Chapter 21
Efficient Options Pricing Using the Fast Fourier Transform

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Abstract We review the commonly used numerical algorithms for option pricing under Levy process via Fast Fourier transform (FFT) calculations. By treating option price analogous to a probability density function, option prices across the whole spectrum of strikes can be obtained via FFT calculations. We also show how the property of the Fourier transform of a convolution product can be used to value various types of option pricing models. In particular, we show how one can price the Bermudan style options under Levy processes using FFT techniques in an efficient manner by reformulating the risk neutral valuation formulation as a convolution. By extending the finite state Markov chain approach in option pricing, we illustrate an innovative FFT-based network tree approach for option pricing under Levy process. Similar to the forward shooting grid technique in the usual lattice tree algorithms, the approach can be adapted to valuation of options with exotic path dependence. We also show how to apply the Fourier space time stepping techniques that solve the partial differential-integral equation for option pricing under Levy process. This versatile approach can handle various forms of path dependence of the asset price process and embedded features in the option models. Sampling errors and truncation errors in numerical implementation of the FFT calculations in option pricing are also discussed.

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21.1 Introduction

The earliest option pricing models originated by Black and Scholes (1973) and Merton (1973) use the Geometric Brownian process to model the underlying asset price process. However, it is well known among market practitioners that the lognormal assumption of asset price returns suffers from serious deficiencies that give rise to inconsistencies as exhibited by smiles (skewness) and term structures in observed implied volatilities. The earlier remedy to resolve these deficiencies is the assumption of state and time dependence of the volatility of the asset price process (see Derman and Kani 1998; Dupire 1994). On the other hand, some researchers recognize the volatility of asset returns as a hidden stochastic process, which may also undergo regime change. Examples of these pioneering works on stochastic volatility models are reported by Stein and Stein (1991), Heston (1993), and Naik (2000). Starting from the seminar paper by Merton (1976), jumps are introduced into the asset price processes in option pricing. More recently, researchers focus on option pricing models whose underlying asset price processes are the Levy processes (see Cont and Tankov 2004; Jackson et al. 2008).

In general, the nice analytic tractability in option pricing as exhibited by Black-Scholes-Merton's Geometric Brownian process assumption cannot be carried over to pricing models that assume stochastic volatility and Levy processes for the asset returns. Stein and Stein (1991) and Heston (1993) manage to obtain an analytic representation of the European option price function in the Fourier domain. Duffie et al. (2000) propose transform methods for pricing European options under the affine jump-diffusion processes. Fourier transform methods are shown to be an effective approach to pricing an option whose underlying asset price process is a Levy process. Instead of applying the direct discounted expectation approach of computing the expectation integral that involves the product of the terminal payoff and the density function of the Levy process, it may be easier to compute the integral of their Fourier transform since the characteristic function (Fourier transform of the density function) of the Levy process is easier to be handled than the density function itself. Actually, one may choose a Levy process by specifying the characteristic function since the Levy-Khinchine formula allows a Levy process to be fully described by the characteristic function.

In this chapter, we demonstrate the effective use of the Fourier transform approach as an effective tool in pricing options. Together with the Fast Fourier transform (FFT) algorithms, real time option pricing can be delivered. The underlying asset price process as modeled by a Levy process can allow for more general realistic structure of asset returns, say, excess kurtosis and stochastic volatility. With the characteristic function of the risk neutral density being known analytically, the analytic expression for the Fourier transform of the option value can be derived. Option prices across the whole spectrum of strikes can be obtained by performing Fourier inversion transform via the efficient FFT algorithms.

This chapter is organized as follows. In the next section, the mathematical formulations for building the bridge that links the Fourier methods with option pricing are discussed. We first provide a brief discussion on Fourier transform and FFT algorithms. Some of the important properties of Fourier transform, like the Parseval relation, are presented. We also present the definition of a Lévy process and the statement of the Lévy-Khintchine formula. In Sect. 21.3, we derive the Fourier representation of the European call option price function. The Fourier inversion integrals in the option price formula can be associated with cumulative distribution functions, similar to the Black-Scholes type representation. However, due to the presence of a singularity arising from non-differentiability in the option payoff function, the Fourier inversion integrals cannot be evaluated by applying the FFT algorithms. We then present various modifications of the Fourier integral representation of the option price using the damped option price method and time value method (see Carr and Madan 1999). Details of the FFT implementation of performing the Fourier inversion in option valuation are illustrated. In Sect. 21.4, we consider the extension of the FFT techniques for pricing multi-asset options. Unlike the finite difference approach or the lattice tree methods, the FFT approach does not suffer from the curse of dimensionality of the option models with regard to an increase in the number of risk factors in defining the asset return distribution (see Dempster and Hong 2000; Hurd and Zhou 2009). In Sect. 21.5, we show how one can price Bermudan style options under Lévy processes using the FFT techniques by reformulating the risk neutral valuation formulation as a convolution. We show how the property of the Fourier transform of a convolution product can be effectively applied in pricing a Bermudan option (see Lord et al. 2008). In Sect. 21.6, we illustrate an innovative FFT-based network approach for pricing options under Lévy processes by extending the finite state Markov chain approach in option pricing. Similar to the forward shooting grid technique in the usual lattice tree algorithms, the approach can be adapted to valuation of options with exotic path dependence (see Wong and Guan 2009). In Sect. 21.7, we derive the partial integral-differential equation formulation that governs option prices under the Lévy process assumption of asset returns. We then show how to apply the Fourier space time stepping techniques that solve the partial differential-integral equation for option pricing under Lévy processes. This versatile approach can handle various forms of path dependence of the asset price process and features/constraints in the option models (see Jackson et al. 2008). We present summary and conclusive remarks in the last section.

21.2 Mathematical Preliminaries on Fourier Transform Methods and Lévy Processes

Fourier transform methods have been widely used to solve problems in mathematics and physical sciences. In recent years, we have witnessed the continual interests in developing the FFT techniques as one of the vital tools in option pricing. In fact, the Fourier transform methods become the natural mathematical tools when we consider option pricing under Lévy models. This is because a Lévy process X_t can be fully described by its characteristic function $\phi_X(u)$, which is defined as the Fourier transform of the density function of X_t.

21.2.1 Fourier Transform and Its Properties

First, we present the definition of the Fourier transform of a function and review some of its properties. Let $f(x)$ be a piecewise continuous real function over $(-\infty, \infty)$ which satisfies the integrability condition:

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

The Fourier transform of $f(x)$ is defined by

$$\mathcal{F}_f(u) = \int_{-\infty}^{\infty} e^{iuy} f(y)\, dy. \tag{21.1}$$

Given $\mathcal{F}_f(u)$, the function f can be recovered by the following Fourier inversion formula:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \mathcal{F}_f(u)\, du. \tag{21.2}$$

The validity of the above inversion formula can be established easily via the following integral representation of the Dirac function $\delta(y-x)$, where

$$\delta(y-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu(y-x)}\, du.$$

Applying the defining property of the Dirac function

$$f(x) = \int_{-\infty}^{\infty} f(y)\delta(y-x)\, dy$$

and using the above integral representation of $\delta(y-x)$, we obtain

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(y) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu(y-x)}\, du dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \left(\int_{-\infty}^{\infty} f(y) e^{iuy}\, dy \right) du. \end{aligned}$$

This gives the Fourier inversion formula (21.2).

