Real options game models of R&D competition between asymmetric firms with spillovers

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Abstract

Using real options game models, we consider the characterization of strategic equilibria associated with an asymmetric R&D race between an incumbent firm and an entrant firm in the development of a new innovative product under market and technological uncertainties. The random arrival time of the discovery of the patent protected innovative product is modeled as a Poisson process. Input spillovers on the R&D effort are modeled by the change in the leader's hazard rate of success of innovation upon the follower's entry into the R&D race. Asymmetry between the two competing firms include sunk costs of investment, stochastic revenue flow rates generated from the product, and hazard rates of arrival of success of R&D efforts of the two firms. Under asymmetric duopoly, we obtain the complete characterization of the three types of Markov perfect equilibria (sequential leader-follower, preemption and simultaneous entry) of the firms' optimal R&D entry decisions with respect to various sets of model parameters. Our model shows that under positive input spillover, preemptive equilibrium does not occur in the R&D race due to the presence of dominant second mover advantage. The two firms choose optimally to enter simultaneously if the sunk cost asymmetry is relatively small; otherwise, sequential equilibrium would occur. When the initial hazard rate is low relative to the level of input spillover, simultaneous entry would occur as an optimal decision, signifying another scenario of dominant second mover advantage. On the other hand, when the initial hazard rate is sufficiently high so that the first mover advantage becomes more significant, simultaneous equilibrium does not occur even under high level of positive input spillover.

1 Introduction

The key elements in the strategic R&D (Research and Development) race between competing firms in the discovery of a new product include market and technological uncertainties, and spillovers in R&D effort. Here, market uncertainty refers to the uncertainty over the future stochastic revenue flow rate generated from the new innovative product. The technological uncertainty is related to the random arrival time of success of the R&D effort in the development of the new product. For spillover effects on R&D, output spillovers are characterized by imperfect appropriability of the revenue generated from the innovation that occurs in the product market after the completion of the R&D race. The use of new knowledge tends to spread through commercial development, though the inventor may want to minimize such spread. For input spillovers, research activities conducted in one firm may influence the

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research activities of other rival firms, and the externality effect can be positive or negative. The knowledge created by one firm through R&D efforts may not be contained within that firm. There are various possible modes of input spillovers under which positive externalities may arise from the research efforts of the rival firms. In terms of knowledge and informational spillovers, the research personnel across various firms may discuss among themselves topics of mutual interest, or research results may be disseminated through various public channels, like publications and seminars. Also, the physical movement of research personnel from one firm to another firm may give rise to knowledge and expertise transfer. In addition, one firm may observe the actions of its competitors and learns from the experience of these actions. On the other hand, negative input spillovers may arise due to congestion effects, say, firms are competing for skilled research personnel. R&D spillovers may increase free-riding incentives and impact on investment spending in research. A good review on the economic analysis of spillovers of research can be found in Jaffe (1996).

The analysis of R&D races with spillovers has been well explored in the literature. Under positive input spillovers, Kamien *et al.* (1992) analyze the effects of R&D cartelization and research joint ventures on firms that are engaged in R&D competitions in the product market. They show that the costs of production tend to decrease with R&D cartelization, thus creating public-good effect. Also, both consumer and producer surpluses are improved, through elimination of duplication effects and positive effects of economies of scale. In a later work, Kamien and Zang (2000) observe that the rate of spillover depends on imitators' level of R&D efforts. In their R&D race model, they assume that a firm's own R&D effort improves its absorptive capacity to realize spillovers from other firms' R&D activity. The positive effect of absorptive capacity may offset the negative effect of providing spillovers to rival firms. The stochastic extension of these earlier deterministic models of input spillovers and imperfect appropriability has been performed by Miyagiwa and Ohno (2002) and Martin (2002), where uncertainty of the arrival of innovation is modeled by a hazard rate process. The strategic aspects of licensing and impact on social welfare are analyzed in these papers. Hauenschild (2003) considers the impact of input and output spillovers when the R&D projects are risky. He argues that since the loser in the R&D race suffers from loss in profit in the product market, so there is a strong incentive to expand R&D effort. In addition, the winner also benefits from the rival's R&D expenditure, so a higher input spillover rate enforces a stronger incentive on R&D. Zhou (2006) examines the effects of uncertainty and spillovers on R&D expenditure. He argues that a higher spillover rate decreases the effectiveness of R&D spending due to the public-good effect. However, since the expected prize of innovations increases with increased R&D efforts, the larger pie effect may offset the public-good effect. These works, however, have not included the consideration of economic uncertainty of the stochastic revenue flows generated from the R&D innovations.

Investment decisions on risky projects with stochastic revenue flows have been commonly analyzed via the real options approach (Dixit and Pindyck, 1994) using an analogy with a financial call option of the right to invest at an optimal timing. Real options games arise when real options of investment decisions are combined with competitive interactions between rival firms. There has been a substantial literature on investment decisions analyzed under the real options game framework. In the pioneering work by Fudenberg and Tirole (1985), they establish the concept of rent equalization in the characterization of preemption equilibrium in a duopoly. Later works by Pawlina and Kort (2006) and Kong and Kwok (2007) deal with real options game models under asymmetric duopoly of investment competitions and subject to economic uncertainty of the future revenue flows. Azevedo and Paxson (2012) consider an asymmetric duopoly preemption real options model to analyze investments in new technologies subject to various forms of uncertainties: market revenues, efficiency after adoption and arrivals of alternative new technologies.

The adoption of the real options game approach in analyzing R&D competition is rather limited in the literature. The first work is initiated by Weeds (2002), in which she considers an irreversible investment on R&D effort between two symmetric firms. Her model assumes stochastic revenue flows and uncertainty of arrival of innovation, with no spillovers of R&D efforts. Depending on the model parameter values, two types of non-cooperative equilibria appear under her symmetric duopoly real options game model. Her model reveals that when preemption equilibrium occurs, real option values of both firms are undermined due to fear of preemption. Otherwise, equilibrium of delayed simultaneous entry of the two firms prevails, where each firm holds back from R&D investing in the fear of starting a keen race. In an extended real options game model of R&D race, Femminis and Martini (2011) incorporate inter-firm spillovers by assuming a reduction of R&D cost for the follower. They show that the follower firm's optimal strategy is to invest on R&D once it can attain the spillover and the resulting spillovers reduce the difference between the leader's and follower's value functions.

Our real options game model of R&D competition extends both the models of Weeds (2002) and Femminis-Martini (2011) in several aspects. First, we assume asymmetric firms instead of identical firms, with asymmetry in sunk costs of R&D investment, stochastic revenue flow rates and hazard rates of success of innovation. Unlike earlier works that model input spillovers by assuming cost reduction in R&D effort or effective research intensity, we incorporate the input spillover effects on the hazard rates of arrival of innovation of both firms. Compared to the spillover assumptions in Femminis and Martini (2011), we consider more elaborate modeling of the Poisson rates of innovation success of the two firms due to spillovers. We assume that the leader's hazard rate of success of innovation jumps to a new value (which can be above or below the original value) upon optimal entry of the follower firm into the R&D race. The positive jump in the hazard rate (positive spillover) indicates that the follower firm's R&D effort contributes to the leader's R&D progress, say, through exchange of information among researchers in the two firms. On the other hand, negative spillover may be resulted when the two firms are competing for research personnel. As an extension to the R&D model of Weeds (2002), we allow the flexibility of choosing different appropriability factors in the stochastic revenue flow rate of the two competing firms upon delivery of the new innovative product. Our model does allow output spillovers of imperfect appropriability of the revenue flow generated by the innovation. Under the restricted assumption of symmetry in costs and hazard rates, Weeds' model reveals only two types of equilibrium: (i) preemption equilibrium where real option values of both firms are reduced due to competition, (ii) simultaneous delayed R&D entry to avoid a keen R&D race.

Compared to Weeds' model, our model helps shed more insight into better understanding of the phenomena in real options games and follower strategies, extending the discussion in Cottrell-Sick (2002) on the second mover advantages in investment games. We show that when the difference in investment costs of two firms is small, the type of equilibrium changes from preemption equilibrium to simultaneous equilibrium as the externality factor of input spillover increases in value (second mover advantage becomes more prevalent). That is, positive input spillover enhances "non-cooperative" simultaneous equilibrium as the two firms hold back from entry into the R&D race. As a result, the negative aspect of value function being undermined due to preemption threat is reduced through positive input spillover. While preemption threat may undermine the second mover advantage, positive input spillover resulted from delayed follower entry enhances the second mover advantage. By allowing asymmetry in costs, revenue flows, hazard rates and input spillover effects, our real R&D race model illustrates that different types of equilibriums (sequential equilibrium, preemptive equilibrium and simultaneous equilibrium) can be resulted from different parameter configurations. In terms of policy implication, our model provides the insight that some form of tacit collusion may help minimize inefficiencies generated in a preemptive game. Suppose one of the two firms is a public firm whose objective is to maximize the social welfare as proxied by the sum of value functions of the two firms, our analysis may justify the adoption of more liberal approach of allowing some form of positive input spillover. Our model also shows that sequential leader-follower equilibrium may be resulted under positive spillovers and high level of cost asymmetry. Moreover, technological uncertainty on the arrival of innovation tends to mitigate the preemption incentive since the first mover that enters into R&D is not guaranteed to be the first one that delivers the innovation.

As a remark, our model assumes "winner-takes-all" in revenue upon the winner's success of innovation. The investment models by Pawlina and Kort (2006) and Kong and Kwok (2007) allow the follower to receive a proportional share of the market revenue (a smaller share when under negative externalities) if it chooses to invest at a later time. The use of the term "follower entry" takes different interpretations under our R&D model (which refers to late entry in initiating R&D) and the other asymmetric duopoly of investment game models (which refers to late entry in investment decision).

This paper is organized as follows. The formulation of the strategic R&D race model in an asymmetric duopoly setting with spillovers is presented in the next section. Both the incumbent firm and entrant firm (as the challenger) have the option to enter into R&D for a new innovative product by investing an upfront sunk cost. In Section 3, we derive the value functions and the corresponding trigger threshold values of the stochastic fundamental of revenue flow at which the firms are optimal to enter into the R&D race either as the leader or follower. Once the value functions and trigger threshold values are known, we deduce the optimal preemption strategies or simultaneous entry by examining the sign properties of the preemption function (defined as the difference between the preemption leader value function and the follower value function). In Section 4, we present the full characterization of the three types of Markov perfect equilibria of their optimal entry decisions into the R&D race. In particular, we examine the impact of various parameter values, like the sunk costs, hazard rate, etc. on the outcomes of the strategic games. In Section 5, we present various plots of the value functions and figures that illustrate the characterization of strategic equilibria under different parameter spaces. The last section contains conclusive remarks of the paper and discuss the potential policy implications and insights that can be deduced from the analysis of strategic equilibria in our R&D real options game models.

2 Model formulation

We consider the formulation of strategic R&D races model in an asymmetric duopoly setting with an incumbent firm (Firm i) and an entrant firm (Firm e) as the challenger. Both firms are assumed to be able to borrow and lend freely at the constant riskfree interest rate r. The incumbent firm is now serving a monopolized market with an existing product. Firm i receives the perpetual stochastic revenue flow rate x_t from operating the incumbent product. The stochastic process x_t is a strong Markov càdlàg (right continuous with left limits) semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ adopted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Under the pricing measure P, x_t follows the Geometric Brownian motion as governed by

$$dx_t = \mu x_t \, dt + \sigma x_t \, dZ_t. \tag{2.1}$$

Here, σ is the constant volatility and Z_t is the standard Brownian motion. By following the usual no-bubble condition, the constant drift rate μ is taken to be less than r (Dixit and Pindyck, 1994).

Both firms are assumed to have the option to operate their R&D effort in the innovation of a new product by investing an instantaneous sunk cost[1]. In this paper, the decision to invest in a R&D project is assumed to be irreversible and the corresponding fixed sunk cost for Firm j is K_j, where $j = i, e$. The two sunk costs, K_i and K_e, are different in general[2]. Both firms strive for the discovery of the same new product. Also, we assume that the new innovative product can be launched without any further cost.

In our model, we assume that the new product is an enhanced version of the incumbent product and it serves a similar set of target customers so that the stochastic revenue flow rates generated by these two products take the same form of the stochastic fundamental, except with different proportional multipliers. The drive for enhanced new products that serve an almost identical group of customers has been quite common in the consumer electronics industry. In general, the combined market size of the incumbent and new products is larger than the original market size of the incumbent product alone; otherwise, there will be no incentive for launching the R&D efforts. To model the output spillover effects, the multipliers (appropriability factors) in the revenue flow rates generated by the two products after the delivery of the new product are chosen so as to reflect the appropriability of the revenue flow rates from the two products to the competing firms. Besides investment cost asymmetry, our model also assumes asymmetry between the two firms in their stochastic revenue flow rates generated from operating the new product. When the incumbent firm (Firm i) wins the R&D race (discovery of the new product and subsequent launching into the product market), the total revenue flow rate received by Firm i from operating the two products is $(1 + \pi_i^+)x_t$, where $\pi_i^+ > 0$. Here, $\pi_i^+ x_t$ represents the additional revenue flow rate from operating the two products for Firm i. On the other hand, suppose the entrant firm wins the R&D race, the revenue flow rate received by Firm e is $\pi_e x_t$, where $\pi_e > 0$. Now the product market is operated in duopoly with two products and this causes a drop in the revenue flow rate of Firm i from x_t to $(1 - \pi_i^-)x_t$, where $0 \leq \pi_i^- < 1$. Here, $\pi_i^- x_t$ represents the drop in the revenue flow rate for Firm i due to loss of monopoly in the product market. It is reasonable to set $\pi_e > \pi_i^-$ so that the combined market size of the two products is larger than the incumbent product alone.

