Pricing Guaranteed Minimum Withdrawal Benefits under Stochastic Interest Rates

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Abstract

We consider the pricing of variable annuities with the Guaranteed Minimum Withdrawal Benefit (GMWB) under the Vasicek stochastic interest rates framework. The holder of the variable annuity contract pays an initial purchase payment to the insurance company, which is then invested in a portfolio of risky assets. Under the GMWB, the holder can withdraw a specified amount periodically over the term of the contract such that the return of the entire initial investment is guaranteed, regardless of the market performance of the underlying asset portfolio. The investors have the equity participation in the reference investment portfolio with protection on the downside risk. The guarantee is financed by paying annual proportional fees. Under the assumption of deterministic withdrawal rates, we develop the pricing formulation of the value function of a variable annuity with the GMWB. In particular, we derive the analytic approximation solutions to the fair value of the GMWB under both equity and interest rate risks, obtaining both the lower and upper bound on the price functions. The pricing behavior of the embedded GMWB under various model parameter values is also examined.

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1 Introduction

A variable annuity is a contract between a policyholder and an insurance company. At initiation of the contract, the policyholder makes a lump-sum purchase payment. In return, the insurer agrees to make periodic payments to the policyholder that start immediately or at some future date. The policyholder can choose to invest the purchase payment in a range of mutual funds, thus the policyholder has the equity participation in a portfolio of risky assets. In other words, the policyholder is exposed to the equity risk of the reference investment portfolio. The value of the personal sub-account of the policyholder in a variable annuity will depend on the performance of the reference investment portfolio. Various forms of guarantees are commonly embedded in variable annuities. In recent years, variable annuities with the guaranteed minimum withdrawal benefit (GMWB) have attracted significant attention and sales. The GMWB allows the policyholders to withdraw funds on an annual or semi-annual basis, and promises to return the entire initial purchase payment over the life of the policy. Thus the guarantee can be viewed as an insurance option. The provision of this option is financed by the proportional fees paid to the insurer by the policyholder. The personal sub-account will be depleted by these periodic partial withdrawals and proportional insurance fees. The current charges for this benefit typically range from 35 to 75 bps per annum. The guarantee kicks in when the personal sub-account falls to zero prior to the policy maturity date. Under the clause of the benefit granted, the insurer continues to provide the guaranteed withdrawal amount until the entire original premium is paid out. When the underlying investment portfolio performs well so that the personal sub-account stays positive at maturity, the whole remaining balance in the personal sub-account is paid to the policyholder at maturity.

In a typical guarantee, withdrawals are taken as a fixed percentage of the premium (say, 5% per annum) until the premium is exhausted. This is called the static withdrawal policy. In a more complicated design of the GMWB that allows dynamic withdrawals, the policyholder may withdraw at a higher rate or exercise complete withdrawal prior to maturity (usually with penalty charges). The pricing models of variable annuities embedded with the GMWB have been studied in several earlier papers. Milevsky and Salisbury (2006) propose the pricing formulations of GMWB with static and dynamic withdrawals under constant interest rate. They analyze the fair proportional fees that should be charged on the provision of the guarantee.

Dai, Kwok and Zong (2008) develop a singular stochastic control model for pricing GMWB under dynamic withdrawal. An efficient finite difference algorithm using the penalty approximation approach is also proposed for solving the singular stochastic control model. Chen, Vetzal and Forsyth (2008) explore the effects of various modeling assumptions on the optimal withdrawal strategy of the policyholder, and examine the impact on the guarantee value under sub-optimal withdrawal behavior. Effective numerical schemes for pricing various types of guaranteed minimum benefits in variable annuities using the impulse control formulation are also proposed by Chen and Forsyth (2008). Bauer, Kling and Russ (2008) adopt a generalization of a finite mesh discretization approach in Monte Carlo method to price GMWB in variable annuities under the optimal policyholder behavior. In all of these papers, the guarantees are priced under the assumption of constant interest rate. Since variable annuities are long-term contracts, the assumption of constant interest rate becomes unrealistic in pricing. More reliable pricing models of variable annuities should allow for stochastic interest rates. Lin and Tan (2003) and Kijima and Wong (2007) consider the pricing of equity-indexed annuities under stochastic interest rates. In this paper, we consider the model formulation of the variable annuities embedded with GMWB under static withdrawal and subject to both equity and interest rate risks. In particular, we examine the various forms of decomposition of the value of the GMWB. For numerical valuation of the guarantee, we show how to obtain the analytic approximation solutions to the fair value of the GMWB by deriving both the lower and upper bound on the value function of the variable annuities contract.

The paper is organized as follows. In the next section, we propose the pricing formulation of variable annuities with GMWB subject to equity and interest rate risks. We illustrate how to decompose the value of the GMWB into a certain-term annuity and a put option on some path dependent function of the value of the personal sub-account. Also, we show how the GMWB value is related to the insurer's liabilities and initial premium. Since the terminal payoff of the GMWB exhibits path dependence of the value process of the sub-account due to static withdrawals, the pricing model does not admit a closed form solution. In Section 3, we apply Roger-Shi's method and Thompson's method to deduce the lower and upper bound on the value of the GMWB, respectively. In Section 4, we report the numerical tests that were performed for checking the accuracy of these numerical bounds. We also examine the pricing behavior of the GMWB under various model parameter values. Conclusive remarks are presented in the last section.

2 Formulations of the value function

The pricing model of a variable annuity contract with the GMWB under constant interest rate has been formulated by Milevsky and Salisbury (2006). We extend the pricing formulation of the GMWB under both equity and interest rate risks. Assumptions on the underlying price process of the reference risky portfolio and the financial market conditions in our continuous pricing models are summarized as follows:

- The process of holder's withdrawal and the payment stream of proportional (insurance) fees to the insurance company are assumed to be deterministic and continuous in time.
- The financial market is complete and free of arbitrage opportunities. There are no transaction costs and no restriction on short selling.
- The value process of the underlying reference portfolio of risky assets follows the Geometric Brownian process with deterministic volatility.
- The stochastic interest rate process is characterized by the Vasicek short rate model.

