ_1^*, τ_2^*) exists only for $1 < \alpha < \alpha_1^*$ and $\tau_m^*(\alpha)$ exists only for $1 < \alpha < \alpha_m^*$. In addition, we have $\alpha_m^* < \alpha_1^*$.

3 Optimal reset policies for options with strike reset right

In this section, we would like to examine the characterization of the optimal reset policies of put options with the strike reset right, where the new strike is given by a multiplicative factor α times the prevailing asset price at the reset moment. This amounts to the determination of the critical asset price $S^*(\tau)$ at which the holder should reset optimally whenever $S \geq S^*(\tau)$. In this section, we show how and why the reset policies are related closely to the sign behaviors of $P'_\alpha(\tau)$ and $F_\alpha(\tau)$.

In our earlier paper (Dai *et al.*, 2004), we have examined the optimal reset policies under the special case of $\alpha = 1$. With $\alpha = 1$, the characterization of the reset boundary that separates the stopping and continuation regions depends on whether $P'_1(\tau)$ stays positive for all $\tau > 0$ or $P'_1(\tau)$ is positive only when $\tau \in [0, \tau^*)$ for some τ^*. In the present case where $\alpha \in (0, \infty)$, we have observed in Sec. 2.2 that $P'_\alpha(\tau) > 0$ occurs either (i) for all $\tau > 0$, or (ii) over $\tau \in (0, \tau^*)$ or (iii) over a finite interval (τ_1^*, τ_2^*). The sign of $P'_\alpha(\tau)$ determines the optimal reset policy. We examine the optimal reset policy under the following three cases (a) $r < 0$ and $0 < \alpha < \infty$; (b) $r > 0$ and $\alpha \leq 1$; (c) $r > 0$ and $\alpha > 1$. Next, we explain why the first two cases are very similar to the special case $\alpha = 1$ considered in our earlier paper.

Firstly, when $r < 0$ and $0 < \alpha < \infty$, we have $P'_\alpha(\tau) > 0$ for all $\tau > 0$. Accordingly, $P_\alpha(\tau) > P_\alpha(0)$ is always satisfied. The two required conditions on the positivity of $P'_\alpha(\tau)$ and $F_\alpha(\tau)$ are fulfilled. By following similar analysis as in Dai *et al.*'s paper (2004), we can deduce that the reset boundary curve $S^*(\tau)$ is defined for all $\tau > 0$, where $S^*(\tau)$ is monotonic in τ with $S^*(0^+) = X/\alpha$ and $S^*(\infty) = \left(1 - \frac{\sigma^2}{2r}\right)\frac{X}{\alpha}$.

Secondly, when $r > 0$ and $\alpha \leq 1$, there exists τ^* such that $P'_\alpha(\tau) > 0$ only for $\tau \in [0, \tau^*)$. It is quite straightforward to show that the optimal reset policy for the case $\alpha < 1$ follows the same pattern as that for $\alpha = 1$, where $S^*(\tau)$ exists only for $\tau \in [0, \tau^*)$ and $S^*(0^+) = X/\alpha$.

New interesting phenomena on the optimal reset policies arise when $r > 0$ and $\alpha > 1$. Recall that only for $\alpha \in (1, \alpha_m^*)$, the positivity conditions on $P'_\alpha(\tau)$ and $F_\alpha(\tau)$ are satisfied for some finite interval (τ_m^*, τ_2^*). When $\alpha \geq \alpha_m^*$, the positivity conditions are not satisfied for any $\tau \geq 0$ (see Theorem 3). In all subsequent exposition in this section, we confine our discussion to the case of $\alpha > 1$ and $r > 0$.

To examine the optimal reset policies of reset put options, we start with the simple case with zero initial strike price (such reset put is commonly called a shout floor), then generalize the results to cases with non-zero initial strike price.

3.1 Optimal reset policies for the shout floors

With the initial strike X being set zero, the shout floor price function becomes linear homogeneous in S since both the terminal payoff and obstacle function are linear homogeneous in S. In this case, we may write the shout floor price function $V(S,\tau;0,r,0)$ as $Sg(\tau)$, for some function $g(\tau)$ to be determined.

By substituting $V = Sg(\tau)$ into the linear complementarity formulation [setting $q = 0$ and $\delta = 0$ in Eqs. (3a,b,c)], we obtain

$$g'(\tau) \geq 0, \quad g(\tau) \geq P_\alpha(\tau), \quad g'(\tau)[g(\tau) - P_\alpha(\tau)] = 0, \quad \text{for } \tau > 0, \tag{12}$$

with initial condition: $g(0) = \alpha - 1$. The linear complementarity formulation dictates either (i) $g'(\tau) > 0$ and $g(\tau) = P_\alpha(\tau)$ or (ii) $g'(\tau) = 0$ and $g(\tau) > P_\alpha(\tau)$. It is possible that $g(\tau)$ satisfies condition (i) over certain time interval but satisfies condition (ii) at other times. Over the time interval where $g'(\tau) = 0$, the holder should never reset. However, at those times where $g'(\tau) > 0$, the holder should choose to reset the shout floor optimally at any asset value level.

Since $g'(\tau) \geq 0$, for $\tau > 0$, we should have $g(\tau) \geq g(0) = \alpha - 1$ for all $\tau > 0$. When $\alpha \geq \alpha_m^*$, we have $P_\alpha(\tau) \leq \alpha - 1$ for all $\tau > 0$, with strict equality satisfied only at $\tau = \tau_m^*$ when $\alpha = \alpha_m^*$. In this case, we cannot have $g(\tau) = P_\alpha(\tau)$. Hence, when $\alpha \geq \alpha_m^*$, we only have $g'(\tau) = 0$; and together with $g(0) = \alpha - 1$, we obtain $g(\tau) = \alpha - 1$. On the other hand, when $1 < \alpha < \alpha_m^*$, we need to solve the problem over successive time intervals: (i) $(0, \tau_1^*)$ where $P_\alpha'(\tau) < 0$ and $P_\alpha(\tau) < \alpha - 1$, (ii) $[\tau_1^*, \tau_m^*]$ where $P_\alpha'(\tau) \geq 0$ but $P_\alpha(\tau) \leq \alpha - 1$, (iii) (τ_m^*, τ_2^*) where $P_\alpha'(\tau) > 0$ and $P_\alpha(\tau) > \alpha - 1$, (iv) $[\tau_2^*, \infty)$ where $P_\alpha'(\tau) \leq 0$ but $P_\alpha(\tau)$ can be greater than, equal to or less than $\alpha - 1$.