In this duopoly R&D race between the incumbent and entrant firms, the two firms face both technological and market uncertainty. The success of innovation by an active firm entering into R&D is assumed to occur according to a Poisson distribution with constant hazard rate. The two Poisson processes are assumed to be mutually independent and independent of the revenue flow process x_t. The earlier entry by a firm into the research phase

[1]Though continual R&D expenses would normally incur during the research phase, the assumption of an instantaneous sunk cost [also adopted by Weeds (2002)] leads to less clumsy calculations in our later analysis of strategic equilibria. Using a R&D flow cost that stops at the time of innovation would add a new term (with dependence on the hazard rates) to the total R&D costs. The analytic procedures for analyzing the strategic equilibriums in both cases of R&D costs assumption are essentially similar.

[2]The assumption of different sunk costs for the two firms does not add complexity into the model. When the respective value functions and revenue flows of each firm are normalized by the corresponding sunk cost, we may set the two sunk costs to assume unit value. The sunk costs are essentially taken to be the numeraires of the respective firm value functions.

may not guarantee the firm to be the eventual winner of the R&D race. The modeling of the arrival of innovation by a simple Poisson process exhibits the undesirable memoryless property. Also, it does not take into account that the firm's knowledge accumulation and continual R&D expenses would affect the hazard rate of arrival of R&D success. Like most of the earlier works on R&D races, we choose to assume a simple Poisson process for the arrival of R&D success for achieving analytic tractability in our analysis.

A firm may enter into the R&D race as the follower (either as the optimal choice of its own or being preempted) provided that discovery of the innovative product has not occurred. There also exists the possibility that the two firms enter simultaneously into the R&D race. Next, we show how to introduce input spillover effects into our model of R&D race. Let h_j denote the constant hazard rate of the Poisson arrival of discovery of Firm j, $j = i, e$, when only one firm is operating in the research phase. When both firms have launched the research efforts into the discovery of the innovative product, our model assumes that the input spillover effects lead to a change in the hazard rate of the Poisson arrival of discovery from h_j to $\hat{h}_j$, $j = i, e$. Note that $\hat{h}_j$ can be lower or higher than h_j, corresponding to positive or negative spillover, respectively.

Compared to the real options game model of R&D race of Weeds (2002) with symmetry in costs and hazard rates, we introduce asymmetries in sunk costs, status of the firms as incumbent and entrant, hazard rates and spillover effects. Following similar assumptions made by Weeds (2002), the initial value of the stochastic revenue flow rate process x_0 is sufficiently low so that an immediate entry leads to negative expected return, thus none of the firms has entered into R&D. Each of the two firms is supposed to make one optimal stopping criterion: the optimal time at which it pays the R&D cost. The optimal stopping times are assumed to be $\mathcal{F}_t$-measurable. Based on the strong Markovian nature of the revenue flow rate process, one may take the range of the process as the state space rather than time itself. Here, the determination of the $\mathcal{F}_t$-measurable stopping time amounts to the determination of the trigger threshold. In other words, the optimal decisions of entry into R&D race are Markov (stationary) trigger strategies, where the strategic moves are time invariant and they are dependent on the current state of x_t and the action taken by the competing firm (specifically, whether the rival has initiated R&D at an earlier time or otherwise). Once a firm has launched the sunk cost of R&D, the research into the innovative discovery continues for all times until the real options game ends with the discovery of the new innovative product by one of the two firms.

3 Value functions and investment thresholds

In this section, we derive the value functions and the trigger threshold values of optimal entry into R&D race of Firm i (incumbent) and Firm e (entrant) under various scenarios. The standard Bellman's optimality approach of solving the associated optimal stopping problems is adopted. As the first step, we find the value functions when the two firms have adopted their respective role as either the leader or follower. Once the leader value function and follower value function are known, we can examine the preemption strategies by analyzing the behavior of the preemption function (defined as the difference of the leader value function and follower value function). We then consider the preemptive leader value function of each firm. Suppose none of the two firms have entered into the R&D phase and the competition for entry is keen, Weeds (2002) shows that one of the two firms may choose to preempt strategically its rival at a threshold level that is below its own optimal leader threshold.

Lastly, we consider the value functions and optimal thresholds under simultaneous entry where the firm would adopt optimal follower entry immediately once the rival firm chooses strategically to invest into R&D. As in most dynamic programming problems, we adopt the backward induction procedure where the value functions are solved backwards in time.

3.1 Value functions

To implement the backward induction procedure, we derive the value functions when both firms have initiated their R&D efforts by paying the corresponding sunk cost of investment but none of them has succeeded in the discovery of the product. Since the two firms are asymmetric in the revenue flows and investment costs, it is necessary to determine the value function of each firm separately.

Let t be the current time and the stochastic state variable x_t assumes the value x. Let $R_i(x)$ and $R_e(x)$ denote the expected revenue value function of Firm i and Firm e, respectively, when both firms are active in R&D but the discovery of the product has not been made by either firm. The value functions are stationary with no dependence on t since perpetuality of the real options game model and time invariant strategic moves are assumed. The arrival of the success of discovery by either firm is assumed to be a Poisson event with constant hazard rate. These two Poisson arrivals of discovery are assumed to be independent of each other and also independent of the stochastic fundamental x_t.

Determination of $R_e(x)$
The value function $R_e(x)$ is computed by finding the expected value of the revenue flow received by Firm e when it is the final winner of the R&D race, which is found to be

$$\begin{aligned} R_e(x) &= \int_t^\infty \int_t^v \hat{h}_i e^{-\hat{h}_i(v-t)} \hat{h}_e e^{-\hat{h}_e(u-t)} \mathbb{E}_t \left[\int_u^\infty e^{-r(s-t)} \pi_e x_s \, ds | x_t = x \right] dudv \\ &= \frac{\hat{h}_e \pi_e x}{(r-\mu)(r-\mu+\hat{h}_i+\hat{h}_e)}. \end{aligned} \tag{3.1}$$

The hazard rate $\hat{h}_e$ appears in the numerator since Firm e succeeds with probability $\hat{h}_e\, dt$ in the innovative discovery over $(t, t+dt]$ and receives the perpetual revenue.

Determination of $R_i(x)$
In a similar manner, we compute $R_i(x)$ by finding the net gain in the expected value of the revenue flow received by Firm i, noting that it may win or lose in the R&D race. Recall that the gain in revenue flow rate is $\pi_i^+ x_s$ when Firm i wins while the corresponding loss is $\pi_i^- x_s$ when it loses. The value function $R_i(x)$ is easily deduced to be

$$R_i(x) = \frac{(\hat{h}_i \pi_i^+ - \hat{h}_e \pi_i^-)x}{(r-\mu)(r-\mu+\hat{h}_i+\hat{h}_e)}. \tag{3.2}$$

3.2 Follower value functions

Suppose the rival firm has entered into the R&D phase as the leader, we would like to determine the corresponding follower value function. The follower value function consists of two parts, depending on whether the follower firm is still waiting for its optimal entry into R&D or it has committed the R&D cost. Suppose Firm j, $j = i, e$, serves as the follower, it enters into the R&D race optimally at the optimal threshold x^*_{jf} at the optimal stopping time t^*_{jf}. The follower value function of Firm j takes the form

$$F_j(x) = \begin{cases} F_j^{(1)}(x), & x < x^*_{jf} \\ R_j(x) - K_j, & x \geq x^*_{jf} \end{cases}, \quad j = i, e. \tag{3.3}$$

Here, $F_j^{(1)}(x)$ is the option value of waiting as follower for Firm j prior to its optimal entry. As in typical optimal stopping models, the continuation value function $F_j^{(1)}(x)$ observes the value matching condition and smooth pasting condition at x^*_{jf}.

Determination of $F_e^{(1)}(x)$ and x^*_{ef}

Based on the strong Markov property and time homogeneity of the underlying geometric Brownian motion x_t, we obtain

$$\begin{aligned} F_e^{(1)}(x) &= \sup_{t_{ef} \geq t} \mathbb{E}_t \left[e^{-(r+h_i)(t_{ef}-t)} [R_e(x_{t_{ef}}) - K_e] \right] \\ &= \mathbb{E}_t \left[e^{-(r+h_i)(t^*_{ef}-t)} \right] \left[R_e(x^*_{ef}) - K_e \right], \quad x < x^*_{ef}. \end{aligned}$$

It can be shown that (Dixit and Pindyck, 1994)

$$\mathbb{E}_t \left[e^{-(r+h_i)(t^*_{ef}-t)} \right] = \left(\frac{x}{x^*_{ef}} \right)^{\beta_i},$$

where β_i is the positive root of the quadratic equation: $\dfrac{\sigma^2}{2}\beta^2 + \left(\mu - \dfrac{\sigma^2}{2} \right) \beta - (r + h_i) = 0$. The optimal threshold x^*_{ef} can be determined by invoking the smooth pasting condition. We obtain

$$x^*_{ef} = \frac{\beta_i}{\beta_i - 1} \frac{K_e}{\hat{h}_e \pi_e} (r - \mu)(r - \mu + \hat{h}_i + \hat{h}_e), \tag{3.4a}$$

and $F_e^{(1)}(x)$ can be simplified to become

$$F_e^{(1)}(x) = \left(\frac{x}{x^*_{ef}} \right)^{\beta_i} \frac{K_e}{\beta_i - 1}, \qquad x < x^*_{ef}. \tag{3.4b}$$

Determination of $F_i^{(1)}(x)$ and x^*_{if}

When Firm e has initiated R&D effort as leader, the revenue flow rate received by Firm i will be undermined by the amount $\pi_i^- x_s$, where $s > \tau_e$, if Firm e is the eventual winner. First, assuming that Firm i never enters into the R&D race, the expected loss of revenue flow received by Firm i conditional on discovery delivered by the rival firm (Firm e) is given by

$$\mathbb{E}_t \left[\int_t^\infty e^{-(h_e+r)(u-t)} \frac{h_e \pi_i^- x_u}{r - \mu} \, du \right] = \frac{h_e \pi_i^- x}{(r - \mu)(r - \mu + h_e)}.$$

We are concerned with the expected loss of revenue faced by Firm i from time t to t^*_{if}, which is then given by

$$\frac{h_e \pi_i^-}{(r - \mu)(r - \mu + h_e)} \left[x - \left(\frac{x}{x^*_{if}} \right)^{\beta_e} x^*_{if} \right],$$

where β_e is the positive root of the quadratic equation: $\dfrac{\sigma^2}{2}\beta^2 + \left(\mu - \dfrac{\sigma^2}{2}\right)\beta - (r + h_e) = 0$. Combining the option value of waiting prior to entry at the optimal threshold x^*_{if} as follower and the expected loss of revenue due to potential R&D success of the rival firm, the follower value function of Firm i prior to entry into R&D investment is given by

$$
\begin{aligned}
F_i^{(1)}(x) &= \sup_{t_{if} \geq t} \mathbb{E}_t \left[- \int_t^{t_{if}} e^{-(r+h_e)(u-t)} \frac{h_e \pi_i^- x_u}{r - \mu} \, du + e^{-(r+h_e)(t_{if}-t)} [R_i(x_{t_{if}}) - K_i] \right] \\
&= \left(\frac{x}{x^*_{if}} \right)^{\beta_e} \frac{K_i}{\beta_e - 1} - \frac{h_e \pi_i^- x}{(r-\mu)(r-\mu+h_e)}, \qquad x \leq x^*_{if}.
\end{aligned} \tag{3.5}
$$

The first term in Eq. (3.5) represents the expected value at time t of the total revenue flow to Firm i when Firm e has not made the discovery of the innovative product by time t^*_{if}, where $t^*_{if} = \inf\{u \geq t : x_u \geq x^*_{if}\}$, The negative to the expected loss faced by Firm i when Firm e has made the discovery before time t^*_{if} (see the second term) is added as part of contribution to the follower value function $F_i^{(1)}(x)$. The optimal threshold x^*_{if} is determined by applying the smooth pasting condition at x^*_{if}, which is found to be

$$
x^*_{if} = \frac{\beta_e K_i}{\beta_e - 1} \frac{1}{\frac{h_e \pi_i^-}{(r-\mu)(r-\mu+h_e)} + \frac{\hat{h}_i \pi_i^+ - \hat{h}_e \pi_i^-}{(r-\mu)(r-\mu+\hat{h}_i+\hat{h}_e)}}. \tag{3.6}
$$

3.3 Leader value functions

We would like to determine the leader value function of each firm where the firm adopts the role as the leader. The derivation of the leader value functions is complicated by the potential entry of the rival firm as the follower at a later time. Once the entry of the rival firm as follower occurs, both firms have initiated R&D and the true R&D race commences. In this case, the value function of Firm j becomes $R_j(x) - K_j$, $j = i, e$. Therefore, the leader value function consists of 3 segments: (i) $x < x^*_{jl}$, (ii) $x^*_{jl} \leq x < x^*_{j'f}$, (iii) $x \geq x^*_{j'f}$, where x^*_{jl} is the optimal leader threshold of Firm j, and $x^*_{j'f}$ is the optimal follower threshold of Firm j'. Note that $j' = e$ when $j = i$ and $j' = i$ when $j = e$. Here, we derive the leader value function under the assumption that $x^*_{jl} < x^*_{j'f}$. The scenario where $x^*_{jl} \geq x^*_{j'f}$ indicates that Firm j has relatively lower first mover advantage when compared to its rival. Under this scenario, it will be shown in the next section that the optimal strategy followed by Firm j is either choosing entry as follower or simultaneous entry with the rival. That is, Firm j will not choose to enter optimally as the leader. In other words, when $x^*_{jl} \geq x^*_{j'f}$, the leader value function of Firm j is not meaningfully defined. The analytic formulas for the leader value functions of both firms are summarized in Proposition 1. The derivation of these results follows a similar procedure as that of the follower value functions.