Sometimes it may be necessary to take u to be complex, with $\operatorname{Im} u \neq 0$. In this case, $\mathcal{F}_f(u)$ is called the generalized Fourier transform of f. The corresponding

Fourier inversion formula becomes

$$f(x) = \frac{1}{2\pi} \int_{i\,\mathrm{Im}u-\infty}^{i\,\mathrm{Im}u+\infty} e^{-iux} \mathcal{F}_f(u)\,du.$$

Suppose the stochastic process X_t has the density function p, then the Fourier transform of p

$$\mathcal{F}_p(u) = \int_{-\infty}^{\infty} e^{iux} p(x)\,dx = E\left[e^{iuX}\right] \tag{21.3}$$

is called the characteristic function of X_t.

The following mathematical properties of $\mathcal{F}_f$ are useful in our later discussion.

1. *Differentiation*

$$\mathcal{F}_{f'}(u) = -iu\mathcal{F}_f(u).$$

2. *Modulation*

$$\mathcal{F}_{e^{\lambda x} f}(u) = \mathcal{F}_f(u - i\lambda), \qquad \lambda \text{ is real}.$$

3. *Convolution*

   Define the convolution between two integrable functions $f(x)$ and $g(x)$ by

$$h(x) = f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy,$$

   then

$$\mathcal{F}_h = \mathcal{F}_f \mathcal{F}_g.$$

4. *Parseval relation*

   Define the inner product of two complex-valued square-integrable functions f and g by

$$< f, g > = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx,$$

   then

$$< f, g > = \frac{1}{2\pi} < \mathcal{F}_f(u), \mathcal{F}_g(u) > .$$

We would like to illustrate an application of the Parseval relation in option pricing. Following the usual discounted expectation approach, we formally write the option price V with terminal payoff $V_T(x)$ and risk neutral density function $p(x)$ as

$$V = e^{-rT} \int_{-\infty}^{\infty} V_T(x)p(x)\,dx = e^{-rT} < V_T(x), p(x) > .$$

By the Parseval relation, we obtain

$$V = \frac{e^{-rT}}{2\pi} < \mathcal{F}_p(u), \mathcal{F}_{V_T}(u) > . \tag{21.4}$$

The option price can be expressed in terms of the inner product of the characteristic function of the underlying process $\mathcal{F}_p(u)$ and the Fourier transform of the terminal payoff $\mathcal{F}_{V_T}(u)$. More applications of the Parseval relation in deriving the Fourier inversion formulas in option pricing and insurance can be found in Dufresne et al. (2009).

21.2.2 Discrete Fourier Transform

Given a sequence $\{x_k\}$, $k = 0, 1, \cdots, N-1$, the discrete Fourier transform of $\{x_k\}$ is another sequence $\{y_j\}$, $j = 0, 1, \cdots, N-1$, as defined by

$$y_j = \sum_{k=0}^{N-1} e^{\frac{2\pi ijk}{N}} x_k, \qquad j = 0, 1, \cdots, N-1. \tag{21.5}$$

If we write the N-dimensional vectors

$$\mathbf{x} = (x_0\ x_1\ \cdots\ x_{N-1})^{\mathbf{T}} \quad \text{and} \quad \mathbf{y} = (y_0\ y_1\ \cdots\ y_{N-1})^{\mathbf{T}},$$

and define a $N \times N$ matrix F^N whose (j,k)th entry is

$$F^N_{j,k} = e^{\frac{2\pi ijk}{N}}, \qquad 1 \le j, k \le N,$$

then $\mathbf{x}$ and $\mathbf{y}$ are related by

$$\mathbf{y} = F^N \mathbf{x}. \tag{21.6}$$

The computation to find $\mathbf{y}$ requires N^2 steps.

However, if N is chosen to be some power of 2, say, $N = 2^L$, the computation using the FFT techniques would require only $\frac{1}{2}NL = \frac{N}{2}\log_2 N$ steps. The idea behind the FFT algorithm is to take advantage of the periodicity property of the Nth root of unity. Let $M = \frac{N}{2}$, and we split vector $\mathbf{x}$ into two half-sized vectors as defined by

$$\mathbf{x}' = (x_0\ x_2\ \cdots\ x_{N-2})^{\mathbf{T}} \quad \text{and} \quad \mathbf{x}'' = (x_1\ x_3\ \cdots\ x_{N-1})^{\mathbf{T}}.$$

We form the M-dimensional vectors

$$\mathbf{y}' = F^M \mathbf{x}' \quad \text{and} \quad \mathbf{y}'' = F^M \mathbf{x}'',$$

where the (j,k)th entry in the $M \times M$ matrix F^M is

$$F^M_{j,k} = e^{\frac{2\pi ijk}{M}}, \qquad 1 \leq j,k \leq M.$$

It can be shown that the first M and the last M components of $\mathbf{y}$ are given by

$$\begin{aligned} y_j &= y'_j + e^{\frac{2\pi ij}{N}} y''_j, \qquad j = 0, 1, \cdots, M-1, \\ y_{j+M} &= y'_j - e^{\frac{2\pi ij}{N}} y''_j, \qquad j = 0, 1, \cdots, M-1. \end{aligned} \tag{21.7}$$

Instead of performing the matrix-vector multiplication $F^N\mathbf{x}$, we now reduce the number of operations by two matrix-vector multiplications $F^M\mathbf{x}'$ and $F^M\mathbf{x}''$. The number of operations is reduced from N^2 to $2\left(\frac{N}{2}\right)^2 = \frac{N^2}{2}$. The same procedure of reducing the length of the sequence by half can be applied repeatedly. Using this FFT algorithm, the total number of operations is reduced from $O(N^2)$ to $O(N \log_2 N)$.

21.2.3 Lévy Processes

An adapted real-valued stochastic process X_t, with $X_0 = 0$, is called a Lévy process if it observes the following properties:

1. *Independent increments*
   For every increasing sequence of times $t_0, t_1, \cdots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.
2. *Time-homogeneous*
   The distribution of $\{X_{t+s} - X_s; t \geq 0\}$ does not depend on s.
3. *Stochastically continuous*
   For any $\epsilon > 0$, $P\left[|X_{t+h} - X_t| \geq \epsilon\right] \to 0$ as $h \to 0$.
4. *Cadlag process*
   It is right continuous with left limits as a function of t.

Lévy processes are a combination of a linear drift, a Brownian process, and a jump process. When the Lévy process X_t jumps, its jump magnitude is non-zero. The Lévy measure w of X_t defined on $\mathbb{R} \setminus \{0\}$ dictates how the jump occurs. In the finite-activity models, we have $\int_{\mathbb{R}} w(dx) < \infty$. In the infinite-activity models, we observe $\int_{\mathbb{R}} w(dx) = \infty$ and the Poisson intensity cannot be defined. Loosely speaking, the Lévy measure $w(dx)$ gives the arrival rate of jumps of size $(x, x + dx)$. The characteristic function of a Lévy process can be described by the Lévy-Khinchine representation

