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ASIAN OPTIONS WITH THE AMERICAN EARLY EXERCISE FEATURE

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By appropriate scaling of the variables, the reduction in the dimensionality of the partial differential equation formulation of an American-style Asian option model is achieved. The integral representation of the early exercise premium can be obtained in a succinct manner. The exercise policy of Asian options with the early exercise provision can then be examined.

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1. Introduction

Asian options are averaging options where the terminal payoffs depend on some form of averaging prices of the underlying asset over a part or the whole life of the option. Averaging options are particularly useful for business involved in trading on thinly traded commodities. These types of options are used by traders who are interested to hedge against the average price of a commodity over a period rather than, say, the price at the end of period.

A wide variety of averaging options have been proposed, and summaries of some of these options can be found in the papers by Boyle [1] and Zhang [11]. The most commonly used sampled average is the discrete *arithmetic average*. However, the pricing of this class of Asian options is almost analytically intractable since the sum of lognormal densities has no explicit representation. On the other hand, if the *Geometric Brownian* motion is assumed for the underlying asset price, the analytic derivation of the price formula for Asian options with geometric averaging is feasible since the product of lognormal prices remains to be lognormal.

The analytic procedures for deriving pricing formula of geometrically averaged Asian options can be broadly classified into two types, one uses the probabilistic approach and the other uses the partial differential equation approach. In the probabilistic approach, one evaluates the price of an Asian option following the risk neutralized discounted expectation approach. The density function of the joint distribution of the asset price and its geometric averaging is derived. This probability approach has been well explored in numerous papers for a wide variety of European-style Asian option models, for examples, in the papers by Kemna and Vorst [8], Conze and Viswanathan [2], Vorst [10], etc. However, the extension of the probabilistic approach to derive pricing formula for American-style Asian options appears to be less straightforward [5]. For the other approach, Dewynne and Wilmott [3, 4] derived the most general partial differential equation formulation of Asian option models. They have also attempted to analyze the properties of the early exercise provision of American-style Asian options. However, they have not come up with the analytical representation of the early exercise premium.

This paper presents the valuation of the floating strike Asian options with American early exercise feature whose payoff functions depend on the continuous geometrical averaging of the asset price. The value of the American Asian option is expressed as the sum of the value of its European counterpart and the early exercise premium, where the premium term is in the form of an integral. The complexity of the derivation is reduced by an appropriate choice of similarity variables, which reduces the dimensionality of the governing equation of the Asian option model. The availability of the integral representation of the early exercise premium leads to an integral equation for the early exercise boundary. The solution of the exercise boundary can be obtained effectively by the recursive integration method [6].

In the next section, we present the partial differential equation formulation of a floating strike American Asian option with continuous geometrical averaging of the asset price.

2. Partial Differential Equation Formulation

Let t denote the current time and T denote the expiration date of the contract of a floating strike Asian call option with continuous geometrical averaging of the asset price. The terminal payoff function of this Asian call option is given by

$$C(S_T, G_T, T) = \max(S_T - G_T, 0)\,, \tag{1}$$

where S_T is the asset price at time T and G_T is the continuous geometrical averaging of the asset price with averaging period starting at the time zero. Accordingly, G_t is defined by

$$G_t = \exp\left(\frac{1}{t}\int_0^t \ln S_u du\right), \quad 0 < t \le T\,. \tag{2}$$

The current asset price S_t is assumed to follow the risk neutral lognormal process:

$$dS_t = (r-q)S_t\ dt + \sigma S_t\ dZ(t)\,. \tag{3}$$

Here, r, q and σ denote the constant riskless interest rate, constant dividend yield and constant volatility, respectively, and $Z(t)$ is the standard Wiener process. By solving the stochastic differential Eq. (3) and using Eq. (2), we obtain

$$\ln S_T = \ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)(T - t) + \sigma[Z(T) - Z(t)]\,, \tag{4}$$

and

$$\ln G_T = \frac{t}{T}\ln G_t + \frac{1}{T}\left[(T-t)\ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2}\right] + \frac{\sigma}{T}\int_t^T [Z(u) - Z(t)]du\,, \tag{5}$$

where it is known that [2]

$$Z(T) - Z(t) = \phi(0, \sqrt{T-t})\,, \tag{6a}$$

$$\int_t^T [Z(u) - Z(t)]du = \phi\left(0, \frac{1}{\sqrt{3}}(T-t)^{\frac{3}{2}}\right)\,. \tag{6b}$$

Here, $\phi(\mu, \sigma)$ denotes the normal distribution with mean μ and standard deviation σ. The above relations reveal that G_T is also lognormally distributed.

