Saddlepoint approximations to tail expectations under non-Gaussian base distributions: option pricing applications

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Abstract

The saddlepoint approximation formulas provide versatile tools for analytic approximation of the tail expectation of a random variable by approximating the complex Laplace integral of the tail expectation expressed in terms of the cumulant generating function of the random variable. We generalize the saddlepoint approximation formulas for calculating tail expectations from the usual Gaussian base distribution to an arbitrary base distribution. Specific discussion is presented on the criteria of choosing the base distribution that fits better the underlying distribution. Numerical performance and comparison of accuracy are made among different saddlepoint approximation formulas. Improved accuracy of the saddlepoint approximations to tail expectations is revealed when proper base distributions are chosen. We also demonstrate enhanced accuracy of the generalized saddlepoint approximation formulas under non-Gaussian base distributions in pricing European options on continuous integrated variance under the Heston stochastic volatility model.

1 Introduction

In many applications in statistics and financial engineering, it is necessary to compute the tail probabilities and tail expectations of random variables using analytic approximation formulas. When the closed form formulas are not available, the saddlepoint approximation methods have been proven to be versatile tools in deriving effective analytic approximation formulas for calculating tail probabilities and tail expectations. The success of the saddlepoint approximation approach relies on the mathematical properties that complex Laplace integrals of the tail probabilities and tail expectations are expressible in terms of cumulant generating functions. Summary of the saddlepoint approximation methods and their applications in statistics can be found in the two comprehensive texts by Jensen (1995) and Butler (2007). Since tail expectation is related to evaluation of discounted expectation of the terminal payoff in a call option, several papers have shown applications of the saddlepoint approximation methods in pricing various types of options (Rogers and Zane, 1999; Xiong *et al.*, 2005; Carr and Madan, 2009; Zheng and Kwok, 2014). Also, the expected shortfall as an effective risk measure is related to the expectation of the tail part of the loss distribution of a credit portfolio. There has been a growing literature on the use of saddlepoint approximation methods in the calculations of risk measures. Broda and Paolella (2012) review the applications of saddlepoint approximations in risk management. Kwok and Zheng (2018) summarize the use

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of saddlepoint approximations in option pricing and credit portfolio calculations. Their text also provides updated literature on the recent research works on these topics.

In the pioneering work by Daniels (1954), the steepest descent method is used to derive the saddlepoint approximation formula for calculating the density function of the mean of a large number of independent and identically distributed (iid) random variables. His saddlepoint formula is expressed as an asymptotic series in powers of the reciprocal of the square root of number of iid random variables. Though the original derivation assumes a large sample size, the saddlepoint approximation formula works well even with a very small sample size, a phenomenon that is called "small sample asymptotic approximation". In his later work, Daniels (1980) shows that the normal, Gamma and inverse normal are the only possible distributions for which the saddlepoint approximation formulas are exact in calculating the density of the mean of iid random variables. For tail probabilities, the saddlepoint approximation formula of Lugannani and Rice (1980) is considered to be the most renowned among all other versions of saddlepoint approximations. Their formula is derived based on the choice of the Gaussian distribution for approximating the distribution of the underlying random variable. Daniels (1987) also derives the saddlepoint approximations to tail probabilities based on the approach of the Edgeworth expansion of the exponentially tilted density recentered at the mean. Wood *et al.* (1993) extend the Lugannani-Rice tail probability approximation formula to the non-Gaussian base distribution. Booth and Wood (1995) illustrate with an example in which the performance of the Lugannani-Rice formula under the Gaussian base distribution can be rather poor. Their modified approximation formula, in which the Gaussian base is replaced by an inverse Gaussian base, gives enhanced accuracy. In the performance assessment of the use of the Lugannani-Rice formula in credit portfolio calculations under the $CreditRisk^+$ model, Annaert *et al.* (2007) raise the concern about the potential failure of the Lugannani-Rice formula when the higher moments of the credit loss distributions become more significant. These works motivate us to explore whether an improved accuracy of the saddlepoint approximation to tail expectation can be achieved when a proper base distribution other than the Gaussian distribution is chosen. An earlier success on the use of non-Gaussian base distribution to option pricing is presented by Carr and Madan (2009). Employing the share measure where the asset price is used as a numeraire, they manage to represent the European call prices as tail probabilities. They then use Wood *et al.*'s saddlepoint approximation formula for tail probabilities together with the choice of the Gaussian less exponential distribution as the base distribution to price European vanilla options under various types of Lévy processes. However, their use of the share measure is only applicable for vanilla options. For more exotic options, like options on continuous integrated variance, option prices can only be expressed as tail expectations rather than tail probabilities.

As one of the earlier works on extending the saddlepoint approximation to pricing options in financial engineering, Martin (2006) proposes a crude saddlepoint approximation formula for calculating tail expectation. Huang and Oosterlee (2011) apply the Edgeworth expansion method to derive the saddlepoint approximations to tail expectations under the Gaussian base distribution (implicitly implied by their use of the local quadratic approximation of the exponential kernel). Zheng and Kwok (2014) derive the saddlepoint approximation to tail expectation via differentiating the Lugannani-Rice formula with respect to the tilted-parameter in the Esscher exponential tilting procedure. Though the two approaches employed in deriving the Huang-Oosterlee formula and Zheng-Kwok formula for tail expectations are very different, it is quite surprising that the analytic expressions of the two saddlepoint approximation formulas are identical under the Gaussian base distribution. However, as shown in our later discussion, the saddlepoint approximation formulas using these two approaches may differ under a non-Gaussian base distribution.

Though the saddlepoint approximations to tail probabilities under arbitrary base distributions have been known in the literature, the corresponding saddlepoint approximation formulas for tail expectations have not been found. In this paper, we generalize the saddlepoint approximation formulas for calculating tail expectations from the usual Gaussian base distribution to an arbitrary base distribution. We also discuss the criteria of choosing the base distribution that fits better the underlying distribution. This paper is organized as follows. In the next section, we derive two different saddlepoint approximation formulas for tail expectations under arbitrary base distributions using the Zheng-Kwok approach and Huang-Oosterlee method. In Section 3, we explain why matching of the fourth order moment is an appropriate criterion for finding the base distribution that matches better the underlying distribution. In Section 4, we present numerical tests that were performed to assess the performance of the saddlepoint approximation formulas for tail expectations under non-Gaussian base distributions. We also show the use of the saddlepoint approximation formulas for tail expectations under non-Gaussian base distributions to price European options on continuous integrated variance under the Heston stochastic volatility model. The non-Gaussian base distributions are chosen to be (i) Gaussian less exponential as proposed by Carr and Madan (2009), (ii) Gamma distribution. Summary and conclusive remarks are presented in the last section.

2 Saddlepoint Approximations to Tail Expectation under Non-Gaussian Base Distribution

Recall that the tail expectation of the random variable X above a fixed threshold K is defined as $E[(X-K)^+]$. In the option pricing literature in finance, the tail expectation is related to the undiscounted call option price with X as the terminal asset price and K as the strike price. Martin (2006) proposes the decomposition of the Laplace integral representation of $E[(X-K)^+]$ as follows:

$$\begin{aligned} E[(X-K)^+] &= E[X\mathbb{1}_{\{X>K\}}] - KP[X>K] \\ &= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \frac{e^{\kappa(z)-zK}}{z}[\kappa'(z)-K]\,dz \\ &= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \frac{e^{\kappa(z)-zK}}{z}(\mu-K)\,dz + \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} e^{\kappa(z)-zK}\frac{\kappa'(z)-\mu}{z}\,dz, \end{aligned} \tag{2.1}$$

where $\kappa(z)$ is the cumulant generating function (cgf) and the mean $\mu = E[X] = \kappa'(0)$. The domain of analyticity of $\kappa(z)$ is the vertical strip $\{z \in \mathbb{C} : \Gamma_- < \operatorname{Re} z < \Gamma_+\}$ and the vertical Bromwich path is chosen such that $0 < z < \Gamma_+$. Let $f(K)$ and $F(K)$ denote the density function and distribution function of X, respectively, and observe that

$$f(K) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} e^{\kappa(z)-zK}\,dz,$$

$$P[X>K] = 1 - F(K) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \frac{e^{\kappa(z)-zK}}{z}\,dz.$$

Provided that $\mu \neq K$, Martin (2006) proposes the approximation of freezing $\frac{\kappa'(z)-\mu}{z}$ at the saddlepoint $\hat{z}$, where $\hat{z}$ solves the saddlepoint equation:

$$\kappa'(z) = K.$$

This leads to the saddlepoint approximation to tail expectation in terms of the density function and distribution function as follows:

$$E[(X-K)^+] \approx (\mu-K)P[X>K] + \frac{K-\mu}{\hat{z}}f(K), \qquad \mu \neq K. \tag{2.2}$$

Higher order saddlepoint approximation to tail expectation under the Gaussian base distribution have been obtained by later works of Huang and Oosterlee (2011) and Zheng and Kwok (2014). We would like to extend their saddlepoint approximation formulas to non-Gaussian base distribution.