Over those times where $P_\alpha'(\tau) < 0$ or $P_\alpha(\tau) < \alpha - 1$, we cannot have $g(\tau) = P_\alpha(\tau)$ since this would violate $g'(\tau) \geq 0$ or $g(\tau) \geq \alpha - 1$, respectively. Hence, we have $g'(\tau) = 0$, that is, $g(\tau)$ is constant in these scenarios. When $\tau \in (\tau_m^*, \tau_2^*)$, we have $P_\alpha'(\tau) > 0$ and $P_\alpha(\tau) > \alpha - 1$ so that Eqs. (12) are automatically satisfied by $g(\tau) = P_\alpha(\tau)$. In summary, we obtain (i) $g(\tau) = \alpha - 1$ when $\tau \in (0, \tau_m^*]$, (ii) $g(\tau) = P_\alpha(\tau)$ when $\tau \in (\tau_m^*, \tau_2^*)$, and (iii) $g(\alpha) = P_\alpha(\tau_2^*)$ when $\tau \in [\tau_2^*, \infty)$. The characteristics of the price function of the shout floor and the corresponding optimal reset policies are summarized in Theorem 4.

Theorem 4 *Let $V(S,\tau;0)$ denote the value of the shout floor (reset put option with zero initial strike). For $\alpha > 1$ and $r > 0$, the shout floor value observes* *the following properties.*

(1) *When $\alpha_m^* \leq \alpha < \infty$, we have $V(S,\tau;0) = (\alpha-1)S$ for all $\tau > 0$. The holder should never reset the shout floor. The long position of the shout floor is equivalent to holding $\alpha - 1$ units of the asset.*

(2) *When $1 < \alpha < \alpha_m^*$, the price function takes different forms over different time intervals, namely,*

$$V(S,\tau;0) = \begin{cases} (\alpha-1)S & \text{for } 0 < \tau \leq \tau_m^* \\ SP_\alpha(\tau) & \text{for } \tau_m^* < \tau < \tau_2^* \\ SP_\alpha(\tau_2^*) & \text{for } \tau \geq \tau_2^* \end{cases}.$$

At the time of initiation of the shout floor, if the time to expiry τ is less than or equal to τ_m^*, then the holder should never reset throughout the life of the contract, except at maturity. However, when $\tau \in (\tau_m^*, \tau_2^*)$, the optimal policy is to reset the shout floor at any asset value level. Lastly, when $\tau > \tau_2^*$, the holder should wait until the time to expiry falls below τ_2^*, then he should reset at any asset value.

Remark

When $\alpha = 1$, we have $\tau_m^* = 0$ so that the holder should reset at any asset value level for $\tau < \tau_2^*$ and never reset for $\tau \geq \tau_2^*$. These results agree with those obtained by Cheuk and Vorst (1997) and Dai *et al.* (2004).

3.2 *Optimal reset policies for the reset puts*

Let $S^*(\tau;X)$ denote the critical asset value at which the reset put option with strike price X should be reset optimally. In Section 3.1, we have seen that $S^*(\tau;0)$ is infinite for all $\tau > 0$ when $\alpha \geq \alpha_m^*$. Since $S^*(\tau;X)$ should be monotonically increasing in X, that is, $S^*(\tau;X) \geq S^*(\tau;0)$, so $S^*(\tau;X)$ is also infinite. Hence, the optimal policy is that the reset put option should never be reset when $\alpha \geq \alpha_m^*$. In this case, the reset put resembles a European put with terminal payoff: $\max(X-S,(\alpha-1)S) = (\alpha-1)S + \alpha \max\left(\frac{X}{\alpha} - S, 0\right)$. The solution to the reset put price function $V(S,\tau;X)$ is easily seen to be

$$V(S,\tau;X) = (\alpha-1)S + \alpha p_E(S,\tau;X/\alpha), \tag{13}$$

where $p_E(S,\tau;X/\alpha)$ is the price of a European vanilla put option with strike price X/α.

When $1 < \alpha < \alpha_m^*$, according to optimal reset policy, the reset put is never reset for $\tau \leq \tau_m^*$ since $S^*(\tau; X) \geq S^*(\tau; 0)$ and $S^*(\tau; 0)$ is infinite. When $\tau \in (\tau_m^*, \tau_2^*)$, the holder should choose to shout at some $S^*(\tau; X) > 0$. Assume the contrary, suppose $S^*(\tau; X)$ does not exist for $\tau \leq \tilde{\tau}$, where $\tilde{\tau} \in (\tau_m^*, \tau_2^*)$, then $V(S, \tau; X) = (\alpha - 1)S + \alpha p_E(S, \tau; X/\alpha)$ for $\tau \leq \tilde{\tau}$. Write $D(S, \tau; X) = V(S, \tau; X) - (\alpha - 1)S$. Accordingly, we observe that $D(S, \tau) = \alpha p_E(S, \tau; X/\alpha)$, which tends to zero as $S \to \infty$. However, for $\tau_m^* < \tau \leq \tilde{\tau}$, we should have $D(S, \tau) \geq S[P_\alpha(\tau) - (\alpha - 1)]$. Since $[P_\alpha(\tau) - (\alpha - 1)]$ is a positive quantity, so $D(S, \tau) \to \infty$ as $S \to \infty$. This leads to a contradiction. Hence, $S^*(\tau; X)$ exists for $\tau \in (\tau_m^*, \tau_2^*)$. Lastly, when $\tau \geq \tau_2^*, S^*(\tau; X)$ does not exist since $S^*(\tau; X) \geq S^*(\tau; 0)$ and $S^*(\tau; 0)$ is infinite. We summarize the above results on the optimal reset policies and the pricing behaviors of the reset put option with non-zero initial strike price in Theorem 5.