Following the riskless hedging approach and applying the no arbitrage argument, the governing equation of the value of the European counterpart of the above Asian call is given by

$$\frac{\partial c}{\partial t} + \left(\frac{G}{t}\ln\frac{S}{G}\right)\frac{\partial c}{\partial G} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2} + (r-q)S\frac{\partial c}{\partial S} - rc = 0, \quad 0 < t < T\,, \tag{7a}$$

with terminal condition:

$$c(S, G, T) = \max(S - G, 0)\,. \tag{7b}$$

Let $S^*(G, t)$ denote the optimal exercise asset price above which it is optimal to exercise the American Asian option. Using the argument of delay exercise compensation as advocated by Jamshidian [7], the governing equation of the above American Asian option is obtained by modifying Eq. (7a) as follows:

$$\frac{\partial C}{\partial t} + \left(\frac{G}{t}\ln\frac{S}{G}\right)\frac{\partial C}{\partial G} + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC = \begin{cases} 0 & \text{if } S \le S^*(G,t) \\ -qS - \dfrac{dG}{dt} + rG & \text{if } S > S^*(G,t) \end{cases}\,. \tag{8}$$

The above partial differential equation formulation involves two spatial variables: S and G, and also the optimal exercise asset price S^* depends on G and t. The non-homogeneous term in Eq. (8) contains the extra term, $\frac{dG}{dt}$, which corresponds to the change of the strike price due to the temporal rate of change of the averaging asset value.

Various attempts have been made to reduce the dimensionality of the governing equation by seeking appropriate scaling of the variables [4, 9]. We propose the following choice for the set of similarity variables:

$$y = t \ln \frac{G}{S}\,, \qquad V(y,t) = \frac{C(S,G,t)}{S}\,, \tag{9}$$

where the asset price S is used as the numeraire. In terms of the new similarity variables, the partial differential equation formulation of the American Asian option model is reduced to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 t^2}{2}\frac{\partial^2 V}{\partial y^2} - \left(r - q + \frac{\sigma^2}{2}\right) t \frac{\partial V}{\partial y} - qV = \begin{cases} 0 & \text{if } y \geq y^*(t) \\ -q + re^{y/t} + \dfrac{y}{t^2} e^{y/t} & \text{if } y < y^*(t) \end{cases}, \tag{10a}$$

and

$$V(y,T) = \max(1 - e^{y/T}, 0)\,. \tag{10b}$$

In the stopping region, the American Asian option value is given by

$$V(y,t) = 1 - e^{y/t}, \quad y \leq y^*(t)\,. \tag{10c}$$

The above new formulation paves the path for the effective derivation of the pricing formula of the American Asian option.

3. Integral Representation of the Early Exercise Premium

The solution for the American call option value obtained from the above pricing model can be formally represented as integrals involving the Green function of the governing equation. Let $G(y,t;Y,T)$ be the Green function which satisfies the following reduced equation:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 t^2}{2}\frac{\partial^2 V}{\partial y^2} - \left(r - q + \frac{\sigma^2}{2}\right) t \frac{\partial V}{\partial y} = 0\,. \tag{11}$$

The Green function is found to be

$$G(y,t;Y,T) = n\left(\frac{Y - y + \mu \int_t^T u\,du}{\sigma\sqrt{\int_t^T u^2 du}}\right), \tag{12}$$

where $\mu = r - q + \frac{\sigma^2}{2}$ and $n(x)$ is the standard normal density function. The solution to the governing Eqs. (10a) and (10b) can be formally represented by

$$V(y,t) = e^{-q(T-t)} \int_{-\infty}^{\infty} \max(1 - e^{Y/T}, 0) G(y,t;Y,T)\, dY$$
$$+ \int_t^T e^{-q(u-t)} \int_{-\infty}^{y^*(u)} \left(q - re^{Y/u} - \frac{Y}{u^2} e^{Y/u} \right) G(y,t;Y,u)\, dY du\,, \quad (13)$$