2.1 Zheng-Kwok approach

Zheng and Kwok (2014) apply the exponential tilting technique to find the relation between tail expectation and tail probability. The distribution $F(x;\theta)$ of the θ-tilted distribution of X is related to $F(x)$ by

$$dF(x;\theta) = e^{\theta x-\kappa(\theta)}dF(x).$$

The cgf of the θ-tilted distribution $\kappa_\theta(z)$ is related to $\kappa(z)$ via

$$\kappa_\theta(z) = \kappa(z+\theta) - \kappa(\theta).$$

When $\theta = 0$, we observe $\kappa(0) = 0$ and recover $F(x;0) = F(x)$. The decomposition in eq.(2.1) reveals an important relation between $E[(X-K)^+]$ and the derivative of $F(K;\theta)$ with respect to θ. First, we observe

$$1 - F(K;\theta) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \frac{e^{\kappa(z+\theta)-\kappa(\theta)-zK}}{z}\,dz, \qquad 0 < \xi < \Gamma_+ - \theta.$$

Differentiating both sides of the above equation with respect to θ and setting $\theta = 0$, we obtain

$$-\frac{\partial F(K;\theta)}{\partial\theta}|_{\theta=0} = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \frac{e^{\kappa(z)-zK}}{z}\left[\kappa'(z) - \kappa'(0)\right]\,dz.$$

Combining these results and observing the decomposition in eq.(2.1), the tail expectation is given by

$$\begin{aligned} E[(X-K)^+] &= [\kappa'(0) - K][1 - F(K)] - \frac{\partial F(K;\theta)}{\partial\theta}|_{\theta=0} \\ &\approx (\mu - K)P[X > K] - \frac{\partial \tilde{F}(K;\theta)}{\partial\theta}|_{\theta=0}. \end{aligned} \tag{2.3}$$

Here, we approximate $F(K;\theta)$ by $\tilde{F}(K;\theta)$, which is taken to be the saddlepoint approximation to the distribution function of the exponentially θ-tilted distribution. It is instructive to compare the respective last term in the two decomposition formulas in eqs.(2.2) and (2.3), which reveal different orders of the saddlepoint approximation. Based on eq.(2.3), the saddlepoint approximation to tail expectation under the Gaussian base distribution can be obtained by differentiation of the corresponding Lugannani-Rice formula for tail probability [see eq.(2.28) in Kwok and Zheng (2018)].

The main objective of this paper is to extend the saddlepoint approximation formula for tail expectation to an arbitrary base distribution, whose cgf is denoted by $\kappa_0(z)$. We follow a similar procedure of applying the modified Legendre-Fenchel transformation based on $\kappa_0(z)$ (Wood *et al.*, 1993), where the transformation of variables from z to w is defined by

$$\kappa_0(w) - w\kappa_0'(\hat{w}) = \kappa(z) - zK, \tag{2.4}$$

where $\hat{w}$ is first determined via the solution of the following equation

$$\kappa_0(w) - w\kappa_0'(w) = \kappa(\hat{z}) - \hat{z}K. \tag{2.5}$$

Here, $\hat{z}$ is the saddlepoint that solves $\kappa'(z) = K$. Correspondingly, $z = \hat{z}$ is matched to $w = \hat{w}$ under the transformation (2.4). According to Wood *et al.* (1993), there are two roots for $\hat{w}$, one positive and the other negative when $E[X] \neq K$. The rule of thumb is to choose $\hat{w}$ to have the same sign as that of $\hat{z}$. When $E[X] = K$, we have the degenerate case where $\hat{w} = 0$. Since we have to deal with the exponentially θ-tilted distribution of X whose cgf is $\kappa_\theta(z) = \kappa(z+\theta) - \kappa(\theta)$, the equation for the transformation of variables has to be modified accordingly based on $\kappa_\theta(z)$.

According to eq.(2.3), it is necessary to compute $\frac{\partial \tilde{F}(K;\theta)}{\partial \theta}$, where $\tilde{F}(K;\theta)$ is the saddlepoint approximation to the distribution function of the exponentially θ-tilted distribution using the base distribution with cgf $\kappa_0(z)$. The expression of $\tilde{F}(K;\theta)$ is given in Appendix A [see eq.(A.2)].

Let $f_0(x)$ and $F_0(x)$ denote the respective density function and distribution function of the base distribution with cgf $\kappa_0(z)$. For $E[X] \neq K$, the saddlepoint approximation to the tail expectation $E[(X-K)^+]$ based on $\kappa_0(z)$ is found to be

$$\begin{aligned}
&E[(X-K)^+] \\
&\approx [\kappa'(0) - K][1 - \tilde{F}(K)] \\
&\quad + f_0\left(\kappa_0'(\hat{w})\right)\left\{[K - \kappa'(0)]\left[\frac{1}{\hat{w}} - \frac{1}{\hat{w}^3 \kappa_0''(\hat{w})} - \frac{\kappa_0'''(\hat{w})}{2\hat{w}\kappa_0''(\hat{w})^{\frac{3}{2}}\hat{\mu}}\right] + \frac{\sqrt{\kappa_0''(\hat{w})}}{\hat{z}\hat{\mu}}\right\} \qquad (2.6)\\
&\quad + f_0'\left(\kappa_0'(\hat{w})\right)[K - \kappa'(0)]\left[\frac{1}{\hat{w}^2} - \frac{\sqrt{\kappa_0''(\hat{w})}}{\hat{w}\hat{\mu}}\right], \qquad E[X] \neq K,
\end{aligned}$$

where $\hat{w}$ is the solution to eq.(2.5), $\hat{\mu} = \hat{z}\sqrt{\kappa''(\hat{z})}$, $\hat{z}$ is the saddlepoint that satisfies $\kappa'(z) = K$ and

$$\tilde{F}(K) = F_0(\kappa_0'(\hat{w})) + f_0(\kappa_0'(\hat{w}))\left\{\frac{1}{\hat{w}} - \frac{1}{\hat{z}}\left[\frac{\kappa_0''(\hat{w})}{\kappa''(\hat{z})}\right]^{\frac{1}{2}}\right\}.$$

Here $\tilde{F}(K)$ is the saddlepoint approximation formula of tail probability proposed in Wood *et al.* (1993). The details of the proof of eq.(2.6) are presented in Appendix A.

When $E[X] = K$, the above saddlepoint approximation formula (2.6) becomes degenerate since $\hat{z} = \hat{w} = 0$. By considering the asymptotic limits under $\hat{w} \to 0$ and $\hat{z} \to 0$, the corresponding saddlepoint approximation formula for tail expectation becomes (see Appendix A for details)

$$\begin{aligned}
E[(X-K)^+] \approx f_0(\kappa_0'(0)) &\left\{\frac{\sqrt{\kappa_0''(0)}}{24\sqrt{\kappa''(0)}}\left[\frac{\kappa'''(0)^2}{\kappa''(0)^2} - \frac{\kappa''''(0)}{\kappa''(0)}\right]\right. \\
&\left. + \frac{\sqrt{\kappa''(0)}}{8\sqrt{\kappa_0''(0)}}\left[\frac{\kappa_0'''(0)^2}{\kappa_0''(0)^2} - \frac{\kappa_0''''(0)}{\kappa_0''(0)}\right] + \frac{1}{12}\frac{\kappa_0'''(0)}{\kappa_0''(0)}\frac{\kappa'''(0)}{\kappa''(0)} + \sqrt{\kappa_0''(0)\kappa''(0)}\right\} \qquad (2.7)\\
&+ \frac{f_0'(\kappa_0'(0))\sqrt{\kappa''(0)}\kappa_0''(0)}{6}\left[\frac{\kappa'''(0)}{\kappa''(0)^{\frac{3}{2}}} - \frac{\kappa_0'''(0)}{\kappa_0''(0)^{\frac{3}{2}}}\right], \qquad E[X] = K.
\end{aligned}$$

2.2 Huang-Oosterlee approach

Huang and Oosterlee (2011) apply the local quadratic approximation to the exponent $\kappa(z) - Kz$ in the Laplace integral to derive the saddlepoint approximation to tail expectation under the Gaussian base distribution. By following a similar procedure of applying the modified Legendre-Fenchel transformation defined in eq.(2.4), it is also possible to extend the Huang-Oosterlee approach to non-Gaussian base distribution. Instead of using the Laplace integral representation of $E[(X-K)^+]$ in eq.(2.1), we consider an alternative Laplace integral for tail expectation [see eq.(1.13c) in Kwok and Zheng (2018)] and performing the modified Legendre-Fenchel transformation (2.4). This gives

$$E[(X-K)^+] = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \frac{e^{\kappa(z)-zK}}{z^2}\,dz = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \frac{e^{\kappa_0(w)-w\kappa_0'(\hat{w})}}{z^2}\frac{dz}{dw}\,dw, \qquad (2.8)$$

where $\hat{w}$ solves eq.(2.5). Differentiating eq.(2.4) by w on both sides once and twice, and observing $K=\kappa'(\hat{z})$, we obtain the following equations:

$$\kappa_0'(w) - \kappa_0'(\hat{w}) = [\kappa'(z)-\kappa'(\hat{z})]\frac{dz}{dw},$$

$$\kappa_0''(w) = [\kappa'(z)-\kappa'(\hat{z})]\frac{d^2z}{dw^2} + \kappa''(z)\left(\frac{dz}{dw}\right)^2.$$