Theorem 5 *Let $V(S, \tau; X)$ denote the price of the reset put with initial strike price X. For $\alpha > 1$ and $r > 0$, $V(S, \tau; X)$ observes the following properties.*

(1) When $\alpha_m^* \leq \alpha < \infty$, the holder should never reset the reset put. The price function has the following analytic representation:

$$V(S, \tau; X) = (\alpha - 1)S + \alpha p_E(S, \tau; X/\alpha) \quad \text{for all } \tau > 0.$$

(2) When $1 < \alpha < \alpha_m^*$, the holder should never reset the reset put for $\tau \in (0, \tau_m^*]$. The price function has the analytic form:

$$V(S, \tau; X) = (\alpha - 1)S + \alpha p_E(S, \tau; X/\alpha), \quad \text{for} \quad \tau \in (0, \tau_m^*].$$

When $\tau \in (\tau_m^*, \tau_2^*)$, there exists some critical asset value $S^*(\tau; X)$ such that the holder should reset optimally when $S \geq S^*(\tau; X)$. When $\tau \geq \tau_2^*$, the holder should never reset the reset put.

Once the optimal reset policies are known, it is relatively straightforward to obtain the integral representation of the reset premium [see Eq. (4.4a) in Dai *et al.'s* paper (2004)]. Also, the optimal reset boundary can be computed either by the recursive integration method or binomial calculations (adopting the dynamic programming procedure at each node to determine either the continuation of the option or exercise of reset right).

Numerical calculations were performed to verify the results in Theorem 5. The parameter values used in the calculations are: $r = 0.04,\ q = 0,\ \sigma = 0.2$ and $X = 1$. We computed the critical asset price $S^*(\tau),\ \tau > 0$, for the reset put option for different values of α (see Figure 3). We observe that $S^*(\tau)$ exists only within the interval (τ_m^*, τ_2^*) when $1 < \alpha < \alpha_m^*$, where $\alpha_m^* = 1.15$. The width of the interval (τ_m^*, τ_2^*) becomes narrower as α increases. In particular, we observe that the width of the interval (τ_m^*, τ_2^*) is quite sensitive to the change of α when α takes values from 1.12 to 1.1485. When $\alpha = 1.12$, the time interval is close to 3 years while the interval shrinks to 0.3 years when α increases slightly to 1.1485.

4 Optimal reset policies of put options with combined reset rights on strike and maturity

In this section, we consider the reset put option models with reset rights on both the strike and maturity, where $\widehat{X} = \alpha S_\xi$ and $\widehat{T} = T + \delta,\ \delta > 0$. The price function of the new option received upon reset is now $SP_\alpha(\tau + \delta)$. Accordingly, the two necessary conditions for the existence of $S^*(\tau)$ are modified to become (i) $P'_\alpha(\tau + \delta) > 0$ and (ii) $P_\alpha(\tau + \delta) > P_\alpha(\delta)$. Like the strike reset put options considered in Section 3, these two conditions play important roles in the determination of the optimal reset policies. The analysis of the optimal reset policies is complicated by the additional parameter δ, the duration of maturity extension.

Here, we limit our analysis to the case of $r > 0$ and $\alpha > 1$. Intuitively, we expect that optimal reset may also commence when $\alpha \leq 1$. We do not extend the analysis of these alternative cases since the analytic procedures are quite similar.

Optimal reset policies of the shout floor

It is more straightforward to analyze the optimal reset policies for the shout floor where the initial strike X is set zero. Similar to the analysis in Section 3.1, the price function $V(S, \tau; 0)$ is seen to be linear homogeneous in S so that $V(S, \tau; 0) = Sg(\tau; \delta)$. The governing formulation for $g(\tau; \delta)$ are given by

$$\begin{aligned} &g'(\tau;\delta) \geq 0, \quad g(\tau;\delta) \geq P_\alpha(\tau+\delta), \\ &g'(\tau;\delta)[g(\tau;\delta) - P_\alpha(\tau+\delta)] = 0, \text{ for } \tau > 0, \end{aligned} \tag{14}$$

with initial condition: $g(0) = P_\alpha(\delta)$. We consider the following separate cases:

(1) $\alpha > \alpha_1^*$
Recall that when $\alpha > \alpha_1^*$, $P'_\alpha(\tau) < 0$ for all $\tau > 0$ so that we cannot have $g(\tau;\delta) = P_\alpha(\tau+\delta)$, $\tau > 0$, since this is a violation of the first inequality in Eq. (14). That is, the holder should never reset for $\tau > 0$. We then have $g'(\tau;\delta) = 0$ for $\tau > 0$, and together with $g(0) = P_\alpha(\delta)$, we obtain $g(\tau;\delta) = P_\alpha(\delta)$. The shout floor is equivalent to $P_\alpha(\delta)$ units of asset, and it is never optimal to reset at any asset value level for all $\tau > 0$.

(2) $\alpha \leq \alpha_1^*$
When $\alpha \leq \alpha_1^*$, there exists an interval (τ_1^*, τ_2^*) such that $P'_\alpha(\tau) > 0$ for $\tau \in (\tau_1^*, \tau_2^*)$ and $P'_\alpha(\tau) \leq 0$ if otherwise. We consider the following

subcases: (a) $\delta \geq \tau_2^*$, (b) $\tau_1^* < \delta < \tau_2^*$, and (c) $\delta \leq \tau_1^*$.

(a) $\delta \geq \tau_2^*$.
In this case, $P'_\alpha(\tau + \delta) < 0$ for $\tau > 0$ so that we cannot have $g(\tau; \delta) = P_\alpha(\tau + \delta)$. Similar to the case $\alpha > \alpha_1^*$, we have $g(\tau; \delta) = P_\alpha(\delta)$.

(b) $\tau_1^* < \delta < \tau_2^*$.
Suppose $\tau_1^* < \delta < \tau_2^*$, we observe that $P'_\alpha(\tau + \delta) > 0$ for $0 < \tau < \tau_2^* - \delta$ but $P'_\alpha(\tau+\delta) \leq 0$ for $\tau \geq \tau_2^* - \delta$. Hence, we have $g(\tau; \delta) = P'_\alpha(\tau+\delta)$ for $0 < \tau < \tau_2^* - \delta$. On the other hand, we have $g'(\tau; \delta) = 0$ for $\tau \geq \tau_2^* - \delta$; that is, $g(\tau; \delta) = P'_\alpha(\tau_2^*)$ for $\tau \geq \tau_2^* - \delta$.

