Distribution of occupation times for CEV diffusions and pricing of α-quantile options

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Abstract

The main results of this paper are the derivation of the distribution functions of occupation times under the constant elasticity of variance (CEV) process. The distribution functions can then be used to price the α-quantile options. We also derive the fixed-floating symmetry relation for α-quantile options when the underlying asset price process follows the Geometric Brownian motion.

Keywords: Distribution of occupation times, constant elasticity of variance process, α-quantile options, fixed-floating symmetry

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1 Introduction

The α-quantile $(0 < \alpha < 1)$ of a stochastic process $\mathcal{S} = \{S_t : t \in [0, T]\}$ is defined by

$$M_T^\alpha(\mathcal{S}) = \inf\{x \in \mathbb{R} : A_T^{x,-}(\mathcal{S}) > \alpha T\}, \tag{1.1}$$

where

$$A_T^{x,-}(\mathcal{S}) = \int_0^T \mathbf{1}_{\{S_t \leq x\}} dt \tag{1.2}$$

is called the occupation time of the process $\mathcal{S}$ staying below the barrier x. It is easily seen that the events $\{M_T^\alpha(\mathcal{S}) > L\}$ and $\{A_T^{L,-}(\mathcal{S}) < \alpha T\}$ are

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equivalent. Therefore, the pricing of α-quantile options whose payoff depends on M_T^α would naturally require the determination of the distribution of the occupation time.

There have been numerous papers on the derivation of the distribution of occupation times for the Brownian motion with drift, $\mathcal{Z} = \{Z_t = W_t + \mu t : t \geq 0, \mu \in \mathbb{R}\}$, where W_t is the standard Wiener process. Akahori (1995) derives the distribution of $A_T^{L,-}(\mathcal{Z})$ by using the Feynman-Kac formula and the strong Markov property of Brownian motion. By adopting various analytic approaches, Takács (1996), Doney and Yor (1998) and Pechtl (1999) obtain an explicit representation of the density of $A_T^{L,-}(\mathcal{Z})$ in terms of the normal density and distribution functions. Linetsky (1999) and Hugonnier (1999) obtain the joint distribution functions of the occupation time and terminal asset value, which are then used to derive pricing formulas of various types of occupation time derivatives. Dassios (1995) obtains a remarkable relationship between the α-quantile of a Brownian motion with drift and the distributions of the maximum and minimum value of the Brownian motion. He shows that

$$M_T^\alpha(\mathcal{Z}) \text{ and } \sup_{\{0 \leq s \leq \alpha T\}} Z_s + \inf_{\{0 \leq s \leq (1-\alpha)T\}} \widetilde{Z}_s$$

are equal in law, where $\widetilde{Z}_s$ is an independent copy of the Brownian motion. An alternative proof of Dassios' result is given by Embrechts *et al.* (1995).

Miura (1992) first introduces the α-quantile option whose payoff depends on the α-quantile of the asset price process $\mathcal{S}$. The terminal payoff functions of the fixed strike call and floating strike put α-quantile options are defined by

$$(M_T^\alpha(\mathcal{S}) - K)^+ \quad \text{and} \quad (M_T^\alpha(\mathcal{S}) - S_T)^+,$$

respectively, where K is the strike price and $x^+ = \max(x, 0)$. By letting α go to 1 (or 0) in an α-quantile option, we obtain the maximum (or minimum) lookback option. Fujita (2000) further extends the α-quantile option to exchange options of α, β-quantiles whose terminal payoff is given by $\left(M_T^\alpha(\mathcal{S}) - M_T^\beta(\mathcal{S})\right)^+$. Besides analytic approaches, there have been numerous papers that deal with the design of numerical algorithms for pricing occupation time derivatives. For example, Kwok and Lau (2001) develop the forward shooting grid method to price the Parisian options and α-quantile options.

In this paper, we would like to derive the distribution of the occupation time, and the joint distribution of the occupation time and terminal asset value under the constant elasticity of variance (CEV) process. The asset price S_t under the CEV process is governed by

$$\begin{cases} dS_t = \mu S_t\, dt + \sigma S_t^{\beta+1}\, dW_t, \quad t \geq 0, S_t \in \widehat{I}, \beta \leq 0, \\ S_0 = S \in \widehat{I}, \end{cases} \tag{1.3}$$

where $\widehat{I} = (0, \infty)$, μ is the constant drift rate, σ is the constant volatility and β is called the elasticity of the local volatility function. When $\beta > 0$, we do not have the existence of an equivalent martingale measure (Davydov and Linetsky, 2001). The CEV process nests a number of some well known processes. It becomes the geometric Brownian motion (GBM) when $\beta = 0$, absolute diffusion when $\beta = -1$ and square root process when $\beta = -1/2$.

The paper is organized as follows. In the next section, we present the derivation of the double Laplace transform of the density function of the occupation time of an asset price process and the joint density of the occupation time and terminal asset value under the CEV process. In Section 3, we illustrate the pricing formulations of α-quantile options. We then present the integral representation of the price of α-quantile options when the asset price process $\mathcal{S}$ follows the CEV process. When the asset price process is GBM, the symmetry relation between the prices of fixed strike call and floating strike put α-quantile options is derived. Summary and conclusive remarks are presented in the last section.