Table 21.1 Characteristic functions of some parametric Levy processes

Lévy process X_t	Characteristic function $\phi_X(u)$
Finite-activity models	
Geometric Brownian motion	$\exp\left(iu\mu t - \frac{1}{2}\sigma^2 t u^2\right)$
Lognormal jump diffusion	$\exp\left(iu\mu t - \frac{1}{2}\sigma^2 t u^2 + \lambda t (e^{iu\mu_J - \frac{1}{2}\sigma_J^2 u^2} - 1)\right)$
Double exponential jump diffusion	$\exp\left(iu\mu t - \frac{1}{2}\sigma^2 t u^2 + \lambda t \left(\frac{1-\eta^2}{1+u^2\eta^2} e^{iu\kappa} - 1\right)\right)$
Infinite-activity models	
Variance gamma	$\exp(iu\mu t)(1 - iu\nu\theta + \frac{1}{2}\sigma^2 \nu u^2)^{\frac{t}{\nu}}$
Normal inverse Gaussian	$\exp\left(iu\mu t + \delta t \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right)$
Generalized hyperbolic	$\exp(iu\mu t)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\frac{\lambda t}{2}}\left(\frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)^t$, where $K_\lambda(z) = \frac{\pi}{2}\frac{I_\nu(z) - I_{-\nu}(z)}{\sin(\nu\pi)}$, $I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\Gamma(\nu+k+1)}$
Finite-moment stable	$\exp\left(iu\mu t - t(iu\sigma)^\alpha \sec\frac{\pi\alpha}{2}\right)$
CGMY	$\exp(C\Gamma(-Y))[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]$, where $C, G, M > 0$ and $Y > 2$

$$
\begin{aligned}
\phi_X(u) &= E[e^{iuX_t}] \\
&= \exp\left(aitu - \frac{\sigma^2}{2}tu^2 + t\int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1}\right) w(dx)\right) \\
&= \exp(t\psi_X(u)),
\end{aligned}
\tag{21.8}
$$

where $\int_{\mathbb{R}} \min(1, x^2)\, w(dx) < \infty$, $a \in \mathbb{R}$, $\sigma^2 \geq 0$. We identify a as the drift rate and σ as the volatility of the diffusion process. Here, $\psi_X(u)$ is called the characteristic exponent of X_t. Actually, $X_t \stackrel{d}{=} tX_1$. All moments of X_t can be derived from the characteristic function since it generalizes the moment-generating function to the complex domain. Indeed, a Lévy process X_t is fully specified by its characteristic function ϕ_X. In Table 21.1, we present a list of Lévy processes commonly used in finance applications together with their characteristic functions.

21.3 FFT Algorithms for Pricing European Vanilla Options

The renowned discounted expectation approach of evaluating a European option requires the knowledge of the density function of the asset returns under the risk neutral measure. Since the analytic representation of the characteristic function rather than the density function is more readily available for Lévy processes, we prefer to express the expectation integrals in terms of the characteristic function. First, we derive the formal analytic representation of a European option price as cumulative distribution functions, like the Black-Scholes type price formula. We then examine the inherent difficulties in the direct numerical evaluation of the Fourier integrals in the price formula.

Under the risk neutral measure Q, suppose the underlying asset price process assumes the form

$$S_t = S_0 \exp(-rt + X_t), \quad t > 0,$$

where X_t is a Lévy process and r is the riskless interest rate. We write $Y = \log S_0 + rT$ and let $\mathcal{F}_{V_T}$ denote the Fourier transform of the terminal payoff function $V_T(x)$, where $x = \log S_T$. By applying the discounted expectation valuation formula and the Fourier inversion formula (21.2), the European option value can be expressed as (see Lewis 2001)

$$\begin{aligned} V(S_t, t) &= e^{-r(T-t)} E_Q[V_T(x)] \\ &= \frac{e^{-r(T-t)}}{2\pi} E_Q \left[\int_{i\mu-\infty}^{i\mu+\infty} e^{-izx} \mathcal{F}_{V_T}(z)\, dz \right] \\ &= \frac{e^{-r(T-t)}}{2\pi} \int_{i\mu-\infty}^{i\mu+\infty} e^{-izx} \phi_{X_T}(-z) \mathcal{F}_{V_T}(z)\, dz, \end{aligned}$$

where $\mu = \operatorname{Im} z$ and $\Phi_{X_T}(z)$ is the characteristic function of X_T. The above formula agrees with (21.4) derived using the Parseval relation.

In our subsequent discussion, we set the current time to be zero and write the current stock price as S. For the T-maturity European call option with terminal payoff $(S_T - K)^+$, its value is given by (see Lewis 2001)

$$\begin{aligned} C(S,T;K) &= \frac{-Ke^{-rT}}{2\pi} \int_{i\mu-\infty}^{i\mu+\infty} \frac{e^{-iz\kappa}\phi_{X_T}(-z)}{z^2 - iz}\, dz \\ &= \frac{-Ke^{-rT}}{2\pi} \left[\int_{i\mu-\infty}^{i\mu+\infty} e^{-iz\kappa} \phi_{X_T}(-z) \frac{i}{z}\, dz \right. \\ &\qquad \left. - \int_{i\mu-\infty}^{i\mu+\infty} e^{-iz\kappa} \phi_{X_T}(-z) \frac{i}{z-i}\, dz \right] \\ &= S \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left(\frac{e^{iu\log\kappa}\phi_{X_T}(u-i)}{iu\phi_{X_T}(-i)} \right) du \right] \\ &\qquad - Ke^{-rT} \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left(\frac{e^{iu\log\kappa}\phi_{X_T}(u)}{iu} \right) du \right], \end{aligned} \tag{21.9}$$

where $\kappa = \log \frac{S}{K} + rT$. This representation of the call price resembles the Black-Scholes type price formula. However, due to the presence of the singularity at $u = 0$ in the integrand function, we cannot apply the FFT to evaluate the integrals. If we expand the integrals as Taylor series in u, the leading term in the expansion for both integral is $O\left(\frac{1}{u}\right)$. This is the source of the divergence, which arises from the discontinuity of the payoff function at $S_T = K$. As a consequence, the Fourier transform of the payoff function has large high frequency terms. Carr and Madan (1999) propose to dampen the high frequency terms by multiplying the payoff by an exponential decay function.

21.3.1 Carr–Madan Formulation

As an alternative formulation of European option pricing that takes advantage of the analytic expression of the characteristic function of the underlying asset price process, Carr and Madan (1999) consider the Fourier transform of the European call price (considered as a function of log strike) and compute the corresponding Fourier inversion to recover the call price using the FFT. Let $k = \log K$, the Fourier transform of the call price $C(k)$ does not exist since $C(k)$ is not square integrable. This is because $C(k)$ tends to S as k tends to $-\infty$.

21.3.1.1 Modified Call Price Method

To obtain a square-integrable function, Carr and Madan (1999) propose to consider the Fourier transform of the damped call price $c(k)$, where

$$c(k) = e^{\alpha k} C(k),$$

for $\alpha > 0$. Positive values of α are seen to improve the integrability of the modified call value over the negative k-axis. Carr and Madan (1999) show that a sufficient condition for square-integrability of $c(k)$ is given by

$$E_Q\left[S_T^{\alpha+1}\right] < \infty.$$

We write $\psi_T(u)$ as the Fourier transform of $c(k)$, $p_T(s)$ as the density function of the underlying asset price process, where $s = \log S_T$, and $\phi_T(u)$ as the characteristic function (Fourier transform) of $p_T(s)$. We obtain

$$\begin{aligned} \psi_T(u) &= \int_{-\infty}^{\infty} e^{iuk} c(k)\, dk \\ &= \int_{-\infty}^{\infty} e^{-rT} p_T(s) \int_{-\infty}^{s} \left[e^{s+\alpha k} - e^{(1+\alpha)k}\right] e^{iuk}\, dkds \\ &= \frac{e^{-rT}\phi_T(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}. \end{aligned} \tag{21.10}$$

The call price $C(k)$ can be recovered by taking the Fourier inversion transform, where

$$
\begin{aligned}
C(k) &= \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \psi_T(u)\, du \\
&= \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-iuk} \psi_T(u)\, du, \qquad (21.11)
\end{aligned}
$$

by virtue of the properties that $\psi_T(u)$ is odd in its imaginary part and even in its real part [since $C(k)$ is real]. The above integral can be computed using FFT, the details of which will be discussed next. From previous numerical experience, usually $\alpha = 3$ works well for most models of asset price dynamics. It is important to observe that α has to be chosen such that the denominator has only imaginary roots in u since integration is performed along real value of u.