where $y^*(u)$ is the critical value of y at time u, such that when $y \le y^*(u)$, the American option value assumes its intrinsic value. The first term in Eq. (13), when multiplied by S, gives the option value of the European counterpart of the present American Asian call option. By brute force integration of the first integral, the value of the European counterpart is found to be

$$c(S,G,t) = Se^{-q(T-t)} N(d_1) - G^{t/T} S^{(T-t)/T} e^{-q(T-t)} e^{-Q} N(d_2)\,, \quad (14)$$

where

$$d_1 = \frac{t \ln \frac{S}{G} + \frac{\mu}{2}(T^2 - t^2)}{\sigma \sqrt{\frac{T^3 - t^3}{3}}}, \quad d_2 = d_1 - \frac{\sigma}{T} \sqrt{\frac{T^3 - t^3}{3}}, \quad Q = \frac{\mu}{2} \frac{T^2 - t^2}{T} - \frac{\sigma^2}{6} \frac{T^3 - t^3}{T^2}\,.$$

We let $e(S,G,t)$ denote the early exercise premium, that is,

$$e(S,G,t) = C(S,G,t) - c(S,G,t)\,. \quad (15)$$

Here, $c(S,G,t)$ is given in Eq. (14) and $C(S,G,t)$ is the solution to Eq. (8). Let the second integral in Eq. (13) be $V_e(y,t)$ so that $e(S,G,t) = SV_e(y,t)$. Again, by performing the integration accordingly, we obtain the integral representation of the early exercise premium as follows:

$$e(S,G,t) = S \int_t^T \Big\{ qe^{-q(u-t)} N(\hat{d}_1)$$
$$- \left(\frac{G}{S} \right)^{t/u} e^{-q(u-t)} e^{\hat{Q}} \left[\left(r + \hat{d}_3 \right) N(\hat{d}_2) - \frac{\hat{\sigma}}{u^2} n(\hat{d}_2) \right] \Big\} du\,, \quad (16)$$

where

$$\hat{d}_1 = \frac{u \ln \frac{G}{S^*(G,u)} - t \ln \frac{G}{S} + \frac{\mu}{2}(u^2 - t^2)}{\hat{\sigma}}, \quad \hat{d}_2 = \hat{d}_1 - \frac{\hat{\sigma}}{u},$$

$$\hat{d}_3 = \frac{t \ln \frac{G}{S} - \frac{\mu}{2}(u^2 - t^2) + \frac{\hat{\sigma}^2}{u}}{u^2},$$

$$\hat{Q} = \frac{\hat{\sigma}^2}{2u^2} - \frac{\mu(u^2 - t^2)}{2u}, \quad \hat{\sigma}^2 = \frac{\sigma^2}{3}(u^3 - t^3)\,.$$

The above early exercise premium integral resembles that of an American vanilla option. The availability of the early exercise premium term in analytic form proves to be valuable in subsequent analysis of the early exercise policy.

4. Early Exercise Boundary

From the integral representation of the early exercise premium, we can deduce the following two properties of the optimal exercise asset value, $S^*(G,t)$.

(1) The optimal exercise asset value is homogeneous in G. In fact,

$$\frac{S^*(G,t)}{G} = e^{-y^*(t)/t}\,, \tag{17}$$

and $e^{-y^*(t)/t}$ is a function of time. This agrees with the homogeneity property of the present floating strike Asian option model.

(2) The asymptotic limit of $y^*(t)$ as $t \to T^-$, at instant right before the expiration time, is given by

$$y^*(T^-) = \min(y^*, 0)\,, \tag{18a}$$

where y^* is the solution to the non-linear algebraic equation

$$q - \left[\frac{y^*}{T^2} + r\right] e^{y^*/T} = 0\,. \tag{18b}$$

The results given by Eqs. (18a) and (18b) are obtained based on the following arguments. Since the payoff of the American Asian call option when exercised prematurely is $S - G$, which must be non-negative; and so correspondingly, $y^*(T^-)$ must be non-positive. In order that the American Asian call option remains alive at time right before the expiration time, the condition $\frac{\partial V}{\partial t}|_{t=T^-} < 0$ must be observed. At the critical value $y = y^*(T^-)$, we should have $\frac{\partial V}{\partial t}|_{t=T^-} = 0$. The critical value $y^*(T^-)$ is then obtained by setting the non-homogeneous term in Eq. (10a) to be zero, thus giving Eq. (18b).