2 Distribution functions of occupation times of CEV process

First, we would like to present the Feynman-Kac equation that governs the double Laplace transform of the density function of the occupation time $A_T^{L,-}(\mathcal{S})$. Consider the stochastic process $\mathcal{S}$ that follows

$$\begin{cases} dS_t = \mu(S_t)\, dt + \sigma(S_t)\, dW_t, \quad t \geq 0, S_t \in I, \\ S_0 = S, S \in I, \end{cases} \tag{2.1}$$

for which the functions $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the global Lipschitz and linear growth conditions as stated in Karatzas and Shreve (1991) and $I \in (-\infty, \infty)$.

Note that the CEV process defined in Eq. (1.3) is a special case of process (2.1). The infinitesimal generator G of the process is defined by

$$G = \frac{\sigma^2(S)}{2}\frac{d^2}{dS^2} + \mu(S)\frac{d}{dS}, \quad S \in I. \tag{2.2}$$

Let $f : I \to \mathbb{R}$ and $k : I \to [0, \infty)$ be piecewise continuous functions. For some fixed constant $\lambda > 0$, we have

$$E_S \int_0^\infty e^{-\lambda t}|f(S_t)|\, dt < \infty \quad \text{for all } S \in I.$$

Define

$$\widehat{Y}(S) = E_S \int_0^\infty f(S_t) \exp(-\lambda t - \int_0^t k(S_u)\, du)\, dt,$$

then $\widehat{Y}(S)$ is piecewise $C^2(I)$. It satisfies the following second order ordinary differential equation (Karatzas and Shreve, 1991, p.366)

$$[\lambda + k(S)]\widehat{Y}(S) = G\widehat{Y}(S) + f(S) \quad \text{on } I\backslash(D_f \cup D_k), \tag{2.3}$$

where D_f and D_k are the set of points of discontinuity of f and k, respectively. By taking $f = \lambda$ and $k(x) = \gamma \mathbf{1}_{\{S \le L\}}$, $\widehat{Y}(S)$ is related to the double Laplace transform of the distribution of $A_t^{L,-}(\mathcal{S})$ and t.

We define

$$Y(S) = E_S[\exp(-\gamma \int_0^\tau \mathbf{1}_{\{S_u \le L\}}\, du)], \tag{2.4}$$

where $\tau \sim \exp(\lambda)$ and τ is independent of $\mathcal{S}$. Here, $Y(S)$ equals λ times the double Laplace transform of the density function of the occupation time with respect to t and occupation time. From Eq. (2.3), $Y(S)$ is seen to satisfy the following Feynman-Kac equation

$$GY(S) - (\lambda + \gamma \mathbf{1}_{\{S \le L\}})Y(S) = -\lambda. \tag{2.5}$$

We show how to solve for $Y(S)$ when the asset price S_t follows the CEV process (see Proposition 2.1).

Proposition 2.1 (distribution function of occupation time)

Assume $\tau \sim \exp(\lambda)$ and τ is independent of $\mathcal{S}$, and suppose the asset price $\{S_t : t \geq 0\}$ follows the CEV process defined in Eq. (1.3) with $\beta < 0$ and $\mu > 0$. For some fixed $L \in \widehat{I}$, we have

$$E_S\left[\exp\left(-\gamma\int_0^\tau \mathbf{1}_{\{S_u\leq L\}}\,du\right)\right] = \begin{cases} \frac{\lambda}{\lambda+\gamma} + c_1\psi_{\lambda+\gamma}(S) & 0 < S \leq L \\ 1 + c_4\phi_\lambda(S) & L < S < \infty \end{cases}, \tag{2.6}$$

where

$$\begin{aligned}
&\psi_\alpha(S) = S^{\beta+\frac{1}{2}}e^{\epsilon x/2}M_{k_\alpha,m}(x)\\
&\phi_\alpha(S) = S^{\beta+\frac{1}{2}}e^{\epsilon x/2}W_{k_\alpha,m}(x)\\
&x = \frac{|\mu|}{\sigma^2|\beta|}S^{-2\beta}, \epsilon = \mathrm{sign}(\mu\beta), \quad m = \frac{1}{4|\beta|},\\
&k_\alpha = \epsilon\left(\frac{1}{2}+\frac{1}{4\beta}\right) - \frac{\alpha}{2|\mu\beta|}, \quad x_L = \frac{|\mu|}{\sigma^2|\beta|}L^{-2\beta},\\
&c_1 = \frac{2m\gamma}{\lambda+\gamma}\frac{M_{k_{\lambda+\frac{1}{2}},m-\frac{1}{2}}(x_L)}{\phi_\lambda(L)W_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L) - 2m\psi_{\lambda+\gamma}(L)M_{k_{\lambda+\frac{1}{2}},m-\frac{1}{2}}(x_L)},\\
&c_4 = -\frac{\gamma}{\lambda+\gamma}\frac{M_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L)}{2m\psi_{\lambda+\gamma}(L)M_{k_{\lambda+\frac{1}{2}},m-\frac{1}{2}}(x_L) + \phi_\lambda(L)W_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L)}.
\end{aligned} \tag{2.7}$$

and $W_{\cdot,\cdot}(\cdot)$ and $M_{\cdot,\cdot}(\cdot)$ are the Whittaker functions (see Appendix).

