Pricing Exotic Discrete Variance Swaps under the 3/2 Stochastic Volatility Models

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Abstract

We consider pricing of various types of exotic discrete variance swaps, like the gamma swaps and corridor variance swaps, under the 3/2-stochastic volatility models with jumps in asset price. The class of stochastic volatility models (SVM) that use a constant-elasticity-of-variance (CEV) process for the instantaneous variance exhibit nice analytical tractability only when the CEV parameter takes just a few special values (namely, 0, 1/2, 1 and 3/2). The popular Heston model corresponds to the choice of the CEV parameter to be 1/2. However, the stochastic volatility dynamics implied by the Heston model fails to capture some important empirical features of the market data. The choice of 3/2 for the CEV parameter in the SVM shows better agreement with empirical studies while it maintains a good level of analytical tractability. By using the partial integro-differential equation formulation, we manage to derive quasi-closed-form pricing formulas for the fair strike prices of various types of exotic discrete variance swaps with various weight processes and different return specifications under the 3/2-model. Pricing properties of these exotic discrete variance swaps with respect to various model parameters are explored.

Keywords: Variance swaps, gamma swaps, corridor variance swaps, 3/2-volatility model

1 Introduction

Variance and volatility derivatives have become more popular in the financial market since their introduction in the late nineties. Since volatility is likely to grow when uncertainty and risk increase, hedge funds and retail investors may use these derivatives to manage their exposure to the volatility risk associated with their trading positions. Speculators can also place their bids on the future movement of the underlying volatility via trading these instruments. Stock options are far from ideal as the instruments to provide exposure to volatility since they have exposure to both volatility and direction of the asset price movement. One may argue that exposure to the asset price in an option can be hedged. However, delta hedging is costly and inaccurate given that volatility cannot be estimated exactly. On the other hand, writers of variance swaps can almost perfectly hedge their positions via replication by using a portfolio of options traded in the markets. The provision of pure exposure of volatility and effective replication by traded options provide the impetus for the growth of the markets for swaps and other derivatives on discrete realized variance. Readers may refer to Carr and Madan (1998) for an introduction to the theory of volatility trading.

Variance swaps are essentially forward contracts on discrete realized variance. In recent years, other variants of variance swaps that target more specific features of variance exposure, like the gamma swaps and corridor variance swaps (commonly called the third generation

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volatility derivatives), are also structured in the financial markets. The product specifications of these exotic variance swap products will be presented in Section 3. The potential uses in hedging and betting the various forms of volatility exposure can be found in numerous articles (Demeterfi *et al.*, 1999; Carr and Lee, 2009; Bouzoubaa and Osseiran, 2010). Most of the earlier works on pricing discrete variance swaps concentrate on the vanilla variance swaps under the Heston stochastic volatility model (Broadie and Jain, 2008; Zhu and Lian, 2011; Rujivan and Zhu, 2012; Elliott and Lian, 2013) and time-changed Lévy processes (Itkin and Carr, 2010). There are two recent papers that consider the pricing of variance swaps with exotic payoff structures on the discrete realized variance. Crosby and Davis (2012) consider the pricing of generalized variance swaps, like the self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps under time-changed Lévy processes. Zheng and Kwok (2014) study the pricing of highly path dependent swaps on discrete corridor realized variance under the Heston stochastic volatility model (Heston, 1993) with simultaneous jumps in the asset price and variance.