(c) $\delta \leq \tau_1^*$
When $\delta \leq \tau_1^*$, we have $P'_\alpha(\tau + \delta) > 0$ for $\tau \in (\tau_1^* - \delta, \tau_2^* - \delta)$ and $P'_\alpha(\tau + \delta) \leq 0$ if otherwise. Recall that there is another necessary condition for the activation of optimal reset: $P_\alpha(\tau + \delta) > g(0; \delta) = P_\alpha(\delta)$. We consider the following two separate cases:

(i) $P_\alpha(\tau_2^*) \leq P_\alpha(\delta)$
Since $P_\alpha(\tau)$ attains its maximum value at $\tau = \tau_2^*$, suppose $P_\alpha(\tau_2^*) \leq P_\alpha(\delta)$, then $P_\alpha(\tau) \leq P_\alpha(\delta)$ for all $\tau > 0$. The condition $P_\alpha(\tau+\delta) > P_\alpha(\delta)$ is never fulfilled so that the holder never reset the shout floor. We then have $g(\tau; \delta) = P_\alpha(\delta)$ for $\tau > 0$.

(ii) $P_\alpha(\tau_2^*) > P_\alpha(\delta)$
Suppose $P_\alpha(\tau_2^*) > P_\alpha(\delta)$, then there exists a unique value $\tau_m^*(\alpha; \delta)$ lying within (τ_1^*, τ_2^*) such that

$$P_\alpha(\tau_m^*(\alpha; \delta)) = P_\alpha(\delta).$$

For $\tau \in (0, \tau_m^*(\alpha; \delta) - \delta]$, we have $P_\alpha(\tau + \delta) < P_\alpha(\delta)$ so that the holder should never shout. For $\tau \in (\tau_m^*(\alpha; \delta) - \delta, \tau_2^* - \delta)$, both necessary conditions for optimal reset are fulfilled, the holder should shout optimally at any asset value. For $\tau \geq \tau_2^* - \delta$, we have $P'_\alpha(\tau + \delta) < 0$ so the holder should never shout. The solution to $g(\tau; \delta)$ can be found to be

$$g(\tau; \delta) = \begin{cases} P_\alpha(\delta), & \text{for } 0 \leq \tau \leq \tau_m^*(\alpha; \delta) - \delta \\ P_\alpha(\tau + \delta), & \text{for } \tau_m^*(\alpha; \delta) - \delta < \tau < \tau_2^* - \delta \\ P_\alpha(\tau_2^*), & \text{for } \tau \geq \tau_2^* - \delta \end{cases}. \quad (15)$$

The analytic forms of the price function and the optimal reset policies of the shout floor with combined reset rights on the strike and maturity are summarized in Theorem 6.

Theorem 6 *Let $V_1(S, \tau; 0)$ denote the price of the shout floor with combined reset rights on strike and maturity. For $\alpha > 1$ and $r > 0, V_1(S, \tau; 0)$ observes the following properties.*

(1) When either (i) $\alpha_1^* \leq \alpha < \infty$, or (ii) $1 < \alpha < \alpha_m^*$ and $\delta \geq \tau_2^*$, or (iii)

$1 < \alpha < \alpha_1^*$, $\delta \leq \tau_1^*$ and $P_\alpha(\tau_2^*) \leq P_\alpha(\delta)$, we have

$$V_1(S, \tau; 0) = SP_\alpha(\delta).$$

The optimal reset policy dictates that the holder should never reset the shout floor at all times, except at maturity.

(2) When $1 < \alpha < \alpha_1^*$ and $\tau_1^* < \delta < \tau_2^*$, we have

$$V_1(S, \tau; 0) = \begin{cases} SP_\alpha(\tau + \delta), & \text{for } 0 \leq \tau < \tau_2^* - \delta \\ SP_\alpha(\tau_2^*), & \text{for } \tau \geq \tau_2^* - \delta \end{cases}.$$

The holder of the shout floor should choose to shout optimally at any asset price level when $\tau \in [0, \tau_2^* - \delta)$, and should not shout when $\tau \geq \tau_2^* - \delta$.

(3) When $1 < \alpha < \alpha_1^*$ and $\delta \leq \tau_1^*$ and $P_\alpha(\tau_2^*) > P_\alpha(\delta)$, we have

$$V_1(S, \tau; 0) = \begin{cases} SP_\alpha(\delta), & \text{for } 0 \leq \tau \leq \tau_m^*(\alpha; \delta) - \delta \\ SP_\alpha(\tau + \delta), & \text{for } \tau_m^*(\alpha; \delta) - \delta < \tau < \tau_2^* - \delta \\ SP_\alpha(\tau_2^*), & \text{for } \tau \geq \tau_2^* - \delta \end{cases}.$$

The holder of the shout floor should choose to shout at any asset price level only when $\tau \in (\tau_m^*(\alpha; \delta) - \delta, \tau_2^* - \delta)$, and should not do so at other times.

Optimal reset policies of reset put

The characterization of the optimal reset policy for a reset put option amounts to the determination of the critical asset price $S^*(\tau; X)$. The critical asset price $S^*(\tau; 0)$ for the shout floor has two possibilities, either it assumes zero value or does not exist. Since $S^*(\tau; X) \geq S^*(\tau; 0)$ for $X > 0$, $S^*(\tau; X)$ does not exist whenever $S^*(\tau; 0)$ is non-existent; and $S^*(\tau; X)$ assumes some finite value when $S^*(\tau; 0) = 0$. The optimal reset policies of the reset put and shout floor are very similar. For a given set of parameter values, at those times when the holder should not reset the shout floor, the same policy should be adopted by the holder of the reset put. On the other hand, at those times when the holder of a shout floor should shout at any asset price, the holder of the corresponding reset put should reset optimally whenever $S \geq S^*(\tau; X)$.