Proof of Proposition 2.1
We write

$$u(S) = E_S\left[\exp\left(-\gamma\int_0^\tau \mathbf{1}_{\{S_u\leq L\}}\,du\right)\right],$$

where $\tau \sim \exp(\lambda)$ and τ is independent of $\mathcal{S}$, then $u(S)$ satisfies

$$\frac{\sigma^2}{2}S^{2\beta+2}\frac{d^2u}{dS^2} + \mu S\frac{du}{dS} - \left(\lambda + \gamma\mathbf{1}_{\{S\leq L\}}\right)u = -\lambda \tag{2.8}$$

subject to the following auxiliary conditions:

(i) $u(L^+) = u(L^-)$ and $u'(L^+) = u'(L^-)$,

(ii) $u(0^+) = \dfrac{\lambda}{\lambda+\gamma}$ and $\lim_{S\to\infty} u(S) < \infty$.

The matching and smooth pasting conditions at L are derived from the requirement of $u(S) \in C^1(I)$. The general solution of Eq. (2.8) is given by

$$u(S) = \begin{cases} \frac{\lambda}{\lambda+\gamma} + c_1\psi_{\lambda+\gamma}(S) + c_2\phi_{\lambda+\gamma}(S), & S \le L, \\ 1 + c_3\psi_\lambda(S) + c_4\phi_\lambda(S), & S > L, \end{cases}$$

where the hypergeometric functions $\psi_\alpha(S)$ and $\phi_\alpha(S)$ are defined in Eq. (2.7).

Next, we determine the arbitrary constants using the auxiliary conditions. First, we deduce $c_3 = 0$ by using the asymptotic property of the Whittaker function $M_{k_\lambda,m}(S)$ as $S \to \infty$ [see Appendix]. By applying the boundary condition $u(0^+) = \dfrac{\lambda}{\lambda+\gamma}$ and the asymptotic expansion of the Kummer function [see Appendix], we obtain $c_2 = 0$. Lastly, we apply the matching and smooth pasting conditions and together using the properties of the derivatives of Whittaker functions, we obtain the solution of c_1 and c_4 as stated in Eq. (2.7).

Proposition 2.2 (joint distribution of occupation time and terminal asset value)
Assume $\tau \sim \exp(\lambda)$ and τ is independent of $\mathcal{S}$, and suppose S_t follows the CEV process. For some fixed $L \in \widehat{I}$, we define

$$U(S) = E_S\left[\exp\left(-\gamma\int_0^\tau \mathbf{1}_{\{S_u \le L\}}\,du\right), S_\tau \in dz\right], \tag{2.9}$$

then $U(S)$ has the following functional forms in different regions:

(i) $0 < z \le L, S \ge L$

$$U(S) = d_2\phi_\lambda(S)$$

where

$$d_2 = \frac{2\lambda e^{\epsilon x_B}}{z^{\beta+3/2}\sigma^2}\left(\frac{\sigma^2|\beta|}{|\mu|}\right)^{2\beta+2} e^{x_B/2}x_B^{1-m}L^{2\beta+1}$$
$$\frac{M_{k_{\lambda+\gamma},m}(x_z)}{2m\phi_\lambda(L)M_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L) + \psi_{\lambda+\gamma}(L)W_{k_\lambda+\frac{1}{2},m-\frac{1}{2}}(x_L)}. \tag{2.10a}$$

(ii) $0 < z \leq L, 0 < S \leq L$

$$U(S) = d_3 \psi_{\lambda+\gamma}(S) + d(S)$$

where

$$\begin{aligned} d_4 &= \frac{\lambda \sigma^2 |\beta| \Gamma\left(m - k_{\lambda+\gamma} + \frac{1}{2}\right) M_{k_{\lambda+\gamma},m}(x_z)}{\sigma^2 \beta |\mu| \Gamma(2m+1) z^{\beta+3/2}} \\ d_3 &= \frac{\phi_\lambda(L) W_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L) - \phi_{\lambda+\gamma}(L) W_{k_\lambda+\frac{1}{2},m-\frac{1}{2}}(x_L)}{2m\phi_\lambda(L) M_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L) + \psi_{\lambda+\gamma}(L) W_{k_\lambda+\frac{1}{2},m-\frac{1}{2}}(x_L)} d_4 \end{aligned}$$

$$d(S) = \begin{cases} d_4 \phi_{\lambda+\gamma}(S), & z \leq S \leq L \\ d_4 \dfrac{\phi_{\lambda+\gamma}(z) \psi_{\lambda+\gamma}(S)}{\psi_{\lambda+\gamma}(z)}, & 0 < S < z \end{cases} \quad , \quad x_z = \frac{|\mu|}{\sigma^2 |\beta|} z^{-2\beta}. \tag{2.10b}$$