Let S_t denote the fund value process of the reference portfolio of risky assets underlying the variable annuity policy before the deduction of the proportional fees. We assume the existence of a risk neutral probability measure Q such that all discounted asset price processes are Q-martingales.

Under the risk neutral measure Q, the joint dynamics of the fund value process S_t and the short rate process r_t is governed by

$$\begin{aligned} dS_t &= r_t S_t\,dt + \sqrt{1-\rho^2}\sigma_S S_t\,dB_{1,t} + \rho\sigma_S S_t\,dB_{2,t} \\ dr_t &= k(\theta - r_t)\,dt + \sigma_r\,dB_{2,t}, \end{aligned} \tag{2.1}$$

where $B_{1,t}$ and $B_{2,t}$ are independent standard Q-Brownian processes, ρ is the instantaneous correlation coefficient between the stochastic processes S_t and r_t, k and θ are constant parameters in the Vasicek model, σ_S and σ_r are constant volatility values of S_t and r_t, respectively. It is well known that the discount bond price function under the Vasicek model is given by

$$D(t,T) = a(t,T)e^{-b(t,T)r_t}, \tag{2.2}$$

where

$$\begin{aligned}
b(t,T) &= \frac{1}{k}\left[1-e^{-k(T-t)}\right], \\
a(t,T) &= \exp\left(\left(\theta-\frac{\sigma_r^2}{2k^2}\right)\left[(b(t,T)-(T-t)\right]-\frac{\sigma_r^2}{4k}b(t,T)^2\right).
\end{aligned}$$

We may write the joint dynamics of S_t and $D(t,T)$ as follows:

$$\begin{aligned}
\frac{dS_t}{S_t} &= r_t\,dt+\boldsymbol{\sigma}_S\,d\boldsymbol{B}_t, \\
\frac{dD(t,T)}{D(t,T)} &= r_t\,dt+\boldsymbol{\sigma}_D(t)\,d\boldsymbol{B}_t,
\end{aligned} \tag{2.3}$$

where

$$\boldsymbol{B}_t=\begin{pmatrix}B_{1,t}\\ B_{2,t}\end{pmatrix}, \boldsymbol{\sigma}_S=(\sqrt{1-\rho^2}\sigma_S \quad \rho\sigma_S) \quad \text{and} \quad \boldsymbol{\sigma}_D=(0 \quad -\sigma_r b(t,T)).$$

Let $\mathcal{F}=\{\mathcal{F}_t : t\geq 0\}$ be the filtration generated by the Brownian processes $B_{1,t}$ and $B_{2,t}$. Under our continuous model framework, the holder's withdrawal process and the payment stream of the proportional insurance fees are assumed to be deterministic and continuous in time. Let W_t denote the value process of the personal annuity sub-account, which is depleted by the continuous static withdrawal at the deterministic rate G_t and the payment of the continuous stream of insurance fees at the constant proportional rate α. Let w_0 denote the initial purchase payment of the annuity contract. The maturity time T and the initial investment amount w_0 are related to the deterministic withdrawal rate G_t by the relation:

$$\int_0^T G_t\,dt=w_0. \tag{2.4}$$

That is, the whole initial lump sum payment w_0 by the policyholder is withdrawn at the rate G_t throughout the life of the contract (no regard is paid to the time value of money).

We define τ_0 to be the first passage time of the value process W_t hitting the zero value, that is,

$$\tau_0=\inf\{t : W_t=0\}.$$

Once W_t hits the zero value, it remains to be zero forever afterwards. That is, the zero value is considered to be an absorbing barrier of W_t.

Under Q, the value process of the personal sub-account is given by

(i) $0 \leq t < \tau_0$

$$dW_t = [(r_t - \alpha)W_t - G_t]\, dt + W_t \boldsymbol{\sigma}_S \, d\boldsymbol{B}_t \tag{2.5a}$$

with $W_0 = w_0$;

(ii) $t \geq \tau_0$

$$W_t = 0. \tag{2.5b}$$

As there is an absorbing barrier at zero for W_t, it is convenient for our later discussion to define the corresponding unrestricted process $\widetilde{W}_t$ to be

$$d\widetilde{W}_t = [(r_t - \alpha)\widetilde{W}_t - G_t]\, dt + \widetilde{W}_t \boldsymbol{\sigma}_s \, d\boldsymbol{B}_t, \quad t \geq 0, \tag{2.6}$$

with $\widetilde{W}_0 = w_0$. It is seen that W_t and $\widetilde{W}_t$ are related by

$$W_t = \widetilde{W}_{t \wedge \tau_0} = \widetilde{W}_t \mathbf{1}_{\{\tau_0 > t\}}. \tag{2.7}$$

The solution to W_t can be shown to have the form (Karatzas and Shreve, 1992)

$$W_t = \max(\widetilde{W}_t, 0), \tag{2.8}$$

where

$$\widetilde{W}_t = X_t \left(w_0 - \int_0^t \frac{G_u}{X_u}\, du \right),$$

$$X_t = \exp\left(\int_0^t \left(r_u - \alpha - \frac{1}{2}\boldsymbol{\sigma}_S \boldsymbol{\sigma}_S^T \right) du + \int_0^t \boldsymbol{\sigma}_S \, d\boldsymbol{B}_u \right).$$

Here, $w_0 X_t$ gives the solution to the sub-account value with proportional fees payment but without static withdrawal. Note that $\dfrac{G_u}{w_0 X_u}\, du$ represents the proportion of the sub-account withdrawn over the differential time interval $(u, u + du)$, so the accumulated depletion of the sub-account due to static withdrawal from time zero to time t is given by $-X_t \displaystyle\int_0^t \frac{G_u}{X_u}\, du$.