Proposition 1 *We write $L_j(x)$ as the leader value function of Firm j, which consists of 3 separate segments:*

$$
L_j(x) = \begin{cases} L_j^{(1)}(x), & x < x^*_{jl} \\ L_j^{(2)}(x), & x^*_{jl} \leq x < x^*_{j'f} \\ R_j(x) - K_j, & x \geq x^*_{j'f} \end{cases}, \quad j = i, e. \tag{3.7}
$$

The leader value function of firm j, $j = i, e$, after its optimal entry but before optimal follower entry of its rival is given by

$$L_j^{(2)}(x) = \left(\frac{x}{x^*_{j'f}}\right)^{\beta_j} d_j x^*_{j'f} + \frac{h_j \pi_j x}{(r-\mu)(r-\mu+h_j)} - K_j,\ x^*_{jl} \le x < x^*_{j'f}, \tag{3.8a}$$

where

$$d_i = \frac{\hat{h}_i \pi_i^+ - \hat{h}_e \pi_i^-}{(r-\mu)(r-\mu+\hat{h}_i+\hat{h}_e)} - \frac{h_i \pi_i^+}{(r-\mu)(r-\mu+h_i)}, \tag{3.8b}$$

$$d_e = \frac{\hat{h}_e \pi_e}{(r-\mu)(r-\mu+\hat{h}_i+\hat{h}_e)} - \frac{h_e \pi_e}{(r-\mu)(r-\mu+h_e)}. \tag{3.8c}$$

The value matching condition (but not the smooth pasting condition) is observed at $x = x^*_{if}$. The option value of waiting $L_j^{(1)}(x)$ prior to the optimal entry at x^*_{jl} is deduced to be

$$L_j^{(1)}(x) = \left(\frac{x}{x^*_{jl}}\right)^{\beta_0} L_j^{(2)}(x^*_{jl}), \qquad x < x^*_{jl}, \tag{3.9}$$

where β_0 is the positive root of the quadratic equation: $\dfrac{\sigma^2}{2}\beta^2 + \left(\mu - \dfrac{\sigma^2}{2}\right)\beta - r = 0$. The entrant's optimal entry threshold x^*_{el} is given by the root of

$$f_j(z) = \frac{d_j(\beta_j - \beta_0)}{(x^*_{j'f})^{\beta_j - 1}} z^{\beta_j} - \frac{(\beta_0 - 1) h_j \pi_j}{(r-\mu)(r-\mu+h_j)} z + \beta_0 K_j = 0. \tag{3.10}$$

Remarks

1. The first term in $L_e^{(2)}(x)$ [see Eq. (3.8a)] represents the expected discount factor $\mathbb{E}_t[e^{-(r+h_e)(t^*_{if}-t)}] = \left(\frac{x}{x^*_{if}}\right)^{\beta_e}$ applied over the time period (t, t^*_{if}), which is then multiplied by the change in value $d_e x^*_{if}$ arising from the potential entry of the incumbent as follower at x^*_{if}. The parameter d_e can be interpreted as Firm e's externality factor that is directly related to the input spillover effect. When $d_e > 0$, the entrant benefits from the follower entry of the incumbent. More precisely, we have

$$d_e > 0 \Leftrightarrow \hat{h}_e - h_e > \frac{h_e \hat{h}_i}{r-\mu}; \tag{3.11a}$$

   so positivity of d_e implies that an increase of the entrant's hazard rate of arrival of discovery arising from the R&D spillover effect outweighs the potential loss in value when the R&D race is lost to the incumbent firm which has entered as the follower.

2. To achieve positivity of Firm i's externality factor d_i, one requires an increase of the incumbent's hazard rate due to R&D spillover effect of sufficient amount as indicated by the following relation:

$$d_i > 0 \Leftrightarrow \hat{h}_i - h_i > \frac{h_i \pi_i^+ + (r-\mu+h_i)\pi_i^-}{(r-\mu)\pi_i^+} \hat{h}_e. \tag{3.11b}$$

3. The optimal entry threshold x_{em}^* of the entrant firm under no competition (monopoly state) is given by

$$x_{em}^* = \frac{\beta_0}{\beta_0 - 1}\frac{K_e(r-\mu)(r-\mu+h_e)}{h_e\pi_e}.$$

By substituting $z = x_{em}^*$ in $f_e(z)$ [see eq. (3.10)], we have

$$f_e(x_{em}^*) = \frac{d_e(\beta_e - \beta_0)}{(x_{if}^*)^{\beta_e - 1}}(x_{em}^*)^{\beta_e}.$$

One can deduce easily that $x_{el}^* > x_{em}^*$ if and only if $d_e > 0$. That is, the entrant chooses to enter at a lower threshold when its externality factor is negative (strong first mover advantage).

3.4 Preemption strategies

It may occur that Firm j is strategically advantageous to preempt its rival by choosing entry as the leader even at level x that is below its leader optimal threshold x_{jl}^* if the rival chooses to enter as a leader at some threshold lower than x_{jl}^*. Accordingly, we define the preemptive leader value function $L_j^{(p)}(x)$ is taken to be the same as $L_j^{(2)}(x)$ while the interval of definition is extended from $[x_{jl}^*, x_{j'f}^*)$ to $[0, x_{j'f}^*)$. Obviously, preemption strategy is adopted only when the firm's leader value is indifferent to or higher than its follower value. To characterize preemption strategies, consider the behavior of the preemption function $\phi_j(x)$ as defined by

$$\phi_j(x) = L_j^{(p)}(x) - F_j(x), \quad j = i, e, \quad 0 \le x < x_{j'f}^*. \tag{3.12}$$

For $d_j \ge 0$, by observing $F_j(x) = F_j^{(1)}(x) \ge R_j(x) - K_j$ for $x < x_{jf}^*$ and $F_j(x) = R_j(x) - K_j$ for $x \ge x_{jf}^*$, we have

$$\phi_j(x) \le L_j^{(2)}(x) - [R_j(x) - K_j] = d_j x \left[\left(\frac{x}{x_{j'f}^*}\right)^{\beta_j - 1} - 1\right] < 0, \quad 0 \le x < x_{j'f}^*.$$

In this case, $\phi_j(x)$ has no root for $0 \le x < x_{j'f}^*$. On the other hand, when $d_j < 0$, we observe

$$\frac{d^2}{dx^2}\phi_j(x) = \beta_j(\beta_j - 1)d_j x_{j'f}^* \frac{x^{\beta_j - 2}}{(x_{j'f}^*)^{\beta_j}} - \beta_{j'} K_j \frac{x^{\beta_{j'} - 2}}{(x_{jf}^*)^{\beta_{j'}}} < 0.$$

Therefore, $\phi_j(x)$ is concave in x. In addition, $\phi_j(0) < 0$ and $\phi_j'(0) > 0$, so either $\phi_j'(x) > 0$ for $x < x_{j'f}^*$; or else there exists $x_0 \in [0, x_{j'f}^*)$ such that $\phi_j'(x) > 0$ for $x < x_0$ and $\phi_j'(x) < 0$ for $x > x_0$. It is straightforward to show that $\phi_j(x)$ has either no root, one root or two roots within $[0, x_{j'f}^*)$. In order that Firm j chooses to preempt its rival at some threshold z, a necessary condition (though not sufficient) is given by $\phi_j(z) > 0$. We consider these 3 separate cases as follows:

(i) No root or one root at $\hat{x}_j$ with $\phi_j'(\hat{x}_j) = 0$
One deduces that

$$L_j^{(p)}(x) \le F_j(x) \text{ for } x \in [0, x_{j'f}^*),$$

so Firm j never chooses to preempt.

(ii) One root at $\underline{x}_{jp}$, where $\phi_j'(\underline{x}_{jp}) \neq 0$ and $0 < \underline{x}_{jp} < x^*_{j'f}$
We have

$$L_j^{(p)}(x) > F_j(x) \text{ for } x \in (\underline{x}_{jp}, x^*_{j'f}).$$

In this case, it may be possible that Firm j chooses to preempt its rival at a lower threshold if the rival chooses to enter at a threshold within $(\underline{x}_{jp}, x^*_{j'f})$ as a leader.

(iii) Two roots at $\underline{x}_{jp}$ and $\overline{x}_{jp}$, where $0 < \underline{x}_{jp} < \overline{x}_{jp} < x^*_{j'f}$
Similarly, it may be possible that Firm j chooses preemption as an optimal strategy if the rival chooses to enter at a threshold within $(\underline{x}_{jp}, \overline{x}_{jp})$ as a leader.

In summary, preemption strategy is never adopted by Firm j if $L_j^{(p)}(x) < F_j(x)$, $0 \leq x < x^*_{j'f}$. For example, when $d_j \geq 0$, one can show that

$$L_j^{(p)}(x) - F_j(x) = d_j x \left[\left(\frac{x}{x^*_{j'f}} \right)^{\beta_i - 1} - 1 \right] < 0, \quad j = i, e, \ 0 \leq x < x^*_{j'f}.$$

Therefore, non-negativity of d_j, $j = i, e$, is seen to be a sufficient condition for Firm j not to adopt preemption strategy at any level x. This result can be explained using economic intuition as follows. When the input spillover effect for Firm j is sufficiently strong (as dictated by $d_j \geq 0$), the second mover advantage prevails for Firm j so it never chooses to adopt preemption strategy.

3.5 Simultaneous entry of both firms

Suppose the input spillover effects are sufficiently strong so that the second mover advantage prevails for both firms, none of the two firms chooses to enter as leader in the R&D race. In this case, the two firms choose to invest into R&D simultaneously as their joint optimal strategies. As the game is non-cooperative, simultaneous entry commences when one firm (Firm j) chooses optimally to invest at level x while the rival firm (Firm j') finds that it is also optimal to invest at the same level. We would like to determine the optimal simultaneous entry threshold x^*_{js} of Firm j, $j = i, e$, given that the conditions for optimal simultaneous entry are met (see Sec. 4.1 for the detailed discussion of these conditions).

Suppose Firm j invests optimally at level z while optimal entry is followed immediately by Firm j', $j' \neq j$, then Firm j's value function at z is given by $R_j(z) - K_j$. The option value of waiting at $x < z$ prior to its optimal simultaneous entry is given by $[R_j(z) - K_j]\left(\frac{x}{z}\right)^{\beta_0}$. Note that the simultaneous entry threshold cannot be lower than $x^*_{j'f}$; otherwise, simultaneous equilibrium cannot be substained since Firm j' chooses not to follow immediately. On the other hand, the simultaneous entry threshold is chosen such that the option value $[R_j(z) - K_j]\left(\frac{x}{z}\right)^{\beta_0}$ is maximized. One can show easily that the choice of the threshold $\frac{\beta_0}{\beta_0 - 1}\frac{K_j}{b_j}$ maximizes $[R_j(z) - K_j]\left(\frac{x}{z}\right)^{\beta_0}$ among $z \in [0, \infty)$. Mathematically, the simultaneous entry threshold x^*_{js} as preferred by Firm j is determined by

$$x^*_{js} = \arg\max_{z \in [x^*_{j'f}, \infty)} [R_j(z) - K_j] \left(\frac{x}{z} \right)^{\beta_0} = \max \left\{ \frac{\beta_0}{\beta_0 - 1} \frac{K_j}{b_j}, x^*_{j'f} \right\}, \tag{3.13}$$

where

$$b_i = \frac{\hat{h}_i \pi_i^+ - \hat{h}_e \pi_i^-}{(r - \mu)(r - \mu + \hat{h}_i + \hat{h}_e)}, \quad b_e = \frac{\hat{h}_e \pi_e}{(r - \mu)(r - \mu + \hat{h}_i + \hat{h}_e)}.$$

One would expect that $x^*_{js} \geq x^*_{j'f}$. These properties are consistent with the conditions for the occurrence of optimal simultaneous entry. Note that $\beta_j > \beta_0$ so that $\dfrac{\beta_0}{\beta_0 - 1} > \dfrac{\beta_j}{\beta_j - 1}$, $j = i, e$. We then have

$$x^*_{js} = \max\left\{\frac{\beta_0}{\beta_0 - 1}\frac{K_j}{b_j}, x^*_{j'f}\right\} \geq \frac{\beta_0}{\beta_0 - 1}\frac{K_j}{b_j} \geq \frac{\beta_{j'}}{\beta_{j'} - 1}\frac{K_j}{b_j} = x^*_{jf}, \quad j = i, e,\ j' \neq j.$$

Together with $x^*_{js} \geq x^*_{j'f}$, we then have

$$x^*_{js} > \max\{x^*_{jf}, x^*_{j'f}\}, \quad j = i, e. \tag{3.14}$$

The corresponding value function of Firm j, $j = i, e$, that follows this joint optimal strategies is seen to be

$$J_j(x) = \begin{cases} \left[R_j(x^*_{js}) - K_j\right]\left(\frac{x}{x^*_{js}}\right)^{\beta_0}, & x < x^*_{js}, \\ R_j(x) - K_j, & x \geq x^*_{js}. \end{cases} \tag{3.15}$$

As a final remark, the two firms actually do not cooperate to enter into R&D race at the same threshold level. Rather, optimal simultaneous entry occurs when one firm enters optimally while the rival firm responds optimally to adopt an immediate entry at the same threshold. When simultaneous equilibrium prevails, both firms jointly invest at the simultaneous entry threshold. Note that the smaller value among x^*_{is} and x^*_{es} is taken due to non-cooperation between the two firms.