Suppose the reset right is rendered useless at all times except at maturity, the reset put with combined reset rights on strike and maturity essentially becomes a European option with the terminal payoff:

$$\begin{aligned} V_1(S, 0; X) &= \max(X - S, SP_\alpha(\delta)) \\ &= SP_\alpha(\delta) + [1 + P_\alpha(\delta)] \max\left(\frac{X}{1 + P_\alpha(\delta)} - S, 0\right). \end{aligned} \tag{16}$$

The value of the European option with the above terminal payoff is given by $SP_\alpha(\delta) + [1 + P_\alpha(\delta)]p_E(S, \tau; \frac{X}{1 + P_\alpha(\delta)})$.

From the knowledge on the optimal reset policies as stated in Theorems 4, 5 and 6, we summarize the optimal reset policies of a reset put with combined reset rights on the strike and maturity in Theorem 7.

Theorem 7 *Let $V_1(S, \tau; X)$ denote the price of the reset put with combined reset rights on the strike and maturity, and with initial strike price X. For $\alpha > 1$ and $r > 0$, $V_1(S, \tau; X)$ observes the following properties.*

(1) When either (i) $\alpha_1^* \leq \alpha < \infty$, or (ii) $1 < \alpha < \alpha_m^*$ and $\delta \geq \tau_2^*$, or (iii) $1 < \alpha < \alpha_1^*$, $\delta \leq \tau_1^*$ and $P_\alpha(\tau_2^*) \leq P_\alpha(\delta)$, we have

$$V_1(S, \tau; X) = SP_\alpha(\delta) + [1 + P_\alpha(\delta)]p_E(S, \tau; \frac{X}{1 + P_\alpha(\delta)}), \ \tau > 0.$$

The holder of the reset put should never reset at all times, except at maturity.

(2) When $1 < \alpha < \alpha_1^*$ and $\tau_1^* < \delta < \tau_2^*$, the holder of the reset put should reset optimally whenever $S \geq S_1^*(\tau; X)$ when $\tau \in [0, \tau_2^* - \delta)$, and should never reset when $\tau \geq \tau_2^* - \delta$.

(3) When $1 < \alpha < \alpha_1^*$, $\delta \leq \tau_1^*$ and $P_\alpha(\tau_2^*) > P_\alpha(\delta)$, it is never optimal for the holder to reset the reset put option for $\tau \in (0, \tau_m^*(\alpha; \delta) - \delta]$. The price function has the analytic form:

$$V_1(S, \tau; X) = SP_\alpha(\delta) + [1 + P_\alpha(\delta)]p_E(S, \tau; \frac{X}{1 + P_\alpha(\delta)}), \ \tau \in (0, \tau_m^*(\alpha; \delta) - \delta].$$

When $\tau \in (\tau_m^*(\alpha; \delta) - \delta, \tau_2^* - \delta)$, there exists some critical asset value $S_1^*(\tau; X)$ such that the holder should reset optimally when $S \geq S_1^*(\tau; X)$. When $\tau \geq \tau_2^* - \delta$, the holder of the reset put should never reset.

Numerical calculations were performed to verify the claims in Theorem 7. The following parameter values were used in the calculations: $r = 0.04$, $q = 0$, $\sigma = 0.2$, $X = 1$ and $\alpha = 1.125$. For this set of parameter values, we obtained $\tau_1^* = 0.2962$, $\tau_2^* = 2.4771$, $\tau_m^* = 0.6358$ and $\alpha_1^* = 1.18$. We chose $\delta = 0.1$ and $\delta = 0.4$ successively in our calculations. When $\delta = 0.4$, we have $1 < \alpha < \alpha_m^*$ and $\tau_1^* < \delta < \tau_2^*$, and this corresponds to the second case in Theorem 7. The theorem states that the critical asset price $S_1^*(\tau; X)$ is defined only for $\tau \in [0, \tau_2^* - \delta)$. For this set of parameters, we have $\tau_2^* - \delta = 2.4771 - 0.4 = 2.0771$. The dotted curve in Figure 4 shows the plot of $S_1^*(\tau; X)$ against τ for $\delta = 0.4$. The plot clearly reveals the validity of the theoretical prediction. Furthermore, the critical asset price starts at $\tau = 0$ with $S_1^*(0^+; X) = X/(1 + P_\alpha(\delta))$. When $\delta = 0.1$, we have $1 < \alpha < \alpha_1^*$, $\delta \leq \tau_1^*$ and $P_\alpha(\tau_2^*) > P_\alpha(\delta)$, and this corresponds to the third case in Theorem 7. From the theorem, one predicts that the critical asset price $S_1^*(\tau; X)$ is only defined for $\tau \in (\tau_m^* - \delta, \tau_2^* - \delta)$, where $\tau_m^* - \delta = 0.7358 - 0.1 = 0.6358$, $\tau_2^* - \delta = 2.4771 - 0.1 = 2.3771$ and τ_m^* satisfies the equation $P_\alpha(\tau_m^*) = P_\alpha(\delta)$. The plot of $S_1^*(\tau; X)$ against τ for $\delta = 0.1$ is shown as the solid curve in Figure 4, and its behaviors agree exactly with those derived from theoretical analysis.

5 Conclusion

The optimal reset policies of a reset option with combined reset rights on the strike and maturity exhibit a myriad of interesting phenomena. The behaviors of the reset policies depend on the relative magnitude of the riskless interest rate r and dividend yield q, the multiplicative factor α in defining the new strike and the duration of maturity extension δ. Since we assume the new strike price upon reset to be a multiple of the prevailing asset price, the price function of the new option becomes linear homogeneous in the asset price. Such linear homogeneity property of the obstacle function in the linear complementarity formulation greatly simplifies the analysis of the free boundary value problem, in particular, the determination of the optimal reset boundary.

We have showed that optimal reset commences only at those times when two necessary conditions are satisfied, namely, the time-derivative of the expectation of the discounted value of the option received upon reset is negative and the reset payoff at the reset moment is higher than the reset payoff at maturity. When both necessary conditions are satisfied, the holder chooses to reset the option optimally when the asset price reaches some critical asset price from below.