(iii) $z > L, 0 < S \leq L$

$$U(S) = e_1 \psi_{\lambda+\gamma}(S)$$

where

$$e_1 = \frac{2\lambda e^{\epsilon x_L}}{z^{\beta+3/2} \sigma^2} \left(\frac{\sigma^2 |\beta|}{|\mu|}\right)^{2\beta+2} e^{x_B/2} x_B^{1-m} S_B^{2\beta+1}$$

$$\frac{W_{k_\lambda,m}(x_z)}{2m\phi_\lambda(L) M_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L) + \psi_{\lambda+\gamma}(L) W_{k_\lambda+\frac{1}{2},m-\frac{1}{2}}(x_L)}. \tag{2.10c}$$

(iv) $z \geq L, S \geq L$

$$U(S) = e_4 \phi_\lambda(S) + e(S)$$

where

$$\begin{aligned} e_3 &= -\frac{\lambda |\beta| \Gamma\left(m - k_\lambda + \frac{1}{2}\right) W_{k_\lambda,m}(x_z)}{z^{\beta+3/2} \beta |\mu| \Gamma(2m+1)} \\ e_4 &= 2me_3 \frac{\psi_{\lambda+\gamma}(L) M_{k_\lambda+\frac{1}{2},m-\frac{1}{2}}(x_L) - \psi_\lambda(L) M_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L)}{2m\phi_\lambda(L) M_{k_{\lambda+\gamma}+\frac{1}{2},m-\frac{1}{2}}(x_L) + \psi_{\lambda+\gamma}(L) W_{k_\lambda+\frac{1}{2},m-\frac{1}{2}}(x_L)} \end{aligned}$$

$$e(S) = \begin{cases} e_3 \psi_\lambda(S) & L \leq S \leq z \\ e_3 \frac{\psi_\lambda(z)}{\phi_\lambda(z)} \phi_\lambda(S) & S > z \end{cases} . \tag{2.10d}$$

Proof of Proposition 2.2
Similar to Eq. (2.8), the governing differential equation for $U(S)$ is given by

$$\frac{\sigma^2}{2}S^{2\beta+2}\frac{d^2U}{dS^2}+\mu S\frac{dU}{dS}-\left(\lambda+\gamma\mathbf{1}_{\{S\leq L\}}\right)U=-\lambda\delta(S-z), \tag{2.11}$$

subject to the following auxiliary conditions

(i) matching conditions at L and z

$$U(L^+)=U(L^-) \quad \text{and} \quad U(z^+)=U(z^-)$$

(ii) smooth pasting condition at L

$$U'(L^+)=U'(L^-)$$

(iii) jump condition across z

$$U'(z^+)-U'(z^-)=-\frac{2\lambda}{\sigma^2 z^{2\beta+2}}$$

(iv) boundary conditions at $S=0$ and $S\to\infty$

$$U(0^+)=0 \quad \text{and} \quad \lim_{S\to\infty}U(S)<\infty.$$

Remark
The jump condition can be deduced by integrating Eq. (2.11) over the infinitesimal interval (z^-,z^+) and applying the continuity condition of $U(S)$ across z.

We consider the solution under the two separate cases (i) $0<z\leq L$ and (ii) $z>L$.

First, for $0<z\leq L$, the general solution to Eq. (2.11) is given by

$$U(S)=\begin{cases} d_1\psi_\lambda(S)+d_2\phi_\lambda(S), & S>L \\ d_3\psi_{\lambda+\gamma}(S)+d_4\phi_{\lambda+\gamma}(S), & z<S\leq L \\ d_5\psi_{\lambda+\gamma}(S)+d_6\phi_{\lambda+\gamma}(S), & 0<S<z \end{cases}.$$

By virtue of the boundary condition at $S\to\infty$, we deduce that $d_1=0$. Also, by applying $U(0^+)=0$ and $\psi_{\lambda+\gamma}(0^+)=0$, we obtain $d_6=0$. The remaining 4 arbitrary constants are determined by applying the matching, smooth pasting and jump conditions. The system of algebraic equations for d_2, d_3, d_4 and d_5 is found to be

$$d_2\phi_\lambda(L) - d_3\psi_{\lambda+\gamma}(L) - d_4\phi_{\lambda+\gamma}(L) = 0 \tag{i}$$

$$d_3\psi_{\lambda+\gamma}(z) + d_4\phi_{\lambda+\gamma}(z) - d_5\psi_{\lambda+\gamma}(z) = 0 \tag{ii}$$

$$d_2\phi'_\lambda(L) - d_3\psi'_{\lambda+\gamma}(L) - d_4\phi'_{\lambda+\gamma}(L) = 0 \tag{iii}$$

$$d_3\psi'_{\lambda+\gamma}(z) + d_4\phi'_{\lambda+\gamma}(z) - d_5\psi'_{\lambda+\gamma}(z) = -\frac{2\lambda}{\sigma^2 z^{2\beta+2}}. \tag{iv}$$

From Eqs. (ii) and (iv), we obtain

$$d_4 = \frac{2\lambda\psi_{\lambda+\gamma}(z)}{\sigma^2 z^{2\beta+2}\left[\phi_{\lambda+\gamma}(z)\psi'_{\lambda+\gamma}(z) - \phi'_{\lambda+\gamma}(z)\psi_{\lambda+\gamma}(z)\right]}.$$