At $w=\hat{w}$ (or equivalently, $z=\hat{z}$), we observe $\kappa'(z)-\kappa'(\hat{z})=0$. Substituting this relation into the last equation gives

$$\frac{dz}{dw}|_{w=\hat{w}} = \frac{\sqrt{\kappa_0''(\hat{w})}}{\sqrt{\kappa''(\hat{z})}}. \qquad (2.9)$$

On the other hand, at $w=0$ (or equivalently, $z=0$), we obtain

$$\frac{dw}{dz}|_{z=0} = \begin{cases} \frac{\sqrt{\kappa''(0)}}{\sqrt{\kappa_0''(0)}} & \text{if } \hat{z}=0 \\ \frac{\kappa'(\hat{z})-\kappa'(0)}{\kappa_0'(\hat{w})-\kappa_0'(0)} & \text{if } \hat{z}\neq 0 \end{cases}. \qquad (2.10)$$

We consider the Laurent expansion at $w=0$ of the factor $\frac{1}{z^2}\frac{dz}{dw}$ inside the Bromwich integral (2.8):

$$\frac{1}{z^2}\frac{dz}{dw} = A(w) + \frac{A_1}{w} + \frac{A_2}{w^2} + \sum_{n=2}^{\infty}\frac{A_n}{w^n},$$

where $A(w)$ is the non-singular part. By choosing a closed contour C around the pole $w=0$ in the w-plane and C' is the transformed contour in the z-plane, and assuming $\hat{z}\neq 0$ (equivalently, $E[X]\neq K$), the Laurent coefficients are found to be

$$A_1 = \frac{1}{2\pi i}\oint_C \frac{1}{z^2}\frac{dz}{dw}\,dw = \frac{1}{2\pi i}\oint_{C'}\frac{1}{z^2}\,dz = 0,$$

$$A_2 = \frac{1}{2\pi i}\oint_C \frac{1}{z^2}\frac{dz}{dw}w\,dw = \frac{1}{2\pi i}\oint_{C'}\frac{w}{z^2}\,dz = \lim_{z\to 0}\frac{w}{z} = \frac{dw}{dz}|_{z=0} = \frac{\kappa'(\hat{z})-\kappa'(0)}{\kappa_0'(\hat{w})-\kappa_0'(0)},$$

The above relations are derived based on the L'Hôspital's rule and eq.(2.10). Similar to the evaluation of A_1, it is straightforward to show that $A_n=0, n>2$.

In the next step to derive the saddlepoint approximation to the tail expectation formula, we freeze the non-singular part $A(w)$ at $w=\hat{w}$. This is justifiable as an acceptable approximation since the principal contribution to the complex integral arises from the singular part. By virtue of eqs.(2.9-2.10), the value of $A(w)$ is approximated by

$$A(w)\approx A(\hat{w}) = \frac{1}{z^2}\frac{dz}{dw}|_{w=\hat{w},z=\hat{z}} - \frac{A_2}{\hat{w}^2} = \frac{\sqrt{\kappa_0''(\hat{w})}}{\hat{z}^2\sqrt{\kappa''(\hat{z})}} - \frac{\kappa'(\hat{z})-\kappa'(0)}{\hat{w}^2\left[\kappa_0'(\hat{w})-\kappa_0'(0)\right]}.$$

Let Y denote the base distribution random variable whose cgf is given by $\kappa_0(w)$, that is, $\kappa_0(w) = \ln E[e^{wY}]$. Combining all the above relations and observing

$$f_0\left(\kappa_0'(\hat{w})\right) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} e^{\kappa_0(w)-w\kappa_0'(\hat{w})}\,dw,$$

$$E\left[\left(Y-\kappa_0'(\hat{w})\right)^+\right]=\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{e^{\kappa_0(w)-w\kappa_0'(\hat{w})}}{w^2}\,dw,$$

we obtain the following saddlepoint approximation formula for tail expectation under the non-Gaussian base distribution whose cgf is $\kappa_0(z)$:

$$\begin{aligned}E[(X-K)^+]&=\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{e^{\kappa_0(w)-w\kappa_0'(\hat{w})}}{z^2}\frac{dz}{dw}\,dw\\&\approx\frac{A(\hat{w})}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}e^{\kappa_0(w)-w\kappa_0'(\hat{w})}\,dw+\frac{A_2}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{e^{\kappa_0(w)-w\kappa_0'(\hat{w})}}{w^2}\,dw\\&=f_0\left(\kappa_0'(\hat{w})\right)\left\{\frac{\sqrt{\kappa_0''(\hat{w})}}{\hat{z}^2\sqrt{\kappa''(\hat{z})}}-\frac{\kappa'(\hat{z})-\kappa'(0)}{\hat{w}^2\left[\kappa_0'(\hat{w})-\kappa_0'(0)\right]}\right\}\\&\quad+E\left[\left(Y-\kappa_0'(\hat{w})\right)^+\right]\frac{\kappa'(\hat{z})-\kappa'(0)}{\kappa_0'(\hat{w})-\kappa_0'(0)},\qquad E[X]\neq K.\end{aligned}\tag{2.11}$$

In the degenerate case $E[X]=K$, we compute the asymptotic limits of $A(\hat{w})$ and A_2 as $\hat{w}\to 0,\hat{z}\to 0$. The asymptotic results are given by

$$\lim_{\hat{z}\to 0,\hat{w}\to 0}A_2=\frac{\sqrt{\kappa''(0)}}{\sqrt{\kappa_0''(0)}},$$

$$\lim_{\hat{z}\to 0,\hat{w}\to 0}A(\hat{w})=\frac{1}{24}\left\{\sqrt{\frac{\kappa''(0)}{\kappa_0''(0)}}\left[\frac{\kappa_0''''(0)}{\kappa_0''(0)}-\frac{\kappa_0'''(0)^2}{\kappa_0''(0)^2}\right]-\sqrt{\frac{\kappa_0''(0)}{\kappa''(0)}}\left[\frac{\kappa''''(0)}{\kappa''(0)}-\frac{\kappa'''(0)^2}{\kappa''(0)^2}\right]\right\}.$$

The corresponding degenerate saddlepoint approximation formula for tail expectation under $\kappa_0(z)$ becomes

$$\begin{aligned}&E[(X-K)^+]\\&\approx\frac{f_0\left(\kappa_0'(0)\right)}{24}\left\{\sqrt{\frac{\kappa''(0)}{\kappa_0''(0)}}\left[\frac{\kappa_0''''(0)}{\kappa_0''(0)}-\frac{\kappa_0'''(0)^2}{\kappa_0''(0)^2}\right]-\sqrt{\frac{\kappa_0''(0)}{\kappa''(0)}}\left[\frac{\kappa''''(0)}{\kappa''(0)}-\frac{\kappa'''(0)^2}{\kappa''(0)^2}\right]\right\}\\&\quad+E\left[\left(Y-\kappa_0'(0)\right)^+\right]\frac{\sqrt{\kappa''(0)}}{\sqrt{\kappa_0''(0)}},\qquad E[X]=K.\end{aligned}\tag{2.12}$$

The saddlepoint approximation formulas for tail expectation derived from the two approaches [Zheng-Kwok formulas (2.6, 2.7) and Huang-Oosterlee formulas (2.11, 2.12)] are in general different under non-Gaussian base distribution. When $\kappa_0(w)=\frac{w^2}{2}$ (Gaussian base distribution), the two saddlepoint approximation formulas to tail expectation derived from the Zheng-Kwok approach and Huang-Oostelee reduce to the same formula. Another interesting case of equality of the two saddlepoint approximation formulas is shown in Section 4.

3 Criteria for Choosing the Base Distribution

The next issue is to explore how to make the judicious choice of the base distribution that has better fit of the underlying distribution so as to achieve improved accuracy of the saddlepoint approximation to tail expectation. We follow a similar criterion for choosing the base distribution as proposed by Wood *et al.* (1993), which is related to two invariant properties of the saddlepoint approximation formulas under linear transformation. For a random variable X, we define its linear transformation to be $\mathcal{L}X = aX + b$ for some constants $a > 0, b \in \mathbb{R}$. Suppose a random variable Y is an appropriate choice as the base distribution, we should expect that $\mathcal{L}Y$ produces the same saddlepoint approximation result. We write $\hat{E}_Y[(X-K)^+]$ as the saddlepoint approximation to $E_Y[(X-K)^+]$. The invariant properties under linear transformation are summarized in the following lemmas.

