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Optimal Policies of Call with Notice Period Requirement

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Abstract When an American warrant or a convertible bond is called by its issuer, the holder is usually given a notice period to decide whether to sell the derivative back to the issuer at the call price or to exercise the conversion right. Several earlier papers have shown that such notice period requirement may substantially affect the optimal call policy adopted by the issuer. In this paper, we perform theoretical studies on the impact of the notice period requirement on issuer's optimal call policy for American warrants and convertible bonds. We also examine how the optimal call policy of the issuer interacts with holder's optimal conversion policy.

1 Introduction

The call (redeemable) provision in an American warrant or a convertible bond gives the right to the issuer to redeem the warrant or convertible bond at a pre-determined call price. When the issuer calls its warrant or convertible bond, it is typical that the holder is given a notice period (say, 30 days) to decide whether to receive the cash or convert into shares. In their pioneering theoretical studies on the optimal call policy of convertible bonds, Ingersoll [8, 9] and Brennan and Schwartz [4] conclude that a callable convertible security should be called as soon as its conversion value equals the prevailing effective call price. Unfortunately, empirical evidence suggests that convertible bonds are usually called late - the stock price at calling far exceeds the theoretical optimal critical stock price. Various explanations, like signaling hypothesis, balance sheet liquidity, yield advantage and after-tax-cash flow considerations, safety premium hypothesis, etc. have been proposed to account for such delayed call phenomena (Jaffee and Shleifer [10], Asquith [3]). While corporate finance considerations would partially explain the delayed call phenomena, some researchers argue that part of the "delayed calling" may be attributed to under estimation of the optimal conversion stock price in earlier contingent claims pricing models of convertible securities (Bühler and Koziol [5], Yigitbasioglu and Alexander [14]).

The holder of a callable convertible is effectively granted a European option upon calling since the holder will make his decision at the end of the notice period. The life of this vested option equals the length of the notice period. Several recent studies report that the notice period requirement at calling may have significant impact on the optimal conversion stock price. Kwok and Wu [11] show that the optimal conversion stock price at which the issuer of a callable American warrant should call may increase quite significantly with the length of the notice period. The percentage increase of the optimal conversion stock price may depend on other factors in the pricing model, like time to expiry, stock price volatility, dividend yield, etc. Similar behaviors of dependence of the optimal call policy on the length of notice period are observed in convertible bond calculations reported by Lau and Kwok [13] and Grau *et al*. [7]. Butler [6] proposes a simplified pricing model to analyze the effect of the notice period on the optimal call policy. He assumes that the convertible bond can be decomposed into a straight debt and a conversion option (European call option). Upon calling, the holder effectively receives the stock plus a European put option whose maturity date coinciding with the ending date of the notice period. The issuer should call at the optimal stock price so as to minimize the net value. Based on Butler's optimal call policy model, Altintig and Butler [1] performed empirical studies on issuers' calling policies using a sample of convertible bonds which were called for redemption between 1986 and 2000. They show that after properly accounting for the call notice period and other factors (like call protection and yield advantage), the median excess call premium in their sample is 3.74%, which is substantially less than 26-44% call premium that previous researchers have documented.

For convenience of presentation, the term “conversion” in this paper refers to the act of exercise in American warrants or the conversion into shares in convertible bonds. The numerical calculations using more refined contingent claims pricing models of callable American warrants and convertible bonds reveal that the optimal calling and conversion policies would be complicated by embedded credit risk, dividend and coupon payment streams (Ayache *et al.*, [2]; Lau and Kwok, [13]). By neglecting credit risk, dividends and coupons, the simplicity adopted in the Butler model allows one to examine the impact of the call notice period on the optimal calling policy. However, a callable convertible security may be terminated prematurely either by early voluntary conversion by holder or calling by issuer. Since Butler assumes no premature conversion by the holder, his model cannot be used to analyze the interaction between the optimal policy (with call notice period) of the issuer and the optimal conversion policy of the holder.

In this paper, we perform theoretical studies on the characterization of issuer's optimal calling policy and holder's optimal conversion policy of the two typical classes of callable convertible securities: American warrants and convertible bonds. First, we present the variational inequalities formulation of the contingent claims pricing models with both call and conversion rights. To simplify our theoretical analyses, we assume zero default risk of the issuer, zero coupon payment, flat interest rate and continuous dividend yield. We demonstrate that the interaction of conversion and call policies depend critically on the payoff structure (warrant or convertible bond), time to expiry, call notice period, call price, dividend yield and other parameters in the pricing model. Numerical calculations were performed to verify the theoretical results.

2 Variational inequalities formulation of pricing models

We would like to derive the variational inequalities formulation of the pricing model of a callable convertible security. We adopt the usual Black-Scholes pricing framework where the underlying stock price S under the risk neutral measure follows the lognormal diffusion process

$$\frac{dS}{S} = (r - q)\, dt + \sigma\, dZ. \tag{1}$$

Here, t is the calendar time, r and q are the riskless interest rate and dividend yield, respectively, σ is the volatility and dZ is the differential of the standard Brownian process. Let τ denote the time to expiry and $V(S,\tau)$ denote the price function of the callable convertible security. Upon optimal conversion by the holder prior to maturity, the payoff of the convertible security is given by $V_0(S;X)$, where X denotes the strike price in a warrant or the par value in a convertible bond. The terminal payoff is represented by $V_T(S;X)$. Let K denote the pre-determined call price and τ_n denote the length of the notice period. Upon calling, the holder essentially receives from the issuer a European option with terminal payoff $\max(V_0(S;X), K)$

and time to expiry τ_n. Let $c_n(S, \tau_n)$ denote the value of this vested European option. Since we emphasize the investigation of interaction of optimal calling and conversion policies in this work, for sake of simplicity, we assume no default risk of the issuer, zero coupon payment, constant interest rate and continuous dividend yield in our pricing models.

First, when there are no conversion and calling rights, the price function $V(S, \tau)$ is governed by the Black-Scholes equation

$$\mathcal{L}V = \left[\frac{\partial}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2}{\partial S^2} - (r-q)S\frac{\partial}{\partial S} + r\right]V = 0, \quad 0 < S < \infty, \tau > 0, \tag{2}$$

and the terminal payoff on the maturity date T is given by

$$V(S, 0) = V_T(S; X), \quad \tau = T - t. \tag{3}$$

When the holder is granted with the early conversion right, the pricing model becomes a free boundary problem. The free boundary is $S^*_{conv}(\tau)$, which is consisted of critical stock price for optimal conversion at varying values of τ. The domain $\mathcal{D} = \{(S, \tau) : 0 < S < \infty, \tau > 0\}$ of the pricing model is divided into two regions: R_{cont} and R_{conv}, which are separated by $S^*_{conv}(\tau)$. In the continuation region: $R_{cont} = \{(S, \tau) : 0 < S < S^*_{conv}, \tau > 0\}$, the security remains alive and the price function $V(S, \tau)$ satisfies $\mathcal{L}V = 0$ and $V > V_0(S; X)$. However, in the conversion region: $R_{conv} = \{(S, \tau) : S \geq S^*_{conv}, \tau > 0\}$, the security is converted into shares and the price function $V(S, \tau)$ satisfies $\mathcal{L}V > 0$ and $V = V_0(S; X)$. The variational inequalities formulation with only conversion right is given by

$$\min(\mathcal{L}V, V - V_0(S;X)) = 0, \tag{4}$$

since both $\mathcal{L}V$ and $V - V_0(S;X)$ are both strictly non-negative in the whole domain $\mathcal{D}$ and one of these two quantities equals zero in either R_{cont} or R_{conv}.

When the holder possesses the early conversion right while the issuer possesses the callable right, the domain $\mathcal{D}$ can be divided into three regions, R_{cont}, R_{conv} and R_{call}. During the life of the convertible security, the contract may be terminated prematurely either by holder's early conversion or issuer's premature redemption. In the call region R_{call}, the security is optimally called by the issuer. The price function $V(S,\tau)$ becomes $c_n(S,\tau_n)$ and $\mathcal{L}V < 0$. The issuer chooses to call optimally when the stock price S hits some threshold level $S^*_{call}(\tau)$.

In most usual circumstances, we have $V_0(S;X) \leq c_n(S,\tau_n)$. However, it may occur that $V_0(S;X) > c_n(S,\tau_n)$. Normally, we allow the issuer's call right to prioritize over holder's conversion right. Therefore, at the stock price level S where $V_0(S;X) > c_n(S,\tau_n)$, the price function V becomes $c_n(S,\tau_n)$ as a consequence of immediate issuer's call. For convenience, we define the effective conversion payoff by

$$\widetilde{V}_0(S,\tau) = \min(V_0(S;X), c_n(S,\tau_n)). \tag{5}$$

Throughout the domain $\mathcal{D}$, we then always have $\widetilde{V}_0(S;X) \leq V(S,\tau) \leq c_n(S,\tau)$, while $\mathcal{L}V$ becomes positive in R_{conv}, zero in R_{cont} and negative in R_{call}. The pricing model constitutes a free boundary value problem with upper obstacle function $c_n(S, \tau_n)$ and lower obstacle function $\widetilde{V}_0(S; X)$. The sign and value of the three quantities, $V - \widetilde{V}_0(S; X), V - c_n(S, \tau_n)$ and $\mathcal{L}V$, can be summarized as follows.