The denominator can be simplified by applying analytic properties of the Whittaker functions as follows:

$$\begin{aligned}
&\phi_{\lambda+\gamma}(z)\psi'_{\lambda+\gamma}(z) - \phi'_{\lambda+\gamma}(z)\psi_{\lambda+\gamma}(z) \\
= &\ -\frac{2\beta e^{\epsilon x_z}|\mu|}{\sigma^2|\beta|}\left[W_{k_{\lambda+\gamma},m}(x_z)M'_{k_{\lambda+\gamma},m}(x_z) - M_{k_{\lambda+\gamma},m}(x_z)W'_{k_{\lambda+\gamma},m}(x_z)\right] \\
= &\ -\frac{2\beta e^{\epsilon x_z}|\mu|}{\sigma^2|\beta|}\frac{\Gamma(2m+1)}{\Gamma\left(m - k_{\lambda+\gamma} + \frac{1}{2}\right)},
\end{aligned}$$

so that the analytic expression for d_4 is obtained [see Eq. (2.10b)]. Here, $\Gamma(x)$ is the Euler Gamma function. Once d_4 is known, we can solve for d_2 and d_3 from the following pair of algebraic equations:

$$\begin{aligned}
d_2\phi_\lambda(L) - d_3\psi_{\lambda+\gamma}(L) &= d_4\phi_{\lambda+\gamma}(L) \\
d_2\phi'_\lambda(L) - d_3\psi'_{\lambda+\gamma}(L) &= d_4\phi'_{\lambda+\gamma}(L).
\end{aligned}$$

The analytic expressions for d_2 and d_3 can be derived in terms of d_4 and the Whittaker functions [see Eqs. (2.10a,b)]. Lastly, d_5 can be expressed in terms of d_3 and d_4, where

$$d_5 = d_3 + d_4\frac{\phi_{\lambda+\gamma}(z)}{\psi_{\lambda+\gamma}(z)}.$$

For the case $z > L$, by following similar derivation procedures as above, we can obtain the analytic expressions for e_1, e_3, e_4 and $e(S)$ as shown in Eqs. (2.10c,d).

Remark
The above two proofs can be extended to study the distribution function of occupation times of other asset price process S_t. The coefficients in the governing Feynman-Kac equation would differ and this leads to different types of hypergeometric functions in the solution. For example, we may consider the Ornstein-Uhlenbeck (OU) process as defined by

$$\begin{cases} dS_t = -\kappa S_t\, dt + \sigma\, dW_t, & t \geq 0, S_t \in I, \\ S_0 = S \in I, \end{cases} \tag{2.12}$$

where $I \in (-\infty, \infty)$ and κ is a parameter. The analytic expressions for the corresponding $u(S)$ and $U(S)$ can be found in the Handbook by Borodin and Salminen (2002).

3 Pricing of α-quantile options

We describe the financial market under which the pricing of options is undertaken. Consider the time horizon $[0, T]$, the uncertain economy is modeled by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, Q)$, where Q is the risk neutral (equivalent martingale) probability measure and the filtration $\mathcal{F}_t$ is generated by the standard Wiener process $\{W_s : 0 \leq s \leq t\}$. Let $\mathcal{S} = \{S_t : t \in [0, T]\}$ be an one-dimensional time-homogeneous diffusion process on $(\Omega, \mathcal{F})$ with state space $I \subseteq R$ (R may be open, semi-open, closed or include ∞ or $-\infty$).

Under the Black-Scholes pricing paradigm, the value of the fixed strike α-quantile call option at $t = 0$ is given by

$$C^{\alpha}_{fix}(S, 0; K) = e^{-rT} E_S[(M^{\alpha}_T(\mathcal{S}) - K)^+], \tag{3.1}$$

where E_S denotes the expectation with respect to the risk neutral measure Q [subscript S indicates that the stochastic process $\mathcal{S}$ starts at $S_0 = S$] and r is the riskless interest rate. The above expectation can be expressed in terms of integration of the distribution function of either $M^{\alpha}_T(\mathcal{S})$ or $A^{L,-}_T(\mathcal{S})$, namely,

$$\begin{aligned} C^{\alpha}_{fix}(S, 0; K) &= e^{-rT} \int_K^{\infty} P[M^{\alpha}_T(\mathcal{S}) > L]\, dL \\ &= e^{-rT} \int_K^{\infty} P[A^{L,-}_T(\mathcal{S}) < \alpha T]\, dL. \end{aligned} \tag{3.2}$$

The pricing of the fixed strike α-quantile call option amounts to the determination of the distribution function $A_T^{L,-}(\mathcal{S})$.