21.3.1.2 FFT Implementation

The integral in (21.11) with a semi-infinite integration interval is evaluated by numerical approximation using the trapezoidal rule and FFT. We start with the choice on the number of intervals N and the stepwidth Δu. A numerical approximation for $C(k)$ is given by

$$
C(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iu_j k} \psi_T(u_j) \Delta u, \qquad (21.12)
$$

where $u_j = (j-1)\Delta u$, $j = 1, \cdots, N$. The semi-infinite integration domain $[0, \infty)$ in the integral in (21.11) is approximated by a finite integration domain, where the upper limit for u in the numerical integration is $N\Delta u$. The error introduced is called the *truncation error*. Also, the Fourier variable u is now sampled at discrete points instead of continuous sampling. The associated error is called the *sampling error*. Discussion on the controls on various forms of errors in the numerical approximation procedures can be found in Lee (2004).

Recall that the FFT is an efficient numerical algorithm that computes the sum

$$
y(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j), \qquad k = 1, 2, \cdots, N. \qquad (21.13)
$$

In the current context, we would like to compute around-the-money call option prices with k taking discrete values: $k_m = -b + (m-1)\Delta k$, $m = 1, 2, \cdots, N$. From one set of the FFT calculations, we are able to obtain call option prices for a range of strike prices. This facilitates the market practitioners to capture the price sensitivity of a European call with varying values of strike prices. To effect the FFT calculations, we note from (21.13) that it is necessary to choose Δu and Δk such that

$$\Delta u \Delta k = \frac{2\pi}{N}. \tag{21.14}$$

A compromise between the choices of Δu and Δk in the FFT calculations is called for here. For fixed N, the choice of a finer grid Δu in numerical integration leads to a larger spacing Δk on the log strike.

The call price multiplied by an appropriate damping exponential factor becomes a square-integrable function and the Fourier transform of the modified call price becomes an analytic function of the characteristic function of the log price. However, at short maturities, the call value tends to the non-differentiable terminal call option payoff causing the integrand in the Fourier inversion to become highly oscillatory. As shown in the numerical experiments performed by Carr and Madan (1999), this causes significant numerical errors. To circumvent the potential numerical pricing difficulties when dealing with short-maturity options, an alternative approach that considers the time value of a European option is shown to exhibit smaller pricing errors for all range of strike prices and maturities.

21.3.1.3 Modified Time Value Method

For notational convenience, we set the current stock price S to be unity and define

$$z_T(k) = e^{-rT} \int_{-\infty}^{\infty} \left[(e^k - e^s)\mathbf{1}_{\{s<k,k<0\}} + (e^s - e^k)\mathbf{1}_{\{s>k,k<0\}} \right] p_T(s)\, ds, \tag{21.15}$$

which is seen to be equal to the T-maturity call price when $K > S$ and the T-maturity put price when $K < S$. Therefore, once $z_T(k)$ is known, we can obtain the price of the call or put that is currently out-of-money while the call-put parity relation can be used to obtain the price of the other option that is in-the-money.

The Fourier transform $\zeta_T(u)$ of $z_T(k)$ is found to be

$$\begin{aligned} \zeta_T(u) &= \int_{-\infty}^{\infty} e^{iuk} z_T(k)\, dk \\ &= e^{-rT} \left[\frac{1}{1+iu} - \frac{e^{rT}}{iu} - \frac{\phi_T(u-i)}{u^2 - iu} \right]. \end{aligned} \tag{21.16}$$

The time value function $z_T(k)$ tends to a Dirac function at small maturity and around-the-money, so the Fourier transform $\zeta_T(u)$ may become highly oscillatory. Here, a similar damping technique is employed by considering the Fourier transform of $\sinh(\alpha k) z_T(k)$ (note that $\sinh \alpha k$ vanishes at $k = 0$). Now, we consider

$$\begin{aligned} \gamma_T(u) &= \int_{-\infty}^{\infty} e^{iuk} \sinh(\alpha k) z_T(k)\, dk \\ &= \frac{\zeta_T(u - i\alpha) - \zeta_T(u + i\alpha)}{2}, \end{aligned}$$

and the time value can be recovered by applying the Fourier inversion transform:

$$z_T(k) = \frac{1}{\sinh(\alpha k)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \gamma_T(u)\, du. \tag{21.17}$$

Analogous FFT calculations can be performed to compute the numerical approximation for $z_T(k_m)$, where

$$z_T(k_m) \approx \tfrac{1}{\pi \sinh(\alpha k_m)} \textstyle\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(m-1)} e^{ibu_j} \gamma_T(u_j) \Delta u, \tag{21.18}$$
$$m = 1, 2, \cdots, N, \quad \text{and} \quad k_m = -b + (m-1)\Delta k.$$

21.4 Pricing of European Multi-Asset Options

Apparently, the extension of the Carr–Madan formulation to pricing European multi-asset options would be quite straightforward. However, depending on the nature of the terminal payoff function of the multi-asset option, the implementation of the FFT algorithm may require some special considerations.

The most direct extension of the Carr–Madan formulation to the multi-asset models can be exemplified through pricing of the correlation option, the terminal payoff of which is defined by

$$V(S_1, S_2, T) = (S_1(T) - K_1)^+ (S_2(T) - K_2)^+. \tag{21.19}$$

We define $s_i = \log S_i$, $k_i = \log K_i$, $i = 1, 2$, and write $p_T(s_1, s_2)$ as the joint density of $s_1(T)$ and $s_2(T)$ under the risk neutral measure Q. The characteristic function of this joint density is defined by the following two-dimensional Fourier transform:

$$\phi(u_1, u_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(u_1 s_1 + u_2 s_2)} p_T(s_1, s_2)\, ds_1 ds_2. \tag{21.20}$$

Following the Carr–Madan formulation, we consider the Fourier transform $\psi_T(u_1, u_2)$ of the damped option price $e^{\alpha_1 k_1 + \alpha_2 k_2} V_T(k_1, k_2)$ with respect to the log strike prices k_1, k_2, where $\alpha_1 > 0$ and $\alpha_2 > 0$ are chosen such that the damped option price is square-integrable for negative values of k_1 and k_2. The Fourier transform $\psi_T(u_1, u_2)$ is related to $\phi(u_1, u_2)$ as follows:

$$\psi_T(u_1, u_2) = \frac{e^{-rT} \phi(u_1 - (\alpha_1 + 1)i, u_2 - (\alpha_2 + 1)i)}{(\alpha_1 + iu_1)(\alpha_1 + 1 + iu_1)(\alpha_2 + iu_2)(\alpha_2 + 1 + iu_2)}. \tag{21.21}$$