4 Analysis of strategic equilibria

Recall that there are 3 types of equilibria of the firms' strategies. The first type is the preemptive equilibrium where one firm chooses to preempt the rival at a threshold lower than its optimal leader threshold due to the preemptive threat of its rival. The other type is the sequential equilibrium where one firm dominates its rival in the sense that it chooses its optimal leader's entry strategy without preemptive threat of its rival. The last type is the simultaneous equilibrium where the two firms optimally choose to enter at the same threshold, one firm's optimal entry is followed immediately by the optimal entry of its rival.

In Sec. 4.1, we consider the categorization of strategic equilibria that is based on the relative magnitudes of the leader and follower thresholds of the two firms. In our strategic R&D race model, we assume that the input spillovers have impact on the follower's R&D hazard rate of discovery but not on follower's R&D cost. However, the existence of upfront R&D cost asymmetry between the two firms does have strong influence on the strategic games. In Sec. 4.2, we characterize the various types of equilibria of the firms' strategies with regard to the upfront R&D costs. One may visualize that the first mover advantage may be lost when the hazard rate of discovery is relatively low. In Sec. 4.3, we examine the impact of hazard rates on the strategic equilibria.

4.1 Optimal entry thresholds and strategic games

We consider the following two mutually exclusive cases (i) at least one firm has dominant first mover advantage over its rival, so simultaneous equilibrium is precluded. This results in a leader-follower game and it may give rise to either preemptive or sequential equilibrium;

(ii) none of the two firms has dominant first mover advantage over its rival. Case (i) occurs when $x^*_{il} < x^*_{ef}$ or $x^*_{el} < x^*_{if}$ or both, while case (ii) occurs when $x^*_{il} \geq x^*_{ef}$ and $x^*_{el} \geq x^*_{if}$.

Leader-follower games resulting in either preemptive or sequential equilibrium When the leader threshold of one firm (say, Firm j) is lower than the rival firm's (Firm j') follower threshold, where $x^*_{jl} < x^*_{j'f}$, it becomes certain that Firm j adopts its optimal leader entry at x^*_{jl} unless preemption strategy has been adopted earlier by itself or the rival firm at some lower threshold. If Firm j' does not choose to be the preemptive leader, then it would delay its follower entry until the higher follower threshold $x^*_{j'f}$ is reached at some later time. The possibility of simultaneous entry where Firm j' enters as follower immediately after leader entry by Firm j is thus precluded. When both firms have the first mover advantage, where $x^*_{il} < x^*_{ef}$ and $x^*_{el} < x^*_{if}$, a similar argument shows that simultaneous equilibrium arising from the action of either firm is precluded. In conclusion, when $x^*_{il} < x^*_{ef}$ or $x^*_{el} < x^*_{if}$ or both, then (i) either one of the two firms enters as the preemptive leader (preemptive equilibrium) or (ii) the two firms enter sequentially as leader and follower (sequential equilibrium) at their respective optimal entry thresholds.

We consider the following two separate cases: (1) only one firm has the first mover advantage (its optimal leader threshold is lower than its rival's optimal follower threshold), (2) both firms have the first mover advantage, that is, $x^*_{il} < x^*_{ef}$ and $x^*_{el} < x^*_{if}$.

1. Only one firm exhibits the first mover advantage: $x^*_{jl} < x^*_{j'f}$, where Firm j can be either Firm i or Firm e

   We examine whether Firm j' has preemption incentive by considering the number and location of roots of $\phi_{j'}(x)$. If Firm j' is shown to have no preemption incentive, then sequential equilibrium is resulted with Firm j as the leader. Otherwise, we compare the preemption incentive of both firms and the one with the stronger preemption incentive becomes the preemptive leader. The loser firm chooses to enter at its optimal follower threshold under preemptive equilibrium.

   (i) When $\phi_{j'}(x)$ has no root or only one root at $\hat{x}_{j'}$ with $\phi'_{j'}(\hat{x}_{j'}) = 0$, then $\phi_{j'}(x) \leq 0$. In this case, Firm j' never chooses to preempt so sequential equilibrium is resulted. That is, Firm j enters optimally as the leader at x^*_{jl} while Firm j' enters later at its optimal follower threshold $x^*_{j'f}$.

   (ii) When $\phi_{j'}(x)$ has one root at $\underline{x}_{j'p}$ and $\phi'_{j'}(\underline{x}_{j'p}) \neq 0$, we compare the relative values of x^*_{jl} and $\underline{x}_{j'p}$. Sequential equilibrium is resulted if $x^*_{jl} \leq \underline{x}_{j'p}$ since Firm j has chosen to enter optimally at x^*_{jl} before Firm j' has incentive to preempt at the higher threshold $\underline{x}_{j'p}$. On the other hand, when $\underline{x}_{j'p} < x^*_{jl}$, preemptive equilibrium is resulted as sequential equilibrium is precluded since Firm j' has incentive to preempt Firm j at some threshold lower than x^*_{jl}. Subsequently, there are four possible forms of preemptive competition, depending on whether Firm j has incentive to preempt further with respect to the relative position of $\underline{x}_{jp}$. The analysis requires the examination of the number of roots of $\phi_j(x)$ and the relative position of x^*_{jl} with respect to these roots.

      (a) Suppose $\phi_j(x)$ has no root or only one root, then Firm j' chooses to epsilon-preempt Firm j by adopting entry at x^*_{jl}, provided that $x^*_{j'f} < x^*_{jf}$.

      (b) Suppose $\phi_j(x)$ has two roots $\underline{x}_{jp}$ and $\overline{x}_{jp}$, then there are 3 possible outcomes:

         · If $\overline{x}_{jp} < x^*_{jl}$, then Firm j' chooses to epsilon-preempt Firm j at x^*_{jl}, as the preemptive leader.

- If $\underline{x}_{j'p} < \underline{x}_{jp} < x^*_{jl} \leq \overline{x}_{jp}$, then Firm j' chooses to epsilon-preempt Firm j at the threshold $\underline{x}_{jp}$.
- If $\underline{x}_{jp} < \underline{x}_{j'p} < x^*_{jl} \leq \overline{x}_{jp}$, then Firm j chooses to epsilon-preempt Firm j' at the threshold $\underline{x}_{j'p}$.

In the last two cases in which Firm j preempts another firm (Firm j') at $\underline{x}_{j'p}$. Firm j' is indifferent between being the leader and follower since $\phi_{j'}(\underline{x}_{j'p}) = 0$. There is no guarantee that Firm j can act as the leader since Firm j' may enter at the same time. To avoid such problem, Firm j may epsilon-preempt its rival at $x_{j'p}$ by acting right before the entry of the rival. However, such preemptive strategy may not be a feasible strategy since optimality of this stopping rule depends on whether the state variable x_t hits the threshold $\underline{x}_{j'p}$ in the next time increment $t + dt$. In order to resolve the difficulty, we follow Thijseen (2012) and impose the following assumption:

Assumption: Provided that $L_j(\underline{x}_{j'p}) > F_j(\underline{x}_{j'p})$, Firm j can act as the leader with probability 1 if it preempts Firm j' at the threshold $\underline{x}_{j'p}$.

Under this assumption, Firm j can wait and preempt its rival until the state variable x_t reaches $\underline{x}_{j'p}$ without facing the risk of being preempted by its rival. According to Thijseen (2012), such preemptive strategy of acting at the stopping time $\tau_j = \inf\{\tau \geq 0 : x_\tau = \underline{x}_{j'p}\}$ is a feasible strategy.

In summary, if Firm j has no incentive to preempt at a lower threshold, then Frim j' will be the preemptive leader and preempt Firm j at x^*_{jl}. On the other hand, if Firm j has an incentive to preempt at a lower threshold, we continue to examine whether Firm j' has an incentive to preempt at a lower threshold and so on. Continuing on this iterative process, one can find that the firm with the lower preemptive threshold will be the preemptive leader and chooses to preempt the rival at rival's preemptive threshold since the rival has no incentive to preempt the competing firm at any threshold lower than its own preemptive threshold. The comprehensive discussion of these various forms of preemptive equilibrium can be found in Leung (2011).

(iii) When $\phi_{j'}(x)$ has two roots at $\underline{x}_{j'p}$ and $\overline{x}_{j'p}$, where $\underline{x}_{j'p} < \overline{x}_{j'p}$, we examine the following 3 cases: $x^*_{jl} \leq \underline{x}_{j'p}$, $\underline{x}_{j'p} < x^*_{jl} < \overline{x}_{j'p}$, or $\overline{x}_{j'p} \leq x^*_{jl}$. When $x^*_{jl} \leq \underline{x}_{j'p}$, sequential equilibrium is resulted. When $\underline{x}_{j'p} < x^*_{jl} < \overline{x}_{j'p}$, we examine the relative values of $\underline{x}_{jp}$ and $\underline{x}_{j'p}$. By following the standard epsilon-preemption arguments in Fudenberg and Tirole (1985), the firm which has the lower preemption threshold is the preemptive leader. Also, the preemptive leader chooses to epsilon-preempt its rival at the rival's preemption threshold (which is higher than its own preemption threshold). Lastly, when $\overline{x}_{j'p} \leq x^*_{jl}$, since $\phi_{j'}(x) \leq 0$ when $x \in [\overline{x}_{j'p}, x^*_{jl})$, so it can be shown that it is non-optimal for Firm j' to preempt Firm j at any threshold lower than x^*_{jl}. Therefore, sequential equilibrium is resulted with Firm j entering as leader at its optimal leader threshold x^*_{jl} and Firm j' entering later as follower at its optimal follower threshold $x^*_{j'f}$.

2. Both firms exhibit the first mover advantage: $x^*_{il} < x^*_{ef}$ and $x^*_{el} < x^*_{if}$

   First, we identify the firm that has the stronger first mover advantage, or equivalently, a lower optimal leader threshold. Let m be the firm such that

$$x^*_{ml} = \min\{x^*_{il}, x^*_{el}\},$$

and Firm m' be its rival. The analysis of strategic competition would be similar to that of case (1), where Firm m plays the same role as Firm j (the only firm that has the first mover advantage). In a similar manner, sequential equilibrium is resulted when Firm m' has no preemption incentive [that is, $\phi_{m'}(x) \leq 0$]. Otherwise, we examine the various forms of preemptive equilibria, depending on the relative value of x_{ml}^* and the nature of roots of $\phi_m(x)$ and $\phi_{m'}(x)$.

Absence of the first mover advantage in both firms resulting in simultaneous equilibrium

Under the scenario where $x_{il}^* \geq x_{ef}^*$ and $x_{el}^* \geq x_{if}^*$, sequential equilibrium is precluded so that either preemptive equilibrium or simultaneous equilibrium is resulted. Now, we establish that neither firm would choose to preempt its rival at any threshold that is lower than the rival's optimal follower threshold. To show the claim, we compare the Firm j's preemptive leader value function at z, where $z < x_{j'f}^*$, with the firm's option value of waiting when firm j's entry is deferred to the higher simultaneous entry threshold level at $\min\{x_{is}^*, x_{es}^*\}$, where $\min\{x_{is}^*, x_{es}^*\} > \max\{x_{if}^*, x_{ef}^*\}$. Since $z < x_{j'f}^* < x_{jl}^*$ and $x_{j'f}^* \leq \min\{x_{is}^*, x_{es}^*\}$, so

$$
\begin{aligned}
L_j^{(p)}(z) &< L_j^{(p)}(x_{j'f}^*)\left(\frac{z}{x_{j'f}^*}\right)^{\beta_0} \\
&= \left[R_j(x_{j'f}^*) - K_j\right]\left(\frac{z}{x_{j'f}^*}\right)^{\beta_0} \\
&\leq \left[R_j(\min\{x_{is}^*, x_{es}^*\}) - K_j\right]\left(\frac{z}{\min\{x_{is}^*, x_{es}^*\}}\right)^{\beta_0}.
\end{aligned}
$$

Therefore, it is always non-optimal for Firm j to preempt its rival at any threshold level z that is lower than $x_{j'f}^*$.

As preemptive equilibrium is precluded, so simultaneous equilibrium prevails. That is, none of the two firms chooses to act as the leader, consistent with the fact that both firms have no dominant first order advantage. Now, the two firms would choose to enter simultaneously at the threshold $\min\{x_{is}^*, x_{es}^*\}$, which is always higher than $\max\{x_{if}^*, x_{ef}^*\}$ [see Eq. (3.14)].

As a remark, the asymmetric investment model in Pawlina and Kort (2006) assumes negative externalties, so $x_{jl}^* < x_{j'f}^*$ is always observed. Our model provides a richer set of feasible equilibrium strategies. In particular, we discuss the nature of equilibrium strategies under the scenario of absence of the first mover advantage (as signified by $x_{jl}^* \geq x_{j'f}^*$).