First, we consider the optimal reset policies for a put option with reset right on the strike price only. When the riskless interest rate is less than or equal to the dividend yield, the two necessary conditions for optimal reset are satisfied at all times, independent of the value of the multiplicative factor. In this case, the optimal reset boundary is defined for all times. On the other hand, when the riskless interest rate is greater than the dividend yield and the multiplicative factor is less than or equal to one, there exists a threshold value such that the optimal reset boundary is defined only for those times where the time to expiry is less than this threshold value. However, when the multiplicative factor α is greater than one, there exists a threshold value α_m^* such that the holder of the reset put should never reset at all times when $\alpha \geq \alpha_m^*$. In this case, the holding of the reset put is equivalent to the holding of a portfolio containing one European put option and certain number of units of the asset. When $1 < \alpha < \alpha_m^*$, the holder should reset optimally only at those times when the time to expiry falls within certain time interval.

The properties of the optimal reset boundary for reset put options with combined reset rights on the strike and maturity are quite similar to those for reset put options with strike reset only. Depending on the relative magnitude of the parameter values in the option pricing model, the optimal reset boundary is defined either (i) for all times, (ii) within certain time interval, (iii) or when the time to expiry is less than certain threshold value. Our theoretical predictions of the optimal reset policies have been verified by numerical calculations.

Appendix

Proof of Lemma 1. To examine the sign behaviors of $P'_\alpha(\tau)$ for $\tau \geq 0$, it is necessary to examine the properties of $f(\infty)$ and $f(0^+)$ and the turning points of the graph of $y = f(\tau)$, where

$$f(\tau) = \frac{P'_\alpha(\tau)}{\alpha e^{-r\tau}} = -rN(-d_-) + n(-d_-)\frac{\sigma}{2\sqrt{\tau}}.$$

First, we consider $f(\infty)$ under different sign properties of γ_-.

(i) When $\gamma_- < 0$, we have $d_- \to -\infty$ as $\tau \to \infty$ so $f(\infty) = -r$.
(ii) When $\gamma_- > 0$, we have $d_- \to \infty$ as $\tau \to \infty$. Consider

$$f(\infty) = \lim_{\tau\to\infty} rN(-d_-)\left[-1 + \frac{n(-d_-)}{N(-d_-)}\frac{\sigma}{2r\sqrt{\tau}}\right]$$

and note that $\lim_{\tau\to\infty} \frac{n(-d_-)}{N(-d_-)}\frac{\sigma}{2r\sqrt{\tau}} = \frac{\gamma_-}{2r}$, we then obtain $f(\infty) = 0^-$.
(iii) When $\gamma_- = 0$, we have $d_- \to 0$ so $f(\infty) = -\frac{r}{2}$.

To examine the turning points of $y = f(\tau)$, we consider the properties of $f'(\tau)$ where

$$f'(\tau) = \frac{\sigma n(-d_-)}{4\tau^{5/2}}\left\{\gamma_-\gamma_+\tau^2 - [\beta(\gamma_- + \gamma_+) + 1]\tau + \beta^2\right\}.$$

Clearly, we have $f'(0^+) = 0^+$. We then consider the roots of $f'(\tau)$ for $\tau > 0$, which give the locations of the turning points of $y = f(\tau)$, for $\tau > 0$. The discriminant of the above quadratic equation in τ is found to be

$$\Delta = \beta^2\sigma^2 + 1 + \frac{4\beta r}{\sigma}.$$

Provided that $\Delta > 0$, the quadratic equation has two real roots τ_1 and τ_2 (say, take $\tau_1 < \tau_2$). As a special case, when $\gamma_- = 0$, the quadratic equation reduces to a linear equation whose root is given by $\tau_0 = \frac{\beta^2\sigma}{2r\beta + \sigma}$.

Note that when $\beta > 0$, we always have $\Delta > 0$. We then consider the following three scenarios.

(i) When $\gamma_- < 0$, only one of the two roots of $f'(\tau)$ is positive. Since $f(0^+) = 0^+$ and $f(\infty) = -r$, and there is only one turning point at $\tau = \tau_2 > 0$ for $y = f(\tau)$, the curve $y = f(\tau)$ crosses the τ-axis at exactly one point $\tau = \tau^* > 0$. Subsequently, $P'_\alpha(\tau) > 0$ for $\tau \in (0, \tau^*)$.

(ii) When $\gamma_- > 0$, both the two roots of $f'(\tau)$ are positive and there are two turning points at $\tau = \tau_1$ and $\tau = \tau_2$ for $y = f(\tau)$. Together with $f(0^+) = 0^+$ and $f(\infty) = 0^-$, the curve of $y = f(\tau)$ crosses the τ-axis at exactly one point $\tau = \tau^* > 0$ and $P'_\alpha(\tau) > 0$ for $\tau \in (0, \tau^*)$.

(iii) Lastly, when $\gamma_- = 0$, there is only one turning point at $\tau = \tau_0 > 0$ for $y = f(\tau)$. This case is similar to case (i) and we again have $P'_\alpha(\tau) > 0$ for $\tau \in (0, \tau^*)$.

Proof of Lemma 2. With $r > 0$ and $\beta < 0$, we have $f(0^+) = -r$ and $f'(0^+) > 0$ while

$$f(\infty) = \begin{cases} -r & \text{if } \gamma_- < 0 \\ -\frac{r}{2} & \text{if } \gamma_- = 0 \\ 0^- & \text{if } \gamma_- > 0 \end{cases} .$$

We consider three separate cases:

(a) $\gamma_- < 0$

The discriminant $\Delta = \beta^2\sigma^2 + 1 + \dfrac{4\beta r}{\sigma} > (\beta\sigma + 1)^2 \geq 0$. Hence, $f'(\tau)$ has only one positive root $\tau_2 > 0$. Since there is only one turning point of the curve $y = f(\tau)$ for $\tau > 0, f(\tau)$ remains negative for all $\tau > 0$ if $f(\tau_2) < 0$. When $f(\tau_2) > 0$, the curve $y = f(\tau)$ crosses the τ-axis twice, say at τ_1^* and τ_2^* (take $\tau_2^* > \tau_1^*$). The sign properties of $P'_\alpha(\tau)$ can be summarized as follows:

(i) When $f(\tau_2) < 0, P'_\alpha(\tau) < 0$ for all $\tau > 0$;

(ii) When $f(\tau_2) > 0, P'_\alpha(\tau) > 0$ for $\tau \in (\tau_1^*, \tau_2^*)$ and $P'_\alpha(\tau) \leq 0$ for $\tau \notin (\tau_1^*, \tau_2^*)$.