To recover $C_T(k_1, k_2)$, we apply the Fourier inversion on $\psi_T(u_1, u_2)$. Following analogous procedures as in the single-asset European option, we approximate the two-dimensional Fourier inversion integral by

$$C_T(k_1, k_2) \approx \frac{e^{-\alpha_1 k_1 - \alpha_2 k_2}}{(2\pi)^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{-i(u_m^1 k_1 + u_n^2 k_2)} \psi_T(u_m^1, u_n^2) \Delta_1 \Delta_2, \tag{21.22}$$

where $u_m^1 = \left(m - \frac{N}{2}\right) \Delta_1$ and $u_n^2 = \left(n - \frac{N}{2}\right) \Delta_2$. Here, Δ_1 and Δ_2 are the stepwidths, and N is the number of intervals. In the two-dimensional form of the FFT algorithm, we define

$$k_p^1 = \left(p - \frac{N}{2}\right) \Delta_1 \qquad \text{and} \qquad k_q^1 = \left(q - \frac{N}{2}\right) \Delta_2,$$

where λ_1 and λ_2 observe

$$\lambda_1 \Delta_1 = \lambda_2 \Delta_2 = \frac{2\pi}{N}.$$

Dempster and Hong (2000) show that the numerical approximation to the option price at different log strike values is given by

$$C_T(k_p^1, k_q^2) \approx \frac{e^{-\alpha_1 k_p^1 - \alpha_2 k_q^2}}{(2\pi)^2} \Gamma(k_p^1, k_q^2) \Delta_1 \Delta_2, \qquad 0 \le p, q \le N, \tag{21.23}$$

where

$$\Gamma(k_p^1, k_q^2) = (-1)^{p+q} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}(mp+nq)} \left[(-1)^{m+n} \psi_T(u_m^1, u_n^2)\right].$$

The nice tractability in deriving the FFT pricing algorithm for the correlation option stems from the rectangular shape of the exercise region Ω of the option. Provided that the boundaries of Ω are made up of straight edges, the above procedure of deriving the FFT pricing algorithm still works. This is because one can always take an affine change of variables in the Fourier integrals to effect the numerical evaluation. What would be the classes of option payoff functions that allow the application of the above approach? Lee (2004) lists four types of terminal payoff functions that admit analytic representation of the Fourier transform of the damped option price. Another class of multi-asset options that possess similar analytic tractability are options whose payoff depends on taking the maximum or minimum value among the terminal values of a basket of stocks (see Eberlein et al. 2009). However, the exercise region of the spread option with terminal payoff

$$V_T(S_1, S_2) = (S_1(T) - S_2(T) - K)^+ \tag{21.24}$$

is shown to consist of a non-linear edge. To derive the FFT algorithm of similar nature, it is necessary to approximate the exercise region by a combination of rectangular strips. The details of the derivation of the corresponding FFT pricing algorithm are presented by Dempster and Hong (2000).

Hurd and Zhou (2009) propose an alternative approach to pricing the European spread option under Lévy model. Their method relies on an elegant formula of the Fourier transform of the spread option payoff function. Let $P(s_1, s_2)$ denote the terminal spread option payoff with unit strike, where

$$P(s_1, s_2) = (e^{s_1} - e^{s_2} - 1)^+.$$

For any real numbers ϵ_1 and ϵ_2 with $\epsilon_2 > 0$ and $\epsilon_1 + \epsilon_2 < -1$, they establish the following Fourier representation of the terminal spread option payoff function:

$$P(s_1, s_2) = \frac{1}{(2\pi)^2} \int_{-\infty+i\epsilon_2}^{\infty+i\epsilon_2} \int_{-\infty+i\epsilon_1}^{\infty+i\epsilon_1} e^{i(u_1 s_1 + u_2 s_2)} \hat{P}(u_1, u_2)\, du_1 du_2, \quad (21.25)$$

where

$$\hat{P}(u_1, u_2) = \frac{\Gamma(i(u_1 + u_2) - 1)\Gamma(-iu_2)}{\Gamma(iu_1 + 1)}.$$

Here, $\Gamma(z)$ is the complex gamma function defined for $\mathrm{Re}(z) > 0$, where

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

To establish the Fourier representation in (21.25), we consider

$$\hat{P}(u_1, u_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(u_1 s_1 + u_2 s_2)} P(s_1, s_2)\, ds_2 ds_1.$$

By restricting to $s_1 > 0$ and $e^{s_2} < e^{s_1} - 1$, we have

$$\begin{aligned}
\hat{P}(u_1, u_2) &= \int_0^\infty e^{-iu_1 s_1} \int_{-\infty}^{\log(e^{s_1}-1)} e^{-iu_2 s_2}(e^{s_1} - e^{s_2} - 1)\, ds_2 ds_1 \\
&= \int_0^\infty e^{-iu_1 s_1}(e^{s_1} - 1)^{1-iu_2} \left(\frac{1}{-iu_2} - \frac{1}{1 - iu_2} \right) ds_1 \\
&= \frac{1}{(1 - iu_2)(-iu_2)} \int_0^1 z^{iu_1} \left(\frac{1-z}{z} \right)^{1-iu_2} \frac{dz}{z},
\end{aligned}$$

where $z = e^{-s_1}$. The last integral can be identified with the beta function:

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 z^{a-1}(1-z)^{b-1}\,dz,$$

so we obtain the result in (21.25). Once the Fourier representation of the terminal payoff is known, by virtue of the Parseval relation, the option price can be expressed as a two-dimensional Fourier inversion integral with integrand that involves the product of $\hat{P}(u_1, u_2)$ and the characteristic function of the joint process of s_1 and s_2. The evaluation of the Fourier inversion integral can be affected by the usual FFT calculations (see Hurd and Zhou 2009). This approach does not require the analytic approximation of the two-dimensional exercise region of the spread option with a non-linear edge, so it is considered to be more computationally efficient.

The pricing of European multi-asset options using the FFT approach requires availability of the analytic representation of the characteristic function of the joint price process of the basket of underlying assets. One may incorporate a wide range of stochastic structures in the volatility and correlation. Once the analytic forms in the integrand of the multi-dimensional Fourier inversion integral are known, the numerical evaluation involves nested summations in the FFT calculations whose dimension is the same as the number of underlying assets in the multi-asset option. This contrasts with the usual finite difference/lattice tree methods where the dimension of the scheme increases with the number of risk factors in the prescription of the joint process of the underlying assets. This is a desirable property over other numerical methods since the FFT pricing of the multi-asset options is not subject to this curse of dimensionality with regard to the number of risk factors in the dynamics of the asset returns.

21.5 Convolution Approach and Pricing of Bermudan Style Options

We consider the extension of the FFT technique to pricing of options that allow early exercise prior to the maturity date T. Recall that a Bermudan option can only be exercised at a pre-specified set of time points, say $\mathcal{T} = \{t_1, t_2, \cdots, t_M\}$, where $t_M = T$. On the other hand, an American option can be exercised at any time prior to T. By taking the number of time points of early exercise to be infinite, we can extrapolate a Bermudan option to become an American option. In this section, we would like to illustrate how the convolution property of Fourier transform can be used to price a Bermudan option effectively (see Lord et al. 2008).

Let $F(S(t_m), t_m)$ denote the exercise payoff of a Bermudan option at time t_m, $m = 1, 2, \cdots, M$. Let $V(S(t_m), t_m)$ denote the time-t_m value of the Bermudan option with exercise point set $\mathcal{T}$; and we write $\Delta t_m = t_{m+1} - t_m$, $m = 1, 2, \cdots, M-1$. The Bermudan option can be evaluated via the following backward induction procedure:

terminal payoff: $V(S(t_M), t_M) = F(S(t_M), t_M)$
For $m = M-1, M-2, \cdots, 1$, compute

$$C(S(t_m), t_m) = e^{-r\Delta t_m} \int_{-\infty}^{\infty} V(y, t_{m+1}) p(y|S(t_m))\, dy$$
$$V(S(t_m), t_m) = \max\{C(S(t_m), t_m), F(S(t_m), t_m)\}.$$

Here, $p(y|S(t_m))$ represents the probability density that relates the transition from the price level $S(t_m)$ at t_m to the new price level y at t_{m+1}. By virtue of the early exercise right, the Bermudan option value at t_m is obtained by taking the maximum value between the time-t_m continuation value $C(S(t_m), t_m)$ and the time-t_m exercise payoff $F(S(t_m), t_m)$.