In a similar manner, the value of the floating strike α-quantile put option at time $t = 0$ is given by

$$P_{f\ell}^{\alpha}(S, 0) = e^{-rT} E_S[(M_T^{\alpha}(\mathcal{S}) - S_T)^+]. \tag{3.3}$$

By observing the following relations

$$\begin{aligned} P\left[M_T^{\alpha}(\mathcal{S}) \in dL, S_T \in dz\right] &= -\frac{\partial}{\partial L}\left\{P\left[M_T^{\alpha}(\mathcal{S}) > L, S_T \in dz\right]\right\} \\ &= -\frac{\partial}{\partial L}\left\{P\left[A_T^{L,-}(\mathcal{S}) < \alpha T, S_T \in dz\right]\right\} \\ &= -\frac{\partial}{\partial L}\int_0^{\alpha T} P\left[A_T^{L,-}(\mathcal{S}) \in d\tau, S_T \in dz\right] d\tau, \end{aligned}$$

the put option value can be expressed in the following form

$$P_{f\ell}^{\alpha}(S, 0) = e^{-rT}\int_0^{\infty}\int_z^{\infty}(z-L)\frac{\partial}{\partial L}\left\{\int_0^{\alpha T} P\left[A_T^{L,-}(\mathcal{S}) \in d\tau, S_T \in dz\right] d\tau\right\} dL dz. \tag{3.4}$$

The valuation of $P_{f\ell}^{\alpha}(S, 0)$ amounts to the determination of the joint density function of $A_T^{L,-}(\mathcal{S})$ and S_T.

Let $\mathcal{L}_\lambda^{-1}$ and $\mathcal{L}_\gamma^{-1}$ denote the Laplace inversion with respect to the Laplace variable λ and γ, respectively, and let t and τ be the respective variables after Laplace inversion. By setting $t = T$ and $\tau = \alpha T$, we have

$$P\left[A_T^{L,-}(\mathcal{S}) < \alpha T\right] = \mathcal{L}_\lambda^{-1}\mathcal{L}_\gamma^{-1}\left[\frac{u(S)}{\lambda\gamma}\right]\Bigg|_{t=T \text{ and } \tau=\alpha T}, \tag{3.5}$$

so that the value of the fixed strike α-quantile call option under the CEV process is given by

$$C_{fix}^{\alpha}(S, 0; K) = e^{-rT}\int_K^{\infty}\mathcal{L}_\lambda^{-1}\mathcal{L}_\gamma^{-1}\left[\frac{u(S)}{\lambda\gamma}\right]\Bigg|_{t=T \text{ and } \tau=\alpha T} dS. \tag{3.6}$$

The above expression gives a formal representation of $C_{fix}^{\alpha}(S, 0; K)$ in terms of double Laplace inversion and an integral.

When the asset price process is the GMB (corresponding to $\beta = 0$), the integral representation of the distribution or density function of $M_T^{\alpha}(\mathcal{Z})$ can be obtained in simpler analytic forms by using various probabilistic approaches (Akahori, 1994; Dassios, 1995). By performing tedious integration procedures, Pechtl (1999) manages to obtain closed form solution to $C^{\alpha}_{fix}(S,0;K)$ by using either Akahori's or Dassios' integral representation.

In a similar manner, the double Laplace inversion of $U(S)$ gives the joint density function of $A_T^{L,-}(\mathcal{S})$ and S_T, where

$$P\left[A_T^{L,-}(\mathcal{S}) \in d\tau, S_T \in dz\right] = \mathcal{L}_\lambda^{-1}\mathcal{L}_\gamma^{-1}\left[\frac{U(S)}{\lambda}\right]\Bigg|_{t=T}$$

so that the value of the floating strike α-quantile put option under the CEV process is given by the following integral representation

$$\begin{aligned} & P^{\alpha}_{f\ell}(S,0) \\ = \; & e^{-rT}\int_0^\infty \int_z^\infty (z-L)\frac{\partial}{\partial L}\left\{\int_0^{\alpha T} \mathcal{L}_\lambda^{-1}\mathcal{L}_\gamma^{-1}\left[\frac{U(S)}{\lambda}\right]\Bigg|_{t=T} d\tau\right\} dL dz \end{aligned} \quad (3.7)$$

Fixed-floating symmetry relation under GBM

When the asset price process follows the GBM, there exists a simple symmetry relation between the prices of fixed strike call and floating strike put for α-quantile options. Under the risk neutral measure Q, the drift rate μ of the GBM is given by $r-q$. Here, q is the constant dividend yield. The symmetry relation has a very simple form, namely,

$$P^{\alpha}_{f\ell}(S,0;r,q) = C^{\alpha}_{fix}(S,0;S,q,r). \quad (3.8)$$

The proof stems from the following identity established by Detemple (2001). We define the process $S_t^* = \frac{S}{S_T}S_t$ and consider

$$\begin{aligned} \frac{S}{S_T}M_T^{\alpha}(\mathcal{S}) &= \frac{S}{S_T}\inf\left\{y\in\mathbb{R} : \int_0^T \mathbf{1}_{\{S_u \le y\}}\,du > \alpha T\right\} \\ &= \inf\left\{\frac{S}{S_T}y\in\mathbb{R} : \int_0^T \mathbf{1}_{\left\{S_u^* \le \frac{S}{S_T}y\right\}}\,du > \alpha T\right\} \\ &= \inf\left\{z\in\mathbb{R} : \int_0^T \mathbf{1}_{\{S_u^* \le z\}}\,du > \alpha T\right\} = M_T^{\alpha}(\mathcal{S}^*), \end{aligned} \quad (3.9)$$

where $M_T^{\alpha}(\mathcal{S}^*)$ is the α-quantile of S_t^*. The above relation is distribution free.