Lemma 3.1 For any linear transformation $\mathcal{L}X = aX + b$, where $a > 0$, the saddlepoint approximation to tail expectation $E[(X-K)^+]$ under the two base distributions Y and $\mathcal{L}Y$ are identical. That is

$$\hat{E}_Y[(X-K)^+] = \hat{E}_{\mathcal{L}Y}[(X-K)^+], \tag{3.1}$$

for any X, Y and K.

Lemma 3.2 The saddlepoint approximation preserves the linear property under the linear transformation $\mathcal{L}X = aX + b$, where

$$\hat{E}_Y[(\mathcal{L}X - \mathcal{L}K)^+] = a\hat{E}_Y[(X-K)^+], \quad a > 0, \tag{3.2}$$

for any X, Y and K. When we scale the underlying random variable under the linear transformation, the saddlepoint approximation formula is scaled accordingly.

The proof of Lemma 3.1 is presented in Appendix B while that of Lemma 3.2 can be done in a similar manner. As a corollary of the lemmas, suppose we match some statistics between a pair of random variables X and Y, like mean, variance or skewness, the target statistic should preserve the invariant properties under the linear transformation $\mathcal{L}$ as stated in the above two lemmas.

We use the Gamma distribution $Gamma(\alpha, \beta)$ and inverse Gaussian $IG(\lambda, \mu)$ distribution as examples to show how to choose the base distribution parameters. These two distributions are chosen due to their wide range of applications and availability of closed form formulas of their density function, cumulative distribution function (cdf), tail probability, tail expectation and cumulant generating function (cgf). The properties of these two two-parameter distributions are listed below:

(i) If $X \sim Gamma(\alpha, \beta)$, then its density $f(x)$, cdf $F(x)$, and cgf $\kappa(z)$ are given by

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, \tag{3.3a}$$

$$F(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha, \frac{x}{\beta}), \tag{3.3b}$$

$$\kappa(z;\alpha,\beta) = -\alpha \ln(1 - \beta z), \tag{3.3c}$$

where $\Gamma(x)$ is the Gamma function and $\Gamma(\alpha, x) = \int_0^x e^t t^{\alpha-1}\, dt$ is the lower incomplete Gamma function.

(ii) If $X \sim IG(\lambda, \mu)$, then its density $f(x)$, cdf $F(x)$, and cgf $\kappa(z)$ are given by

$$f(x;\lambda,\mu) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}, \tag{3.4a}$$

$$F(x;\lambda,\mu) = \Phi\left(\sqrt{\frac{\lambda}{x}}(\frac{x}{\mu} - 1)\right) + e^{\frac{2\lambda}{\mu}} \Phi\left(-\sqrt{\frac{\lambda}{x}}(\frac{x}{\mu} + 1)\right), \tag{3.4b}$$

$$\kappa(z;\lambda,\mu) = \frac{\lambda}{\mu}\left(1 - \sqrt{1 - \frac{2\mu^2 z}{\lambda}}\right), \tag{3.4c}$$

where $\Phi(.)$ is the cdf of the standard normal distribution.

The choice of the scale parameter (β for the Gamma distribution and μ for the IG distribution) can be arbitrary, which can be seen from the following property. For any $c > 0$, we observe

$$\text{if } X \sim Gamma(\alpha, \beta), \text{ then } Y = cX \sim Gamma(\alpha, c\beta);$$
$$\text{if } X \sim IG(\lambda, \mu), \text{ then } Y = cX \sim IG(c\lambda, c\mu).$$

By virtue of Lemma 3.1, the saddlepoint approximation results remain unchanged for different scale parameters. We are more concerned with the choice of the shape parameter. One possible approach is to match the higher order standardized cumulants $\xi_i(z) = \frac{\kappa^{(i)}(z)}{\kappa''(z)^{\frac{i}{2}}}$, for $i \geq 3$, where $\kappa^{(i)}(z)$ is the i^{th} order derivatives of $\kappa(z)$. The choice of the matching target comes from two considerations. First, it should be independent of a linear transformation due to the invariant property stated in Lemma 3.1. We motivate the argument using the following intuition. When $z = 0$, $\xi_3(0)$ and $\xi_4(0)$ are the skewness and excess kurtosis of the corresponding random variable respectively. When $z = 0$ or $z = \hat{z}$, $\xi_i(z)$ remains unchanged after a linear transformation. We prefer $\xi_4(\hat{z})$ to $\xi_3(\hat{z})$ as the choice of the matching criterion.

We now present the matching method used in our numerical tests, where the Gamma distribution or the inverse Gaussian distribution is chosen as the base distribution. For the Gamma distribution, its fourth order standardized cumulant is given by $\xi_4(z) = \frac{6}{\alpha}$, which is independent of z. The choice of α should be taken to be

$$\alpha = \frac{6\kappa''(\hat{z})^2}{\kappa''''(\hat{z})}, \tag{3.5}$$

where $\kappa(z)$ is the cgf of underlying distribution and $\hat{z}$ is the solution of the saddlepoint equation: $\kappa'(z) = K$. The choice of β can be any arbitrary positive number. Recall that the support of the cgf of the Gamma distribution is $(-\infty, \frac{1}{\beta})$, so it is convenient to fix β so that we have a fixed boundary in our numerical procedure.

To compute the solution to eq.(2.5), we introduce the Lambert W-function, where $W(x)$ is the solution of $We^W = x$. The two real valued branches of $W(x)$ are the principal branch $W_0(x)$ and the lower branch $W_{-1}(x)$, both are defined for real number $x > -\frac{1}{e}$. For $x \geq 0$, there exist a unique solution for the equation $We^W = x$, which is given by $W_0(x)$. There are two roots for the case $-\frac{1}{e} < x < 0$, one of which is $W_0(x) \geq -1$ and the other one is $W_{-1}(x) < -1$. In terms of W_0 and W_{-1}, the solution to (2.5) is given by

$$\hat{w} = \begin{cases} \frac{1}{\beta} + \frac{1}{\beta W_{-1}\left(-\frac{1}{e^{\frac{c}{\alpha}+1}}\right)} & \text{if } \hat{z} > 0, \\ \frac{1}{\beta} + \frac{1}{\beta W_0\left(-\frac{1}{e^{\frac{c}{\alpha}+1}}\right)} & \text{if } \hat{z} \leq 0, \end{cases} \tag{3.6}$$

where $c = \hat{z}K - \kappa(\hat{z})$.

For the inverse Gaussian distribution, we have $\xi_4(z) = \frac{15\mu}{\lambda}\frac{1}{\sqrt{1-\frac{2\mu^2 z}{\lambda}}}$. The solution to eq.(2.5) is given by

$$\hat{w} = \frac{\lambda(1-x^2)}{2\mu^2}, \tag{3.7}$$

where $c = \hat{z}K - \kappa(\hat{z})$ and

$$x = \frac{\lambda + \mu c - \text{sgn}(\hat{z})\sqrt{(\lambda + \mu c)^2 - \lambda^2}}{\lambda}.$$

The matching equation is given by

$$\xi_4(\hat{w}) = \frac{15}{a + c - \mathrm{sgn}(\hat{z})\sqrt{(a+c)^2 - a^2}} = \frac{\kappa''''(\hat{z})}{\kappa''(\hat{z})^2}, \tag{3.8}$$

where $a = \frac{\lambda}{\mu}$ and $\kappa(z)$ is the cgf of the underlying distribution. When the right-hand side is positive, the above equation admits a unique positive solution for a, which serves as the choice of the shape parameter λ for a fixed value of the scale parameter μ. The positivity of a indicates that the underlying distribution would be fat tailed. The requirement is reasonable since neither the Gamma distribution nor the inverse Gaussian distribution would generate a random variable with thin tail.

4 Numerical Tests

To illustrate enhanced numerical accuracy of the two saddlepoint formulas for calculating tail expectation under non-Gaussian base distribution, we choose the Gamma distribution and inverse Gaussian distribution alternatively as the underlying random variable and base distribution. These choices take advantage that the Gamma distribution and inverse Gaussian distribution admit closed form formulas for density, tail probability, and tail expectation. As a remark, though Daniels (1980) shows that the saddlepoint approximation to density function becomes exact for the three distributions: Gaussian, Gamma and inverse Gaussian, our numerical tests show that exactness of the saddlepoint approximation to tail expectation fails under these three distributions. We recall that Wood *et al.* (1993) performed similar numerical tests on assessment of accuracy of saddlepoint approximation to tail probabilities under various choices of the base distributions.