In particular, when $q = 0$, Eq. (18b) can be solved analytically to give

$$y^*(T^-) = -rT^2\,. \tag{19}$$

Hence, it is still optimal to exercise the present American Asian call option at sufficiently high asset price even when the underlying asset is non-dividend paying, a property not shared by the American vanilla call option.

In order to solve for the critical exercise boundary, we derive the following integral equation for the critical value $y^*(t)$ above which the option value assumes the intrinsic value. Setting $V(y^*,t) = 1 - e^{y^*/t}$ along the critical boundary $(y^*(t),t), t \le T$, we obtain

$$1 - e^{y^*/t} = V_E(y^*(t),t) + \int_t^T f(y^*(t),t;y^*(u),u)du\,, \tag{20}$$

where $f(y^*(t),\, t; y^*(u), u)$ denotes the integrand in the integral representing $V_e(y^*(t), t)$. The solution of $y^*(t), t \leq T$, can be effected by applying the following numerical procedure, which has been coined as the recursive integration method [6].

In the recursive integration method, one attempts to find the numerical approximation of $y^*(t)$ at discrete instants $t_k, k = 0, 1, \ldots, n$, where $t_0 = t,\ t_n = T$ and $\Delta t = \frac{T-t}{n}$. Let y_k^* denote the numerical approximation of $y^*(t_k), k = 0, 1, \ldots, n$. By approximating the integral in Eq. (20) using the trapezoidal rule in numerical integration, we obtain the following non-linear algebraic equation for y_k^*:

$$\begin{aligned} 1 - e^{y_k^*/t_k} &= V_E(y_k^*, t_k) \\ &\quad + \frac{\Delta t}{2}\left[f(y_k^*, t_k; y_k^*, t_k) + f(y_k^*, t_k; y_n^*, t_n) + 2\sum_{i=k+1}^{n-1} f(y_k^*, t_k; y_i^*, t_i)\right]. \quad (21) \end{aligned}$$

Provided that $y_i^*, i = k+1, \ldots, n$ are known, one can solve for y_k^* from the above equation by any iterative method.

The following procedure is adopted to solve for $y_n^*,\ y_{n-1}^*,\ y_{n-2}^*, \ldots, y_0^*$ in sequential manner. First, y_n^* is obtained by solving Eqs. (18a) and (18b). Next, we solve for y_{n-1}^* from the equation obtained by taking $k = n-1$ in Eq. (21). Once y_{n-1}^* is known, we take $k = n-2$ in Eq. (21) and again solve for y_{n-2}^* from the corresponding equation. The same procedure is repeated until we have found y_n^*, $y_{n-1}^*, \ldots, y_0^*$ sequentially.

5. Numerical Examples

We apply the above recursive integration method to determine the early exercise boundary of a floating strike American-style Asian call option whose payoff function is given by Eq. (1). The other parameter values of the Asian model are (i) annualized riskless interest rate $r = 4\%$, (ii) annualized dividend yield q is set to be 0, 4%, and 8% successively, (iii) annualized volatility $\sigma = 20\%$, (iv) averaging period from $t = 0$ till the expiration time, $t = 1.5$. In Fig. 1, we plot $S^*(G,t)/G = e^{-y^*(t)/t}$ against time t for varying values of dividend yield q. At any moment when the asset price S is above $S^*(G,t)$, the American Asian option should be optimally exercised. It is observed that the function $S^*(G,t)/G$ is not a monotonic function in t. The higher fluctuation level of G to changes in asset price S at the earlier time of the averaging period compared to that at the later time may explain the concavity property of the plot of $S^*(G,t)/G$ shown in Fig. 1. Premature exercise becomes more attractive when the dividend yield becomes higher, as evidenced by decreasing value of $S^*(G,t)/G$ with increasing dividend yield (see Fig. 1). Even when the underlying asset is non-dividend paying, that is, $q = 0$, the American floating strike Asian call option will be exercised prematurely when it is sufficiently deep-in-the-money, a property not shared by the American vanilla call option. This