1. In the conversion region R_{conv}, we have $V - \widetilde{V}_0(S; X) = 0, \mathcal{L}V > 0$ and $V - c_n(S, \tau_n) < 0$.
2. In the call region R_{call}, we have $V - c_n(S, \tau_n) = 0, \mathcal{L}V < 0$ and $V - \widetilde{V}_0(S; X) > 0$.
3. In the continuation region R_{cont}, we have $\mathcal{L}V = 0, V - \widetilde{V}_0(S; X) > 0$ and $V - c_n(S, \tau_n) < 0$.

We can express the variational inequalities formulation of the pricing model of a callable convertible security as

$$\max(V - c_n(S, \tau_n), \min(\mathcal{L}V, V - \widetilde{V}_0(S; X)) = 0. \tag{6a}$$

The terminal payoff is given by

$$V(S, 0) = \min(V_T(S; X), c_n(S, \tau_n)). \tag{6b}$$

Similar formulation of the variational inequalities can be found in the paper by Grau *et al*. (2003).

In the above variational inequalities formulation, we consider the following two mutually exclusive and exhaustive events corresponding to calling or no calling. When $(S, \tau) \notin R_{call}$, the call is not activated and $V < c_n(S, \tau_n)$. In this case, the price function $V(S, \tau)$ is governed by the variational inequalities formulation with early conversion right [see Eq. (4)]. This leads to zero value for $\min(\mathcal{L}V, V - V_0(S;X))$ and so Eq. (5) is satisfied. On the other hand, when $(S,\tau) \in R_{call}$, we have $V - c_n(S,\tau_n) = 0, \mathcal{L}V < 0$ and $V - V_0(S;X) > 0$ so that Eq. (5) is again satisfied.

3 Callable American warrants

In this section, we would like to perform detailed analysis of the impact of the call notice period requirement on the interaction of the callable and conversion rights in callable American warrants. Some preliminary studies have been reported by Kwok and Wu [11]. For example, they show that the issuer should call the warrant when the stock price S reaches $K+X$ if there is no notice period requirement.

For an American warrant with strike price X, the conversion payoff $V_0(S;X)$ and terminal payoff $V_T(S;X)$ are, respectively, given by

$$V_0(S;X) = (S-X)^+ \quad \text{and} \quad V_T(S;X) = (S-X)^+, \tag{7}$$

where the symbol x^+ denotes $\max(x,0)$. From the payoff functions defined in Eq. (7), we can deduce that the lower bound on the warrant value is $(S-X)^+$. The upper bound on the warrant value is given by $c_n(S,\tau_n)$, which is the value of the vested European option received by the holder upon calling. By observing that

$$\max(S-X,K) = K + (S-K-X)^+,$$

we deduce that

$$c_n(S,\tau_n) = Ke^{-r\tau_n} + c(S,\tau_n;K+X), \tag{8}$$

where $c(S, \tau_n; K + X)$ represents the price function of a European call with time to expiry τ_n and strike price $K + X$.

The optimal call policy of a callable American warrant is strongly dependent on the dividend policy of the stock price, the details are presented in the subsections below.

3.1 Underlying stock is non-dividend paying

When the underlying stock is non-dividend paying, the holder of a callable American warrant should never exercise the warrant prematurely. When the underlying stock is non-dividend paying, the usual argument of receiving no gain from earlier possession of shares through early exercise still applies even with the inclusion of the callable feature. When the conversion right is forfeited by the holder, premature termination of the warrant can only be attributed to optimal calling. With the presence of the call notice period requirement, the behaviors of the optimal call policy are summarized in Proposition 1.

Proposition 1 *Assume that the underlying stock of a callable American warrant pays no dividend. The critical stock price $S^*_{call}(\tau)$ exhibits the following properties.*

1. *$S^*_{call}(\tau)$ does not exist for $\tau \leq \tau_n$, that is, the issuer never calls the warrant when the time to expiry is shorter than or equal to the call notice period.*

2. *$S^*_{call}(\tau)$ always exists for $\tau > \tau_n$. The issuer should call the warrant when the stock price reaches $S^*_{call}(\tau)$ from below.*

3. *$S^*_{call}(\tau)$ is monotonically decreasing with respect to τ, also $S^*_{call}(\tau) \to \infty$ as $\tau \to \tau_n^+$ and $S^*_{call}(\infty)$ is finite. Indeed, $S^*_{call}(\infty)$ is determined by the following algebraic equation*

$$N\left(\frac{\ln \frac{S^*_{call}(\infty)}{K+X} + \left(r - \frac{\sigma^2}{2}\right)\tau_n}{\sigma\sqrt{\tau_n}}\right) = \frac{K}{K+X}. \tag{9}$$

The proof of Proposition 1 is presented in Appendix A. In Figure 1a, we show the plot of $S^*_{call}(\tau)$ against τ of a callable American warrant on a non-dividend paying stock. The parameter values of the pricing model used in the calculations are: $X = 1, K = 0.5, \tau_n = \dfrac{1}{12}, \sigma = 0.3, r = 0.05$. By solving Eq. (9), we obtain $S^*_{call}(\infty) = 1.4445$. All the time dependent behaviors of $S^*_{call}(\tau)$ as stated in Proposition 1 are revealed by the plot in Figure 1a.

3.2 Underlying stock pays continuous dividend yield

When the underlying stock pays continuous dividend yield, the callable American warrant may be terminated prematurely by either early conversion or calling. Recall that the price function $W(S,\tau)$ of the callable American warrant is bounded by the floor value $(S-X)^+$ and the cap value $c_n(S,\tau_n)$, that is,

$$(S-X)^+ \le W(S,\tau) \le c_n(S,\tau_n). \tag{10}$$

The floor value function and cap value function intersect at $\widehat{S}$, where $\widehat{S}$ is the unique solution to the algebraic equation: $c_n(S, \tau_n) = (S - X)^+$. This equation can be simplified into the form

$$c_n(\widehat{S}, \tau_n) = \widehat{S} - X, \tag{11}$$

since $c_n(S, \tau_n) > 0$. First, we observe that solution to Eq. (11) does not exist when $q = 0$ since $c_n(S, \tau_n) > (S - X)^+$ for all S. On the other hand, solution to Eq. (11) always exists when $q > 0$. To show the claim, suppose we define

$$f(S) = c_n(S, \tau_n) - (S - X). \tag{12}$$

When $q > 0$, we observe that $f'(S) < 0$ so that $f(S)$ is monotonically decreasing with respect to S. Together with $f(X) > 0$ and $f(\infty) < 0$, we can deduce that there exists unique value $\widehat{S}$ such that $f(\widehat{S}) = 0$. Also, it is quite straightforward to visualize that $\widehat{S}$ is an increasing function of K and Ineqs. (10) hold only for $S \leq \widehat{S}$.

As a remark, when $S > \widehat{S}$, it becomes optimal for both "issuer to call" and "holder to exercise". It is seen that $c_n(S, \tau_n) < S - X$ for $S > \widehat{S}$ when $q > 0$. Suppose the bond indenture allows issuer's call to prioritize over holder's conversion, then the warrant value becomes $c_n(S, \tau_n)$ when $S > \widehat{S}$ as a consequence of immediate issuer's call. In this case, the effective holder's conversion payoff is $\min(S - X, c_n(S, \tau_n))$ [see Eq. (5)].

Upper bounds on $S^*_{conv}(\tau)$ and $S^*_{call}(\tau)$

We claim that $S^*_{conv}(\tau)$ is bounded from above by $\widehat{S}$, that is,

$$S^*_{conv}(\tau) \leq \widehat{S} \quad \text{for all } \tau. \tag{13}$$

Assume the contrary, suppose $\widehat{S} < S^*_{conv}(\tau)$ so that $f(S^*_{conv}(\tau)) < 0$, we then have

$$W(S^*_{conv}, \tau) = S^*_{conv} - X > c_n(S^*_{conv}, \tau_n), \tag{14}$$

a contradiction to Ineqs. (10). In a similar manner, we can also show

$$S^*_{call}(\tau) \leq \widehat{S} \quad \text{for all } \tau. \tag{15}$$

Critical stock price of non-callable American warrant

Let $S^*(\tau)$ denote the critical stock price at the occurrence of premature exercise of the non-callable counterpart (usual American call). It is well known that $S^*(\tau)$ is a monotonically increasing function of τ with $S^*(0^+) = X \max\left(1, \frac{r}{q}\right)$ and $S^*(\infty) = \frac{\mu_+}{\mu_+ - 1}X$, where μ_+ is the positive root of the equation: $\frac{\sigma^2}{2}\mu^2 + \left(r - q - \frac{\sigma^2}{2}\right)\mu - r = 0$ (see Kwok's text [12]).

We claim that the optimal calling and conversion policies exhibit different behaviors according to whether (i) $\widehat{S} \geq S^*(\infty)$, (ii) $\widehat{S} \leq S^*(0^+)$ or (iii) $S^*(0^+) < \widehat{S} < S^*(\infty)$, the details of the justification are presented below.