4.1 Tail expectation of the inverse Gaussian distribution

We take the underlying random variable to follow the inverse Gaussian distribution, $IG(\lambda, \mu)$. The analytic formula for its tail expectation is given by

$$E[(X-K)^+] = \mu \left[\Phi\left(-\sqrt{\frac{\lambda}{K}}(\frac{K}{\mu} - 1) \right) + e^{\frac{2\lambda}{\mu}} \Phi\left(-\sqrt{\frac{\lambda}{K}}(\frac{K}{\mu} + 1) \right) \right] - K[1 - F(K; \lambda, \mu)], \tag{4.1}$$

where the distribution function $F(K; \lambda, \mu)$ is given by eq.(3.4b). In our numerical tests to demonstrate numerical accuracy of the saddlepoint approximation formulas, we choose the Gamma distribution as our base distribution. The tail expectation of the Gamma distribution $Gamma(\alpha, \beta)$ is given by

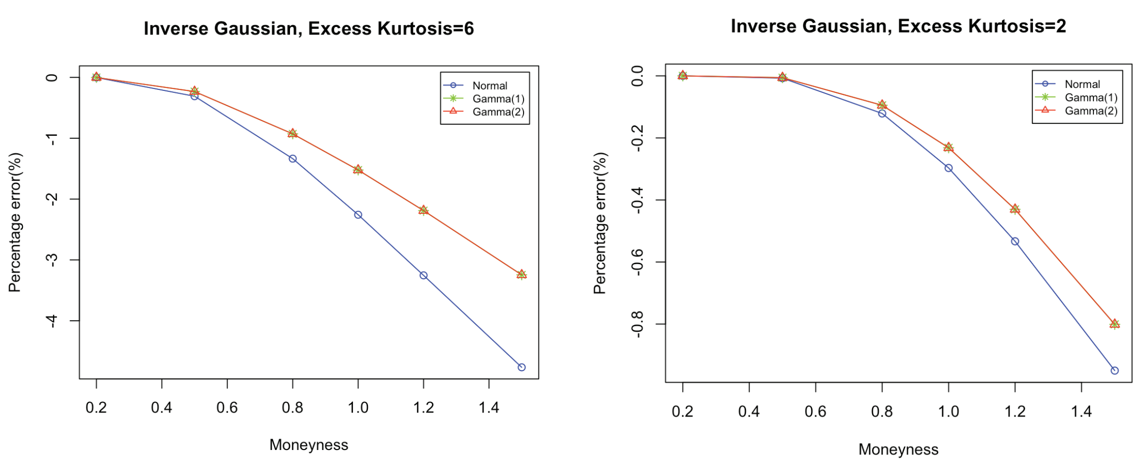
$$E[(X-K)^+] = \alpha\beta[1 - F(K; \alpha+1, \beta)] - K[1 - F(K; \alpha, \beta)], \tag{4.2}$$

where the distribution function $F(K; \alpha, \beta)$ is given by eq.(3.3b). In our numerical calculations, the parameters of the inverse Gaussian random variable are chosen to be $\mu = 2$ and $\lambda = 5$ or $\lambda = 15$. The corresponding excess kurtosis equals 6 when $\lambda = 5$ and equals 2 when $\lambda = 15$. The scale parameter β of the Gamma distribution is fixed at $\beta = 1$, while the shape parameter α is determined using eq.(3.5) according to the matching criterion of the base distribution. In Table 1, we present the corresponding $\xi_4(\hat{z})$, matched value of the shape parameter α, and saddlepoint approximation to tail expectation of the inverse Gaussian distribution at varying levels of moneyness and excess kurtosis using various base distributions. Moneyness is defined to be the ratio of K to the mean μ of the underlying. The relative errors of calculating tail expectation at different levels of moneyness and excess kurtosis using various base distributions are plotted in Figure 1.

Table 1: Saddlepoint approximation to tail expectation of the inverse Gaussian distribution at different levels of moneyness and excess kurtosis. "Exact" denotes the exact value of $E[(X-K)^+]$, "Gaussian" denotes the saddlepoint approximation results calculated using the standard Gaussian distribution as the base, "Gamma1" and "Gamma2" denote the saddlepoint approximation results calculated using formulas (2.6-2.7) and (2.11-2.12) using the Gamma distribution, respectively.

Moneyness	Exact	$\xi_4(\hat{z})$	α	Gaussian	Gamma1	Gamma2
Excess Kurtosis: 6						
0.2	1.600192	1.2	5	1.600161	1.600166	1.600166
0.5	1.045480	3	2	1.042241	1.043045	1.043045
0.8	0.646020	4.8	1.25	0.637393	0.640023	0.640023
1.0	0.464652	6	1	0.454163	0.457579	0.454579
1.2	0.333945	7.2	0.8333	0.323080	0.326633	0.326633
1.5	0.204054	9	0.667	0.194332	0.197432	0.197432
Excess Kurtosis: 2						
0.2	1.600000	0.4	15	1.600000	1.600000	1.600000
0.5	1.004326	1	6	1.004253	1.004267	1.004267
0.8	0.501472	1.6	3.75	0.500864	0.500996	0.500996
1.0	0.282473	2	3	0.281634	0.281818	0.281818
1.2	0.149724	2.4	0.2.5	0.148926	0.149080	0.149080

Figure 1: Comparison of the relative errors for calculating tail expectation of the inverse Gaussian distribution with respect to different levels of moneyness. "Normal" is the relative error using the standard Gaussian base, "Gamma (1)" and "Gamma (2)" are the relative errors computed by formulas (2.6-2.7) and (2.11-2.12) using the Gamma base distribution, respectively.



(a) Excess Kurtosis: 6 (b) Excess Kurtosis: 2

The numerical results in Table 1 and the plots of the relative errors in Figure 1 reveal enhanced accuracy of the saddlepoint approximation to tail expectation using the Gamma base distribution that fits better the underlying distribution when compared with the saddlepoint approximation using the usual Gaussian base distribution. Numerical accuracy of the saddlepoint approximation is well within 1% unless moneyness is high and/or kurtosis is significant. It is interesting to observe that the numerical results calculated from eqs. (2.6-2.7) and (2.11-2.12) are the same, though the analytic forms of the two saddlepoint approximation formulas look very differently. Besides the choice of the Gaussian distribution as the base distribution, the numerical tests reveal that the Gamma distribution is another choice of the base distribution that the two saddlepoint approximation formulas for calculating tail expectation give the same numerical results.

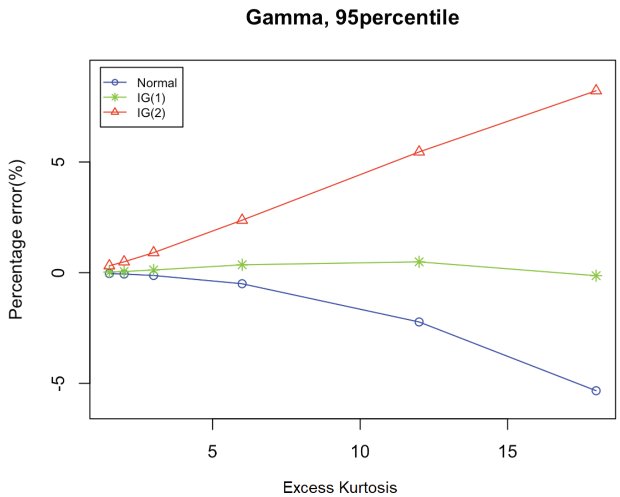
4.2 Tail expectation of the Gamma distribution

Next, we choose the Gamma distribution as the underlying distribution and the inverse Gaussian distribution as the base distribution. The moneyness, defined as the ratio of K to the mean, is chosen in such a way that K equals 95% percentile of the underlying. Moneyness is calculated by a root finding procedure according to the cdf of $Gamma(\alpha, \beta)$. Recall that the excess kurtosis of $Gamma(\alpha, \beta)$ is given by $\frac{\alpha}{6}$, which is independent of $\hat{z}$. The scale parameters of the Gamma distribution and inverse Gaussian distribution are fixed at $\beta = 2$ and $\mu = 1$, respectively. The shape parameter λ of the inverse Gaussian base distribution is determined according to eq.(3.8). In Table 2, we present the moneyness, the matched value of the shape parameter λ, the saddlepoint approximation to tail expectation of the Gamma distribution using various base distributions at varying levels of kurtosis and α. The relative errors of calculating tail expectation at different levels of excess kurtosis using various base distributions are plotted in Figure 2.

Table 2: Saddlepoint approximation to tail expectation of the Gamma distribution at different levels of kurtosis. "Kurtosis" is the excess kurtosis of the underlying random variable. "Exact" denotes the exact value of $E[(X-K)^+]$, and "Gaussian" denotes the saddlepoint approximation with the standard Gaussian base. Also, "IG1" and "IG2" denote the saddlepoint approximation results calculated based on formulas (2.6-2.7) and (2.11-2.12) using the inverse Gaussian base distribution, respectively.