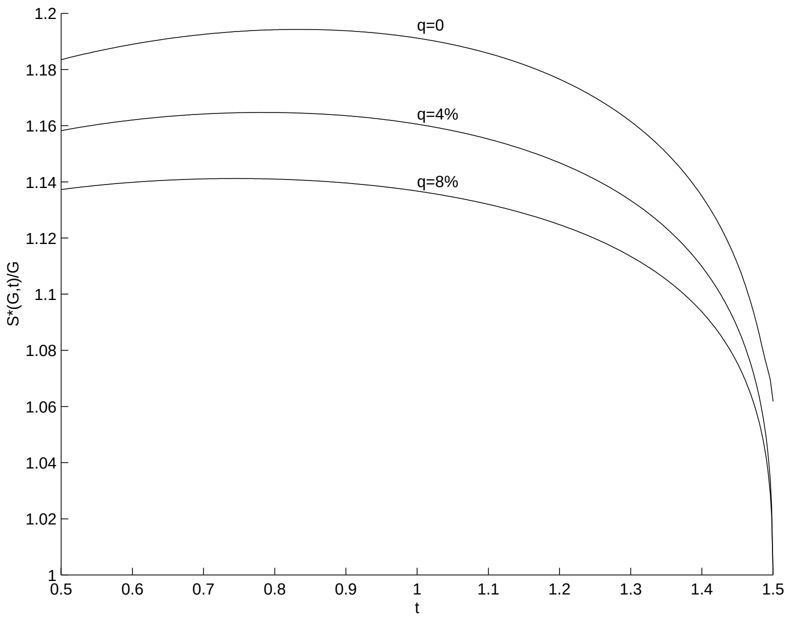

Fig. 1. Plot of $S^*(G,t)/G$ against time t.

is because the strike price, which equals the averaging price G, is changing at all times in the present floating strike option.

We would also like to explore the effects of interest rate and dividend yield on the early exercise policy of the American Asian options and examine how these may differ from those of the American vanilla options. The American value, $C(S,t)$, consists of two parts: the early exercise premium, $e(S,t)$, and the value of the European counterpart, $c(S,t)$. The ratio $R(S,t) = e(S,t)/c(S,t)$ somewhat reveals the value of the early exercise privilege, where higher value of the ratio would indicate that the potential of taking advantage of premature exercise is higher. We consider an American vanilla call option and an American floating strike Asian call option with continuous geometrically averaging, and compute the corresponding $R(S,t)$ in both models. In the numerical calculations, the asset price and the time to expiry are chosen to be $S = 100$ and $\tau = 1.0$, respectively, in both options. The strike price for the vanilla option is $X = 100$ and the geometrical average value is $G = 100$. The annualized volatility σ is chosen to be 20% in all calculations. We take the dividend yield q to be $r/2$, r and $2r$ successively. Figures 2 and 3 show the plots of the ratio $R(S,t)$ against the interest rate r for the American vanilla call option and the American Asian call option, respectively.

We observe that the ratio $R(S,t)$ normally has a higher value for the American Asian option comparing with that of the American vanilla option in similar situation. This is attributed to the positive correlation between S and G, so that the intrinsic value $S - G$ of the American Asian option fluctuates at a lower level

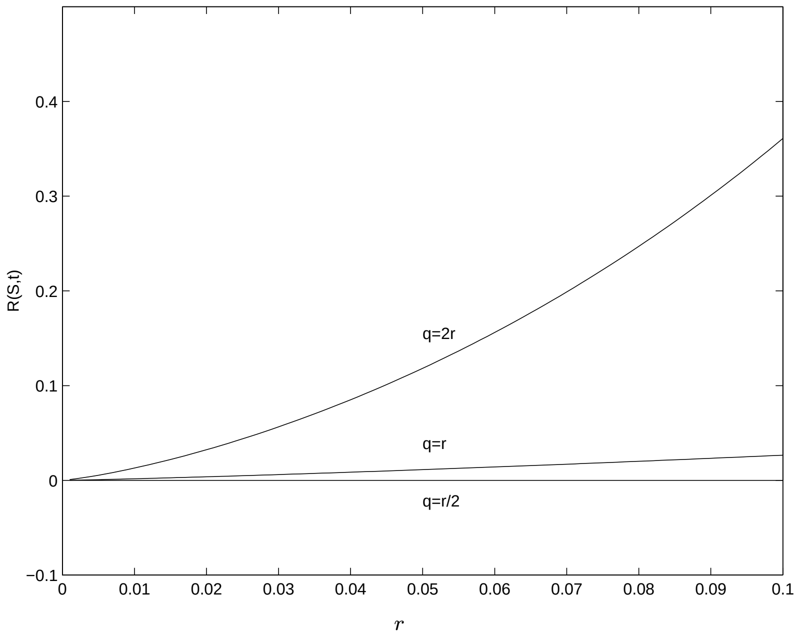

Fig. 2. Plot of $R(S,t)$ against r for the American vanilla call option model.