The evaluation of $C(S(t_m), t_m)$ is equivalent to the computation of the time-t_m value of a t_{m+1}-maturity European option. Suppose the asset price process is a monotone function of a Lévy process (which observes the independent increments property), then the transition density $p(y|x)$ has the following property:

$$p(y|x) = p(y-x). \tag{21.26}$$

If we write $z = y - x$, then the continuation value can be expressed as a convolution integral as follows:

$$C(x, t_m) = e^{-r\Delta t_m} \int_{-\infty}^{\infty} V(x+z, t_{m+1}) p(z)\, dz. \tag{21.27}$$

Following a similar damping procedure as proposed by Carr and Madan (1999), we define

$$c(x, t_m) = e^{\alpha x + r\Delta t_m} C(x, t_m)$$

to be the damped continuation value with the damping factor $\alpha > 0$. Applying the property of the Fourier transform of a convolution integral, we obtain

$$\mathcal{F}_x\{c(x, t_m)\}(u) = \mathcal{F}_y\{v(y, t_{m+1})\}(u)\phi(-(u - i\alpha)), \tag{21.28}$$

and $\phi(u)$ is the characteristic function of the random variable z.

Lord et al. (2008) propose an effective FFT algorithm to calculate the following convolution:

$$c(x, t_m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{v}(u)\phi(-(u - i\alpha))\, du, \tag{21.29}$$

where $\hat{v}(u) = \mathcal{F}\{v(y, t_m)\}$. The FFT calculations start with the prescription of uniform grids for u, x and y:

$$u_j = u_0 + j\Delta u, \quad x_j = x_0 + j\Delta x, \quad y_j = y_0 + j\Delta y, \qquad j = 0, 1, \cdots, N-1.$$

The mesh sizes Δx and Δy are taken to be equal, and Δu and Δy are chosen to satisfy the Nyquist condition:

$$\Delta u \Delta y = \frac{2\pi}{N}.$$

The convolution integral is discretized as follows:

$$c(x_p) \approx \frac{e^{-iu_0(x_0+p\Delta y)}}{2\pi} \Delta u \sum_{j=0}^{N-1} e^{-ijp\frac{2\pi}{N}} e^{ij(y_0-x_0)\Delta u} \phi(-(u_j - i\alpha)) \hat{v}(u_j), \quad (21.30)$$

where

$$\hat{v}(u_j) \approx e^{iu_0 y_0} \Delta y \sum_{n=0}^{N-1} e^{ijn2\pi/N} e^{inu_0 \Delta y} w_n v(y_n),$$

$$w_0 = w_{N-1} = \frac{1}{2}, \quad w_n = 1 \qquad \text{for } n = 1, 2, \cdots, N-2.$$

For a sequence x_p, $p = 0, 1, \cdots, N-1$, its discrete Fourier transform and the corresponding inverse are given by

$$\mathcal{D}_j\{x_n\} = \sum_{n=0}^{N-1} e^{ijn2\pi/N} x_n, \quad \mathcal{D}_n^{-1}\{x_j\} = \frac{1}{N} \sum_{j=0}^{N-1} e^{-ijn2\pi/N} x_j.$$

By setting $u_0 = -\frac{N}{2}\Delta u$ so that $e^{inu_0\Delta y} = (-1)^n$, we obtain

$$c(x_p) \approx e^{iu_0(y_0-x_0)} (-1)^p \mathcal{D}_p^{-1}\{e^{ij(y_0-x_0)\Delta u} \phi(-(u_j - i\alpha)) \mathcal{D}_j\{(-1)^n w_n v(y_n)\}\}. \quad (21.31)$$

In summary, by virtue of the convolution property of Fourier transform, we compute the discrete Fourier inversion of the product of the discrete characteristic function of the asset returns $\phi(-(u_j - i\alpha))$ and the discrete Fourier transform of option prices $\mathcal{D}_j\{(-1)^n w_n v(y_n)\}$. It is seen to be more efficient when compared to the direct approach of recovering the density function by taking the Fourier inversion of the characteristic function and finding the option prices by discounted expectation calculations (see Zhylyevsky 2010).

21.6 FFT-Based Network Method

As an extension to the usual lattice tree method, an FFT-based network approach to option pricing under Lévy models has been proposed by Wong and Guan (2009). The network method somewhat resembles Duan-Simonato's Markov chain approximation method (Duan and Simonato 2001). This new approach is developed for option pricing for which the characteristic function of the log-asset value is known. Like the lattice tree method, the network method can be generalized to valuation of path dependent options by adopting the forward shooting grid technique (see Kwok 2010).

First, we start with the construction of the network. We perform the space-time discretization by constructing a pre-specified system of grids of time and state: $t_0 < t_1 < \cdots < t_M$, where t_M is the maturity date of the option, and $x_0 < x_1 < \cdots < x_N$, where $\mathcal{X} = \{x_j | j = 0, 1, \cdots, N\}$ represents the set of all possible values of log-asset prices. For simplicity, we assume uniform grid sizes, where $\Delta x = x_{j+1} - x_j$ for all j and $\Delta t = t_{i+1} - t_i$ for all i. Unlike the binomial tree where the number of states increases with the number of time steps, the number of states is fixed in advance and remains unchanged at all time points. In this sense, the network resembles the finite difference grid layout. The network approach approximates the Lévy process by a finite state Markov chain, like that proposed by Duan and Simonato (2001). We allow for a finite probability that the log-asset value moves from one state to any possible state in the next time step. This contrasts with the usual finite difference schemes where the linkage of nodal points between successive time steps is limited to either one state up, one state down or remains at the same state. The Markov chain model allows greater flexibility to approximate the asset price dynamics that exhibits finite jumps under Lévy model with enhanced accuracy. A schematic diagram of a network with seven states and three time steps is illustrated in Fig. 21.1.

After the construction of the network, the next step is to compute the transition probabilities that the asset price goes from one state x_i to another state x_j under the Markov chain model, $0 \le i, j \le N$. The corresponding transition probability is defined as follows:

$$p_{ij} = P[X_{t+\Delta t} = x_j | X_t = x_i], \tag{21.32}$$

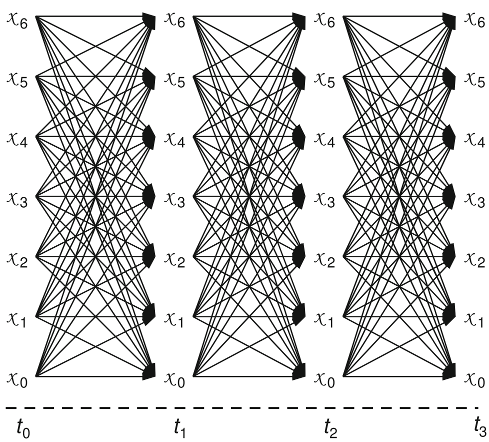

Fig. 21.1 A network model with three time steps and seven states

which is independent of t due to the time homogeneity of the underlying Lévy process. We define the corresponding characteristic function by

$$\phi_i(u) = \int_{-\infty}^{\infty} e^{iuz} f_i(z|x_i)\, dz,$$

where $f_i(z|x_i)$ is the probability density function of the increment $X_{t+\Delta t} - X_t$ conditional on $X_t = x_i$. The conditional probability density function can be recovered by Fourier inversion:

$$f_i(x_j|x_i) = \mathcal{F}_u^{-1}\{\phi_i(u)\}(x_j). \tag{21.33}$$

If we take the number of Markov chain states to be $N+1 = 2^L$ for some integer L, then the above Fourier inversion can be carried out using the FFT techniques. The FFT calculations produce approximate values for $f_i(x_j|x_i)$ for all i and j. We write these approximate conditional probability values obtained from the FFT calculations as $\tilde{f}_i(x_j|x_i)$. The transition probabilities among the Markov chain states are then approximated by

$$\tilde{p}_{ij} \approx \frac{\tilde{f}_i(x_j|x_i)}{\sum_{i=0}^{N} \tilde{f}_i(x_j|x_i)}, \qquad 0 \le i, j \le N. \tag{21.34}$$