Let $V(W, r, t)$ denote the no-arbitrage value of the variable annuity with the GMWB subject to both equity and interest rate risks. In terms of $\widetilde{W}_t$, the variable annuity value is given by

$$\begin{aligned} V(W, r, t) &= E_Q\left[e^{-\int_t^T r_u \, du} \widetilde{W}_{T \wedge \tau_0} + \int_t^T e^{-\int_t^u r_s \, ds} G_u \, du \middle| \mathcal{F}_t \right] \\ &= E_Q\left[e^{-\int_t^T r_u \, du} \max(\widetilde{W}_T, 0) + \int_t^T e^{-\int_t^u r_s \, ds} G_u \, du \middle| \mathcal{F}_t \right]. \end{aligned} \tag{2.9}$$

The mathematical justification of replacing $\widetilde{W}_{T\wedge\tau_0}$ by the optionality payoff $\max(\widetilde{W}_T, 0)$ is shown in Appendix A. At $t = 0$, we have

$$\begin{aligned} V(W, r, 0) &= \int_0^T G_u D(0, u)\, du \\ &\quad + w_0 E_Q \left[e^{-\int_0^T r_u\, du} X_T \max\left(1 - \int_0^T \frac{G_u}{w_0 X_u} du, 0\right)\right]. \end{aligned} \tag{2.10}$$

It is convenient to define

$$A_t = \int_0^t \frac{G_u}{w_0 X_u}\, du. \tag{2.11}$$

Hence, the time-0 value of the variable annuity with the GMWB can be expressed as

$$V(W, r, 0) = \int_0^T G_u D(0, u)\, du + w_0 E_Q \left[e^{-\int_0^T r_u\, du} X_T \max(1 - A_T, 0)\right]. \tag{2.12}$$

This representation formula indicates that the variable annuity with the GMWB can be decomposed into a term-certain annuity paying G_t per annum over the life of the contract and a "generalized" put option on some path dependent function of the value of the personal sub-account. The path dependent state variable A_t captures the depletion of the personal sub-account due to the continuous withdrawal process.

The valuation of the above put option term requires the joint dynamics of r_t, X_t and A_t under Q. Fortunately, the expectation calculation procedure can be much simplified under the corresponding new measure Q_S with S_t as the numeraire. Let M_t denote the money market account process. The corresponding Radon-Nikodym derivative associated with the change of measure from Q to Q_S is given by

$$\left.\frac{dQ_S}{dQ}\right|_{\mathcal{F}_T} = \frac{S_T/S_0}{M_T/M_0},$$

so that the value of the "generalized" put option is given by

$$\begin{aligned} & E_Q \left[e^{-\int_0^T r_u\, du} X_T \max(1 - A_T, 0)\right] \\ = \;& E_{Q_S} \left[\frac{M_0}{M_T} \left(\frac{M_T}{M_0} \Big/ \frac{S_T}{S_0} \right) X_T \max(1 - A_T, 0)\right] \\ = \;& e^{-\alpha T} E_{Q_S} \left[\max(1 - A_T, 0)\right]. \end{aligned} \tag{2.13}$$

By the Girsanov Theorem, $\boldsymbol{B}_t^{Q_S}$ is Q_S-Brownian. Also, $\boldsymbol{B}_t$ and $\boldsymbol{B}_t^{Q_S}$ are related by

$$d\boldsymbol{B}_t^{Q_S} = d\boldsymbol{B}_t - \boldsymbol{\sigma}_S\, dt.$$

The dynamics of $\dfrac{D(t,T)}{S_t}$ under Q_S can be shown to be

$$d\left(\frac{D(t,T)}{S_t}\right) = \frac{D(t,T)}{S_t}\left[\boldsymbol{\sigma}_D(t,T) - \boldsymbol{\sigma}_S\right] d\boldsymbol{B}_t^{Q_S}. \tag{2.14}$$

We write

$$\boldsymbol{\sigma}_{Q_S}(t,T) = \boldsymbol{\sigma}_D(t,T) - \boldsymbol{\sigma}_S,$$

and observe $D(t,t) = 1$, so we obtain

$$\frac{1}{X_t} = e^{\alpha t}\frac{S_0}{S_t} = D(0,t)e^{\alpha t}e^{-\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T\, du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S}}. \tag{2.15}$$

Decomposition of the value of the GMWB

From the perspective of the insurer, she receives the proportional fees until the personal sub-account hits the zero value at the random time τ_0. After then, under the GMWB, the insurer has to pay the guaranteed withdrawal stream to the policyholder until maturity T. Therefore, the time-0 value of the liability to the insurer associated with the GMWB is given by

$$L = E_Q\left[\int_{\tau_0\wedge T}^{T} e^{-\int_0^t r_u\, du} G_t\, dt - \int_0^{\tau_0\wedge T} \alpha e^{-\int_0^t r_u\, du} W_t\, dt\right],$$

where W_t is the time-t value of the sub-account. Interestingly, it can be shown that (see Appendix B)

$$L \;=\; V(W,r,0) - w_0. \tag{2.16}$$

This agrees with the financial intuition that the time-0 value of the GMWB is equal to the initial premium w_0 plus the time-0 value of the insurer's liability.

3 Analytic approximation formulas

It has been shown in the last section that pricing of the GMWB with both equity and interest rate risks amounts to the evaluation of the put option value: $E_{Q_S}[\max(1-A_T,0)]$, whose closed form solution does not exist. Using similar techniques that have been developed for finding the analytic approximation solutions to Asian option models, we show how to obtain the lower and upper bound of the above put option value.