Call right rendered worthless

When $\widehat{S} \geq S^*(\infty)$, we would like to show that the issuer will not call the American warrant throughout the whole life of the warrant. This corresponds to the case where the call price K assumes a sufficiently high value relative to X. Note that the warrant value with the callable feature is always less than or equal to the non-callable counterpart. When $\frac{\partial W_{non}}{\partial S}(S,\tau) > \frac{\partial c_n}{\partial S}(S,\tau_n)$ is observed, we can argue that the value of the non-callable counterpart $W_{non}(S,\tau)$ is always less than $c_n(S,\tau_n)$ for $S < S^*(\tau)$ and all values of τ, and so it is always sub-optimal for the issuer to call. At any given time to expiry τ, given that $\widehat{S} \geq S^*(\infty) > S^*(\tau)$ so that $f(S^*(\tau)) > 0$, we then have $c_n(S^*(\tau),\tau_n) > S^*(\tau) - X$. One deduces that $W_{non}(S,\tau)$ is less than $c_n(S,\tau_n)$ for $S < S^*(\tau)$ provided that $\frac{\partial W_{non}}{\partial S}(S,\tau) > \frac{\partial c_n}{\partial S}(S,\tau_n)$. In this case, the call by the issuer would increase the warrant value to $c_n(S,\tau_n)$, thus it is non-optimal. Since the call right is rendered worthless, the callable American warrant behaves like its non-callable counterpart. As the callable American warrant is terminated prematurely only by the holder's early exercise, we obtain $S^*_{conv}(\tau) = S^*(\tau)$. Once S reaches $S^*(\tau)$ from below, the callable American warrant would be exercised optimally by the holder.

In Figure 1b, we plot $S^*_{conv}(\tau)$ against τ, where $S^*_{conv}(\tau)$ denotes the critical stock price at which the callable American warrant should be exercised by its holder. In our numerical calculations, we take the following parameter values in the pricing model: $X = 1, K = 0.5, \tau_n = \frac{1}{12}, \sigma = 0.3, r = 0.05$ and $q = 0.1$. The asymptotic limit of S^*_{conv} at infinite time to expiry is found to be 1.6463 and $\widehat{S} = 1.6547$.

Conversion right rendered worthless

Next, suppose $\widehat{S} \leq S^*(0^+)$, which corresponds to the scenario where the call price K assumes a low value relative to X. Note that $r > q$ is one of the necessary conditions for $\widehat{S} \leq S^*(0^+)$. Assume the contrary, suppose $r \leq q$, then $S^*(0^+) = X$ but $\widehat{S}$ is always greater than X so it is impossible to have $\widehat{S} \leq S^*(0^+)$. For $\widehat{S} \leq S^*(0^+)$, we argue that the holder should never exercise the warrant prematurely. Assume the contrary, suppose the conversion region R_{conv} exists, then there exists $(S, \tau) \in R_{conv}$ where $S \leq \widehat{S} \leq S^*(0^+)$ such that $V = S - X$. It is known from an established result in American call option model that $\mathcal{L}V = \mathcal{L}(S - X) \leq 0$ when $S \leq S^*(0^+) = \dfrac{r}{q}X$ and $r > q$. This contradicts with the requirement that $\mathcal{L}V > 0$ in R_{conv}. When the early exercise right is rendered worthless, the optimal call policy adopted by the issuer is similar to that of the callable warrant on a non-dividend paying asset (see Proposition 1), except that $S^*_{call}(\tau)$ is bounded from above by a finite value $\widehat{S}$ [see Ineq. (15)]. Recall that when $q = 0$, $S^*_{call}(\tau)$ may become infinite [see part (3) in Proposition 1]. This does not lead to inconsistency since $\widehat{S}$ becomes infinite when $q = 0$. The time dependent properties of $S^*_{call}(\tau)$ are summarized in Proposition 2.

Proposition 2 *Assume that the underlying stock of a callable American warrant pays continuous dividend yield* q *and* $\widehat{S} < S^*(0^+)$. *It is always non-optimal for the holder to exercise the warrant prematurely. The critical stock price* $S^*_{call}(\tau)$ *at optimal calling exhibits the following properties.*

1. $S^*_{call}(\tau)$ *always exists for* $\tau \geq 0$, *and* $S^*_{call}(\tau)$ *is a decreasing function of* τ.

2. $S^*_{call}(\tau)$ *is bounded from above by* $\widehat{S}$ *and below by* $S^*_{call}(\infty)$*. Here,* $S^*_{call}(\infty)$ *is determined by the following algebraic equation*

$$Ke^{-r\tau_n} + \left(1 - \frac{1}{\mu_+}\right) S^*_{call}(\infty) e^{-q\tau_n} N(d_1) - (K+X)e^{-r\tau_n} N(d_2) = 0, \tag{16}$$

where

$$d_1 = \frac{\ln \frac{S^*_{call}(\infty)}{K+X} + \left(r - q + \frac{\sigma^2}{2}\right)\tau_n}{\sigma\sqrt{\tau_n}} \quad \textit{and} \quad d_2 = d_1 - \sigma\sqrt{\tau_n}.$$

The proof of Proposition 2 is presented in Appendix B. In Figure 1c, we show the plot of $S^*_{call}(\tau)$ against τ of a callable American warrant on an underlying stock that pays continuous dividend yield. The parameter values used in the calculations are: $X = 1, K = 0.5, \tau_n = \dfrac{1}{12}, \sigma = 0.2, r = 0.04$ and $q = 0.02$. For this set of parameter values, we obtain $S^*(0^+) = 2$ and $\widehat{S} = 1.9983$ so that $\widehat{S} < S^*(0^+)$. The asymptotic lower bound $S^*_{call}(\infty)$ is found to be 1.4958. The plot reveals that $S^*_{call}(\tau)$ is a decreasing function of τ and

$$\widehat{S} \geq S^*_{call}(\tau) \geq S^*_{call}(\infty) \quad \text{for all} \quad \tau \geq 0.$$

There exists a finite time interval near expiry such that $S^*_{call}(\tau) = \widehat{S}$, and in particular, $S^*_{call}(0^+) = \widehat{S}$.

Interaction of optimal call and conversion rights

We consider the last case where $S^*(0^+) < \widehat{S} < S^*(\infty)$. Since $S^*(\tau)$ is monotonically increasing with respect to τ, there exists unique value $\widehat{\tau}$ such that $S^*(\widehat{\tau}) = \widehat{S}$. We consider the two separate cases, (i) $\tau < \widehat{\tau}$ and (ii) $\tau \geq \widehat{\tau}$.

For $\tau < \widehat{\tau}$, the issuer would not call the warrant when $S < S^*(\tau)$ since the warrant value $W(S,\tau)$ is less than $c_n(S,\tau_n)$ [see similar argument presented for the first case where $\widehat{S} > S^*(\infty)$]. On the other hand, the warrant will be exercised optimally by the holder when S reaches $S^*(\tau)$ from below. When $\tau \geq \widehat{\tau}$, the warrant remains unexercised when S reaches $\widehat{S}$. If the warrant is not called when S goes beyond $\widehat{S}$, then the warrant value will be above $S - X$ and in turns greater than $c_n(S,\tau_n)$. Since $c_n(\widehat{S},\tau_n) = \widehat{S} - X$, the issuer should call the warrant when S reaches $\widehat{S}$ so as to cap the warrant value not to shoot above $c_n(\widehat{S},\tau_n)$. Therefore, when $\tau \geq \widehat{\tau}$, the warrant is terminated prematurely due to calling. As τ increases, the critical stock price $S^*_{call}(\tau)$ stays at $\widehat{S}$ for some period of time and eventually decreases monotonically with τ. As $\tau \to \infty, S^*_{call}(\tau)$ tends to an asymptotic limit whose value is determined by solving Eq. (16).

In Figure 1d, we plot the warrant value against S for varying values of τ. The parameter values used in the calculations are: $X = 1, K = 0.5, \tau_n = \dfrac{1}{12}, \sigma = 0.3, r = 0.01$ and $q = 0.015$. Recall that the price curve of a warrant always stays between the curve of $c_n(S,\tau_n)$ and the intrinsic value line $S-X$. When $\tau = 0.25$ (less than $\widehat{\tau} = 0.97$), the price curve intersects the intrinsic value line signifying that premature termination is due to early exercise by the holder. As τ increases to $\widehat{\tau}$, the price curve of the warrant, intrinsic value line and curve of $c_n(S,\tau_n)$ all intersect at $\widehat{S}$. For the given set of parameter values, we obtain $\widehat{S} = 1.776$. The phenomenon of intersection of the three curves at $S = \widehat{S}$ remains for some time period beyond $\widehat{\tau}$ (see the price curve at $\tau = 1$). When τ increases further, the warrant's price curve intersects the curve of $c_n(S,\tau)$ at $S^*_{call}(\tau)$, whose value is less than $\widehat{S}$ (see the price curve at $\tau = 5$).

Using the same set of parameter values, we computed $S^*_{conv}(\tau)$ and $S^*_{call}(\tau)$ and they are plotted against τ in Figure 1e. Over the time period $0 \leq \tau \leq \widehat{\tau}$, the warrant is terminated prematurely by holder's early exercise and the plot of $S^*_{conv}(\tau)$ is shown as the dotted-dashed curve. When $\tau > \widehat{\tau}$, the warrant is terminated prematurely by calling and the plot of $S^*_{call}(\tau)$ is shown as the solid curve. We observe that $S^*_{call}(\tau)$ stays at the constant value $\widehat{S}$ for certain time period beyond $\widehat{\tau}$, then decreases monotonically with respect to τ and tends to the lower asymptotic limit $S^*_{call}(\infty) = 1.4855$ as $\tau \to \infty$. The behaviors exhibited by the plots of $S^*_{conv}(\tau)$ and $S^*_{call}(\tau)$ clearly reveal the interaction between optimal conversion and calling policies in a callable American warrant. The monotonicity properties with respect to τ of both $S^*_{conv}(\tau)$ and $S^*_{call}(\tau)$ are resulted from the monotonic increasing property of the warrant value.