Kurtosis	α	Moneyness	Exact	λ	Gaussian	IG1	IG2
1.5	4	1.93841	0.141685	14.70357	0.141632	0.141726	0.142129
2	3	2.09860	0.130593	11.50996	0.130514	0.130666	0.131236
3	2	2.37193	0.117409	8.188189	0.117260	0.117553	0.118472
6	1	2.99575	0.099999	4.619616	0.099501	0.100356	0.102372
12	0.5	3.84148	0.087028	2.617306	0.085091	0.087455	0.091776
18	0.333	4.41982	0.080686	1.869813	0.076383	0.080576	0.087318

Figure 2: Comparison of the relative errors for calculating tail expectation against different levels of kurtosis. "Normal" is the relative error using the standard Gaussian base, "IG (1)" and "IG(2)" are the relative errors computed by formulas (2.6-2.7) and (2.11-2.12) using the inverse Gaussian base distribution, respectively.



As revealed in Table 2 and Figure 2, the Zheng-Kwok saddlepoint approximation formulas (2.6-2.7) provide the best numerical accuracy compared to other saddlepoint approximation formulas. In this numerical example, we concentrate on the tail event ($K = 95\%$ percentile) and the excess kurtosis of the underlying distribution is relatively high. Under this scenario, the choice of the matching criterion is quite crucial for achieving high level of numerical accuracy. Numerical tests show that matching $\xi_4(0)$ rather than $\xi_4(\hat{z})$ results in worse accuracy when formulas (2.11-2.12) are used even when compared with the use of the standard Gaussian base. The Zheng-Kwok saddlepoint approximation formulas (2.6-2.7) provide very good accuracy even under high level of kurtosis. Though the analytic form of the Huang-Oosterlee saddlepoint approximation formulas (2.11-2.12) are simpler, the saddlepoint approximation formulas do not provide enhanced accuracy.

4.3 Pricing of European call option on integrated variance

We demonstrate the use of the saddlepoint approximation formulas (2.6-2.7) for tail expectation under non-Gaussian base to price European call option on continuous integrated variance under the Heston stochastic volatility model (Sepp, 2008). The terminal payoff of the European call option is given by $\max(I_T - K, 0)$, where K is the strike price and I_T is the continuous integrated variance of the stock price process S_t over $[0, T]$ as defined by

$$I_T = \int_0^T v_t \, \mathrm{d}t. \tag{4.3}$$

Here, v_t is the instantaneous variance of the stock price process.

To price an option on continuous integrated variance, it is necessary to specify the joint dynamics of the stock price process S_t and its instantaneous variance v_t under a risk neutral measure Q. In this numerical example, we adopt the Heston stochastic volatility model. The stochastic dynamic equations for S_t and v_t under Q are specified by

$$\frac{\mathrm{d}S_t}{S_t} = r\ \mathrm{d}t + \sqrt{v_t}(\rho\ \mathrm{d}W_t^v + \sqrt{1-\rho^2}\ \mathrm{d}W_t) \tag{4.4a}$$

$$\mathrm{d}v_t = \kappa(\theta - v_t)\ \mathrm{d}t + \varepsilon\sqrt{v_t}\ \mathrm{d}W_t^v, \tag{4.4b}$$

where ρ is the correlation coefficient between the Brownian motions W_t^v and W_t, θ is the mean reversion level, κ is the mean reversion speed, r is the riskfree interest rate and ε is the volatility of variance. By the risk neutral valuation principle, the time-t value of the European call option on integrated variance is given by

$$c_t = e^{-r(T-t)} E_t^Q \left[\max(I_T - K, 0)\right]. \tag{4.5}$$

Here, I_t is the integrated variance. We define the moment generating function of the integrated variance by

$$U(v_t, I_t, t; z, T) = E_t^Q[e^{zI_T}], \quad z \text{ is a complex-valued parameter.} \tag{4.6a}$$

We write $\tau = T - t$ as the time to expiry. By the Feynman-Kac Theorem, $U(v_t, I_t, t; z, T)$ satisfies the following partial differential equation:

$$\frac{\partial U}{\partial \tau} = \kappa(\theta - v)\frac{\partial U}{\partial v} + \frac{\varepsilon^2 v}{2}\frac{\partial^2 U}{\partial v^2} + v\frac{\partial U}{\partial I}. \tag{4.6b}$$

Thanks to the affine structure of the Heston stochastic volatility model, $U(v_t, I_t, t; z, T)$ admits solution in the exponential affine form, where

$$U(v_t, I_t, t; z, T) = \exp(B(\tau)v_t + zI_t + \Gamma(\tau)), \quad \tau = T - t.$$

The solution to $B(\tau)$ and $\Gamma(\tau)$ are found to be [see eq.(B.4) in Zheng and Kwok (2014)]

$$B(\tau) = \frac{2z(1 - e^{-\zeta\tau})}{\xi_+ e^{-\zeta\tau} + \xi_-} \tag{4.7a}$$

$$\Gamma(\tau) = -\frac{\kappa\theta}{\varepsilon^2}\left(\xi_+\tau + 2\ln\frac{\xi_+ e^{-\zeta\tau} + \xi_-}{2\zeta}\right), \tag{4.7b}$$

where $\zeta = \sqrt{\kappa^2 - 2\varepsilon^2 u}$ and $\xi_\pm = \zeta \mp \kappa$. Once we have obtained the moment generating function of I_T, its cgf is given by

$$\kappa_I(z) = \log U(v_t, I_t, t; z, T). \tag{4.8}$$

Once $\kappa_I(z)$ is available, we can use the saddlepoint approximation formulas (2.6-2.7) to price the European call option on continuous integrated variance.

We may express the call price formula as a complex Bromwich integral. Suppose that $\kappa_I(z)$ is analytic in some vertical strip: $\{z \in \mathbb{C} : \alpha_- < \mathrm{Re}\ z < \alpha_+\}$, where $\alpha_- < 0$ and $\alpha_+ > 0$. It can be shown that the call option price c_t admits the following integral representation [see eq.(1.13c) in Kwok and Zheng (2018)]:

$$c_t = \frac{e^{-r(T-t)}}{2\pi}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\kappa_I(z) - z\kappa}}{z^2}\ \mathrm{d}z, \quad \gamma \in (0, \alpha_+). \tag{4.9}$$

In our numerical test on the performance of accuracy of the saddlepoint approximation formula (2.6) in pricing European call option on continuous integrated variance, we used the set of parameter values shown in Table 3.

Table 3: List of parameter values in the Heston stochastic volatility model used in pricing European call option on continuous integrated variance.

κ	θ	ε	ρ	$\sqrt{v_t}$
3.46	$(0.0894)^2$	0.14	-0.82	0.087

In addition, we take $S_0 = 1$, $\tau = 1$ and $r = 3.19\%$.

We performed numerical valuation of the value of the European call option on continuous integrated variance using the saddlepoint approximation formulas (2.6-2.7) under the standard Gaussian base, Gamma base and Gaussian less exponential base (Carr and Madan, 2009). The Gaussian less exponential base takes the form

$$Z + \frac{1}{\lambda} - Y,$$

where Z is a standard Gaussian variable and Y is a positive exponential with parameter λ. We chose the Gamma distribution with $\alpha = 5$ and $\beta = 1$ and the Gaussian less exponential distribution with $\lambda = 3$. We also computed the option value using direct numerical integration of the complex integral price formula (4.9) as the benchmark comparison. As revealed by the pricing results of the one-year European call option on continuous realized variance in Table 4, we observe enhanced numerical accuracy using the saddlepoint approximation under non-Gaussian base distribution when compared with the results under the Gaussian base distribution. The percentage errors are typically within 0.1%. The Gaussian less exponential base proposed by Carr and Madan (2009) performs slightly better than the Gamma base.

Table 4: Comparison of pricing results of the one-year European call option on continuous integrated variance using the saddlepoint approximation with various base distributions: (i) Gaussian distribution, (ii) Gamma distribution, (iii) Gaussian less exponential. The numerical results from the direct numerical integration of the complex integral price formula are used as the benchmark comparison for numerical accuracy. The market convention for quoting call price on continuous integrated variance is percentage point squared; that is, the option value and strike price have been multiplied by 100^2.