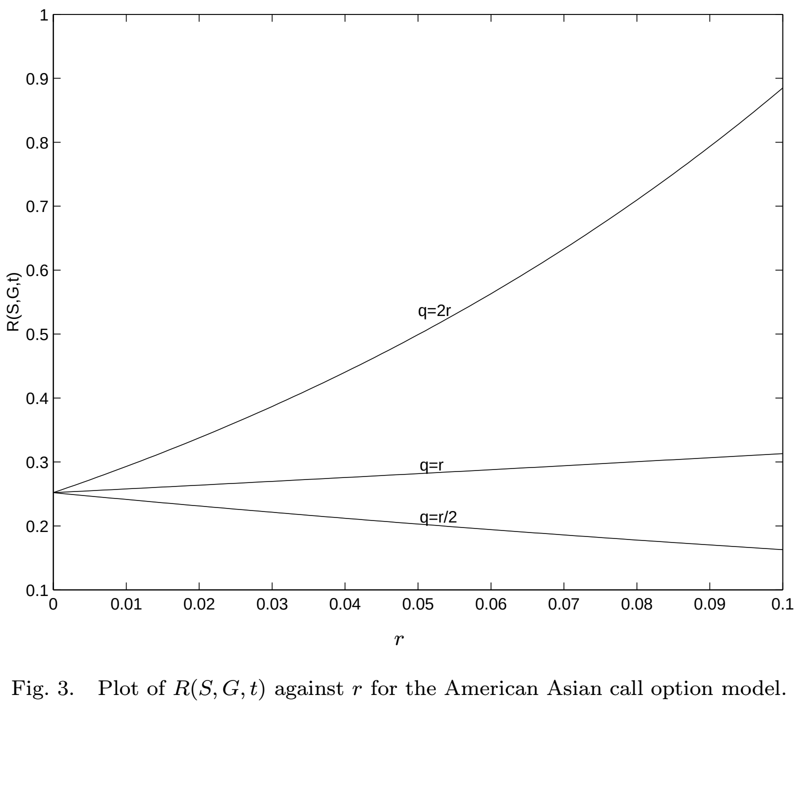

Fig. 3. Plot of $R(S,G,t)$ against r for the American Asian call option model.

compared to $S - X$ of the American vanilla option. The lower level of fluctuation leads to higher potential of taking the early exercise advantage and so higher value of $R(S,t)$. It is interesting to observe that when $q > r$, the ratio $R(S,t)$ increases steeply with increasing r in both the Asian and vanilla options. In the risk neutral world, the drift rate becomes negative when $q > r$ so that the asset value has higher tendency to drop then to rise. Correspondingly, the early exercise privilege becomes more valuable when the drift rate becomes more negative and so the ratio $R(S,t)$ increases with increasing value of r.

6. Conclusion

The apparent difficulties of analyzing the pricing models of American Asian options stem from the dependence of the option value on the stochastic movements of both the asset price and its averaging value. In this paper, we illustrate that the option value, when normalized by the asset price, depends only on a single stochastic variable. This stochastic variable is the ratio of the averaging price to the asset price. The early exercise policy of the American Asian options can be analyzed by solving an integral equation for the exercise boundary, the complexity of which resembles that of an American vanilla option. Some interesting properties of the early exercise policy unique to averaging options have been obtained in the present analysis. Compared to the American vanilla options, it is shown that the ratio of the early exercise premium to the value of the European counterpart is higher for the American Asian options.

One may query that this paper only deals with the pricing of a floating strike American Asian option with continuous geometrical averaging, which is considered to be one of the simplest option model among the whole class of American Asian options. However, the methodology discussed here can be applied to price other types of American-style Asian options, including models whose averaging is sampled discretely, and with other terminal payoff structures. In addition, more efficient numerical algorithms may also be constructed from the more succinct partial differential equation formulation derived using the scaling technique proposed in this paper.

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