Once the transition probabilities are known, we can perform option valuation using the usual discounted expectation approach. The incorporation of various path dependent features can be performed as in usual lattice tree calculations. Wong and Guan (2009) illustrate how to compute the Asian and lookback option prices under Lévy models using the FFT-based network approach. Their numerical schemes are augmented with the forward shooting grid technique (see Kwok 2010) for capturing the asset price dependency associated with the Asian and lookback features.

21.7 Fourier Space Time Stepping Method

When we consider option pricing under Lévy models, the option price function is governed by a partial integral-differential equation (PIDE) where the integral terms in the equation arise from the jump components in the underlying Lévy process. In this section, we present the Fourier space time stepping (FST) method that is based on the solution in the Fourier domain of the governing PIDE (see Jackson et al. 2008). This is in contrast with the usual finite difference schemes which solve the PIDE in the real domain. We discuss the robustness of the FST method with regard to its symmetric treatment of the jump terms and diffusion terms in the PIDE and the ease of incorporation of various forms of path dependence in the option models. Unlike the usual finite difference schemes, the FST method does not require time stepping calculations between successive monitoring dates in pricing Bermudan options and discretely monitored barrier options. In the numerical implementation procedures, the FST method does not require the analytic expression for the Fourier transform of the terminal payoff of the option so it can deal easier with more exotic forms of the payoff functions. The FST method can be easily extended to multi-asset option models with exotic payoff structures and pricing models that allow regime switching in the underlying asset returns.

First, we follow the approach by Jackson et al. (2008) to derive the governing PIDE of option pricing under Lévy models and consider the Fourier transform of the PIDE. We consider the model formulation under the general multi-asset setting. Let $\mathbf{S}(t)$ denote a d-dimensional price index vector of the underlying assets in a multi-asset option model whose T-maturity payoff is denoted by $V_T(\mathbf{S}(T))$. Suppose the underlying price index follows an exponential Lévy process, where

$$\mathbf{S}(t) = \mathbf{S}(0)e^{\mathbf{X}(t)},$$

and $\mathbf{X}(t)$ is a Lévy process. Let the characteristic component of $\mathbf{X}(t)$ be the triplet $(\boldsymbol{\mu}, M, \boldsymbol{\nu})$, where $\boldsymbol{\mu}$ is the non-adjusted drift vector, M is the covariance matrix of the diffusion components, and $\boldsymbol{\nu}$ is the d-dimensional Lévy density. The Lévy process $\mathbf{X}(t)$ can be decomposed into its diffusion and jump components as follows:

$$\mathbf{X}(t) = \boldsymbol{\mu}(t) + M\mathbf{W}(t) + \mathbf{J}^l(t) + \lim_{\epsilon\to 0}\mathbf{J}^\epsilon(t), \tag{21.35}$$

where the large and small components are

$$\begin{aligned}
\mathbf{J}^l(t) &= \int_0^t \int_{|\mathbf{y}|\geq 1} \mathbf{y}\, m(d\mathbf{y}\times ds) \\
\mathbf{J}^\epsilon(t) &= \int_0^t \int_{\epsilon\leq|\mathbf{y}|<1} \mathbf{y}\,[m(d\mathbf{y}\times ds) - \nu(d\mathbf{y}\times ds)],
\end{aligned}$$

respectively. Here, $\mathbf{W}(t)$ is the vector of standard Brownian processes, $m(d\mathbf{y}\times ds)$ is a Poisson random measure counting the number of jumps of size $\mathbf{y}$ occurring at time s, and $\nu(d\mathbf{y}\times ds)$ is the corresponding compensator. Once the volatility and Lévy density are specified, the risk neutral drift can be determined by enforcing the risk neutral condition:

$$E_0[e^{\mathbf{X}(1)}] = e^r,$$

where r is the riskfree interest rate. The governing partial integral-differential equation (PIDE) of the option price function $V(\mathbf{X}(t), t)$ is given by

$$\frac{\partial V}{\partial t} + \mathcal{L}V = 0 \tag{21.36}$$

with terminal condition: $V(\mathbf{X}(T), T) = V_T(\mathbf{S}(0), e^{\mathbf{X}(T)})$, where $\mathcal{L}$ is the infinitesimal generator of the Lévy process operating on a twice differentiable function $f(\mathbf{x})$ as follows:

$$\mathcal{L}f(\mathbf{x}) = \left(\boldsymbol{\mu}^{\mathbf{T}}\frac{\partial}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{x}}^{\mathbf{T}} M \frac{\partial}{\partial \mathbf{x}}\right) f(\mathbf{x}) \\ + \int_{\mathbb{R}^n \setminus \{\mathbf{0}\}} \{[f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})] - \mathbf{y}^{\mathbf{T}} \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}) \mathbf{1}_{|\mathbf{y}|<1}\} \, \nu(d\mathbf{y}). \quad (21.37)$$

By the Lévy-Khintchine formula, the characteristic component of the Lévy process is given by

$$\psi_{\mathbf{X}}(\mathbf{u}) = i\boldsymbol{\mu}^{\mathbf{T}}\mathbf{u} - \frac{1}{2}\mathbf{u}^{\mathbf{T}} M \mathbf{u} + \int_{\mathbb{R}^n} \left(e^{i\mathbf{u}^{\mathbf{T}}\mathbf{y}} - 1 - i\mathbf{u}^{\mathbf{T}}\mathbf{y}\mathbf{1}_{|\mathbf{y}|<1}\right) \nu(d\mathbf{y}). \quad (21.38)$$

Several numerical schemes have been proposed in the literature that solve the PIDE (21.36) in the real domain. Jackson et al. (2008) propose to solve the PIDE directly in the Fourier domain so as to avoid the numerical difficulties in association with the valuation of the integral terms and diffusion terms. An account on the deficiencies in earlier numerical schemes in treating the discretization of the integral terms can be found in Jackson et al. (2008).