Lower bound using Rogers-Shi's method

We follow Rogers-Shi's method (1995) to deduce the lower bound of the expectation term: $E_{Q_S}[\max(1 - A_T, 0)]$. By Jensen's inequality, we have

$$\begin{aligned} & E_{Q_S}[\max(1 - A_T, 0)] \\ = \ & E_{Q_S}\left[E_{Q_S}\left[\max(1 - A_T, 0)|Z\right]\right] \\ \geq \ & E_{Q_S}\left[\max\left(E_{Q_S}\left[1 - A_T|Z\right], 0\right)\right], \end{aligned} \tag{3.1}$$

where Z is a conditional variable. On the other hand, we can deduce

$$\begin{aligned} & E_{Q_S}[\max(1 - A_T, 0)|Z] - E_{Q_S}[\max(E_{Q_S}[1 - A_T|Z], 0] \\ \leq \ & \frac{1}{2}E_{Q_S}[\sqrt{\mathrm{var}(A_T|Z)}]. \end{aligned}$$

The quality of the above lower bound

$$\ell_Z = E_{Q_S}\left[\max(E_{Q_S}[1 - A_T|Z], 0)\right] \tag{3.2}$$

is highly dependent on the choice of Z. Here, we choose Z such that $E_{Q_S}[\sqrt{\mathrm{var}(A_T|Z)}]$ is minimized. Following a similar choice as shown in Rogers and Shi (1995), we choose

$$Z = \frac{1}{\Sigma}\int_0^T \left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u, t)\, d\boldsymbol{B}_u^{Q_S}\right) dt, \tag{3.3}$$

where

$$\Sigma^2 = \mathrm{var}_{Q_S} \int_0^T \left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u, t)\, d\boldsymbol{B}_u^{Q_S}\right) dt.$$

It can be shown that Z is a standard normal distribution under Q_S.

Theorem 1

A lower bound on the value of the "generalized" put option defined in Eq. (3.2) is given by

$$\ell_Z = N(z_2) - \frac{1}{w_0}\int_0^T G_t D(0, t) e^{\alpha t} N(z_2 - m(t))\, dt,$$

where z_2 is the larger root of the following equation:

$$F(z) = 1 - \frac{1}{w_0} \int_0^T G_t D(0,t) e^{\alpha t} g(z)\, dt = 0.$$

Here, $g(z) = e^{m_t z - \frac{m_t^2}{2}}$ and

$$m_t = \frac{1}{\sum} \int_0^t \left(\boldsymbol{\sigma}_{Q_S}(u,t) \int_u^T \boldsymbol{\sigma}_{Q_S}(u,s)\, ds \right) du.$$

The proof of Theorem 1 is shown in Appendix C.

Upper bound using Thompson's method

We follow Thompson's method (1999) to deduce the upper bound on the expectation term: $E_{Q_S}\,[\max(1 - A_T, 0)]$. Suppose f_t is a random function such that $\frac{1}{T}\int_0^T f_t\, dt = 1$ and the withdrawal rate G_t is constant, then an upper bound can be deduced as follows:

$$\begin{aligned} E_{Q_S}[\max(1 - A_T, 0)] &= \frac{1}{T} E_{Q_S}\left[\max\left(\int_0^T \left(f_t - \frac{1}{X_t}\right) dt, 0\right)\right] \\ &\leq \frac{1}{T}\int_0^T E_{Q_S}\left[\max\left(f_t - \frac{1}{X_t}, 0\right)\right] dt. \end{aligned} \tag{3.4}$$

As suggested by Lord (2005), we choose

$$f_t = \mu_t + \beta \left[\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S} - \frac{1}{T}\int_0^T \left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S} \right) dt \right], \tag{3.5}$$

where β is a deterministic parameter and μ_t is a deterministic function which satisfies

$$\frac{1}{T}\int_0^T \mu_t\, dt = 1.$$

Also, we choose

$$\eta_t = \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S} - \frac{1}{T}\int_0^T \left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S} \right) dt \tag{3.6}$$

and a normal variable

$$\widehat{Z} = \frac{1}{\widehat{\Sigma}_t} \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S}$$

with

$$\widehat{\Sigma}_t^2 = \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T \, du.$$

Conditioning on $\widehat{Z}$, $f_t - \dfrac{1}{X_t}$ is known to be distributed as

$$\mu_t - \frac{1}{X_t} + \beta \left(E_{Q_S}[\eta_t|\widehat{Z}] + \sqrt{\mathrm{var}_{Q_S}(\eta_t|\widehat{Z})\epsilon} \right),$$

where ϵ is a standard normal variable. Let $n(\cdot)$ denote the density function of a standard normal random variable. As a result, we manage to obtain an estimate on the upper bound of the put value, the result of which is summarized in Theorem 2.

Theorem 2

An upper bound on $E_{Q_S}[\max(1 - A_T, 0]$, as defined in Eq. (3.4), is given by

$$\begin{aligned} & E_{Q_S}\left[\max(1 - A_T, 0)\right] \\ \leq \; & \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} \left[a(t,z) N\left(\frac{a(t,z)}{b(t,z)}\right) + b(t,z) n\left(\frac{a(t,z)}{b(t,z)}\right)\right] n(z)\, dz\, dt, \end{aligned}$$

where $a(t,z)$ and $b(t,z)$ are defined by

$$\begin{aligned} a(t,z) &= \mu_t - D(0,t) e^{\alpha t - \frac{\widehat{\Sigma}_t^2}{2} + \widehat{\Sigma}_t z} + \beta \left(\widehat{\Sigma}_t - \frac{\Sigma m_t}{T \widehat{\Sigma}_t} \right) z \\ b(t,z) &= \beta \sqrt{\mathrm{var}(\eta_t|\widehat{Z})} = \frac{\beta \Sigma}{T} \sqrt{1 - \frac{m_t^2}{\widehat{\Sigma}_t^2}}. \end{aligned}$$