As explained earlier in this subsection, suppose the bond indenture allows the issuer's call to prioritize over holder's conversion, the warrant value is seen to be equal to the call payoff $c_n(S,\tau_n)$ when $S > \widehat{S}$. Hence, the region $\{(\tau, S) : S > \widehat{S}\}$ is contained in the call region.

3.3 Underlying asset pays discrete dividends

We also computed $S^*_{conv}(\tau)$ and $S^*_{call}(\tau)$ of a callable American warrant on a discrete dividend paying stock using the following set of parameter values: $X = 1, K = 0.5, \tau_n = \dfrac{1}{12}, \sigma = 0.3, r = 0.01$ and proportional discrete dividends of annualized yield of 1.5%. The discrete dividends are paid at $\tau = 0.5, 1.0, 1.5, 2.0, 2.5$. The discrete dividends are paid semi-annually. When the underlying stock pays discrete dividends, like the non-callable counterpart, the holder chooses to exercise the warrant only at times right before the dividend dates. In Figure 1f, we use '△' symbols to show the critical stock price S^*_{conv} at which the warrant should be exercised optimally at times right before the dividend dates. These critical stock prices for early exercise are seen to be monotonically increasing with respect to τ, again due to the monotonic increasing property of the warrant value. Within the time interval between successive dividend dates, the warrant may be terminated prematurely by issuer's call. The solid curves in Figure 1f show the plot of $S^*_{call}(\tau)$ against τ over successive time intervals. We observe that $S^*_{call}(\tau)$ is monotonically decreasing with respect to τ and $S^*_{call}(\tau)$ does not exist over the time interval of width τ_n right after the dividend date. These phenomena can be explained by similar arguments presented in Proposition 1 for American warrants on non-dividend paying stock. When the time to expiry of the warrant is sufficiently long, we observe in Figure 1f that S^*_{conv} does not exist at $\tau = 2$ and $\tau = 2.5$. This is because the warrant should have been called by the issuer at a lower critical stock price.

4 Callable convertible bonds

Consider a callable convertible bond with face value X, with the conversion number assumed to be unity, the payoff function upon conversion $B_0(S;X)$ and the terminal payoff $B_T(S;X)$ of the bond are given by

$$B_0(S;X) = S \quad \text{and} \quad B_T(S;X) = \max(X,S) = X + (S-X)^+, \qquad (17)$$

respectively. Upon calling by the issuer, the bondholder can choose to receive either one unit of stock or cash of amount $K + X$ at the end of the notice period. Here, we use K to denote the excess of the call price over the face value. The vested European option received by the bondholder upon calling has a life span of τ_n and terminal payoff $\max(K+X, S)$, so its price function $\widetilde{c}_n(S,\tau_n)$ is given by

$$\widetilde{c}_n(S,\tau_n) = (K+X)e^{-r\tau_n} + c(S,\tau_n;K+X). \qquad (18)$$

From the conversion and terminal payoff functions defined in Eq. (17), we can deduce that the lower bound on the bond value is given by $\max(Xe^{-r\tau}, S)$. Due to the time dependence of the lower bound value, the convertible bond value no longer observes monotonicity in τ, unlike that of the American warrant. This also leads to loss of monotonicity in τ of the critical stock prices $\widetilde{S}^*_{conv}(\tau)$ and $\widetilde{S}^*_{call}(\tau)$ for a callable convertible bond. On the other hand, the bond value is bounded from above by $\widetilde{c}_n(S,\tau_n)$. In summary, the price function $B(S,\tau)$ of a callable convertible bond is bounded by the floor value function $\max(Xe^{-r\tau}, S)$ and the cap value function $\widetilde{c}_n(S,\tau_n)$, that is,

$$\max(Xe^{-r\tau}, S) \le B(S,\tau) \le \widetilde{c}_n(S,\tau_n). \qquad (19)$$

Similar to the callable American warrant, the above inequalities hold only for $S \leq \widetilde{S}$, where $\widetilde{S}$ is the unique solution to the following algebraic equation

$$\widetilde{c}_n(\widetilde{S}, \tau_n) = \max(Xe^{-r\tau}, \widetilde{S}), \quad \tau > 0. \tag{20a}$$

Since $\widetilde{c}_n(S, \tau_n) > Xe^{-r\tau}$ for all values of τ, the above equation can be simplified into the form

$$\widetilde{c}_n(\widetilde{S}, \tau_n) = \widetilde{S}. \tag{20b}$$

It can be shown that $\widetilde{S}$ exists when $q > 0$, but there is no solution to Eq. (20b) if $q = 0$. Similar to the callable American warrant, both $\widetilde{S}^*_{conv}(\tau)$ and $\widetilde{S}^*_{call}(\tau)$ are bounded above by $\widetilde{S}$, provided that they exist. Since $e^{-r\tau_n}$ is in general a small quantity and K is of the same order of magnitude as X, so we have $(K + X)e^{-r\tau_n} \geq X$. It is then easily seen that $\widetilde{S} > X$ since

$$\widetilde{S} = \widetilde{c}_n(\widetilde{S}, \tau_n) > (K + X)e^{-r\tau_n} \geq X.$$

Unlike the American call option, the optimal conversion policy of a non-callable convertible bond is less well known. We write $S^*_b(\tau)$ as its critical stock price at optimal conversion. Like the American warrant, the properties of $\widetilde{S}^*_{conv}(\tau)$ and $\widetilde{S}^*_{call}(\tau)$ are related to the relative magnitude of $\widetilde{S}$ and $S^*_b(\tau)$. The time dependent behaviors of $S^*_b(\tau)$ are summarized in Proposition 3.

Proposition 3 *Assume that the underlying stock pays continuous dividend yield. The critical stock price $S^*_b(\tau)$ for optimal conversion of a non-callable convertible bond observes the following properties.*

1. *Suppose we set the strike price of an American call to be the same as the face value of a non-callable convertible bond and let* $S^*(\tau)$ *denote its critical stock price of early exercise, then*

$$S_b^*(\tau) \leq S^*(\tau) \quad \textit{for all} \quad \tau \geq 0. \tag{21}$$

2. *At time close to expiry,* $S_b^*(0^+) = X$*; and at infinite time to expiry,* $S_b^*(\infty) = 0$.

The proof of Proposition 3 is presented in Appendix C. Similar to the analysis performed in Sec. 3 for the callable American warrants, we analyze the optimal conversion policy and calling policy and their possible interaction for callable convertible bonds under the following three mutually exclusive and exhaustive scenarios. Firstly, it occurs that it is always non-optimal for the holder to convert prematurely so that the conversion right becomes non-effective. Secondly, it is always non-optimal for the issuer to call so that the bond becomes essentially a non-callable convertible bond. Third, both optimal conversion and calling occur. Specifically, over some part of the bond's life, optimal call may occur prior to optimal conversion, but vice versa for other times.

The first scenario always occurs when the underlying stock is non-dividend paying. When the underlying asset pays continuous dividend yield, either the second or third scenario occurs. Which one does occur is dependent on the relative magnitude of $\widetilde{S}$ and the maximum value of $S_b^*(\tau)$ over the life of the bond.

Non-effective conversion right

When the underlying stock is non-dividend paying, the conversion right is rendered worthless. This is because early conversion would lead to loss on the insurance value associated with the embedded optionality but no gain from the earlier procession of shares. In Figure 2a, we show the plot of $\widetilde{S}^*_{call}(\tau)$ against τ of a callable convertible bond on a non-dividend paying stock. The parameter values chosen for the bond model are: $X = 1, K = 0.5, \tau_n = \dfrac{1}{12}, \sigma = 0.3, r = 0.05$. Similar to $S^*_{call}(\tau)$ of the callable American warrant, we observe that $\widetilde{S}^*_{call}(\tau)$ does not exist for $\tau \leq \tau_n$ and it always exists for $\tau > \tau_n$. However, there is no monotonicity in τ for $\widetilde{S}^*_{call}(\tau)$.

For a callable American warrant, it has been discussed in Sec. 3.2 that the conversion right becomes non-effective when $\widehat{S}$ is less than $S^*(0^+)$, where $S^*(0^+)$ is the minimum of the critical stock price of optimal conversion. However, we have seen that $S^*_b(\tau) \to 0$ as $\tau \to \infty$ when the underlying stock pays continuous dividend yield q. Since $\widetilde{S}$ cannot be less than the minimum value of $S^*_b(\tau)$, the scenario of conversion right being rendered worthless does not occur in convertible bond when $q > 0$. In other words, conversion right in a convertible bond becomes non-effective if and only if the underlying stock is non-dividend paying.