(b) $\gamma_- > 0$

When $\beta(\gamma_- + \gamma_+) + 1 \leq 0$, we observe that $f'(\tau) > 0$ for all $\tau > 0$ so that $P'_\alpha(\tau) < 0$ for $\tau > 0$. On the other hand, when $\beta(\gamma_- + \gamma_+) + 1 > 0$, we consider the two separate cases: $\Delta \leq 0$ and $\Delta > 0$. (i) When $\Delta \leq 0$, we have $f'(\tau) \geq 0$ and $P'_\alpha(\tau) < 0$ for all $\tau > 0$. (ii) When $\Delta > 0$, there are two positive roots for $f'(\tau)$. We deduce that $P'_\alpha(\tau) < 0$ for all $\tau > 0$ if $f(\tau_1) \leq 0$ and $P'_\alpha(\tau) > 0$ for some interval (τ_1^*, τ_2^*) if $f(\tau_1) > 0$.

(c) $\gamma_- = 0$

In this case, $f'(\tau)$ has no positive root if $\beta(\gamma_- + \gamma_+ + 1) \leq 0$ and one positive root $\tau_0 = \dfrac{\beta^2\sigma}{2r\beta + \sigma}$ if $\beta(\gamma_- + \gamma_+) + 1 > 0$. Hence, $P'_\alpha(\tau) > 0$ for some interval (τ_1^*, τ_2^*) if $\beta(\gamma_- + \gamma_+ + 1) > 0$ and $f(\tau_0) > 0$, and $P'_\alpha(\tau) < 0$ for all $\tau > 0$ if otherwise.

Proof of Theorem 3.

1. We would like to show that $\tau_2^*(\alpha) > 0$ does not exist when α is sufficiently large. It suffices to show that the conditions stated in Part 1 of Lemma

2 cannot be fulfilled when α is sufficiently large. Firstly, when $\gamma_- < 0$, $d_-(\tau_2) = \dfrac{-\ln\alpha + \gamma_-\tau_2}{\sigma\sqrt{\tau_2}}$ which tends to $-\infty$ as $\alpha \to \infty$, so $f(\tau_2) < 0$ for sufficiently large value of α. Secondly, when $\gamma_- \geq 0, \beta(\gamma_- + \gamma_+) + 1$ becomes negative when α becomes sufficiently large. Since the width of the interval (τ_1^*, τ_2^*) decreases monotonically with increasing α, and τ_2^* does not exist for sufficiently large α, then there exists a threshold α_1^* such that $\tau_1^*(\alpha_1^*) = \tau_2^*(\alpha_1^*)$ and both $\tau_1^*(\alpha)$ and $\tau_2^*(\alpha)$ do not exist for $\alpha > \alpha_1^*$.

2. Since $\tau_m^*(\alpha)$ exists for α sufficiently close to but greater than one and the existence of $\tau_m^*(\alpha)$ implies the existence of $\tau_1^*(\alpha)$ and $\tau_2^*(\alpha)$, together with the result in Part 1, we can conclude that there exists a threshold value α_m^* such that $\tau_m^*(\alpha)$ exists for $1 < \alpha < \alpha_m^*$. This would imply that

$$P_\alpha(\tau) - (\alpha - 1) < 0 \quad \text{for all } \tau > 0 \text{ when } \alpha > \alpha_m^*.$$

By the continuity property of $P_\alpha(\tau)$ on α, we expect that $P_{\alpha_m^*}(\tau) \leq \alpha_m^* - 1$ and $P_{\alpha_m^*}(\tau)$ attains its maximum value at $\tau = \tau_m^*(\alpha_m^*)$. We then have $P_{\alpha_m^*}(\tau_m^*(\alpha_m^*)) = \alpha_m^* - 1$, $P'_{\alpha_m^*}(\tau_m^*(\alpha_m^*)) = 0$ and $P'_{\alpha_m^*}(\tau_2^*(\alpha_m^*)) = 0$; so we obtain $\tau_m^*(\alpha_m^*) = \tau_2^*(\alpha_m^*)$.

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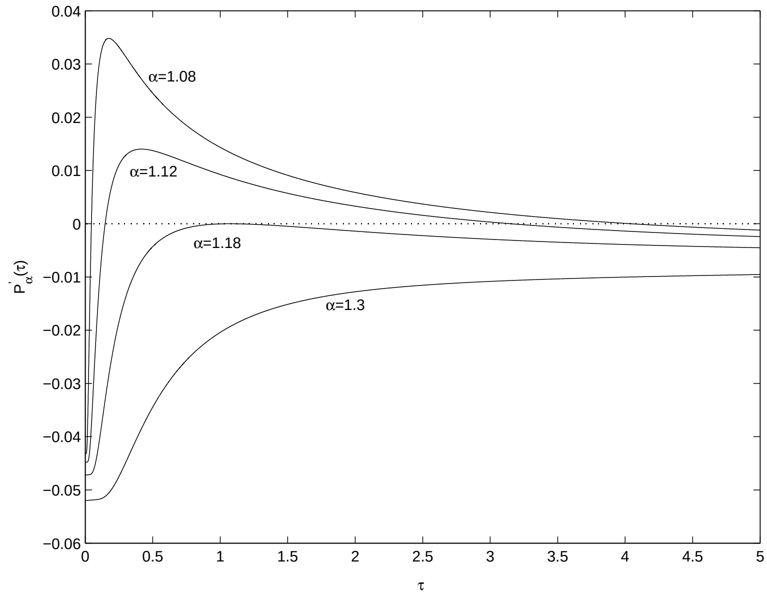

Fig. 1. The figure shows the plot $P'_\alpha(\tau)$ against τ for different values of α, where $\alpha > 1$. The parameter values used in the calculations are: $r = 0.04$, $q = 0$ and $\sigma = 0.2$. The interval (τ_1^*, τ_2^*) within which $P'_\alpha(\tau)$ stays positive becomes narrower when α increases in value. The threshold value α_1^* for which (τ_1^*, τ_2^*) does not exist is estimated to be 1.18.