Remark

The tightness of the upper bound depends sensibly on the choice of the parameter β and the function μ_t. We would like to find β and μ_t such that

$$\frac{1}{T}\int_0^T E_{Q_S}\left[\max\left(f_t - \frac{1}{X_t}, 0\right)\right] dt$$

is minimized, in addition to the observation of the constraint

$$\frac{1}{T}\int_0^T \mu_t \, dt = 1.$$

Accordingly, we define the Lagrangian

$$L(\mu_t, \beta; \lambda) = \frac{1}{T} \int_0^T E_{Q_S} \left[\max \left(f_t - \frac{1}{X_t}, 0 \right) \right] dt - \lambda \left(\frac{1}{T} \int_0^T \mu_t \, dt - 1 \right). \tag{3.7}$$

The first order condition for the optimality of μ_t gives

$$Q_S \left[\frac{1}{X_t} - \beta \eta_t \leq \mu_t \right] = \lambda.$$

By applying the approximation: $e^x \approx 1 + x$, we use

$$\widetilde{Y} = D(0,t) e^{\alpha t - \frac{\widehat{\Sigma}_t^2}{2}} \left(1 + \widehat{\Sigma}_t \widehat{Z} \right) - \beta \eta_t$$

to approximate $\dfrac{1}{X_t} - \beta \eta_t$. Note that $\widetilde{Y}$ is normally distributed with mean $D(0,t) e^{\alpha t - \frac{\widehat{\Sigma}_t^2}{2}}$ and its variance is given by

$$\mathrm{var}(\widetilde{Y}) = c_t^2 \widehat{\Sigma}_t^2 + \frac{2\beta c_t \Sigma m_t}{T} + \frac{\beta^2 \Sigma^2}{T^2},$$

where

$$c_t = D(0,t) e^{\alpha t - \frac{\widehat{\Sigma}_t^2}{2}} - \beta.$$

We write

$$\zeta = N^{-1}(\lambda) = \frac{\mu_t - D(0,t) e^{\alpha t - \frac{\widehat{\Sigma}_t^2}{2}}}{\sqrt{\mathrm{var}(\widetilde{Y})}}$$

so that

$$\mu_t = D(0,t) e^{\alpha t - \frac{\widehat{\Sigma}_t^2}{2}} + \zeta \sqrt{\mathrm{var}(\widetilde{Y})}. \tag{3.8}$$

The constant ζ is determined by the condition:

$$\frac{1}{T} \int_0^T \mu_t \, dt = 1.$$

Next, we derive the optimal condition for the determination of the parameter β. Similarly, we apply the first order derivative condition of $L(\mu_t, \beta; \lambda)$ with respect to β and obtain

$$
\begin{aligned}
0 &= \frac{1}{T}\int_0^T E_{Q_S}\left[\eta_t \mathbf{1}_{\{\mu_t+\beta\eta_t-\frac{1}{X_t}>0\}}\right] dt \\
&= \int_0^T E_{Q_S}\left[\eta_t N\left(\frac{\ln\left(\frac{\mu_t+\beta\eta_t}{D(0,t)e^{\alpha t}}\right) - E_{Q_S}\left[\xi_t|\eta_t\right]}{\sqrt{\mathrm{var}_{Q_S}(\xi_t|\eta_t)}}\right)\right] dt, \qquad (3.9)
\end{aligned}
$$

where ξ_t is defined in Eq. (C.1) in Appendix C, and

$$
\begin{aligned}
E_{Q_S}[\xi_t|\eta_t] &= -\frac{\widehat{\Sigma}_t^2}{2} + \frac{\mathrm{cov}(\xi_t,\eta_t)}{\mathrm{var}(\eta_t)}\eta_t \\
\mathrm{var}_{Q_S}(\xi_t|\eta_t) &= \widehat{\Sigma}_t^2 - \frac{\mathrm{cov}(\xi_t,\eta_t)^2}{\mathrm{var}(\eta_t)} \\
\mathrm{cov}(\xi_t,\eta_t) &= \widehat{\Sigma}_t^2 - \frac{\Sigma m_t}{T} \\
\mathrm{var}(\eta_t) &= \widehat{\Sigma}_t^2 - \frac{2\Sigma m_t}{T} + \frac{\Sigma^2}{T^2}.
\end{aligned}
$$

4 Numerical performance of the analytic approximation formulas and pricing behavior of the GMWB

First, we present the numerical experiments that were performed to access the tightness of the lower and upper bound on the put option value embedded in the GMWB. Recall that the put option value is given by $E_{Q_S}[\max(1-A_T, 0)]$. The basic set of parameter values employed in the numerical calculations are: $\theta = 0.05, k = 0.0349, \alpha = 0.006$. In our calculations, we vary the instantaneous correlation coefficient ρ, maturity T, interest rate volatility σ_r and fund value volatility σ_S. We also compute the put option value using direct Monte Carlo simulation with $100,000$ simulation trials. The standard deviation of the Monte Carlo simulation results is typically less than 0.1% of the option value. The simulation results are used to serve as the benchmark for comparing the numerical results obtained from the analytic approximation formulas of finding the lower and upper bound. The lower bound is easier and more efficient to be computed since the calculations involve one-dimensional integrals only. It is less computationally efficient to compute the upper bound since two-dimensional integrals are involved in the calculations. Also, the formulation of the Thompson's upper bound is limited to uniform withdrawal rate.

In Table 1, we list the numerical values of the put option value obtained from the analytic approximation formulas and Monte Carlo simulation with varying values of the different parameters in the pricing model. The withdrawal rate is assumed to be uniform throughout the life of the contract. The lower bound values are seen to be highly accurate with percentage error less than 1%. The upper bound values are less tight when compared to the corresponding lower bound values. The percentage error may increase as high as 4% when $T = 15$ and $\sigma_S = 0.4$. In general, the accuracy of the analytic approximation values decreases with increasing fund value volatility σ_S, interest rate volatility σ_r and maturity T. We conclude that the Rogers-Shi approach of computing the lower bound generates sufficiently accurate approximation solutions for practical valuation of the fair value of the annuities. The valuation of Rogers-Shi's approximation to the put value is computationally efficient compared to the Monte Carlo simulation. The valuation of option values in our subsequent analysis of pricing properties of the GMWB had been performed using Rogers-Shi's lower bound approximation.