Strike price	Gaussian distribution	Gamma distribution	Gaussian less exponential	numerical integration
63.1	19.035	19.451	19.235	19.306
64.6	18.219	18.393	18.321	18.323
66.1	17.289	17.376	17.378	17.381
67.7	16.307	16.401	16.377	16.381
69.3	15.316	15.464	15.416	15.419
70.9	14.396	14.569	14.496	14.499
72.4	13.702	13.714	13.716	13.718
74.1	12.664	12.899	12.774	12.776
75.6	11.988	12.121	12.071	12.073
77.2	11.168	11.381	11.306	11.307
78.7	10.613	10.714	10.681	10.679
80.3	9.924	10.011	9.985	9.985

5 Conclusion

We have derived saddlepoint approximation formulas for calculating tail expectation under non-Gaussian base distribution using two different approaches: the Esscher exponentially tilting technique (Zheng-Kwok, 2014) and the local approximation of the exponent in the Laplace integral (Huang-Oosterlee, 2011). Our numerical tests show higher level of numerical accuracy in the Zheng-Kwok formula, even in extreme scenarios of high kurtosis when other saddlepoint approximation formulas do not perform well. We also propose an effective set of criteria for matching better the underlying distribution. The fourth order standardized cumulant is the best statistics used for matching the underlying distribution and base distribution in determining the shape parameter when the Gamma distribution or inverse Gaussian distribution is used as the base distribution. We performed numerical tests to reveal enhanced numerical accuracy of the saddlepoint approximation formulas under non-Gaussian base distribution in pricing European options on continuous integrated variance under the Heston stochastic volatility model. The use of the Gaussian less exponential base performs better than the Gaussian base and provides very high numerical accuracy in option value calculations (typically within 0.1% in percentage error).

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Appendix A Proof of formulas (2.6) and (2.7)

We consider the Legendre-Fenchel transformation of the exponentially θ-tilted distribution of X, which involves the transformation from the variable z to the new variable w_θ. Let $\hat{z}$ denote the saddlepoint for the original random variable X, where $\kappa'(\hat{z}) = K$. The saddlepoint equation for the exponentially θ-tilted distribution based on the base distribution with cgf $\kappa_0(z)$ is given by [see eqs.(2.4) and (2.5)]

$$w_\theta \kappa_0'(w_\theta) - \kappa_0(w_\theta) = (\hat{z} - \theta)K - \kappa(\hat{z}) + \kappa(\theta). \tag{A.1}$$

We obtain the following saddlepoint approximation to the θ-tilted tail probability (Wood *et al.*, 1993):

$$\tilde{F}(K;\theta) = F_0\left(\kappa_0'(\hat{w}_\theta)\right) + f_0\left(\kappa_0'(\hat{w}_\theta)\right)\left\{\frac{1}{\hat{w}_\theta} - \frac{1}{\hat{z}-\theta}\left[\frac{\kappa_0''(\hat{w}_\theta)}{\kappa''(\hat{z})}\right]^{\frac{1}{2}}\right\}, \tag{A.2}$$

where $\hat{w}_\theta$ is a root of eq.(A.1) that is chosen to have the same sign as that of $\hat{z}$. The differentiation of $\tilde{F}(K;\theta)$ with respect to θ gives

$$\begin{aligned}
&\frac{\partial \tilde{F}(K;\theta)}{\partial\theta} \\
&= f_0\left(\kappa_0'(\hat{w}_\theta)\right)\left\{\left[K - \kappa'(\theta)\right]\left[\frac{1}{\hat{w}_\theta^3\kappa_0''(\hat{w}_\theta)} - \frac{1}{\hat{w}_\theta} + \frac{\kappa_0'''(\hat{w}_\theta)}{2\hat{w}_\theta\kappa_0''(\hat{w}_\theta)^{\frac{3}{2}}\hat{\mu}_\theta}\right] - \frac{\sqrt{\kappa_0''(\hat{w}_\theta)}}{(\hat{z}-\theta)\hat{\mu}_\theta}\right\} \\
&\quad + f_0'\left(\kappa_0'(\hat{w}_\theta)\right)\left[K - \kappa'(\theta)\right]\left[\frac{\sqrt{\kappa_0''(\hat{w}_\theta)}}{\hat{w}_\theta\hat{\mu}_\theta} - \frac{1}{\hat{w}_\theta^2}\right],
\end{aligned} \tag{A.3}$$

where $\hat{\mu}_\theta = (\hat{z} - \theta)\sqrt{\kappa''(\hat{z})}$. Lastly, we set $\theta = 0$ in $\frac{\partial \tilde{F}(K;\theta)}{\partial\theta}$ and put all terms together to obtain formula (2.6). Note that $\hat{w}$ satisfies eq.(2.5), $\hat{\mu} = \hat{z}\sqrt{\kappa''(\hat{z})}$ and $\hat{z}$ satisfies $\kappa'(\hat{z}) = K$.

The derivation of the degenerate case, $E[X] = K$, requires tedious calculation of the power series expansion of various terms. For notational simplicity, we use the shorthand: $\hat{w}_\theta$ as $\hat{w}$. We start with the expansion of the right hand side of (A.1) into a power series of $\theta - \hat{z}$ around the point $z = \hat{z}$ as follows:

$$\kappa(\theta) = \kappa(\hat{z}) + \kappa'(\hat{z})(\theta - \hat{z}) + \frac{1}{2}\kappa''(\hat{z})(\theta-\hat{z})^2 + \frac{1}{6}\kappa'''(\hat{z})(\theta-\hat{z})^3 + \frac{1}{24}\kappa''''(\hat{z})(\theta-\hat{z})^4 + O\left((\theta-\hat{z})^5\right).$$

We define

$$b(\hat{z}) = \frac{\kappa'''(\hat{z})}{\kappa''(\hat{z})} \quad \text{and} \quad c(\hat{z}) = \frac{\kappa''''(\hat{z})}{\kappa''(\hat{z})},$$

and rewrite the right-hand side of (A.1) by

$$\frac{1}{2}\kappa''(\hat{z})(\hat{z}-\theta)^2 \left[1 - \frac{1}{3}b(\hat{z})(\hat{z}-\theta) + \frac{1}{12}c(\hat{z})(\hat{z}-\theta)^2 + O\left((\hat{z}-\theta)^3\right)\right].$$

Observing $\kappa_0(0) = 0$ and applying a similar procedure to the left-hand side of (A.1), we derive the following expansion around $w = \hat{w}$:

$$\frac{1}{2}\kappa_0''(\hat{w})\hat{w}^2 \left[1 - \frac{1}{3}b_0(\hat{w}) + \frac{1}{12}c_0(\hat{w})\hat{w}^2 + O(\hat{w}^3)\right],$$

where

$$b_0(\hat{w}) = \frac{\kappa_0'''(\hat{w})}{\kappa_0''(\hat{w})} \quad \text{and} \quad c_0(\hat{w}) = \frac{\kappa_0''''(\hat{w})}{\kappa_0''(\hat{w})}.$$

Combining the above results together, we obtain the following equation:

$$\begin{aligned}
&\kappa_0''(\hat{w})\hat{w}^2 \left[1 - \frac{1}{3}b_0(\hat{w}) + \frac{1}{12}c_0(\hat{w})\hat{w}^2 + O(\hat{w}^3)\right] \\
&= \kappa''(\hat{z})(\hat{z}-\theta)^2 \left[1 - \frac{1}{3}b(\hat{z})(\hat{z}-\theta) + \frac{1}{12}c(\hat{z})(\hat{z}-\theta)^2 + O\left((\hat{z}-\theta)^3\right)\right].
\end{aligned}$$

One can deduce that $\hat{w}$ is the same order as $\hat{z} - \theta$. Similarly, the expansion of $K - \kappa'(\theta)$ as a power series of $\theta - \hat{z}$ is given by

$$\begin{aligned}
K - \kappa'(\theta) &= \kappa'(\hat{z}) - \kappa'(\theta) \\
&= \kappa''(\hat{z})(\hat{z}-\theta) \left[1 - \frac{1}{2}b(\hat{z})(\hat{z}-\theta) + \frac{1}{6}c(\hat{z})(\hat{z}-\theta)^2 + O\left((\hat{z}-\theta)^3\right)\right].
\end{aligned}$$