Suppose Q^* denote the equivalent martingale measure where the asset price is used as the numeraire, then

$$\frac{dQ^*}{dQ} = \frac{S_T}{S} e^{-(r-q)T}. \tag{3.10}$$

Using the Girsanov Theorem, $W_t^* = W_t - \sigma t$ is a Brownian motion under Q^*. The process $S_t^* = \dfrac{S}{S_T} S_t$ is related to W_t^* by

$$\begin{aligned} S_t^* &= S \exp\left(\sigma(W_t - W_T) + \left(r - q - \frac{\sigma^2}{2}\right)(t-T)\right) \\ &= S \exp\left(\sigma(W_t^* - W_T^*) + \left(r - q + \frac{\sigma^2}{2}\right)(t-T)\right). \end{aligned} \tag{3.11a}$$

Let $\widetilde{W}_t = -W_t^*$, which is a reflected Q^*-Brownian motion. Note that $W_t^* - W_T^* \overset{\text{law}}{=} \widetilde{W}_{T-t}$ so that

$$S_t^* \overset{\text{law}}{=} S \exp\left(\sigma \widetilde{W}_{T-t} + \left(q - r - \frac{\sigma^2}{2}\right)(T-t)\right). \tag{3.11b}$$

The price formula of $P_{f\ell}^{\alpha}(S, 0)$ can be rewritten in the form of a fixed strike α-quantile call option when the asset price process is expressed in terms of $\mathcal{S}^*$, where

$$\begin{aligned} P_{f\ell}^{\alpha}(S, 0; r, q) &= e^{-rT} E_S\left[(M_T^{\alpha}(\mathcal{S}) - S_T)^+\right] \\ &= E_S\left[\frac{S_T}{S} e^{-(r-q)T} e^{-qT}\left(\frac{S}{S_T} M_T^{\alpha}(\mathcal{S}) - S\right)^+\right] \\ &= e^{-qT} E_S^*\left[(M_T^{\alpha}(\mathcal{S}^*) - S)^+\right], \end{aligned} \tag{3.12}$$

where E_S^* is the expectation under Q^* with $S_0 = S$. Equation (3.12) represents an interesting symmetry relation between a floating strike α-quantile put option with state variable process $\mathcal{S}$ and a fixed strike α-quantile call with state variable process $\mathcal{S}^*$ in a market with interest rate q.

To proceed further, by considering the reversal of time via the change of variable $\widetilde{t} = T - u$, by virtue of Eq. (3.11), we obtain

$$\begin{aligned} M_T^{\alpha}(\mathcal{S}^*) &= \inf\left\{z \in \mathbb{R} : \int_0^T \mathbf{1}_{\{S_u^* \le z\}}\, du > \alpha T\right\} \\ &= \inf\left\{z \in \mathbb{R} : \int_0^T \mathbf{1}_{\left\{S \exp\left(\sigma \widetilde{W}_{\widetilde{t}} + \left(q - r - \frac{\sigma^2}{2}\right)\widetilde{t}\right) \le z\right\}}\, d\widetilde{t} > \alpha T\right\} \end{aligned} \tag{3.13}$$

If we let $\widetilde{S}_t = S \exp\left(\sigma \widetilde{W}_t + \left(q - r - \frac{\sigma^2}{2}\right) t\right)$, then $\widetilde{S}_t$ is governed by

$$\frac{d\widetilde{S}_t}{\widetilde{S}_t} = (q - r)\, dt + \sigma\, d\widetilde{W}_t, \quad S_0 = S. \tag{3.14}$$

We write $M_T^\alpha(\widetilde{\mathcal{S}})$ as the α-quantile of $\widetilde{S}_t$, then Eq. (3.13) reveals that $M_T^\alpha(\mathcal{S}^*) = M_T^\alpha(\widetilde{\mathcal{S}})$. Together with the property that $\widetilde{S}_t$ follows the GBM under Q^* with drift rate $q - r$, we obtain

$$\begin{aligned} e^{-qT} E_S^* \left[(M_T^\alpha(\mathcal{S}^*) - S)^+ \right] &= e^{-qT} E_S^* \left[\left(M_T^\alpha(\widetilde{\mathcal{S}}) - S \right)^+ \right] \\ &= C_{fix}^\alpha(S, 0; S, q, r). \end{aligned} \tag{3.15}$$

Hence, the symmetry relation in Eq. (3.8) is established.

Remark
The fixed-floating symmetry relation in Eq. (3.15) does not hold when the volatility is state dependent, like the CEV process. This is because when $\sigma = \sigma(S_t, t)$, the process S_t^* is related to $W_t^* = W_t - \int_0^t \sigma(S_u, u)\, dt$ as follows:

$$S_t^* = S \exp\left(-\int_t^T \left[r(S_u, u) - q(S_u, u) + \frac{\sigma^2}{(S_u, u)} \right] du - \int_t^T \sigma(S_u, u)\, dW_u^* \right). \tag{3.16}$$

Apparently, the dynamics of S_t^* depends on the dynamics of S_t beyond the current time t. In this case, $\widetilde{S}_t$ cannot be defined when we attempt to do the time reversal.