Next, we explore the pricing behavior of the GMWB with respect to different parameter values in the pricing model. In particular, we would like to examine the dependence of the put option value on varying values of the interest rate volatility σ_r and instantaneous correlation coefficient ρ. In Figure 1, we plot the put option value against σ_r with $\rho = 0.2$, $\rho = 0$ and $\rho = -0.2$, respectively. The withdrawal rate is uniform and the other model parameter values are: $\sigma_S = 0.2$, $T = 10$, $\theta = 0.05$, $k = 0.0349$, $\alpha = 0.006$. With non-negative instantaneous correlation coefficient ($\rho = 0$ and $\rho = 0.2$), the put option value is an increasing function of σ_r. However, when the instantaneous correlation coefficient is negative ($\rho = -0.2$), the put option value first decreases with increasing σ_r until a minimum value is reached, then subsequently increases with increasing σ_r. A simple explanation to the above phenomenon can be offered by examining the dependence of $\boldsymbol{\sigma}_{Q_S}(u,t)$ on ρ and σ_r. Recall that

$$\begin{aligned}\boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T &= \|\boldsymbol{\sigma}_{Q_S}(u,t)\|^2 \\ &= \sigma_S^2 + 2\rho b(u,t)\sigma_S\sigma_r + b(u,t)^2\sigma_r^2.\end{aligned}$$

When ρ is non-negative, $\|\boldsymbol{\sigma}_{Q_S}(u,t)\|^2$ is always an increasing function of σ_r. However, when ρ is negative, $\|\boldsymbol{\sigma}_{Q_S}(u,t)\|^2$ is a decreasing function of σ_r if $\sigma_r < -\dfrac{\rho\sigma_S}{b(u,t)}$, and it becomes an increasing function if $\sigma_r > -\dfrac{\rho\sigma_S}{b(u,t)}$.

Since $\|\boldsymbol{\sigma}_{Q_S}(u,t)\|^2$ is an increasing function of ρ, so we expect that the put option value is an increasing function of ρ. This property is confirmed by the plot of the put option value against ρ in Figure 2. The option value under constant interest rate (corresponds to $\sigma_r = 0$) is independent of ρ. It is seen that when ρ is negative, the put option value decreases with increasing value of σ_r. The put option value increases with increasing σ_r when ρ becomes positive.

We analyze the impact of varying static withdrawal policies on the fair value of the put option value against interest rate volatility σ_r. We choose the following withdrawal policies in our calculations (i) uniform withdrawal over the life of a 9-year annuity contract, (ii) zero withdrawal in the first 3 years and uniform withdrawal in the remaining 6 years, (iii) steady increase in the withdrawal rate, where $G(1) = 3\%$, $G(2) = 5\%$, $G(3) = 7\%$, $\cdots$, $G(8) = 17\%$ and $G(9) = 20\%$. It is revealed in Figure 3 that the put option value has a higher value and becomes less dependent on σ_r under the third withdrawal policy. When more is withdrawn at the later life of the contract, the chance that the put option being in-the-money is higher and so susceptibility to interest rate fluctuations become less.

Lastly, we compute the fair rate of proportional fees α to be charged to cover the embedded put option in the GWMB. The three curves in Figure 4 show the plot of α against maturity T with constant interest rate (solid curve) and stochastic interest rates (the upper dotted curve corresponds to $\sigma_r = 0.3$ and the middle dashed curve corresponds to $\sigma_r = 0.2$.). When faced with higher interest rate risk, we expect that α should be higher. Under constant interest rate, the insurer charges a lower rate of proportional fees to cover the embedded put option when the life of the annuity policy is lengthened. However, when the interest rate volatility is sufficiently high, the rate of proportional fees may increase with increasing maturity.

5 Conclusion

We have considered the pricing of the embedded Guaranteed Minimum Withdrawal Benefit (GMWB) in variable annuities with both equity and interest rate risks under static withdrawal policies. The value of the GMWB can be decomposed into a term-certain annuity and a put option. Also, the fair value of the contract is shown to be equal to the sum of the insurer's liabilities and the initial premium. We apply the extension of Rogers-Shi's technique and Thompson's method to deduce the respective lower and the upper bound of the option value, respectively. The numerical accuracy of these analytic approximation formulas is found to be sufficiently accurate even under long maturity and high volatility. The pricing properties of the GMWB value under varying values of interest rate volatility and instantaneous correlation coefficient (between equity and interest rate risks) are examined. Interestingly, when the instantaneous correlation coefficient is negative, the value of the embedded put option may first decrease with increasing interest rate volatility and then becomes increasing at sufficiently high level of interest rate volatility. Also, we have shown that the GMWB value is highly dependent on the withdrawal policies adopted by the policyholder. In addition, we analyze the impact of stochastic interest rates on the fair value of the proportional fees to be charged for the provision of the benefit.

References

Bauer, D., A. Kling and J. Russ (2008), "A universal pricing framework for guaranteed minimum benefits in variable annuities," *Astin Bulletin*, vol. 38(2), 621-651.

Chen, Z. and P.A. Forsyth (2008), "A numerical scheme for the impulse control formulation for pricing variable annuities with a guaranteed minimum withdrawal benefit (GMWB)," *Numerische Mathematik*, vol. 109, 535-569.

Chen, Z., K. Vetzal and P.A. Forsyth (2008), "The effect of modelling parameters on the value of GMWB guarantee," *Insurance: Mathematics and Economics*, vol. 43(1), 165-173.

Dai, M., Y.K. Kwok and J. Zong (2008), "Guaranteed minimum withdrawal benefit in variable annuities," *Mathematical Finance*, vol. 8(6), 561-569.