We observe that the asymptotic behaviors of $\hat{w}$ at $\hat{w} = 0$ and $\hat{z}$ at $\hat{z} = \theta$ share the same order. We define the remainder term $R_n(\hat{w}, \hat{z} - \theta)$ as the collection of the power terms of $\hat{w}$ and $\hat{z} - \theta$ with their sum of exponents greater or equal to n. We can derive the asymptotic limit via the Laurent expansion of the different terms in eq.(A.3) as follows:

$$\begin{aligned}
\mathcal{A} &= \frac{K-\kappa'(\theta)}{\hat{w}^3\kappa_0''(\hat{w})} = \frac{[K-\kappa'(\theta)]\sqrt{\kappa_0''(\hat{w})}}{\left[\hat{w}\sqrt{\kappa_0''(\hat{w})}\right]^3} \\
&= \frac{[K-\kappa'(\theta)]\sqrt{\kappa_0''(\hat{w})}}{\left[\sqrt{\kappa_0''(\hat{z})}(\hat{z}-\theta)\right]^3} \frac{\left[1-\frac{1}{3}b(\hat{z})(\hat{z}-\theta)+\frac{1}{12}c(\hat{z})(\hat{z}-\theta)^2+O\left((\hat{z}-\theta)^3\right)\right]^{-\frac{3}{2}}}{\left[1-\frac{1}{3}b_0(\hat{w})\hat{w}+\frac{1}{12}c_0(\hat{w})\hat{w}^2+O(\hat{w}^3)\right]^{-\frac{3}{2}}} \\
&= \frac{\sqrt{\kappa_0''(\hat{w})}}{\sqrt{\kappa_0''(\hat{z})}(\hat{z}-\theta)^2} \frac{1+\frac{1}{2}b(\hat{z})(\hat{z}-\theta)+[\frac{5}{24}b(\hat{z})^2-\frac{1}{8}c(\hat{z})](\hat{z}-\theta)^2+O\left((\hat{z}-\theta)^3\right)}{\left[1-\frac{1}{3}b_0(\hat{w})\hat{w}+\frac{1}{12}c_0(\hat{w})\hat{w}^2+O(\hat{w}^3)\right]^{-\frac{3}{2}}} \\
&\quad \left[1-\frac{1}{2}b(\hat{z})(\hat{z}-\theta)+\frac{1}{6}c(\hat{z})(\hat{z}-\theta)^2+O\left((\hat{z}-\theta)^3\right)\right] \\
&= \frac{\sqrt{\kappa_0''(\hat{w})}}{\sqrt{\kappa_0''(\hat{z})}(\hat{z}-\theta)^2} \left\{1+\left[\frac{1}{24}c(\hat{z})-\frac{1}{24}b(\hat{z}^2)\right](\hat{z}-\theta)^2+O\left((\hat{z}-\theta)^3\right)\right\} \\
&\quad \left\{1-\frac{1}{2}b_0(\hat{w})\hat{w}+\left[\frac{1}{24}b_0(\hat{w})^2+\frac{1}{8}c_0(\hat{w})\right]\hat{w}^2+O(\hat{w}^3)\right\} \\
&= \frac{\sqrt{\kappa_0''(\hat{w})}}{\sqrt{\kappa''(\hat{z})}(\hat{z}-\theta)^2} \left\{1-\frac{1}{2}b_0(\hat{w})\hat{w}+\left[\frac{1}{24}c(\hat{z})-\frac{1}{24}b(\hat{z})^2\right](\hat{z}-\theta)^2\right. \\
&\quad \left.+\left[\frac{1}{24}b_0(\hat{w})^2+\frac{1}{8}c_0(\hat{w})\right]\hat{w}^2+R_3(\hat{w},\hat{z}-\theta)\right\}.
\end{aligned}$$

Here, we calculate the power series expansion up to the third order so that the remainder terms converge to 0 as $\hat{w}\to 0$ and $\hat{z}\to\theta$. Also, we have arranged the singular terms in negative power terms of $\hat{z}-\theta$. In a similar manner, we obtain:

$$\begin{aligned}
\mathcal{B} &= \frac{(K-\kappa'(\theta))\kappa_0'''(\hat{w})}{2\hat{w}\kappa_0''(\hat{w})^{\frac{3}{2}}(\hat{z}-\theta)\sqrt{\kappa''(\hat{z})}} = \frac{(K-\kappa'(\theta))b_0(\hat{w})}{2\hat{w}\sqrt{\kappa_0''(\hat{w})}(\hat{z}-\theta)\sqrt{\kappa''(\hat{z})}} \\
&= \frac{b_0(\hat{w})}{2(\hat{z}-\theta)}\left[1-\frac{1}{3}b(\hat{z})(\hat{z}-\theta)-\frac{1}{6}b_0(\hat{w})\hat{w}+R_2(\hat{w},\hat{z}-\theta)\right], \\
\mathcal{C} &= \mathcal{A}+\mathcal{B}-\frac{\sqrt{\kappa_0''(\hat{w})}}{(\hat{z}-\theta)^2\sqrt{\kappa''(\hat{z})}} \\
&= \frac{1}{24}\sqrt{\frac{\kappa_0''(\hat{w})}{\kappa''(\hat{z})}}\left[c(\hat{z})-b(\hat{z})^2\right]+\frac{1}{8}\sqrt{\frac{\kappa''(\hat{z})}{\kappa_0''(\hat{w})}}\left[c_0(\hat{w})-b_0(\hat{w})^2\right] \\
&\quad -\frac{1}{12}b_0(\hat{w})b(\hat{z})+R_1(\hat{w},\hat{z}-\theta), \\
\mathcal{D} &= \frac{K-\kappa'(\theta)}{\hat{w}} = \sqrt{\kappa''(\hat{z})\kappa_0''(\hat{w})}+R_1(\hat{w},\hat{z}-\theta), \\
\mathcal{E} &= \left[K-\kappa'(\theta)\right]\left[\frac{\sqrt{\kappa_0''(\hat{w})}}{\sqrt{\kappa''(\hat{z})}(\hat{z}-\theta)\hat{w}}-\frac{1}{\hat{w}^2}\right] = \mathcal{D}\left[\frac{\sqrt{\kappa_0''(\hat{w})}}{\sqrt{\kappa''(\hat{z})}(\hat{z}-\theta)}-\frac{1}{\hat{w}}\right] \\
&= \frac{\sqrt{\kappa''(\hat{z})\kappa_0''(\hat{w})}}{6}\left[\frac{b_0(\hat{w})}{\sqrt{\kappa_0''(\hat{w})}}-\frac{b(\hat{z})}{\sqrt{\kappa''(\hat{z})}}\right]+R_1(\hat{w},\hat{z}-\theta).
\end{aligned}$$

Collecting all terms together and taking the limits of $\hat{w}\to 0$ and $\hat{z}\to\theta$, we obtain the following asymptotic limit:

$$
\begin{aligned}
\frac{\partial \tilde{F}(K;\theta)}{\partial \theta} &= \lim_{\hat{w}\to 0,\hat{z}\to\theta} \left[f_0(\kappa_0'(\hat{w}))(\mathcal{C}-\mathcal{D}) + f_0'(\kappa_0'(\hat{w}))\mathcal{E} \right] \\
&= f_0(\kappa_0'(0)) \left\{ \frac{1}{24}\sqrt{\frac{\kappa_0''(0)}{\kappa''(\theta)}} \left[\frac{\kappa''''(\theta)}{\kappa''(\theta)} - \frac{\kappa'''(\theta)^2}{\kappa''(\theta)^2} \right] \right. \\
&\quad \left. + \frac{1}{8}\sqrt{\frac{\kappa''(\theta)}{\kappa_0''(0)}} \left[\frac{\kappa_0''''(0)}{\kappa_0''(0)} - \frac{\kappa_0'''(0)^2}{\kappa_0''(0)^2} \right] - \frac{1}{12}\frac{\kappa_0'''(0)}{\kappa_0''(0)}\frac{\kappa'''(\theta)}{\kappa''(\theta)} - \sqrt{\kappa''(\theta)\kappa_0''(0)} \right\} \\
&\quad + \frac{f_0'(\kappa_0'(0))\sqrt{\kappa''(\theta)}\kappa_0''(0)}{6} \left[\frac{\kappa_0'''(0)}{\kappa_0''(0)^{\frac{3}{2}}} - \frac{\kappa'''(\theta)}{\kappa''(\theta)^{\frac{3}{2}}} \right].
\end{aligned}
$$

Lastly, we set $\theta = 0$ to obtain eq.(2.7).

Appendix B Proof of Lemma 3.1

We let $\kappa_X(z)$ denote the cgf of the random variable X, $\hat{z}_X$ denote the corresponding saddlepoint solution determined by $\kappa_X(z) = K$. Recall the linear transformation: $\mathcal{L}X = aX + b$, $a > 0$. Let f_X and F_X denote the density and distribution function of X, respectively. It is seen that

$$\kappa_{\mathcal{L}X}(z) = bz + \kappa_X(az).$$

The corresponding saddlepoint, cdf and density of $\mathcal{L}X$ are related to those of X via the relations:

$$
\begin{aligned}
a\hat{z}_{\mathcal{L}X} &= \hat{z}_X, \\
F_X(K) &= F_{\mathcal{L}X}(\mathcal{L}K), \\
f_X(K) &= \frac{d}{dK}F_X(K) = af_{\mathcal{L}X}(\mathcal{L}K).
\end{aligned}
$$

The first order derivative of $\kappa_{\mathcal{L}X}(\hat{z}_X)$ is given by

$$\kappa_{\mathcal{L}X}'(\hat{z}_{\mathcal{L}X}) = b + a\kappa_X'(\hat{z}_X) = \mathcal{L}\kappa_X'(\hat{z}_X),$$

while the n^{th} order derivatives are given by

$$\kappa_{\mathcal{L}X}^{(n)}(\hat{z}_{\mathcal{L}X}) = a^n\kappa_X^{(n)}(\hat{z}_X)\,,\, n \geq 2.$$

By substituting the above relations into eq.(2.6) and eq.(2.11), we obtain the invariant property (3.1) in Lemma 3.1.

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