On the other hand, when $\sigma = \sigma(t)$ is time dependent, the stochastic differential equation for $\widetilde{S}_t$ is then modified as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_t} = [q(T - t) - r(T - t)]\, dt + \sigma(T - t)\, d\widetilde{W}_t, \quad S_0 = S. \tag{3.17}$$

For the price formulas of the α-quantile options, the corresponding modifications to the discount terms and variance rate terms are given by

$$e^{-\int_0^T [q(T-t) - r(T-t)]\, dt} = e^{-\int_0^T [q(t) - r(t)]\, dt}$$

and

$$\frac{1}{T}\int_0^T \sigma^2(T-t)\,dt = \frac{1}{T}\int_0^T \sigma^2(t)\,dt,$$

respectively. Hence, the symmetry relation as in Eq. (3.15) remains valid when r, q and σ are time dependent.

4 Conclusion

The Feynman-Kac approach is an effective tool to study the law of Brownian functionals. In this paper, we apply the Feynman-Kac approach to derive analytic expressions for the double Laplace transform of the density function of occupation time and the joint density function of occupation time and terminal asset value under the CEV process. The derivation procedures involve the solution of the Sturm-Liouville equation and the resulting analytic formulas are expressed in terms of the Whittaker functions. We then illustrate how to use the occupation time density functions to derive the integral representations of α-quantile options. When the underlying asset price process follows the Geometric Brownian motion, we obtain the symmetry relation between the prices of floating strike put and fixed strike call α-quantile options.

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Appendix – Whittaker functions

The Whittaker functions satisfy the following second order ordinary differential equation (commonly called the Whittaker equation)

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\frac{1}{4} - m^2}{x^2} + \frac{k}{x} - \frac{1}{4}\right) u = 0.$$

The Whittaker functions are defined by

(i) $M_{k,m}(x) = e^{-x/2} x^{m+\frac{1}{2}} {}_1F_1\left(\frac{1}{2} + m - k, 1 + 2m, x\right)$, where the confluent hypergeometric function ${}_1F_1$ is given by

$${}_1F_1(a, b, x) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n) x^n}{\Gamma(b+n) n!};$$

(ii) $W_{k,m}(x) = e^{-x/2} x^{m+\frac{1}{2}} \mathcal{T}\left(\frac{1}{2} + m - k, 1 + 2m, x\right)$, where the Tricomi function is given by

$$\mathcal{T}(a, b, x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-xt} t^{a-1} (1+t)^{b-a-1}\, dt.$$

The derivatives of $M_{k,m}(x)$ and $W_{k,m(x)}$ are given by

$$\frac{d^n}{dx^n}\left(e^{-\frac{x}{2}} x^{m-\frac{1}{2}} M_{k,m}(x)\right) = (-1)^n (-2m)_n e^{-\frac{x}{2}} x^{m-\frac{n}{2}-\frac{1}{2}} M_{k+\frac{n}{2}, m-\frac{n}{2}}(x)$$
$$\frac{d^n}{dx^n}\left(e^{-\frac{x}{2}} x^{m-\frac{1}{2}} W_{k,m(x)}\right) = (-1)^n e^{-\frac{x}{2}} x^{m-\frac{1}{2}-\frac{n}{2}} W_{k+\frac{n}{2}, m-\frac{n}{2}}(x).$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1), (a)_0 = 1$.

In addition, we have the following relation:

$$M'_{k,m}(x) W_{k,m}(x) - M_{k,m}(x) W'_{k,m}(x) = \frac{\Gamma(2m+1)}{\Gamma\left(m - k + \frac{1}{2}\right)}.$$

Some asymptotic results of ${}_1F_1(a, b, x)$ and $\mathcal{T}(a, b, x)$ for large x and small x are presented below:

(i) For large $x, (a, b$ fixed),

(a) $_1F_1(a,b,x) = \dfrac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}(1 + O(x^{-1}))$;

(b) $\mathcal{T}(a,b,x) = x^{-a}(1 + O(x^{-1}))$.

(ii) For small $x, (a, b$ fixed)

(a) $_1F_1(a,b,0) = 1$, for $b > 0$;

(b)
$$\mathcal{T}(a,b,x) = \begin{cases} \frac{\Gamma(1-b)}{\Gamma(1+a+b)} + O(|x|) & \text{for } b < 0 \\ \frac{1}{\Gamma(1+a)} + O(|x\ln x|) & \text{for } b = 0 \\ \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + O(|x|^{1-b}) & \text{for } 0 < b < 1 \\ -\frac{1}{\Gamma(a)}\left[\ln x + \frac{\Gamma'(a)}{\Gamma(a)} - 2\frac{\Gamma'(1)}{\Gamma(1)}\right] + O(|x\ln x|) & \text{for } b = 1 \\ \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} + O(1) & \text{for } 1 < b < 2 \\ \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} + O(|\ln x|) & \text{for } b = 2 \\ \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} + O(|x|^{b-2}) & \text{for } b > 2 \end{cases}.$$