Karatzas, I. and S.E. Shreve (1992), *Brownian motion and stochastic calculus*, second edition, Springer, New York.

Kijima, M. and T. Wong (2007), "Pricing of ratchet equity-indexed annuities under stochastic interest rates," *Insurance: Mathematics and Economics*, vol. 41(3), 317-338.

Lin, X.S. and K.S. Tan (2003), "Valuation of equity-indexed annuities under stochastic interest rates," *North American Actuarial Journal*, vol. 6, 72-91.

Lord, R. (2006), "Partially exact and bounded approximations for arithmetic Asian options," *Journal of Computational Finance*, vol. 10(2), 1-52.

Milevsky, M.A. and T.S. Salisbury (2006), "Financial valuation of guaranteed minimum withdrawal benefits," *Insurance: Mathematics and Economics*, vol. 38(1), 21-38.

Rogers, L.C.G. and Z. Shi (1995), "The value of an Asian option," *Journal of Applied Probability*, vol. 32, 1077-1088.

Thompson, G.W.P. (1999), "Fast narrow bounds on the value of Asian options," Working paper of University of Cambridge.

Appendix A – proof of Eq. (2.10)

Intuitively, once the unrestricted process $\widetilde{W}_t$ becomes negative, it will never return to the positive region again. This is because once $\widetilde{W}_t$ hits the zero value, the drift becomes negative and the random term becomes zero, so $\widetilde{W}_t$ is pulled back into the negative region immediately. It suffices to show that $\tau_0 > T$ if and only if $\widetilde{W}_T > 0$.

"$\Longrightarrow$" part
Suppose $\tau_0 > T$, by the definition of the first passage time, we then have $\widetilde{W}_T > 0$.

"$\Longleftarrow$" part
Recall that

$$\widetilde{W}_t = X_t \left(w_0 - \int_0^t \frac{G_u}{X_u}\, du \right)$$

so that

$$\widetilde{W}_t > 0 \quad \text{if and only if} \quad \int_0^t \frac{G_u}{X_u}\, du < w_0.$$

Suppose $\widetilde{W}_T > 0$, this implies that

$$\int_0^T \frac{G_u}{X_u}\, du < w_0.$$

Since $X_u \geq 0$, so for any $t < T$, we have

$$\int_0^t \frac{G_u}{X_u}\, du \leq \int_0^T \frac{G_u}{X_u}\, du < w_0.$$

Therefore, if $\widetilde{W}_T > 0$, then $\widetilde{W}_t > 0$ for any $t < T$.

Appendix B – proof of Eq. (2.16)

From the dynamics of W_t, we have

$$\alpha W_t\,dt = r_t W_t\,dt - dW_t - G_t\,dt + W_t \boldsymbol{\sigma}_S\,d\boldsymbol{B}_t, \quad 0 \le t < \tau_0.$$

Multiplying by the discount factor $e^{-\int_0^t r_u\,du}$ and integrating from 0 to $\tau_0 \wedge T$, we obtain

$$\begin{aligned}
\int_0^{\tau_0\wedge T} e^{-\int_0^t r_u\,du}\alpha W_t\,dt &= -\int_0^{\tau_0\wedge T} d(e^{-\int_0^t r_u\,du}W_t) - \int_0^{\tau_0\wedge T} e^{-\int_0^t r_u\,du}G_t\,dt \\
&\quad + \int_0^{\tau_0\wedge T} e^{-\int_0^t r_u\,du}W_t\boldsymbol{\sigma}_S\,d\boldsymbol{B}_t \\
&= w_0 - W_{\tau_0\wedge T}e^{-\int_0^{\tau_0\wedge T} r_u\,du} - \int_0^{\tau_0\wedge T} e^{-\int_0^t r_u\,du}G_t\,dt \\
&\quad + \int_0^{\tau_0\wedge T} e^{-\int_0^t r_u\,du}W_t\boldsymbol{\sigma}_S\,d\boldsymbol{B}_t.
\end{aligned}$$

Rearranging the above terms and observing

$$W_{\tau_0\wedge T}e^{-\int_0^{\tau_0\wedge T} r_u\,du} = W_T e^{-\int_0^T r_u\,du},$$

we obtain

$$\begin{aligned}
&\int_{\tau_0\wedge T}^{T} e^{-\int_0^t r_u\,du}G_t\,dt - \int_0^{\tau_0\wedge T} \alpha e^{-\int_0^t r_u\,du}W_t\,dt \\
=\ & W_T e^{-\int_0^T r_u\,du} - w_0 + \int_0^T e^{-\int_0^t r_u\,du}G_t\,dt - \int_0^{\tau_0\wedge T} e^{-\int_0^t r_u\,du}W_t\boldsymbol{\sigma}_S\,d\boldsymbol{B}_t.
\end{aligned}$$

Lastly, by taking the expectation under Q, we obtain the result in Eq. (2.16).

Appendix C – proof of Theorem 1

We define

$$\xi_t = -\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T\,du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\,d\boldsymbol{B}_u^{Q_S}, \qquad \text{(C.1)}$$

which is also normal. For a pair of joint normal random variables Z_1 and Z_2, the well known Projection Theorem states that

$$E[Z_1|Z_2] = E[Z_1] + \frac{\mathrm{cov}(Z_1,Z_2)}{\mathrm{var}(Z_2)}(Z_2 - E[Z_2])$$

and

$$\mathrm{var}(Z_1|Z_2) = \mathrm{var}(Z_1) - \frac{\mathrm{cov}(Z_1,Z_2)^2}{\mathrm{var}(Z_2)}.$$