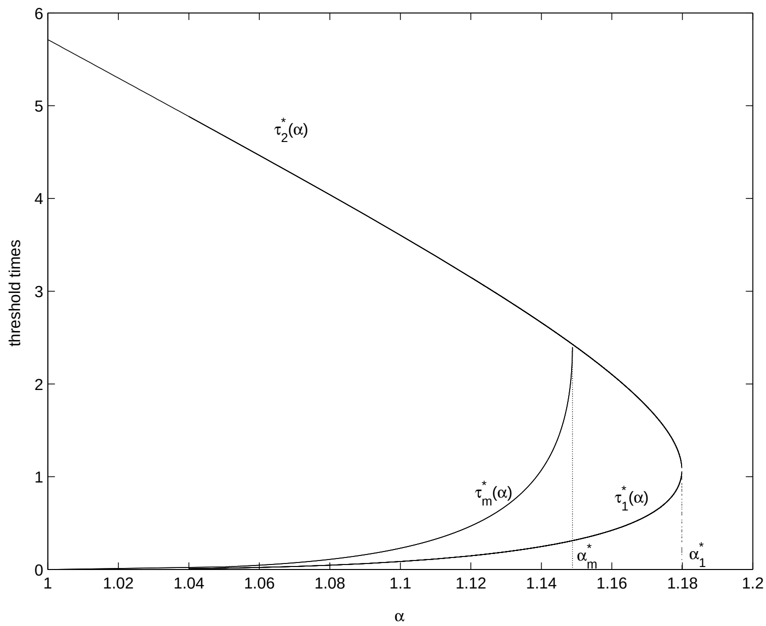

Fig. 2. We plot the threshold times $\tau_1^*(\alpha)$, $\tau_2^*(\alpha)$ and $\tau_m^*(\alpha)$ against α. The parameter values used in the calculations are the same as those used for Figure 1. Both $\tau_1^*(\alpha)$ and $\tau_m^*(\alpha)$ are monotonically increasing with respect to α while $\tau_2^*(\alpha)$ decreases monotonically with increasing α. The curves of $\tau_1^*(\alpha)$ and $\tau_2^*(\alpha)$ intersect at $\alpha_1^* = 1.18$, while the curves of $\tau_m^*(\alpha)$ and $\tau_2^*(\alpha)$ intersect at $\alpha_m^* = 1.15$.

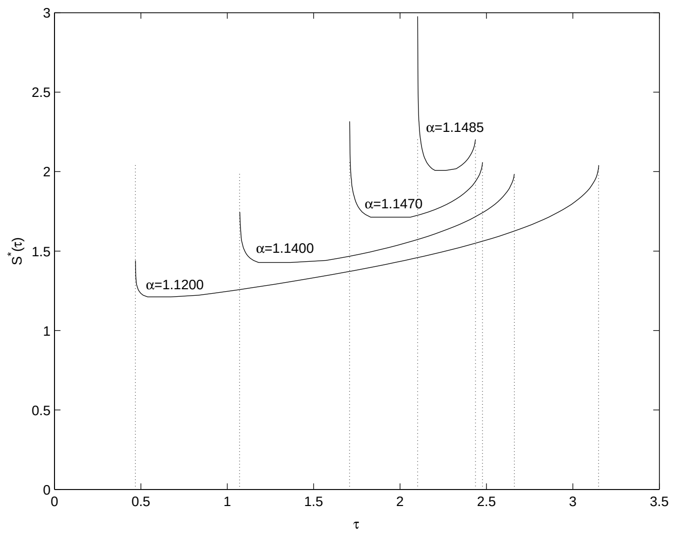

Fig. 3. The curves show the plot of the critical asset price $S^*(\tau)$ against τ for the strike reset put options corresponding to different values of α. The parameter values used in the calculations are: $r = 0.04,\ q = 0,\ \sigma = 0.2$ and $X = 1$. For those cases with α satisfying $1 < \alpha < \alpha_m^*$, where $\alpha_m^* = 1.15$, $S^*(\tau)$ exists only within the interval (τ_m^*, τ_2^*). Also, the interval (τ_m^*, τ_2^*) shrinks in size as α increases.

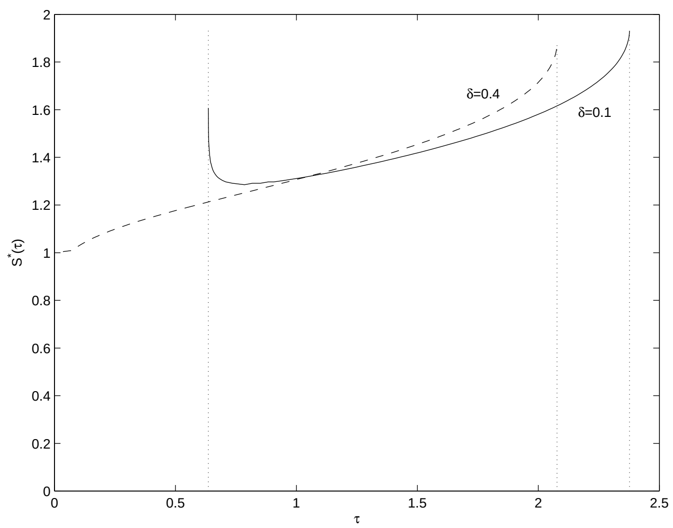

Fig. 4. The curves show the plots of the critical asset price $S^*(\tau)$ against τ for the reset put option with combined reset rights on strike and maturity. The parameter values used in the calculations are $r = 0.04$, $q = 0$, $\sigma = 0.2$ and $\alpha = 1.125$. For this choice of α, we obtained $\tau_1^* = 0.2962$, $\tau_2^* = 2.4771$ and $\tau_m^*(\alpha,\delta) = 0.6358$. When $\delta = 0.1 < \tau_1^*$ (see the solid curve), $S^*(\tau)$ is defined only for $\tau \in (\tau_m^*(\alpha,\delta), \tau_2^* - \delta) = (0.6358, 2.3771)$. This is the time interval within which the holder should reset optimally when $S \geq S^*(\tau)$. When $\delta = 0.4 \in (\tau_1^*, \tau_2^*)$ (see the dotted curve), $S^*(\tau)$ is defined for $\tau \in (0, \tau_2^* - \delta) = (0, 2.0771)$.