Non-effective call privilege

Let $S^*_{b,\max}$ denote $\max\limits_{\tau \in [0,\infty)} S^*_b(\tau)$. The call privilege would be forfeited by the issuer when $\widetilde{S} > S^*_{b,\max}$. This corresponds to the scenario where the call price is of sufficiently high value relative to the face value. Like the case

$\widehat{S} > S^*(\infty)$ in callable American warrant (see Sec. 3.2), one can show that optimal conversion by holder always occurs prior to optimal calling by the issuer. Under such scenario, the value of the convertible bond always stays below $\widetilde{c}_n(S, \tau_n)$ so that calling by issuer is always sub-optimal. When the call privilege is non-effective, the bond becomes effectively a non-callable convertible bond, so we have $\widetilde{S}^*_{conv}(\tau) = S^*_b(\tau)$ for all τ.

In Figure 2b, we show the plot of $\widetilde{S}^*_{conv}(\tau)$ against τ for a convertible bond where the call privilege is non-effective. The parameter values used in the model are: $X = 1, K = 0.5, \tau_n = \dfrac{1}{12}, \sigma = 0.3, r = 0.06, q = 0.04$. For this given set of parameter values, $\widetilde{S}$ is found to be 1.6832, which is greater than $S^*_{b,\max}$. The plot of $\widetilde{S}^*_{conv}(\tau)$ observes the properties that $\widetilde{S}^*_{conv}(0^+) = X$, and $\widetilde{S}^*_{conv}(\tau)$ first increases to some maximum value then decreases to zero as $\tau \to \infty$.

Interaction of optimal call and conversion rights

When $\widetilde{S} < S^*_{b,\max}$, both optimal calling and conversion can occur during the life of a callable convertible bond. Recall that $S^*_b(\tau)$ starts at $S^*(0^+) = X$, increases to a maximum peak value $S^*_{b,max}$ then decreases to zero as $\tau \to \infty$ (see Figure 2b). Together with $\widetilde{S} > X$, we can deduce that when $\widetilde{S} < S^*_{b,\max}$, there exist two values of τ such that $S^*_b(\tau) = \widetilde{S}$. Let the smaller of these two critical values of τ be denoted by $\widehat{\tau}_{small}$, and note that $\widetilde{S} \geq S^*_b(\tau)$ for $\tau \leq \widehat{\tau}_{small}$. Similar to the case $S^*(0^+) < \widehat{S} < S^*(\infty)$ in a callable American warrant, we argue that for $\tau \leq \widehat{\tau}_{small}$, the issuer would not call the convertible bond when $S < S^*_b(\tau)$ since the bond value is less than

$\widetilde{c}_n(S, \tau_n)$. In this case, the bond will be converted into shares optimally by the holder when S reaches $S_b^*(\tau)$. When τ increases beyond $\widehat{\tau}_{small}$, we have $\widetilde{S} < S_b^*(\tau)$. Under this scenario, optimal calling by issuer occurs prior to premature conversion. This phenomenon persists over some time interval. However, since $S_b^*(\tau)$ tends to zero as $\tau \to \infty$, we expect that optimal conversion prior to optimal calling occurs again when the time to expiry continues to increase beyond some sufficiently high value.

To verify the above claim, we performed sample calculations using the following set of parameter values: $X = 1, K = 0.5, \tau_n = \dfrac{1}{12}, \sigma = 0.3, r = 0.03$ and $q = 0.018$ for the callable convertible bond model. The plot of $\widetilde{S}^*_{conv}(\tau)$ and $\widetilde{S}^*_{call}(\tau)$ against τ is shown in Figure 2c. For the given set of parameter values, the value of $\widetilde{S}$ is found to be 1.7484. Both $\widetilde{S}^*_{conv}(\tau)$ and $\widetilde{S}^*_{call}(\tau)$ are observed to be bounded above by $\widetilde{S}$. For times close to expiry, early conversion occurs prior to calling and $\widetilde{S}^*_{conv}(\tau)$ is monotonically increasing with respect to τ over this period. For intermediate values of time to expiry, optimal calling occurs prior to early conversion. We observe that $\widetilde{S}^*_{call}(\tau)$ stays at the constant value $\widetilde{S}$ for some time beyond $\widehat{\tau}_{small}$, then decreases in value with increasing τ and eventually increases to the level $\widetilde{S}$ again. When τ increases further to some high threshold value, optimal conversion prior to calling occurs again and $\widetilde{S}^*_{conv}(\tau)$ becomes monotonically decreasing with respect to τ.

In Figure 2d, we plot the price function of a callable convertible bond against S for varying values of τ. The same set of parameter values as those in Figure 2c are used in the calculations. The bond price curves are seen to be bounded between the cap value curve $\widetilde{c}_n(S, \tau_n)$ and floor value curve $\max(Xe^{-r\tau}, S)$. At small value of τ, the price curve for $\tau = 0.25$ intersects the conversion value line tangentially signifying that early conversion occurs prior to calling. As τ increases, the price curve for $\tau = 1.5$ intersects the cap value curve at $S = \widetilde{S}$ (which is the point of intersection of the cap and floor value curves). When τ increases further, say $\tau = 10$, the bond price curve intersects the cap value curve at $\widetilde{S}^*_{call}$ less than $\widetilde{S}$.

5 Conclusion

We have performed theoretical analysis on the optimal policies of holder's conversion and issuer's calling of two types of callable convertible securities, namely, the callable American warrants and callable convertible bonds. We present the variational inequalities formulations for pricing models of derivative products with embedded conversion and calling rights. The pricing model of a derivative with both conversion and calling rights is essentially a free boundary value problem with two-sided obstacles. In particular, we consider the impact of the notice period requirement on the optimal calling policy.

The calling right allows the issuer to place a cap on the derivative value, where the cap value is the payoff received by the holder upon calling. With the presence of the notice period requirement, the payoff upon calling has dependence on the stock price since the holder is essentially given a vested

European option with maturity date being set at the end of the notice period. A richer set of patterns of interaction of optimal calling and conversion policies are exhibited due to the dependence of the cap value on the stock price level.

For American warrants and convertible bonds, both the cap and floor values have stock price dependence when there is a notice period requirement for calling. We show that the critical stock prices at optimal conversion or optimal calling are bounded above by some threshold stock price, whose value is equal to the stock price at which the cap and floor values are equal. Also, we demonstrate that the optimal calling and conversion policies and their interaction depend on the dividend policy of the underlying stock and the relative magnitude of the call price with respect to the strike price (or face value). When the stock is non-dividend paying, premature conversion by the holder is always sub-optimal so that the conversion right is rendered worthless. Conversion privilege also becomes essentially non-effective when the call price is sufficiently low so that optimal calling by the issuer always occurs prior to premature conversion. On the other hand, when the call price is too high so that optimal holder's conversion always occurs prior to issuer's calling, this would render the call right worthless. When the call price assumes a value that is intermediate between the upper and lower threshold levels as stated in the above two cases, both optimal conversion and calling may occur during the life of the derivative.

For the sake of analytic tractability, default risk of the callable securities has been neglected in our analysis. Also, we assume constant interest rate and continuous dividend yield and zero coupon. The presence of discrete coupon and dividend payments will lead to jumps in the critical stock price before and after payment dates. However, the underlying principles that determine the optimal policies of premature conversion and calling remain intact, except that the influences of discrete coupons and dividends on the optimal policies have to be incorporated.

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Appendix A — Proof of Proposition 1

1. When $q = 0$, the callable American warrant is never exercised prematurely so that $W(S,\tau) \leq c(S,\tau)$, where $c(S,\tau)$ is the price function of the European call counterpart. For $\tau \leq \tau_n$, we have $c_n(S,\tau_n) > c(S,\tau) \geq W(S,\tau)$ so that the issuer never calls the warrant prematurely.
2. To show that $S^*_{call}(\tau)$ always exists for $\tau > \tau_n$, we prove by contradiction. Assume that there exists some $\tau_0 > \tau_n$ such that $W(S,\tau) < c_n(S,\tau_n)$ for all S and $\tau_n < \tau \leq \tau_0$, that is, the warrant is never called. Combining with the result in part (1), the callable American warrant then becomes a European call option for $\tau \leq \tau_0$ so that

$$W(S,\tau) = c(S,\tau;X) \quad \text{for} \quad \tau \leq \tau_0.$$

It suffices to show that $W(S,\tau) > c_n(S,\tau_n)$ for some S and $\tau_n < \tau \leq \tau_0$. For $\tau_n < \tau \leq \tau_0$, the put-call parity relation gives

$$\begin{aligned} & c_n(S,\tau_n) - W(S,\tau) \\ =\ & c_n(S,\tau_n) - c(S,\tau) \\ =\ & Ke^{-r\tau_n} + c(S,\tau_n;K+X) - c(S,\tau;X) \\ =\ & X(e^{-r\tau} - e^{-r\tau_n}) + p(S,\tau_n;K+X) - p(S,\tau;X), \end{aligned}$$

where $p(S,\tau;X)$ denotes the price function of a European put with strike price X. Since the put price function tends to zero as S tends to infinity, $P(S,\tau;X)$ and $P(S,\tau_n;K+X)$ become sufficiently small when S is sufficiently large. Since we have

$$\lim_{S\to\infty}[c_n(S,\tau_n) - W(S,\tau)] = X(e^{-r\tau} - e^{-r\tau_n}) < 0 \quad \text{for} \quad \tau > \tau_n,$$

we then deduce that there exist some sufficiently large value of S such that

$$W(S,\tau) > c_n(S,\tau_n) \quad \text{for} \quad \tau_n < \tau \leq \tau_0.$$

A contradiction is encountered since we have assumed $W(S,\tau) < c_n(S,\tau_n)$ for all S and $\tau_n < \tau \leq \tau_0$. This would imply that there exists critical stock price S^*_{call} such that when S reaches the level $S^*_{call}, W(S,\tau)$ becomes equal to $c_n(S,\tau_n)$.