4.2 Impact of cost asymmetry on the strategic games

First, it may be instructive to recall some of the earlier results obtained by Pawlina and Kort (2006) and Kong and Kwok (2007) on the impact of investment cost asymmetry on the optimal strategies in duopolistic investment games. Under cost asymmetry and symmetry in all other model parameters, Pawlina and Kort (2006) comment that the lower-cost firm has higher first mover advantage so it tends to act either as the dominant leader or preemptive leader. Kong and Kwok (2007) consider duopolistic investment games under a more general setting, where positive externalities correspond to returns in the duopoly state exceed that in the monopoly state, and vice versa for negative externalities. It is seen that negative externalities induce keen competition between the two rival firms. Simultaneous equilibrium is resulted under positive externalities when there is no dominant first mover advantage of both firms. On the other hand, under negative externalities, preemptive equilibrium is resulted when the cost-profit ratio is low (keen competition) while sequential leader-follower equilibrium is attained when the cost-profit ratio becomes sufficiently high (competition is less keen).

Referring to our R&D model, positive and negative externalities of Firm j, $j = i, e$, are seen to correspond respectively to positivity and negativity in the sign of d_j, $j = i, e$, [see Eqs.(3.8b) and (3.8c)]. We would like to examine the impact of cost asymmetry on the strategic games under positive and negative externalities. To simplify our analysis, we set all other model parameters except the sunk costs to be the same for both firms. That is, we set

$$\pi_i^+ = \pi_e = \pi, \ \pi_i^- = 0, \ h_i = h_e = h, \text{ and } \hat{h}_i = \hat{h}_e = \hat{h}.$$

Under the above assumption of the model parameter values, d_i and d_e are seen to be equal, and we write

$$d = d_i = d_e.$$

Now, since $h_i = h_e$, we have equality of β_i and β_e. For convenience, we write $\hat{\beta} = \beta_i = \beta_e$.

The following two propositions state the pattern of strategic equilibria under positive and negative externalities, respectively, in the K_i-K_e parameter space of the sunk costs of R&D investment.

Proposition 2 *Under positive externalities, where $d > 0$, there exists $k_l \in (0, K_e)$ and $k_u \in (K_e, \infty)$ such that simultaneous equilibrium is resulted when $K_i \in [k_l, k_u]$. Otherwise, when $K_i < k_l$ (or $K_i > k_u$), Firm i (or Firm e) is the leader in the resulting sequential leader-follower equilibrium.*

The proof of Proposition 2 is relegated to Appendix A. Recall that the input spillover has to be sufficiently strong in order to induce positive externalities [see Eqs. (3.11a,b)]. The results in Proposition 2 reveal that under positive externalities, simultaneous equilibrium is resulted when cost asymmetry between the two firms is small. Otherwise, sequential leader-follower equilibrium prevails when the cost asymmetry is significant. The firm with the lower cost then serves as the leader. Interestingly, the sequential leader-follower equilibrium represents the more desirable scenario of tactic collusion (with no fear of preemption).

The next proposition characterizes the strategic equilbria under negative externalities, where $d < 0$. The pattern of strategic equilibria depends on whether $d^* < d < 0$ or $d < d^* < 0$, where the critical threshold d^* is given by

$$d^* = -\frac{h\pi(\hat{\beta} - \beta_0)}{(r-\mu)(r-\mu+h)(\hat{\beta}^2 - \beta_0)} < 0. \tag{4.1}$$

Proposition 3 *Under negative externalities, where $d < 0$, the pattern of strategic equilibria of the two firms can be characterized as follows:*

(a) $d^* < d < 0$
There exist $k_l^{(1)} \in (0, K_e)$ and $k_u^{(1)} \in (K_e, \infty)$ such that simultaneous equilibrium is resulted when $K_i \in [k_l^{(1)}, k_u^{(1)}]$. Otherwise, when $K_i < k_l^{(l)}$ (or $K_i > k_u^{(1)}$), Firm i (or Firm e) is either the preemptive or sequential leader in the resulting leader-follower equilibrium.

(b) $d < d^* < 0$
There exist $k_l^{(2)} \in (0, K_e)$ *and* $k_u^{(2)} \in (K_e, \infty)$ *such that preemptive equilibrium is resulted when* $K_i \in [k_l^{(2)}, k_u^{(2)}]$*, where the firm with the lower sunk cost serves as the preemptive leader. Otherwise, when* $K_i < k_l^{(2)}$ *(or* $K_i > k_u^{(2)}$*), Firm i (or Firm e) is the leader in the resulting sequential leader-follower equilibrium.*

The proof of Proposition 3 is relegated to Appendix B. The standard leader-follower game is resulted under negative externalities when d is sufficiently negative in value. Preemptive equilibrium emerges when cost asymmetry is not significant, where the firm with the lower sunk cost serves as the preemptive leader. Otherwise, the sequential leader-follower equilibrium is resulted when cost asymmetry becomes more significant. On the other hand, when d is negative but larger than some threshold value d^*, the pattern of strategic equilibria is somewhat similar to that under positive externalities where simultaneous equilibrium is resulted when cost asymmetry is not significant. Otherwise, either preemptive or sequential leader-follower equilibrium may result when cost asymmetry is significant.

4.3 Hazard rates and strategic equilibria

It would be instructive to examine the impact of the initial hazard rates of the R&D investment of the two firms on the pattern of strategic equilibria. A lower value of the initial hazard rate h_j of Firm j indicates a lower chance of innovative success before the entry of the rival firm, given that Firm j enters as the leader in the R&D race. In other words, the scenario represents a weaker first mover advantage of Firm j. Alternatively, we observe that d_j, $j = i, e$, [see Eqs. (3.8b) and (3.8c)] are both decreasing functions with respect to h_j. This is because the second mover advantage decreases as h_j increases in value. In other words, Firm j may enjoy positive externalities with $d_j > 0$ at a lower value of h_j but subject to negative externalities with $d_j < 0$ at some sufficiently high value of h_j.

Propositions 2 and 3 show that when cost asymmetry is small, the two firms choose optimally to invest simultaneously when $d_j > d^*$, where $d^* < 0$; otherwise, they adopt the leader-follower equilibrium. We then expect that the two firms under symmetry conditions (same set of model parameters and same initial status) tend to invest simultaneously when the common initial hazard rate h is low while they tend to preempt each other when h is sufficiently high. Taking the assumption that $\pi_i^+ = \pi_e = \pi$, $K_i = K_e = K$, $h_i = h_e = h$, $\hat{h}_i = \hat{h}_e = \hat{h}$ and $\pi_i^- = 0$, we summarize the impact of hazard rates on the pattern of strategic equilibria of the two symmetric firms in the following proposition.

Proposition 4 *Assuming that the two rival firms are symmetric, the common hazard rates* h *and* $\hat{h}$ *exhibit the following properties on the pattern of strategic equilibrium.*

(a) *Suppose* $h < r - \mu$*, preemptive equilibrium is resulted if* $\hat{h} < \hat{h}^*$ *while simultaneous equilibrium is resulted if* $\hat{h} \geq \hat{h}^*$*, where*

$$\hat{h}^* = \frac{h\hat{\beta}(\hat{\beta} - 1)(r - \mu)}{(r - \mu)(\hat{\beta}^2 - \beta_0) - h(\hat{\beta}^2 - 2\hat{\beta} + \beta_0)}.$$

(b) *There exists some threshold* h^**, where* $h^* > r - \mu$*, such that the preemptive leader-follower equilibrium is always resulted when* $h > h^*$*.*

The proof of Proposition 4 is presented in Appendix C. Given that $h < r - \mu$, Proposition 4(a) states precisely the condition on the common updated hazard rate $\hat{h}$ such that simultaneous equilibrium is resulted ($\hat{h}$ has to be above some threshold value $\hat{h}^*$). This corresponds to the scenario where the first mover advantage is low (small value of h) and the second mover advantage is substantial as dictated by the condition: $\hat{h} > \hat{h}^*$. The result is seen to be similar to that of Proposition 2 where simultaneous equilibrium is resulted when the two firms are under positive externalities and low cost asymmetry. On the other hand, when the common initial hazard rate h is above certain threshold value, Proposition 4(b) states that simultaneous equilibrium is always ruled out due to significant first mover advantage (even in the presence of strong positive spillovers). This result echoes that of part (b) in Proposition 3 when one considers the scenario where the two firms face sufficiently deep negative externalities.

5 Numerical examples

In this section, we would like to illustrate through various numerical examples that demonstrate how the hazard rate and spillover effects may impact on the strategic equilibria in the R&D races of the two firms. First, we show the plot of the value functions of the two firms under various types of strategic equilibriums. We then illustrate the dependence of the entry threshold values of the two firms on their hazard rates. We also characterize the types of strategic equilibria in the parameter space of various pairs of model parameters.

5.1 Plots of value functions

In Figures 1(a-d), we show various plots of the value functions of the two firms under different strategic equilibria. The common set of parameter values in the numerical calculations for plotting the value functions are chosen to be: $r = 0.05$, $\mu = 0.01$, $\sigma = 0.3$, $\pi_i^+ = 0.8$, $\pi_i^- = 0$, $K_i = 8$. Other model parameters, like h_i, h_e, $\hat{h}_i$, $\hat{h}_e$, π_e and K_e, assume different set of values in each figure.

In Figure 1(a), we demonstrate the behavior of various value functions under sequential equilibrium with Firm e as the leader. This scenario corresponds to case 1(i) considered in Sec. 4.1. The other parameter values used in generating the plots in the figure are taken to be: $h_i = 0.1$, $h_e = 0.2$, $\hat{h}_i = 1$, $\hat{h}_e = 0.25$, $\pi_e = 0.9$ and $K_e = 5$. Note that Firm i enjoys a strong positive spillover since the hazard rate jumps from $h_i = 0.1$ (in the monopoly state) to $\hat{h}_i = 1$ (in the duopoly state). The parameter values give $d_i > 0$, thus Firm i has no preemption incentive. This agrees with $L_i(x) < F_i(x)$ for $x < x_{ef}^*$ as shown in the figure. Indeed, our numerical calculations give $x_{il}^* = 1.85$, $x_{el}^* = 0.46$, $x_{if}^* = 0.81$, $x_{ef}^* = 2.06$, which show $x_{el}^* < x_{if}^*$. Since $x_{il}^* < x_{ef}^*$ and $x_{el}^* < x_{if}^*$, both firms possess the dominant first mover advantage. According to the discussion in Sec. 4.1, one has to check whether Firm i has incentive to preempt Firm e at a threshold lower than x_{el}^* since Firm e has lower leader threshold. Since $L_i(x) < F_i(x)$ for all values of x and $\phi_i(x)$ has no root, so preemption incentive does not exist for Firm i. As a result, it is never optimal for Firm i to preempt Firm e at any threshold below x_{el}^*. Based on the result discussed in case 1(i) in Sec. 4.1, Firm e enters into the R&D race optimally at x_{el}^* as leader while Firm i enters optimally at x_{if}^* as follower.

In Figure 1(b), we plot the value functions of the two firms under preemptive equilibrium with Firm e preempting its rival at $\underline{x}_{ip}$. This scenario corresponds to case 1(iii) in Sec.

4.1. The relevant parameter values used in generating the plots are taken to be: $h_i = 0.3$, $h_e = 0.4$, $\hat{h}_i = 0.5$, $\hat{h}_e = 0.5$, $\pi_e = 0.9$ and $K_e = 7$. The threshold values of the two firms are found to be: $x^*_{il} = 0.72$, $x^*_{el} = 0.68$, $x^*_{if} = 1.16$, $x^*_{ef} = 0.94$, $\underline{x}_{ep} = 0.36$, $\underline{x}_{ip} = 0.53$, $\overline{x}_{ip} = 0.90$. Both firms hold dominant first mover advantage since $x^*_{il} < x^*_{ef}$ and $x^*_{el} < x^*_{if}$, and they also share negative externalities as the parameter values give $d_i < 0$ and $d_e < 0$. Since Firm e has lower leader threshold, one has to check whether Firm i has incentive to preempt Firm e according to the guiding principles discussed in case 2 in Sec. 4.1. The leader function $L_i(x)$ and follower function $F_i(x)$ intersect twice at $x = \underline{x}_{ip}$ and $x = \overline{x}_{ip}$. Both firms face keen competition as $\underline{x}_{ip} < x^*_{el} < \overline{x}_{ip}$. It is necessary to consider the relative magnitude of $\underline{x}_{ep}$ and $\underline{x}_{ip}$ in order to determine the preemptive leader [see case 1(iii) in Sec. 4.1]. Since $\underline{x}_{ep} < \underline{x}_{ip}$, we conclude that preemptive equilibrium is resulted and Firm e chooses to preempt its rival at the rival's preemption threshold $\underline{x}_{ip}$.

To generate the plots in Figure 1(c), we modify the hazard rate parameter $\hat{h}_e$ from $\hat{h}_e = 0.5$ used in Figure 1(b) to the new value $\hat{h}_e = 0.9$ while keeping all other parameter values the same. This scenario corresponds to case 1(ii)(a) in Sec. 4.1. From the leader and follower value functions of the two firms shown in Figure 1(c), the corresponding threshold values of the two firms are found to be: $x^*_{il} = 0.59$, $x^*_{el} = 0.82$, $x^*_{if} = 1.60$, $x^*_{ef} = 0.72$, $\underline{x}_{ep} = 0.36$. Both firms remain to hold dominant first mover advantage since $x^*_{il} < x^*_{ef}$ and $x^*_{el} < x^*_{if}$, and they both face negative externalities. Since Firm i has lower leader threshold, one needs to check whether Firm e has incentive to preempt Firm i. Note that $\phi_e(x)$ has one root $\underline{x}_{ep}$ with $\phi'_e(\underline{x}_{ep}) \neq 0$ and $\underline{x}_{ep} < x^*_{il}$, Firm e has incentive to preempt Firm i at $x \in [\underline{x}_{ep}, x^*_{il})$. On the other hand, $L_i(x) < F_i(x)$ for all x and $\phi_i(x)$ has no root, so Firm i has no preemption incentive. According to the guiding principles discussed in case 1(ii)a in Sec. 4.1, we conclude that Firm e is the preemptive leader and chooses to epsilon-preempt Firm i at x^*_{il}.