By taking the Fourier transform on both sides of the PIDE, the PIDE is reduced to a system of ordinary differential equations parametrized by the d-dimensional frequency vector $\mathbf{u}$. When we apply the Fourier transform to the infinitesimal generator $\mathcal{L}$ of the process $\mathbf{X}(t)$, the Fourier transform can be visualized as a linear operator that maps spatial differentiation into multiplication by the factor $i\mathbf{u}$. We define the multi-dimensional Fourier transform as follows (a slip of sign in the exponent of the Fourier kernel is adopted here for notational convenience):

$$\mathcal{F}[f](\mathbf{u}) = \int_{-\infty}^{\infty} f(\mathbf{x}) e^{-i\mathbf{u}^{\mathbf{T}}\mathbf{x}} \, d\mathbf{x}$$

so that

$$\mathcal{F}^{-1}[\mathcal{F}_f](\mathbf{u}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_f e^{i\mathbf{u}^{\mathbf{T}}\mathbf{x}} \, d\mathbf{u}.$$

We observe

$$\mathcal{F}\left[\frac{\partial}{\partial \mathbf{x}} f\right] = i\mathbf{u}\mathcal{F}[f] \qquad \text{and} \qquad \mathcal{F}\left[\frac{\partial^2}{\partial \mathbf{x}^2} f\right] = i\mathbf{u}\,\mathcal{F}[f]\, i\mathbf{u}^{\mathbf{T}}$$

so that

$$\mathcal{F}[\mathcal{L}V](\mathbf{u}, t) = \psi_{\mathbf{X}}(\mathbf{u})\mathcal{F}[V](\mathbf{u}, t). \quad (21.39)$$

The Fourier transform of $\mathcal{L}V$ is elegantly given by multiplying the Fourier transform of V by the characteristic component $\psi_{\mathbf{X}}(\mathbf{u})$ of the Lévy process $\mathbf{X}(t)$. In the Fourier domain, $\mathcal{F}[V]$ is governed by the following system of ordinary differential equations:

$$\frac{\partial}{\partial t}\mathcal{F}[V](\mathbf{u},t) + \psi_{\mathbf{X}}(\mathbf{u})\mathcal{F}[V](\mathbf{u},t) = 0 \tag{21.40}$$

with terminal condition: $\mathcal{F}[V](\mathbf{u},T) = \mathcal{F}_{V_T}(\mathbf{u},T)$.

If there is no embedded optionality feature like the knock-out feature or early exercise feature between t and T, then the above differential equation can be integrated in a single time step. By solving the PIDE in the Fourier domain and performing Fourier inversion afterwards, the price function of a European vanilla option with terminal payoff V_T can be formally represented by

$$V(\mathbf{x},t) = \mathcal{F}^{-1}\left\{\mathcal{F}[V_T](\mathbf{u},T)e^{\psi_{\mathbf{X}}(\mathbf{u})(T-t)}\right\}(\mathbf{x},t). \tag{21.41}$$

In the numerical implementation procedure, the continuous Fourier transform and inversion are approximated by some appropriate discrete Fourier transform and inversion, which are then effected by FFT calculations. Let $\mathbf{v}_T$ and $\mathbf{v}_t$ denote the d-dimensional vector of option values at maturity T and time t, respectively, that are sampled at discrete spatial points in the real domain. The numerical evaluation of $\mathbf{v}_t$ via the discrete Fourier transform and inversion can be formally represented by

$$\mathbf{v}_t = \mathcal{FFT}^{-1}[\mathcal{FFT}[\mathbf{v}_T]e^{\psi_{\mathbf{X}}(T-t)}], \tag{21.42}$$

where $\mathcal{FFT}$ denotes the multi-dimensional FFT transform. In this numerical FFT implementation of finding European option values, it is not necessary to know the analytic representation of the Fourier transform of the terminal payoff function. This new formulation provides a straightforward implementation of numerical pricing of European spread options without resort to elaborate design of FFT algorithms as proposed by Dempster and Hong (2000) and Hurd and Zhou (2009) (see Sect. 21.4).

Suppose we specify a set of preset discrete time points $\mathcal{X} = \{t_1, t_2, \cdots, t_N\}$, where the option may be knocked out (barrier feature) or early exercised (Bermudan feature) prior to maturity T (take $t_{N+1} = T$ for notational convenience). At these time points, we either impose constraints or perform optimization based on the contractual specification of the option. Consider the pricing of a discretely monitored barrier option where the knock-out feature is activated at the set of discrete time points $\mathcal{X}$. Between times t_n and t_{n+1}, $n = 1, 2, \cdots, N$, the barrier option behaves like a European vanilla option so that the single step integration can be performed from t_n to t_{n+1}. At time t_n, we impose the contractual specification of the knock-out feature. Say, the option is knocked out when S stays above the up-and-out barrier B. Let R denote the rebate paid upon the occurrence of knock-out, and $\mathbf{v}^n$ be the vector of option values at discrete spatial points. The time stepping algorithm can be succinctly represented by

$$\mathbf{v}^n = H_B(\mathcal{FFT}^{-1}[\mathcal{FFT}[\mathbf{v}^{n+1}]e^{\psi_{\mathbf{x}}(t_{n+1}-t_n)}]),$$

where the knock-out feature is imposed by defining H_B to be (see Jackson et al. 2008)

$$H_B(\mathbf{v}) = \mathbf{v}\mathbf{1}_{\left\{x<\log \frac{B}{S(0)}\right\}} + R\mathbf{1}_{\left\{x\geq\log \frac{B}{S(0)}\right\}}.$$

No time stepping is required between two successive monitoring dates.

21.8 Summary and Conclusions

The Fourier transform methods provide the valuable and indispensable tools for option pricing under Lévy processes since the analytic representation of the characteristic function of the underlying asset return is more readily available than that of the density function itself. When used together with the FFT algorithms, real time pricing of a wide range of option models under Lévy processes can be delivered using the Fourier transform approach with high accuracy, efficiency and reliability. In particular, option prices across the whole spectrum of strikes can be obtained in one set of FFT calculations.

In this chapter, we review the most commonly used option pricing algorithms via FFT calculations. When the European option price function is expressed in terms of Fourier inversion integrals, option pricing can be delivered by finding the numerical approximation of the Fourier integrals via FFT techniques. Several modifications of the European option pricing formulation in the Fourier domain, like the damped option price method and time value method, have been developed so as to avoid the singularity associated with non-differentiability of the terminal payoff function. Alternatively, the pricing formulation in the form of a convolution product is used to price Bermudan options where early exercise is allowed at discrete time points. Depending on the structures of the payoff functions, the extension of FFT pricing to multi-asset models may require some ingeneous formulation of the corresponding option model. The order of complexity in the FFT calculations for pricing multi-asset options generally increases with the number of underlying assets rather than the total number of risk factors in the joint dynamics of the underlying asset returns. When one considers pricing of path dependent options whose analytic form of the option price function in terms of Fourier integrals is not readily available, it becomes natural to explore various extensions of the lattice tree schemes and finite difference approach. The FFT-based network method and the Fourier space time stepping techniques are numerical approaches that allow greater flexibility in the construction of the numerical algorithms to handle various form of path dependence of the underlying asset price processes through the incorporation of the auxiliary conditions that arise from modeling the embedded optionality features. The larger number of branches in the FFT-based network approach can provide better accuracy to approximate the Lévy process with jumps when compared to the usual trinomial tree approach. The Fourier space time stepping method solves the governing partial integral-differential equation of option pricing under Lévy model in the Fourier domain. Unlike usual finite difference schemes, no time stepping procedures are required between successive monitoring instants in option models with discretely monitored features.

In summary, a rich set of numerical algorithms via FFT calculations have been developed in the literature to perform pricing of most types of option models under Lévy processes. For future research topics, one may consider the pricing of volatility derivatives under Lévy models where payoff function depends on the realized variance or volatility of the underlying price process. Also, more theoretical works should be directed to error estimation methods and controls with regard to sampling errors and truncation errors in the approximation of the Fourier integrals and other numerical Fourier transform calculations.

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