3. The monotonically decreasing property of $S^*_{call}(\tau)$ is derived from the monotonically increasing property of the price function $W(S,\tau)$ with respect to τ. To show the unboundedness of $S^*_{call}(\tau_n^+)$, we prove by contradiction. Suppose $S^*_{call}(\tau_n^+)$ is finite, by continuity of the price function, we have

$$W(S,\tau_n) = c_n(S,\tau_n) \quad \text{for} \quad S > S^*_{call}(\tau_n^+).$$

This leads to a contradiction since the issuer should not call the warrant at $\tau = \tau_n$. Hence, $S^*_{call} \to \infty$ as $\tau \to \tau_n^+$. By setting $q = 0$ in the asymptotic formula for $S^*_{call}(\infty)$ in Proposition 2 [see Eq. (16)] and observing that $\mu_+ = 1$ when $q = 0$, we obtain Eq. (9).

Appendix B — Proof of Proposition 2

The proofs of the existence of $S^*_{call}(\tau)$ for all τ and the decreasing property of $S^*_{call}(\tau)$ can be established using similar arguments as presented in Appendix A.

We compute the asymptotic limit $S^*_{call}(\infty)$ by solving the price function $W_\infty(S)$ of the perpetual callable American warrant. The governing equation of $W_\infty(S)$ is given by

$$\frac{\sigma^2}{2}S^2\frac{d^2W_\infty}{dS^2} + (r-q)S\frac{dW_\infty}{dS} - rW_\infty = 0, \quad 0 < S < S^*_{call}(\infty),$$

with boundary conditions

$$W_\infty(0) = 0 \quad \text{and} \quad W_\infty(S^*_{call}(\infty)) = c_n(S^*_{call}(\infty), \tau_n)$$

and smooth pasting condition

$$\frac{dW_\infty}{dS}(S^*_{call}(\infty)) = e^{-q\tau_n}N(d_1(S^*_{call}(\infty))),$$

where

$$d_1(S) = \frac{\ln\frac{S}{K+X} + \left(r - q + \frac{\sigma^2}{2}\right)\tau_n}{\sigma\sqrt{\tau_n}}.$$

The general solution to the price function $W_\infty(S)$ takes the form (Kwok, [12])

$$W_\infty(S) = \alpha S^{\mu_+},$$

where α is an arbitrary constant and μ_+ is the positive root of the auxiliary equation:

$$\frac{\sigma^2}{2}\mu^2 + \left(r - q - \frac{\sigma^2}{2}\right)\mu - r = 0.$$

The arbitrary constant α and $S^*_{call}(\infty)$ are determined by solving simultaneously

$$\alpha[S^*_{call}(\infty)]^{\mu_+} = c_n(S^*_{call}(\infty), \tau_n)$$

$$\alpha\mu_+[S^*_{call}(\infty)]^{\mu_+-1} = e^{-q\tau_n}N(d_1(S^*_{call}(\infty))).$$

By eliminating α and using the price formula of $c_n(S,\tau_n)$ in Eq. (8), the asymptotic lower bound $S^*_{call}(\infty)$ is determined by solving

$$Ke^{-r\tau_n} + \left(1 - \frac{1}{\mu_+}\right) S^*_{call}(\infty)e^{-q\tau_n} N(d_1(S^*_{call}(\infty)) - (K+X)e^{-r\tau_n} N(d_2(S^*_{call}(\infty))) = 0,$$

where

$$d_2(S) = d_1(S) - \sigma\sqrt{\tau_n}.$$

By setting $q = 0$, the above equation for $S^*_{call}(\infty)$ reduces to Eq. (9).

Appendix C — Proof of Proposition 3

1. Suppose we let $\widetilde{W}(S,\tau) = B(S,\tau) - X$, where B is the price of a non-callable convertible bond. Without the embedded callable right in the bond, we always have $\mathcal{L}B \geq 0$. It then follows that $\widetilde{W}(S,\tau)$ satisfies the following linear complementarity formulation

$$\mathcal{L}\widetilde{W} \geq -rX \quad \text{and} \quad \widetilde{W} \geq S - X$$
$$(\mathcal{L}\widetilde{W} + rX)[\widetilde{W} - (S - X)] = 0$$

and

$$\widetilde{W}(S,0) = (S - X)^+.$$

The above formulation differs from that of an American call only by the source term $-rX$. By using the comparison principle, we deduce that $\widetilde{W}(S,\tau) \leq C(S,\tau)$, where $C(S,\tau)$ is the price function of the American call. The price curve of $\widetilde{W}(S,\tau)$ always stays below that of $C(S,\tau)$ so that it intersects the intrinsic value line $S - X$ at a lower critical stock price, hence $S_b^*(\tau) \leq S^*(\tau)$ for all $\tau \geq 0$.

2. It is obvious that $S_b^*(\tau) \geq Xe^{-r\tau}$ since the lower bound of the bond value is given by $\max(Xe^{-r\tau}, S)$. As $\tau \to 0^+$, we have

$$S_b^*(0^+) \geq X. \qquad (i)$$

On the other hand, suppose $S_b^*(0^+) > X$, then there exists some stock price level S satisfying $X < S < S_b^*(0^+)$ such that the bond remains alive at $\tau \to 0^+$. By continuity, the bond value is equal to S as $\tau \to 0^+$. Substituting $B(S, 0^+) = S$ into the Black-Scholes equation, we have $\frac{\partial B}{\partial \tau}(S, 0^+) = -qS < 0$. A contradiction is encountered since this implies that the bond value falls below the intrinsic value. Hence, we deduce that

$$S_b^*(0^+) \leq X. \qquad (ii)$$

Combining the above two inequalities, we obtain $S_b^*(0^+) = X$. When $\tau \to \infty$, the convertible bond becomes essentially equivalent to the American warrant with zero strike price. Recall that $S^*(\infty) = \frac{\mu_+}{\mu_+ - 1}X$, so we obtain $S_b^*(\infty) = 0$ since X is taken to be 0.

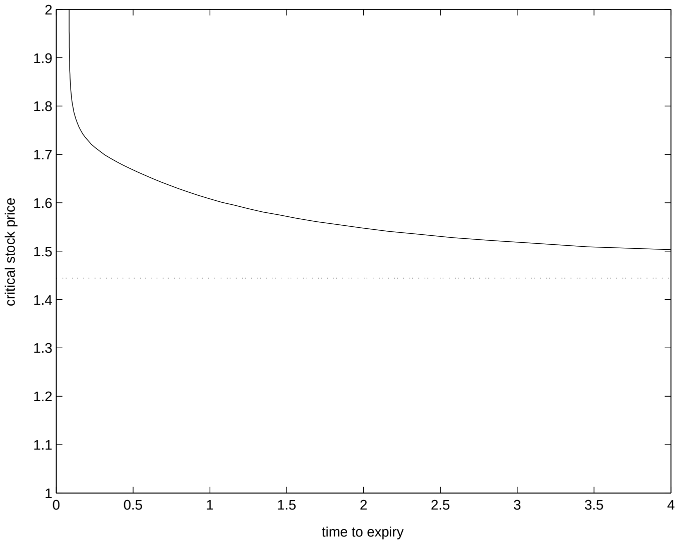

Fig. 1a The critical stock price $S^*_{call}(\tau)$ of a callable American warrant on a non-dividend paying stock is plotted against τ. The plot reveals the following properties of $S^*_{call}(\tau)$: (i) $S^*_{call}(\tau)$ is defined for all $\tau > \tau_n$, (ii) $S^*_{call}(\tau)$ is a monotonic decreasing function of τ, where $S^*_{call}(\tau) \to \infty$ as $\tau \to \tau_n^+$ and $S^*_{call}(\tau)$ tends to a finite limit as $\tau \to \infty$.

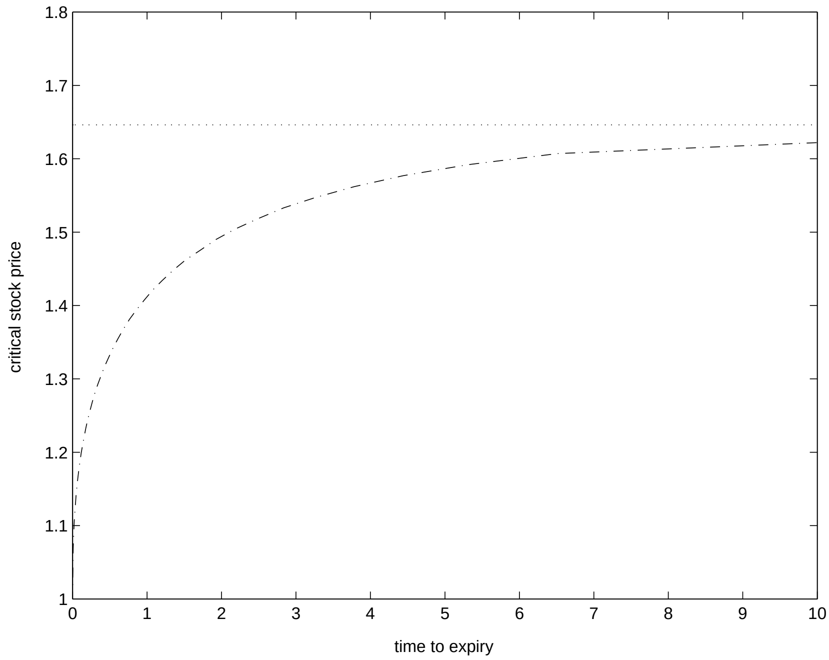

Fig. 1b The critical stock price $S^*_{conv}(\tau)$ is plotted against τ for a callable American warrant, corresponding to $\widehat{S} > S^*(\infty)$. In this case, the callable right of the warrant is rendered worthless. The asymptotic value $S^*_{conv}(\infty)$ is found to be 1.6463.