Lastly, we choose larger values of $\hat{h}_i$ and $\hat{h}_e$ in order to generate strong positive spillovers among the two firms. This is the scenario where first mover advantage is absent in both firms, thus resulting in simultaneous equilibrium. As revealed by the plots in Figure 1(d), the new set of relevant parameter values are taken to be: $h_i = h_e = 0.03$, $\hat{h}_i = \hat{h}_e = 1$, $\pi_e = 0.8$, $K_e = 7$. The leader and follower threshold values of the two firms are found to be: $x^*_{il} = 2.41$, $x^*_{el} = 2.11$, $x^*_{if} = 1.87$, $x^*_{ef} = 1.63$. Note that both firms face positive externalities as $d_i > 0$ and $d_e > 0$, and there exist no dominant first advantage in both firms as $x^*_{il} \geq x^*_{ef}$ and $x^*_{el} \geq x^*_{if}$. According to the analysis in Sec. 4.1, under the scenario of absence of first mover advantage in both firms, simultaneous equilibrium is resulted at which both firms enter at the some threshold that equals $\min\{x^*_{is}, x^*_{es}\}$. The plots of the value functions of joint optimal entry in Figure 1(d) indicate that $x^*_{is} = 2.41$ and $x^*_{es} = 2.11$, so the common simultaneous threshold is given by $\min\{2.41, 2.11\} = 2.11$.

5.2 Impact of spillovers on optimal entry threshold values

In Figures 2(a) and 2(b), we show plots of the optimal entry threshold values of the two firms with respect to $\hat{h}_i$ and $\hat{h}_e$, respectively. These plots help understand the impact of spillovers on the strategic equilibria, as depicted by the optimal entry threshold values of the two firms either as the leader or follower. To generate these plots of the entry threshold values, we adopt the common set of model parameters in the calculations for generating the plots in Figures 1(a-d), except for some changes for the parameter values of the hazard rates and sunk costs of the two firms.

In Figure 2(a), the entry threshold values of the incumbent firm (Firm i) and entrant firm (Firm e) are plotted against $\hat{h}_i$ with common hazard rate in the monopoly state (that is, $h_i = h_e$). The hazard rates and sunk costs are chosen to be: $h_i = 0.2$, $h_e = 0.2$, $\hat{h}_e = 0.2$, $K_i = 5$, $K_e = 5$, and $\pi_i^+ = \pi_e = 0.8$. For this given set of model parameter values, preemptive equilibrium is always resulted, though the nature of the preemptive equilibrium differs under different levels of $\hat{h}_i$. When $\hat{h}_i$ is sufficiently low, where $\hat{h}_i \leq \hat{h}_i^*$ ($\hat{h}_i^*$ is found to be 0.103 based on this given set of parameters), Firm e chooses to preempt Firm i at Firm i's leader threshold. In other words, Firm i enters optimally as the follower at its optimal follower threshold x_{if}^* while Firm e enters as the preemptive leader at its Firm i's optimal leader threshold x_{il}^* [as illustrated in Figure 2(a) by $x_e^{(l)} = x_{el}^*$ and $x_i^{(f)} = x_{if}^*$ when $\hat{h}_i \leq \hat{h}_i^*$].

It is seen that an increase in $\hat{h}_i$ would cause x_{il}^* to assume a higher value since Firm i's second mover advantage is strengthened with a higher hazard rate under duopoly. When $\hat{h}_i$ increases beyond $\hat{h}_i^*$, a new form of preemption equilibrium arises in the leader-follower game. At an intermediate level of $\hat{h}_i$, where $\hat{h}_i^* < \hat{h}_i < \hat{h}_i^{**}$ (our calculations give $\hat{h}_i^* = 0.103$ and $\hat{h}_i^{**} = 0.355$), the competition is relatively keen, so the firm with a higher hazard rate under duopoly (Firm j) preempts its rival at the rival's preemption threshold $\underline{x}_{j'p}$, $j \neq j'$. When $\hat{h}_i$ increases further, the preemptive incentive of Firm e is weakened. Once $\hat{h}_i > \hat{h}_i^{**}$, Firm i chooses to preempt at the rival's leader threshold x_{el}^*. As illustrated in Figure 2(a), we have $x_i^{(l)} = x_{el}^*$ and $x_e^{(f)} = x_{ef}^*$ when $\hat{h}_i > \hat{h}_i^{**}$. Also, Figure 2(a) reveals that the difference of the entry thresholds converges as $\hat{h}_i$ first increases from a low value, then diverges when $\hat{h}_i$ further increases beyond 0.2. When $\hat{h}_i$ increases from low value to 0.2, Firm i is the follower since $\hat{h}_i < \hat{h}_e = 0.2$. Within this range of $\hat{h}_i$, $x_e^{(l)}$ is not quite sensitive to increasing value of $\hat{h}_i$ while $x_i^{(f)} = x_{if}^*$ decreases quite significant with increasing value of $\hat{h}_i$. This is expected since Firm i responds more strongly by entering at a lower follower entry threshold when the increase of the incumbent's hazard rate is higher. This gives the converging trend of the difference of the thresholds when $\hat{h}_i$ increases from low value to 0.2. When $\hat{h}_i > \hat{h}_e = 0.2$, Firm i becomes the leader. As $\hat{h}_i$ increases beyond 0.2, $x_i^{(l)}$ is not quite sensitive to increasing value of $\hat{h}_i$. However, the incentive for entrant's entry as the follower is much weakened as $\hat{h}_i$ increases. Firm e chooses to enter at a relatively higher follower threshold. This leads to the diverging trend of the difference of the thresholds when $\hat{h}_i$ increases beyond 0.2.

It is instructive to compare our result with that of Weeds (2002). At $\hat{h}_i = \hat{h}_e = h_i = h_e = 0.2$, we obtain symmetric duopoly similar to the model of Weeds (with zero spillover). There exist two possible preemption equilibria: (i) Firm i acts as the preemptive leader at $x_i^{(l)} = \underline{x}_{ep}$ (same value as $\underline{x}_{ip}$ due to symmetry) and Firm e acts optimally as the follower at $x_e^{(f)} = x_{ef}^*$. (ii) Firm e acts as the preemptive leader at $x_e^{(l)} = \underline{x}_{ip} = \underline{x}_{ep}$ and Firm i acts optimally as the follower at $x_i^{(f)} = x_{if}^*$. This result is consistent with that of Weeds (2002).

In Figure 2(b), the entry threshold values of the two firms are plotted against $\hat{h}_i$ with common hazard rate in the monopoly state (that is, $h_i = h_e$). The hazard rates and sunk costs are chosen to be: $h_i = 0.05$, $h_e = 0.05$, $\hat{h}_e = 0.3$, $K_i = 5$, $K_e = 5$, and $\pi_i^+ = 0.8$, $\pi_e = 0.8$. Here, the hazard rates in the monopoly state are chosen to assume a small value since we would like to demonstrate the occurrence of simultaneous equilibrium under low hazard rates. When $\hat{h}_i$ is lower than some threshold level $\hat{h}_i^*$ ($\hat{h}_i^* = 0.23$ is obtained based on this set of parameter values), only Firm e has first mover advantage while Firm i has no preemptive incentive. As a result, Firm e is the leader in the resulting sequential leader-follower equilibrium. At an intermediate level of $\hat{h}_i$, where $\hat{h}_i^* < \hat{h}_i < \hat{h}_i^{**}$ (our calculations give $\hat{h}_i^* = 0.23$ and $\hat{h}_i^{**} = 0.42$), both firms do not exhibit first mover advantage, so simultaneous equilibrium is resulted. The simultaneous entry threshold value shared by both firms are given by $x_i^{(s)} = x_e^{(s)} = \min(x_{is}^*, x_{es}^*)$. As $\hat{h}_i$ increases beyond $\hat{h}_i^{**}$, only Firm i has first mover advantage while Firm e has no preemptive incentive, so Firm i is the leader in the resulting sequential leader-follower equilibrium.

5.3 Pattern of strategic equilibria

Lastly, we perform characterization of the strategic equilibria in the parameter space of (i) d and K_i (where d is the common externality factor), and (ii) h and $\hat{h}_i$ (where h is the common hazard rate under monopoly). The corresponding patterns of strategic equilibria are illustrated in Figures 3(a) and 3(b), respectively.

In generating the plot in Figure 3(a), the model parameter values are chosen to be $h_i = h_e = 0.05$, $K_e = 10$, $\pi_i^+ = \pi_e = 0.8$. We take $\hat{h}_i = \hat{h}_e$, and these two parameters assume values between 0 to 0.3 to generate the range of values for d as shown in the figure. According to Proposition 2, under the assumption of negative externalities with $d < 0$, the pattern of strategic equilibria can be characterized according to $d < d^*$ or $d > d^*$, where d^* is some threshold value. In our calculations, the critical threshold d^* is found to be -2.09. When $d < d^*$, keen competition arises when K_i is chosen to be close to K_e, where K_e is chosen to be 10. Under this scenario, we obtain preemptive equilibrium with the lower cost firm as the preemptive leader. When the difference in sunk costs becomes wider, sequential equilibrium is resulted. On the other hand, when $d > d^*$, both firms do not have first mover advantage when the difference in the sunk costs is small, thus leads to simultaneous equilibrium. However, the strategic equilibrium pattern changes to sequential equilibrium when the two sunk costs differ widely. All these observations, as illustrated in Figure 3(a), agree with the results stated in Proposition 2.

Figure 3(b) shows the pattern of strategic equilibria in the parameter space of h and $\hat{h}_i$. The model parameter values are chosen to be the same as those in generating Figure 2(a), expect that $K_i = K_e = 5$ and $\hat{h}_e = 0.3$. Here, we set the sunk costs of the two firms to be the same. When the common hazard rate under monopoly h is less than some threshold value h^* (in our calculations, h^* is found to be 0.056), it becomes much likely that both firms do not have first mover advantage. In particular, this occurs when the two hazard rates under duopoly of the two firms do not differ widely. Under this scenario, simultaneous equilibrium is resulted. Otherwise, when the difference in $\hat{h}_e$ and $\hat{h}_i$ becomes more significant, sequential equilibrium is resulted, where the firm with the higher hazard rate under duopoly becomes the preemptive leader. On the other hand, when $h > h^*$, preemptive equilibrium is resulted when $\hat{h}_e$ and $\hat{h}_i$ do not differ widely, and the firm with the higher hazard rate under duopoly becomes the preemptive leader. Otherwise, sequential equilibrium is resulted when the difference in $\hat{h}_e$ and $\hat{h}_i$ becomes sufficiently large.

6 Conclusions

Using the real options game approach, we perform analysis of strategic equilibria of optimal entries into an asymmetric duopoly R&D race in the development of a new product with both market and technological uncertainty. The types of Markov perfect equilibria include sequential leader-follower equilibrium, preemptive equilibrium and simultaneous equilibrium. The relative ordering of the various trigger thresholds with reference to the appropriate actions taken by the rival firms determines the type of equilibrium that prevails. The final outcome of equilibrium is related to the interplay between leader's first mover advantage and follower's second mover advantage. The positivity of the relevant externality factors, like input spillovers, plays an important role in determining the optimal actions taken by the rival firms.

Under preemptive equilibrium, real option values are reduced by fear of preemption since the preemptive leader chooses to enter at the threshold that is below its optimal entry threshold that is without preemption threat. The two competing forces are characterized by the loss of real option value due to preemption and delay entry as a follower to take advantage of the positive input spillover. When positive input spillover is present, it is interesting to observe that a higher sunk cost of R&D investment or a lower hazard rate of arrival of innovation of the incumbent firm value may increase its firm value due to the change from preemptive equilibrium to simultaneous equilibrium. In this sense, delay of entry into R&D under simultaneous equilibrium is more desirable since keen competition between the competing firms is avoided.

The analysis of the real options game R&D race model reveals several interesting phenomena. When the input spillover stays positive, preemptive equilibrium is always ruled out due to the presence of dominant second mover advantage. Also, we show that the two firms choose optimally to enter simultaneously if the sunk cost asymmetry between them is relatively small while sequential equilibrium is resulted if otherwise. Dominant second mover advantage is seen to prevail when the initial hazard rate is low while the input spillover is sufficiently high, resulting in simultaneous equilibrium. However, the first mover advantage may become significant when the initial hazard rate becomes sufficiently high. In this case, simultaneous equilibrium is ruled out even under very high positive input spillover. Suppose the incumbent's hazard rate is held fixed while the entrant's hazard rate increases gradually, it may occur that preemption action taken optimally by the incumbent is changed to sequential follower entry since a stronger incumbent's second mover advantage is resulted.

We have observed how the equilibrium strategies of R&D investment may change from one type to another type depending on the level of spillovers. Also, we have shown how the value functions may be enhanced through the avoidance of keen competition (for example, preemptive entry is not adopted as an optimal entry decision). Our model may provide insight on finding the optimal level of spillovers that enhance social welfare (like maximizing the sum of the value functions of the competing firms) while the drive for innovation is not significantly undermined due to delay in launching the R&D investment.