Applying the above relations for conditional expectation of normal random variables, we obtain

$$\begin{aligned} E_{Q_S}[\xi_t|Z] &= -\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T\,du + Zm_t \\ \mathrm{var}_{Q_S}(\xi_t|Z) &= \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T\,du - m_t^2, \end{aligned}$$

where

$$m_t = E_{Q_S}[\xi_t Z] = \frac{1}{\Sigma}\int_0^t \left[\boldsymbol{\sigma}_{Q_S}(u,t)\int_u^T \boldsymbol{\sigma}_{Q_S}(u,s)\,ds\right] du.$$

We then obtain $E_{Q_S}\left[\frac{1}{X_t}\middle| Z\right]$ in terms of m_t and Z as follows:

$$\begin{aligned} E_{Q_S}\left[\frac{1}{X_t}\middle| Z\right] &= D(0,t)e^{\alpha t}E_{Q_S}\left[e^{\xi_t}|Z\right] \\ &= D(0,t)e^{\alpha t}e^{Zm_t - \frac{m_t^2}{2}}. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} \ell_Z &= E_{Q_S}\left[\max(E_{Q_S}[1 - A_T|Z], 0\right] \\ &= E_{Q_S}\left[\max\left(1 - \frac{1}{w_0}\int_0^T G_t D(0,t)e^{\alpha t}g(Z)\,dt, 0\right)\right], \end{aligned}$$

where

$$g(z) = e^{m_t z - \frac{m_t^2}{2}}.$$

The function $g(z)$ is an increasing (decreasing) convex function for $m_t > 0\,(m_t < 0)$. Since the sum of convex functions remains to be convex, so the following equation:

$$F(z) = 1 - \frac{1}{w_0}\int_0^T G_t D(0,t) e^{\alpha t} g(z)\, dt = 0$$

either has zero, unique or two solutions. Let z_1 and z_2, $z_1 < z_2$, denote the two possible solutions of $F(z) = 0$. We then have

$$F(z) > 0 \quad \text{for} \quad z_1 < z < z_2.$$

For notational convenience, if $F(z) = 0$ has no solution, we set $z_1 = z_2 = \infty$; and if only one solution exists, then either we set $z_1 = -\infty$ or $z_2 = \infty$ depending on the sign of $F'(z)$. The roots of $F(z)$ can be found easily using any root-finding algorithm. In terms of z_1 and z_2, the lower bound ℓ_Z can be evaluated as follows:

$$\begin{aligned}
\ell_Z &= E_{Q_S}\left[\mathbf{1}_{\{z_1 < z < z_2\}}\left[1 - \frac{1}{w_0}\int_0^T G_t D(0,t) e^{\alpha t} g(z)\, dt\right]\right] \\
&= N(z_2) - N(z_1) - \frac{1}{w_0}\int_0^T G_t D(0,t) e^{\alpha t} \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-m_t)^2}{2}}\, dz\, dt \\
&= N(z_2) - N(z_1) - \frac{1}{w_0}\int_0^T G_t D(0,t) e^{\alpha t}\left[N(z_2 - m(t)) - N(z_1 - m(t))\right] dt,
\end{aligned}$$

where $N(\cdot)$ is the standard normal distribution function. Suppose $m_t > 0$ for $t \in [0, T]$, then $F(z)$ has unique root and z_1 is set to be $-\infty$. This property on m_t is commonly observed since B_t and Z are positively correlated. Under this condition, we obtain the following lower bound

$$\begin{aligned}
& E_{Q_S}\left[\max(1 - A_T, 0)\right] \\
\geq\ & N(z_2) - \frac{1}{w_0}\int_0^T G_t D(0,t) e^{\alpha t} N(z_2 - m(t))\, dt.
\end{aligned}$$

T	σ_r	σ_S	ρ	optimal β	lower bound	percentage error of lower bound	upper bound	percentage error of upper bound	Monte Carlo simulation result
10	0.01	0.2	-0.2	0.8741	0.2401	0.13%	0.2418	0.56%	0.2404
			0	0.8741	0.2436	0.14%	0.2454	0.61%	0.2440
			0.2	0.8741	0.2470	0.26%	0.2490	0.56%	0.2476
		0.3	-0.2	0.7615	0.2933	0.33%	0.2985	1.45%	0.2942
		0.4	-0.2	0.6501	0.3476	0.47%	0.3590	2.81%	0.3492
	0.02	0.2	-0.2	0.8607	0.2368	0.08%	0.2386	0.70%	0.2370
		0.3	-0.2	0.7524	0.2892	0.32%	0.2946	1.52%	0.2902
	0.03	0.2	-0.2	0.8454	0.2338	0.10%	0.2361	0.89%	0.2340
		0.3	-0.2	0.7487	0.2851	0.23%	0.2908	2.00%	0.2851
15	0.01	0.2	-0.2	0.8063	0.3096	0.21%	0.3132	0.95%	0.3102
			0	0.7804	0.3147	0.22%	0.3188	1.06%	0.3154
			0.2	0.7632	0.3197	0.18%	0.3242	1.23%	0.3203
		0.3	-0.2	0.6620	0.3655	0.33%	0.3752	2.31%	0.3667
		0.4	-0.2	0.5414	0.4234	0.63%	0.4413	3.57%	0.4261
	0.02	0.2	-0.2	0.7774	0.3021	0.31%	0.3068	1.22%	0.3031
		0.3	-0.2	0.6574	0.3576	0.33%	0.3680	2.58%	0.3588
	0.03	0.2	-0.2	0.7405	0.2946	0.20%	0.3012	2.02%	0.2952
		0.3	-0.2	0.6375	0.3486	0.42%	0.3608	3.06%	0.3501

Table 1 Comparison of numerical accuracy of the lower bound and upper bound on the put option values. The numerical results obtained from Monte Carlo simulation serve as the benchmark for comparison. The basic set of parameter values used in the pricing model are: $\theta = 0.05, k = 0.0349, \alpha = 0.006$. The percentage errors of the lower bound values are less than 1% while the percentage errors of the upper bound values may reach as high as 4% under long maturity and high volatility values.