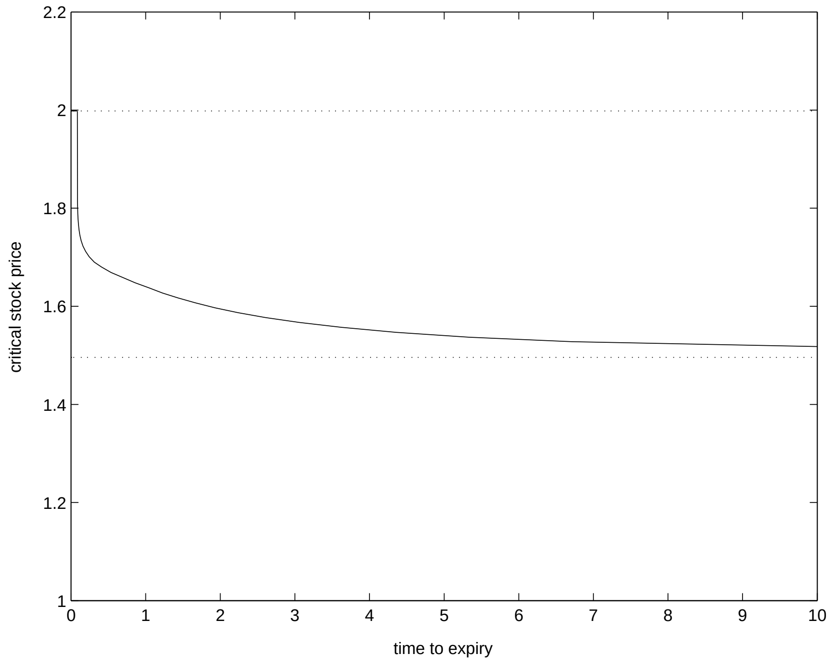

Fig. 1c The critical stock price $S^*_{call}(\tau)$ is plotted against τ for a callable American warrant, corresponding to $\widehat{S} < S^*(0^+)$. In this case, it is always non-optimal for the holder to exercise the warrant prematurely. The asymptotic lower bound $S^*_{call}(\infty)$ is found to be 1.4958. For a small time interval near expiry, $S^*_{call}(\tau)$ becomes equal to the upper bound $\widehat{S}$ (whose value is found to be 1.9983).

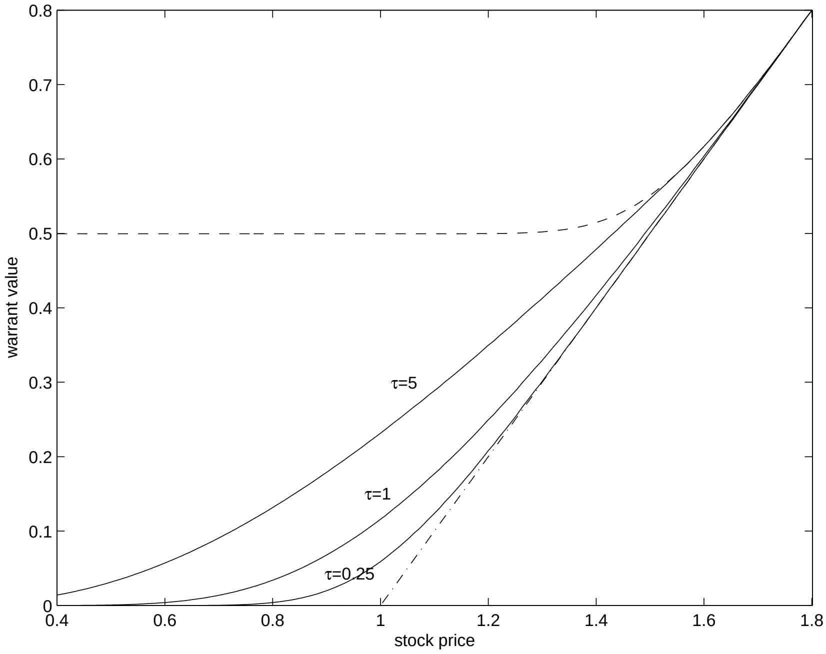

Fig. 1d The price function of a callable American warrant is plotted against S for varying values of τ, corresponding to $S^*(0^+) < \widehat{S} < S^*(\infty)$. When $\tau = 0.25$, the price curve intersects the intrinsic value line signifying that premature termination is due to early optimal conversion. When $\tau = 1$ and $\tau = 5$, the price curve intersects the curve of $c_n(S, \tau_n)$ signifying that premature termination is due to optimal calling by the issuer.

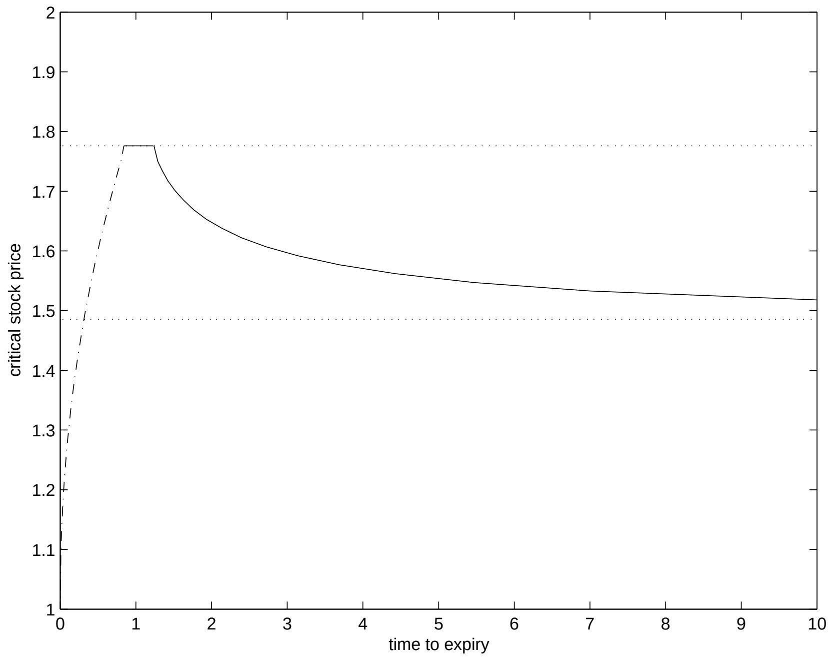

Fig. 1e The critical stock price $S^*_{conv}(\tau)$ of early exercise (shown as dotted-dashed curve) and $S^*_{call}(\tau)$ of call (shown as solid curve) are plotted against τ for a callable American warrant, corresponding to $S^*(0^+) < \widehat{S} < S^*(\infty)$. The upper and lower bounds of $S^*_{call}(\tau)$ are found to be 1.776 and 1.4855, respectively.

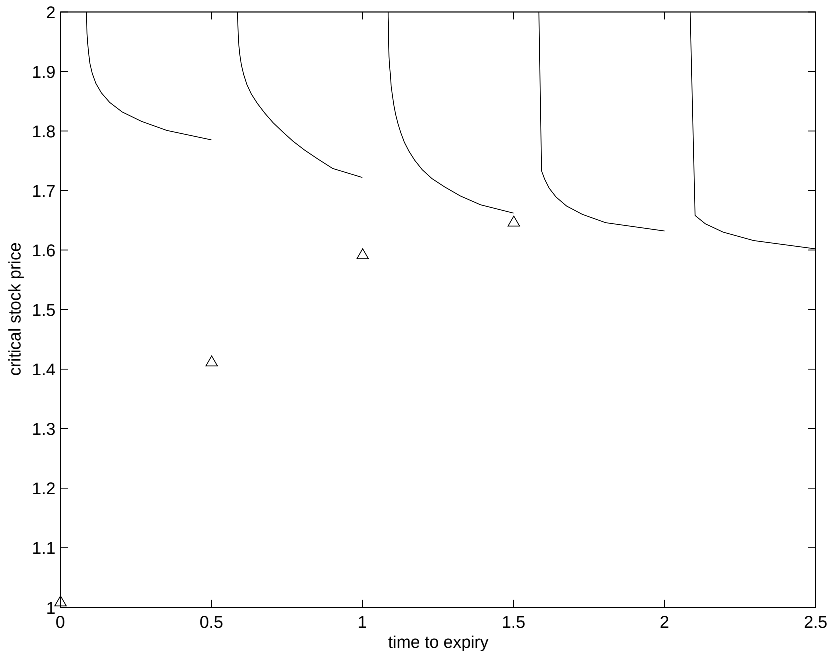

Fig. 1f The solid curves show the plot of the critical stock price $S^*_{call}(\tau)$ for a callable American warrant on a discrete dividend paying stock. Within each time interval between two successive dividend dates, $S^*_{call}(\tau)$ is monotonically decreasing with respect to τ. The '$\triangle$' symbols indicate the critical stock price S^*_{conv} at which the holder should exercise the warrant prematurely at times right before the dividend dates.