Our real options game model can be extended in several directions. Normally, R&D investment may occur in several stages with results on partial success of innovation released at each stage. The competing firms may modify their strategies based on the relevant updated information on the potential of successful innovation. Modeling of multistage R&D races together with information updating would pose interesting challenges. Also, we may consider a mixed duopoly of R&D race where one firm is a welfare maximizing public sector firm while the other firm is a profit maximizing private firm. If we allow costless imitation of the research results from the public sector firm, then this may result in too little research by the private firm. We may consider various forms of input and output spillovers, and their appropriate level such that it is socially optimal. That is, there is no over-investment in R&D under competition on one hand and no under-investment in the economy on the other hand. The natural question: does the occurrence of the sequential leader-follower equilibrium represent an ideal outcome of the R&D race, where natural market forces are in full action without the social planner's intervention? Next comes the challenge: how to achieve that?

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Appendix A - Proof of Proposition 2

All the model parameters have been set the same for both firms except the sunk costs. We concentrate our analysis under $K_i \le K_e$, while the analysis under $K_i > K_e$ can be performed in a similar manner.

It is instructive to investigate whether there exists dominant first mover advantage, that is, $x^*_{il} < x^*_{ef}$ and/or $x^*_{el} < x^*_{if}$. First, we argue that Firm e has no dominant first mover advantage by establishing $x^*_{el} \ge x^*_{if}$ when $K_i \le K_e$. This can be shown easily by observing that the following equation [see also eq. (3.10)]:

$$\frac{d(\hat{\beta}-\beta_0)}{[x^*_{if}(K_i)]^{\hat{\beta}-1}} z^{\hat{\beta}} - \frac{(\beta_0 - 1)h\pi}{(r-\mu)(r-\mu+h)} z + \beta_0 K_e = 0$$

has no root in $[0, x^*_{if}(K_i)]$ when $K_i \le K_e$.

Next, we show that Firm i holds dominant first mover advantage when K_i is sufficiently low. Recall that x^*_{il} satisfies the following equation [see also eq. (3.10)]:

$$g(z; K_i) = \frac{d(\hat{\beta}-\beta_0)}{(x^*_{ef})^{\hat{\beta}-1}} z^{\hat{\beta}} - \frac{(\beta_0 - 1)h\pi}{(r-\mu)(r-\mu+h)} z + \beta_0 K_i = 0.$$

It is easily seen that $g(z; K_i)$ possesses the following properties:

(i) $g(z; K_i)$ is increasing with respect to K_i;

(ii) when $K_i = K_e$, $g(z) > 0$ for all $z \in [0, x^*_{ef})$;

(iii) when $K_i = 0$, $g(z) \le 0$ for some $z \in [0, x^*_{ef})$.

One can then deduce that there exists $k_l \in (0, K_e)$ such that

(i) when $0 < K_i \le k_l$, $g(z; K_i) = 0$ has at least one root in $[0, x^*_{ef})$;

(ii) when $k_l < K_i \le K_e$, $g(z; K_i) > 0$ for all $z \in [0, x^*_{ef})$.

We then have (a) $x^*_{il} < x^*_{ef}$ when $0 < K_i \le k_l$; and (b) $x^*_{il} \ge x^*_{ef}$ when $k_l < K_i \le K_e$.

When $K_i > K_e$, by performing a similar analysis, we deduce that there exists $k_u \in (K_e, \infty)$ such that (a) $x^*_{el} < x^*_{if}$ when $K_e < k_u < K_i$, and (b) $x^*_{el} > x^*_{if}$ when $K_e < K_i < k_u$.

From the above results, the strategic equilibrium can be deduced as follows:

(i) $0 < K_i \le k_l$, where $k_l \in (0, K_e)$
Firm i exhibits dominant first mover advantage as $x^*_{il} < x^*_{ef}$. Preemptive equilibrium is ruled out under positivity of d, so sequential equilibrium is resulted with Firm i acting as the dominant leader (see Sec. 4.1).

(ii) $k_l < K_i < k_u$, where $k_l \in (0, K_e)$ and $k_u \in (K_e, \infty)$
Both firms do not hold dominant first mover advantage, so simultaneous equilibrium is resulted (see Sec. 4.2). Both firms choose to enter into R&D investment at $\min\{x^*_{is}, x^*_{es}\}$.

(iii) $K_i > k_u$, where $k_u \in (K_e, \infty)$
Sequential equilibrium is resulted with Firm e acting as the dominant leader.

Appendix B - Proof of Proposition 3

It is necessary to consider the two separate cases: (i) $d^* < d < 0$ and (ii) $d < d^* < 0$. The proof for the results in case (i) follows a similar analysis as depicted in Appendix A. However, it is necessary to consider the possibility of ϵ-preemption in the resulting leader-follower equilibrium due to the existence of the preemption trigger threshold. The detailed discussion on the occurrence of either the preemptive equilibrium or sequential equilibrium with various selected ranges of various choices of the cost parameters can be found in Leung (2011).

For case (ii), we start the proof by showing that dominant first mover advantage always exists in at least one firm so the R&D game always results in leader-follower equilibrium. We then consider the two separate cases, either only one firm has dominant first mover advantage or both firms hold dominant first mover advantage. In the first case, the analysis that determines whether preemption equilibrium or sequential equilibrium occurs is similar to part (a). In the second case, it is necessary to determine which firm emerges as the eventual leader by analyzing the relative positions of the preemption thresholds and leader entry thresholds of both firms with respect to the cost parameters. Detailed discussion of the relevant procedures can be found in Leung (2011).

Appendix C - Proof of Proposition 4

When the two firms are symmetric, we deduce from Propositions 1 and 2 that (i) simultaneous equilibrium is resulted if $d \geq d^*$, and (ii) preemptive equilibrium is resulted if $d < d^*$. We would like to examine the conditions on h and $\hat{h}$ that lead to the above two cases.

Here, we write the functional dependence of d on $\hat{h}$ as $d(\hat{h})$. First, we establish the following results [see Leung (2011)]:

(a) If $\frac{h}{r-\mu+h}\frac{\hat{\beta}^2-\hat{\beta}}{\hat{\beta}^2-\beta_0} \geq \frac{1}{2}$, then $d(\hat{h}) < d^*$ for $\hat{h} \geq 0$.

(b) If $\frac{h}{r-\mu+h}\frac{\hat{\beta}^2-\hat{\beta}}{\hat{\beta}^2-\beta_0} < \frac{1}{2}$, then there exists $\hat{h}^*$ (as defined in Proposition 3) such that $d(\hat{h}) \geq d^*$ if and only if $\hat{h} \geq \hat{h}^*$.

Next, for $h < r - \mu$, it is easily seen that $\frac{h}{r-\mu+h}\frac{\hat{\beta}^2-\hat{\beta}}{\hat{\beta}^2-\beta_0} < \frac{1}{2}$, so by the result in part (b) above, we obtain the result in Proposition 3(a). Lastly, by observing

$$\lim_{h\to\infty} \frac{h}{r-\mu+h}\frac{\hat{\beta}^2-\hat{\beta}}{\hat{\beta}^2-\beta_0} = 1 \text{ and } \frac{h}{r-\mu+h}\frac{\hat{\beta}^2-\hat{\beta}}{\hat{\beta}^2-\beta_0}\Big|_{h=0} = 0,$$

we deduce that there exists h^*, where $h^* > r - \mu$, such that $\frac{h}{r-\mu+h}\frac{\hat{\beta}^2-\hat{\beta}}{\hat{\beta}^2-\beta_0} \geq \frac{1}{2}$. By the result in part (a) above, we obtain Proposition 4(b).

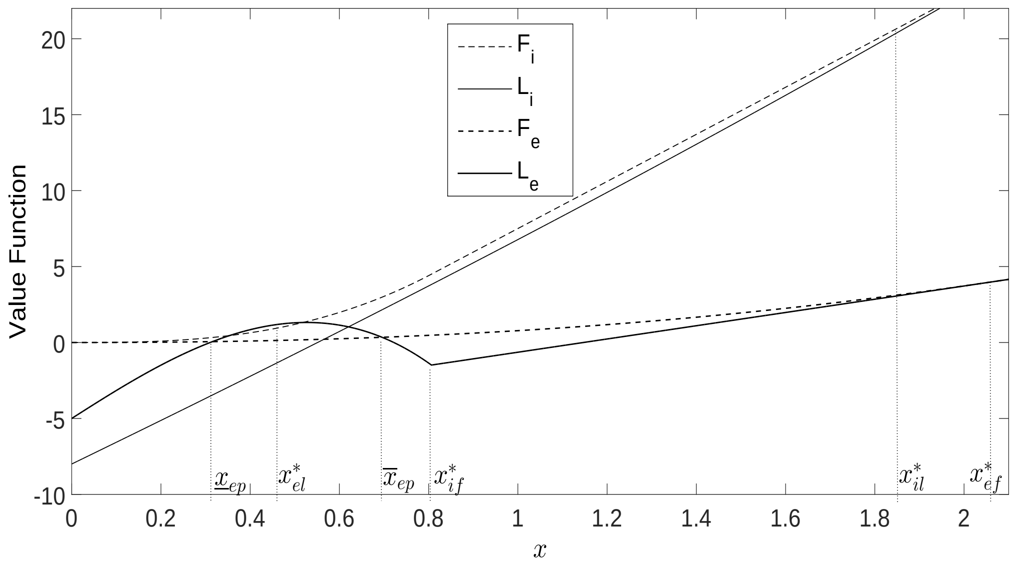

Figure 1a: Plot of the leader and follower value functions under sequential equilibrium with Firm e as the leader. Though both firms hold dominant first mover advantage as revealed by $x^*_{il} < x^*_{ef}$ and $x^*_{el} < x^*_{if}$, Firm i has no preemption incentive as demonstrated by $L_i(x) < F_i(x)$ for $x < x^*_{ef}$. As a result, Firm e enters optimally at x^*_{el} as leader while Firm i enters optimally at x^*_{if} as follower.

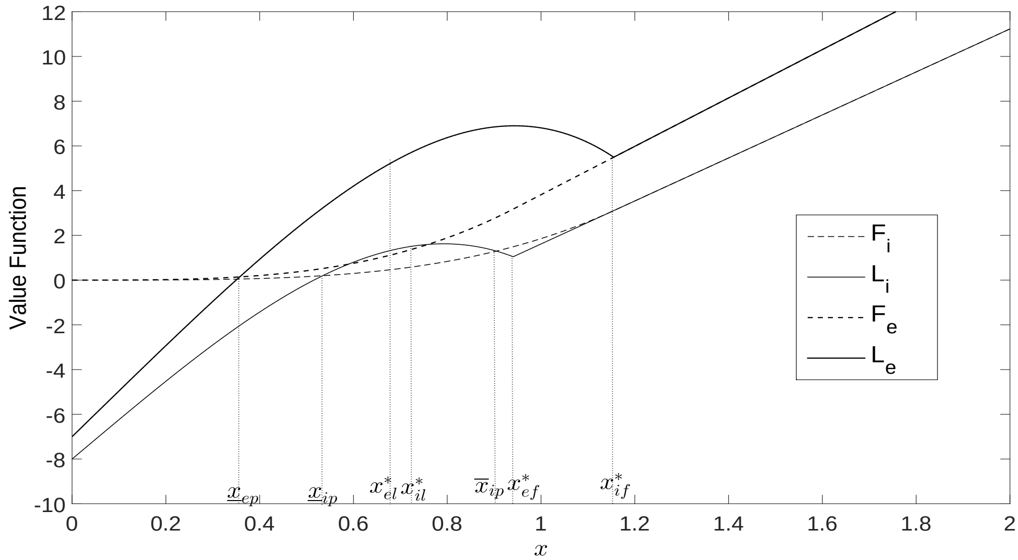

Figure 1b: Plot of the leader and follower value functions under preemptive equilibrium with Firm e as the preemptive leader. Both firms have preemption incentive since preemption thresholds exist for both firms. Since Firm e has lower preemption threshold, where $\underline{x}_{ep} < \underline{x}_{ip}$, Firm e chooses to preempt its rival at the rival's preemption threshold $\underline{x}_{ip}$.

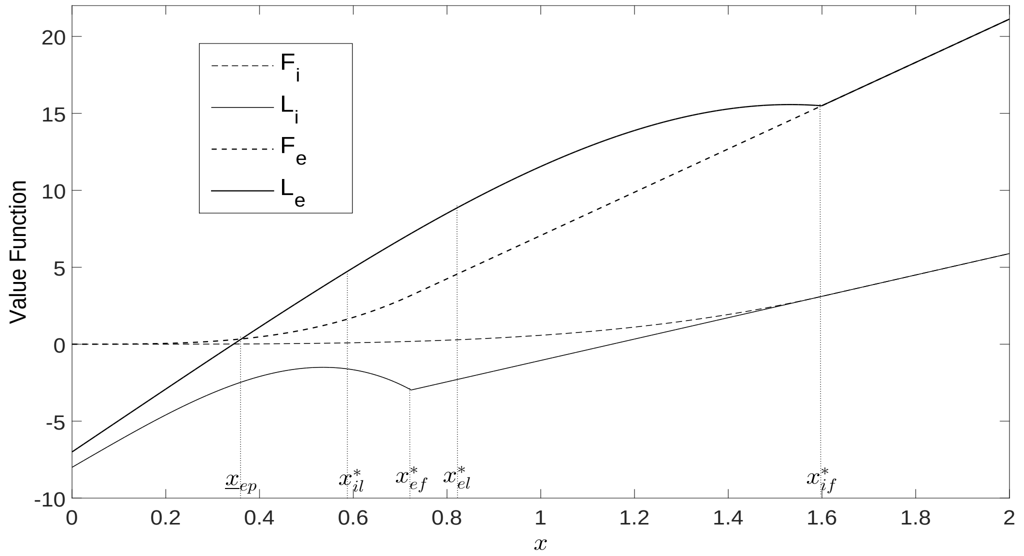

Figure 1c: Plot of the leader and follower value functions under preemptive equilibrium with Firm e preempting the rival firm at the rival's leader threshold x^*_{il}. The functions $L_i(x)$ and $F_i(x)$ do not intersect while the functions $L_e(x)$ and $F_e(x)$ intersects only once at $\underline{x}_{ep}$. The competition for leader's entry is less keen since preemption incentive exists only in Firm e.

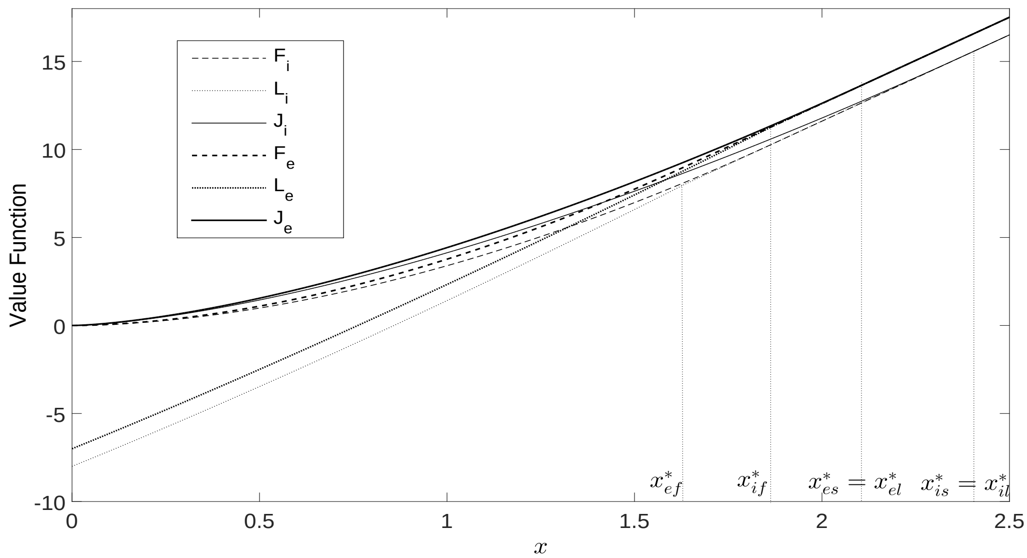

Figure 1d: Plot of the leader and follower value functions of each of two firms, and the value functions of joint optimal entry of the two firms under simultaneous equilibrium.

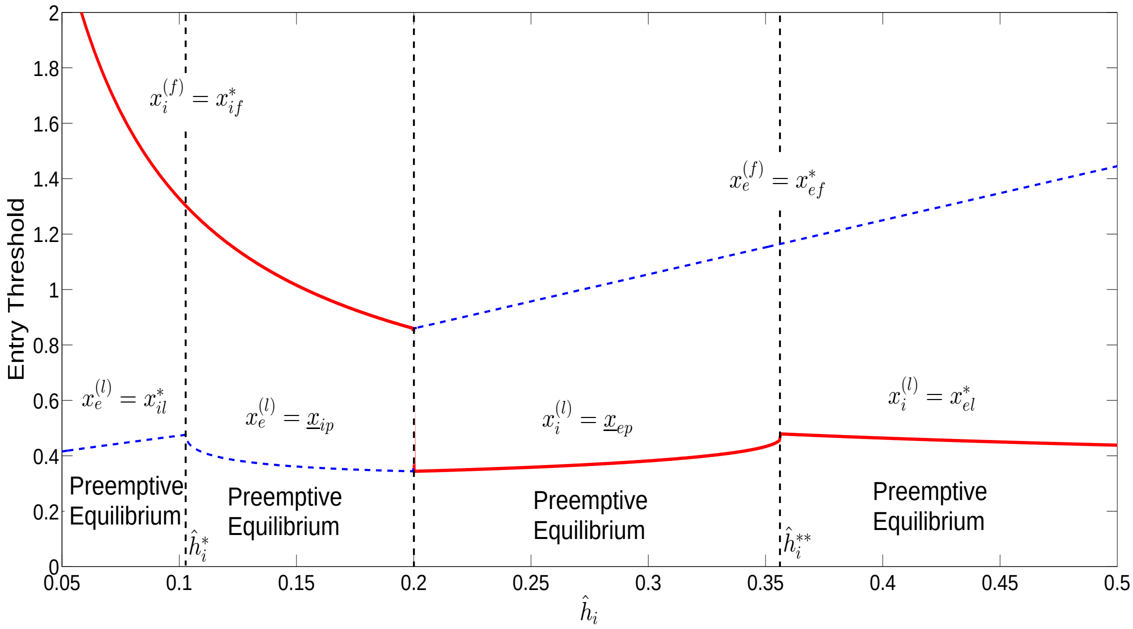

Figure 2a: Plot of the optimal entry threshold values of the two firms with respect to $\hat{h}_i$ with common hazard rate in the monopoly state, $h_i = h_e$. The thick curves (dotted curves) show the entry threshold values of the incumbent (entrant). At a lower value of $\hat{h}_i$, the entrant enters as the preemptive leader at either Firm i's optimal leader threshold x_{il}^* or Firm i's preemption threshold $\underline{x}_{ip}$. As $\hat{h}_i$ increases, the incumbent becomes the preemptive leader. At $\hat{h}_i = \hat{h}_e = h_i = h_e = 0.2$, we recover the symmetric duopoly model of Weeds; and preemption equilibrium prevails in this case. Due to symmetry, either firm may become the preemptive leader entering at the rival's preemption threshold; the other firm serving as the follower would enter at its own optimal follower threshold.

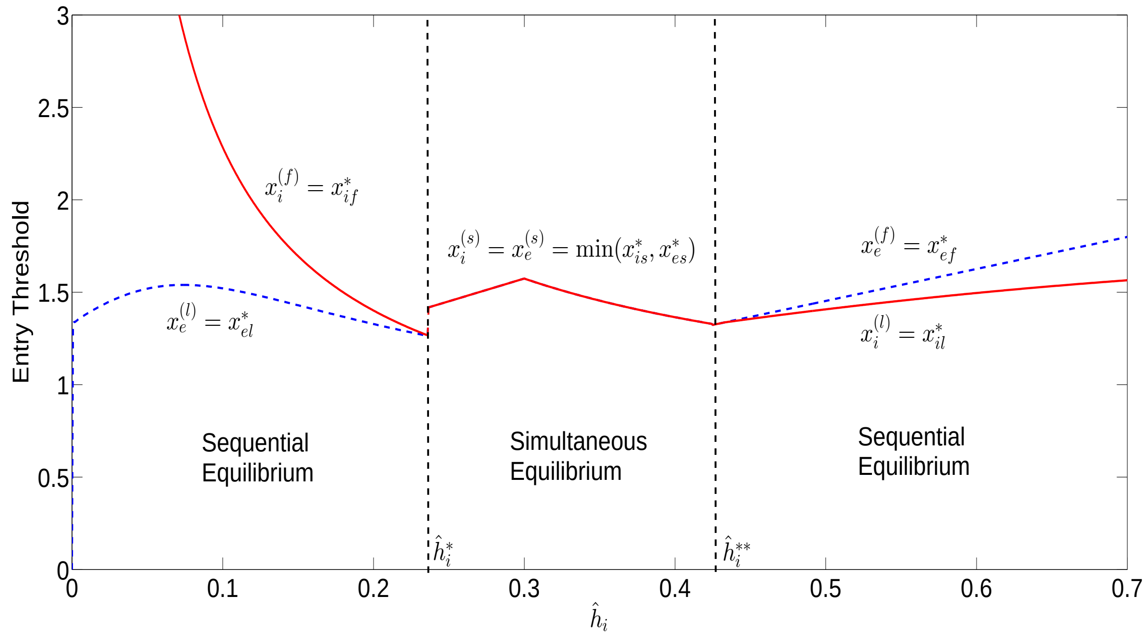

Figure 2b: Plot of the optimal entry threshold values of the two firms with respect to $\hat{h}_i$ with common hazard rate in the monopoly state, $h_i = h_e$. The thick curves (dotted curves) show the entry threshold values of the incumbent (entrant). At a lower value of $\hat{h}_i$, Firm e is the leader under the resulting sequential leader-follower equilibrium since it has stronger first mover advantage. At an intermediate value of $\hat{h}_i$, both firms do not exhibit first mover advantage, so simultaneous equilibrium is resulted. At a higher value of $\hat{h}_i$, Firm i is the leader under the resulting sequential leader-follower equilibrium since only Firm i has first mover advantage.

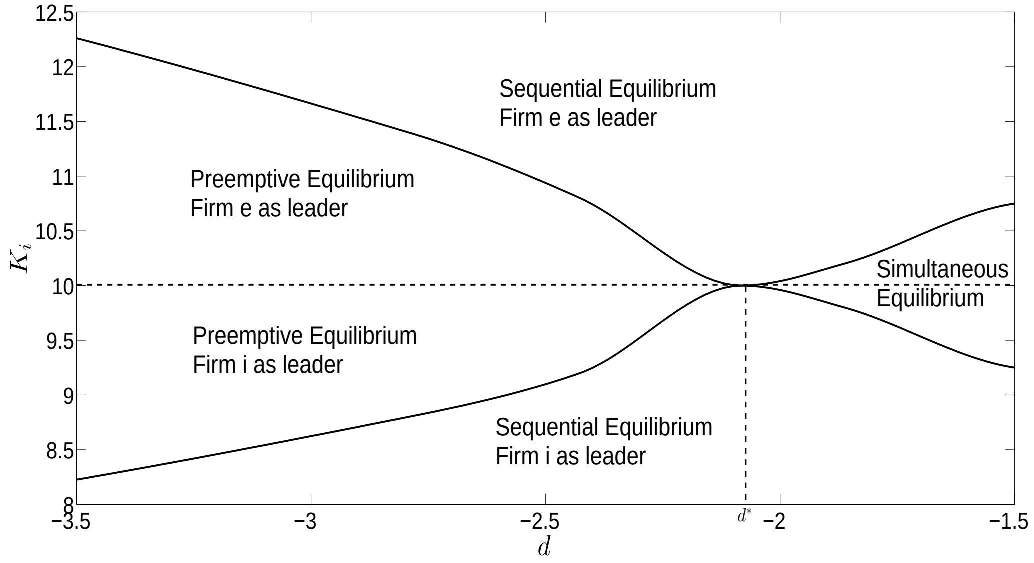

Figure 3a: Characterization of the pattern of strategic equilibria in the d-K_i plane. When $d < d^*$, where $d^* = -2.09$, we obtain preemptive equilibrium (with the lower cost firm as the leader) when the sunk costs are close to each other (representing keen competition). Otherwise, sequential equilibrium is resulted when the sunk costs become wider apart. On the other hand, when $d > d^*$, simultaneous (sequential) equilibrium is resulted when the sunk costs differ narrowly (widely).

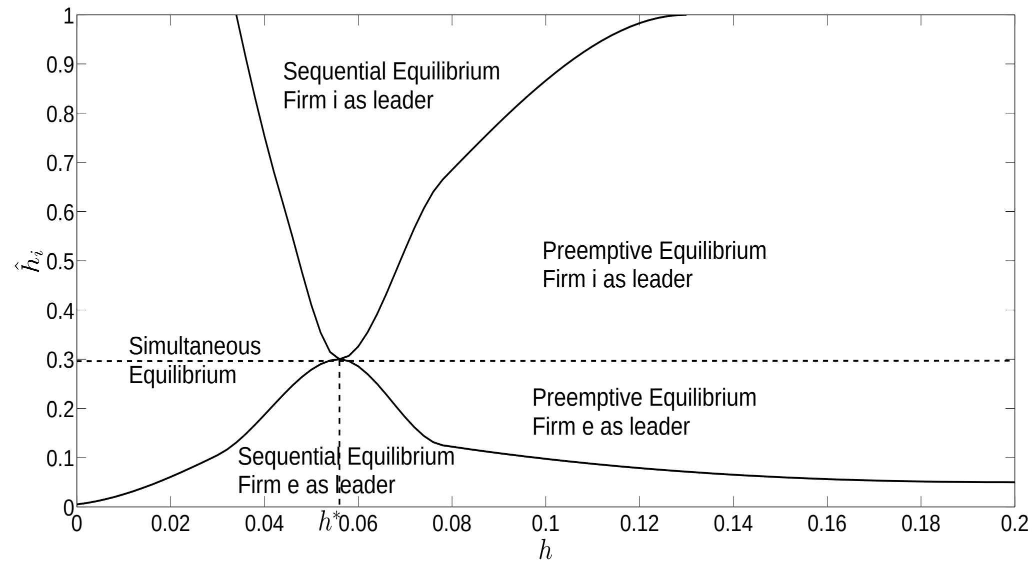

Figure 3b: Characterization of the pattern of strategic equilibria in the h-$\hat{h}_i$ plane. When $h < h^*$, where $h^* = 0.056$, we obtain simultaneous equilibrium when $\hat{h}_i$ is close to $\hat{h}_e$, where $\hat{h}_e$ is set to 0.3 (both firms have no first mover advantage). Otherwise, sequential equilibrium is resulted and the firm with the higher hazard rate under duopoly becomes the leader. On the other hand, when $h > h^*$, preemptive equilibrium (sequential) equilibrium is resulted when the two hazard rates under duopoly differ slightly (significantly).