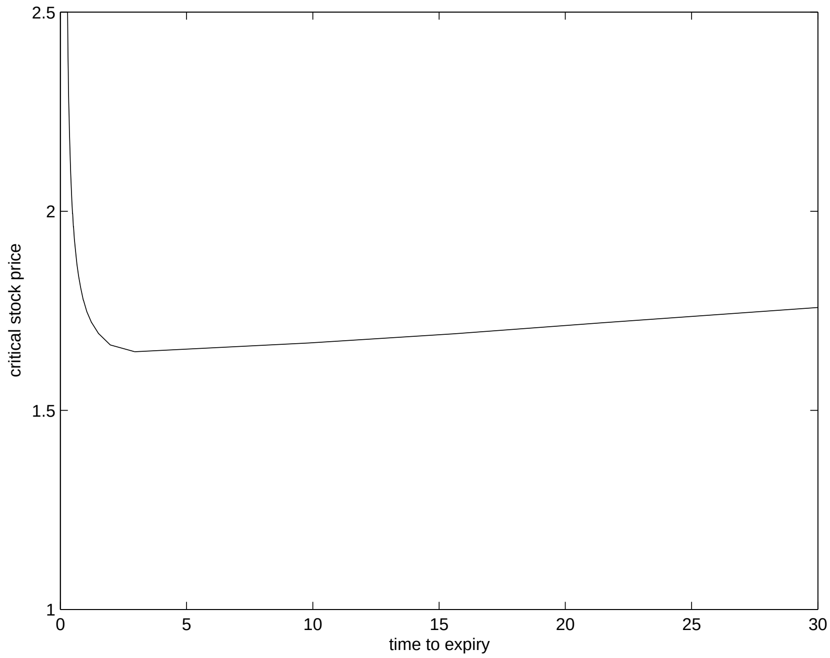

Fig. 2a The critical stock price $\widetilde{S}^*_{call}(\tau)$ is plotted against τ for a callable convertible bond, where the underlying asset is non-dividend paying. Unlike the callable American warrant, $\widetilde{S}^*_{call}(\tau)$ does not exhibit monotonicity in τ.

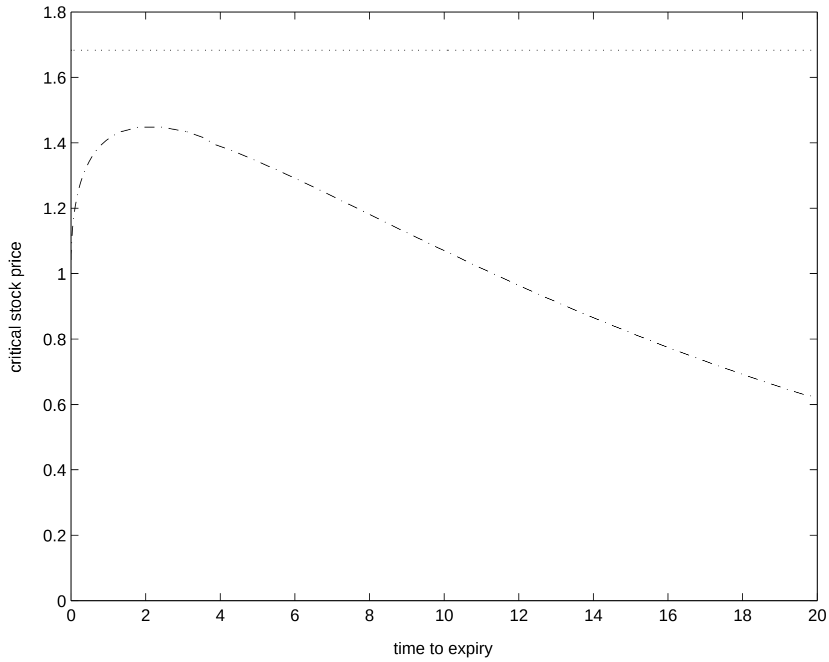

Fig. 2b The critical stock price $\widetilde{S}^*_{conv}(\tau)$ is plotted against τ for a callable convertible bond, corresponding to $\widetilde{S} > S^*_{b,max}$. In this case, the callable right of the bond is rendered worthless. Note that $\widetilde{S}^*_{conv}(\tau)$ always stays below $\widetilde{S}$ (see the dotted line), where $\widetilde{S}$ is found to be 1.6832.

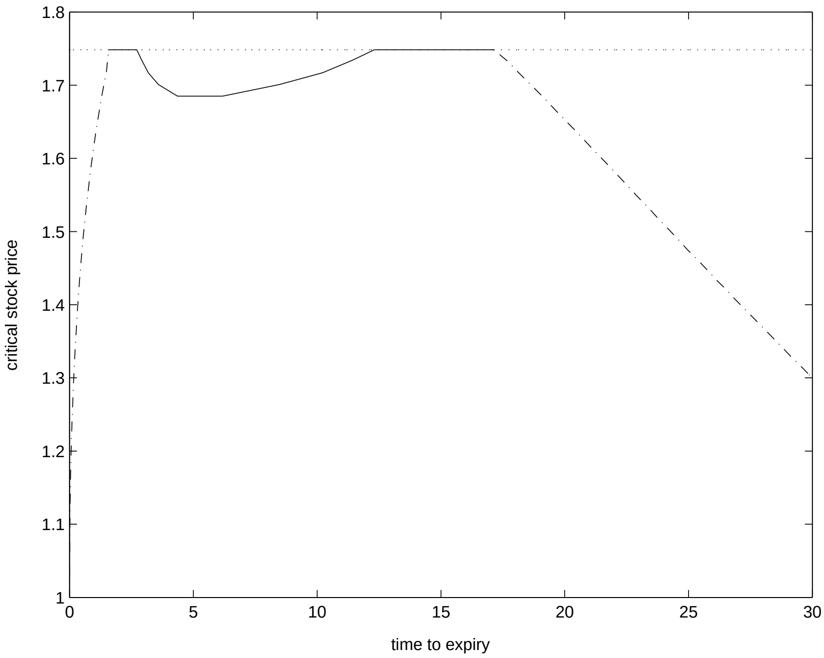

Fig. 2c The critical stock price $\widetilde{S}^*_{conv}(\tau)$ of early conversion (shown as dotted-dashed curves) and $\widetilde{S}^*_{call}(\tau)$ of premature calling (shown as solid curve) are plotted against τ for a callable convertible bond. Both $\widetilde{S}^*_{conv}(\tau)$ and $\widetilde{S}^*_{call}(\tau)$ are bounded above by $\widetilde{S}$ (see the dotted line), whose value is found to be 1.7484.

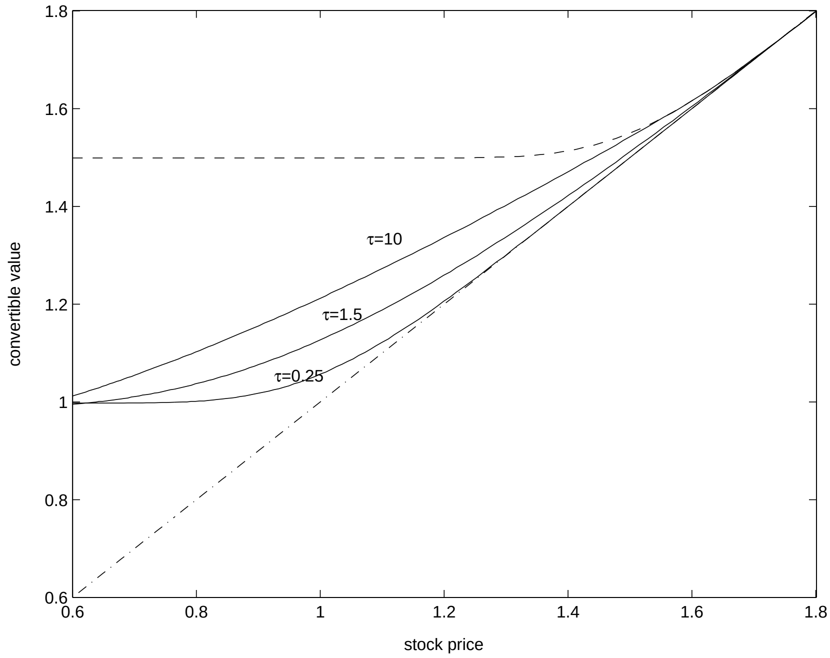

Fig. 2d The price function of a callable convertible bond is plotted against S for varying values of τ. When $\tau = 0.25$ and $\tau = 10$, the bond price curves intersect the conversion value line (shown as dotted-dashed line) and the cap value curve $\widetilde{c}_n(S, \tau_n)$ (shown as dashed line), respectively. When $\tau = 1.5$, the price curve of the bond ends at the intersection point of the conversion value line and the cap value curve.