The Heston stochastic volatility model has been commonly used for pricing variance and volatility derivatives due to its affine structure that lead to nice analytic tractability. However, the Heston model does not receive as much support as the 3/2-model from empirical studies. Itkin (2013) considers the class of stochastic volatility models (SVM) whose instantaneous variance is modeled by a constant-elasticity-of-variance (CEV) process. The number of analytically tractable models is rather limited, which is limited to only 4 specific values: 0, 1/2, 1 and 3/2 of the CEV parameter. The Heston model (square root process) corresponds to the choice of $\gamma = 1/2$, and it enjoys the best analytical tractability. Indeed, the Heston model belongs to the class of affine models and the methodologies for finding the corresponding joint moment generating functions are well established (Duffie *et al.*, 2000). Unfortunately, the Heston model has been shown to be inconsistent with observations in the variance markets. In particular, it leads to downward sloping volatility of variance smiles, contradicting with empirical findings from market data (Drimus, 2012). On the other hand, the 3/2-model (choice of $\gamma = 3/2$) exhibits better agreement with empirical studies while maintains some level of analytical tractability. For example, based on S&P 100 implied volatilities, Jones (2003) and Bakshi *et al.* (2006) estimate that γ should be around 1.3, which is close to 3/2 over 1/2 (Heston model). In addition, Jones concludes that jumps are needed in the underlying process for fitting short maturity options. Under the statistical measure, Javaheri (2004) analyzes the CEV type instantaneous variance process with the exponent of the volatility of variance process either being 1/2, 1 or 3/2 by using the time series data of S&P 500 daily returns and finds that the 3/2-power performs the best. Ishida and Engle (2002) estimate the power to be 1.71 for S&P 500 daily return for a 30-year period. Chacko and Viceira (2003) employ the technique of the Generalized Method of Moments on a 35-year period of weekly return and a 71-year period of monthly return. They estimate the power to be 1.10 and 1.65, respectively, over the two periods. By using S&P 500 index options over a period of 7 years, Poteshman (1998) concludes that the drift of the instantaneous variance is not affine as assumed in the Heston model.

There have been several recent works that use the 3/2-model for pricing variance and volatility derivatives. Drimus (2012) compute the fair strike prices for the continuously monitored variance and volatility derivatives under the 3/2-model without jumps. Instead of modeling the dynamics of the instantaneous volatility directly, Carr and Sun (2007) adopt a new approach by assuming continuous dynamics for the variance swap rate, taking advantage of the liquidity of the variance swap market. They argue that the 3/2 specification for the instantaneous variance is a direct consequence of the model consistency requirement. They manage to derive an analytic closed form formula for the joint conditional Fourier-Laplace transform of the log-asset price and its quadratic variation for the 3/2-model. Chan and

Platen (2012) derive exact pricing and hedging formulas of continuously monitored long dated variance swaps under the 3/2-model using the benchmark approach, a pricing concept that provides minimal fair strike prices for variance swaps when an equivalent risk neutral probability measure does not exist. Itkin and Carr (2010) derive closed form pricing formulas for discrete variance swaps and options on quadratic variation under a class of time-changed Lévy models, which includes the 3/2-power clock change.

In this paper, we propose a two-step partial integro-differential equation (PIDE) approach for pricing exotic discrete variance swaps with two different return specifications (actual rate of return and log rate of return) under the 3/2-model. The major contributions of our work are three-fold. First of all, we develop a unified PIDE approach for pricing European contingent claims with terminal payoffs depending on the asset prices monitored at discrete time instants. The use of the partial differential equation (PDE) was first proposed by Little and Pant (2001) for pricing discrete variance swaps under local volatility models and later extended to the Heston model by Zhu and Lian (2011). We further extend this approach to the 3/2-model with jumps and different return specifications of the discrete realized variance. Secondly, we use the PIDE approach to derive analytic pricing formulas for swaps on discrete (weighted) realized variance, including the variance swaps, gamma swaps, and corridor variance swaps. As an important step in the derivation procedure, we derive an analytic formula for the Fourier transform of the joint density of the log asset price and the instantaneous variance. Once the joint density is known analytically, quasi-closed-form (up to a double integration) pricing formulas for these exotic discrete variance swaps can be obtained. Thirdly, we perform sensitivity analysis of the prices of these exotic discrete variance swaps with respect to model parameters under the 3/2-model and report some interesting findings on the pricing behaviors of these swap products.

It is worth noting that there are alternative methods for pricing discrete variance swaps in the literature. For instance, Itkin and Carr (2010) price discrete variance swaps using the forward characteristic function approach under the general stochastic volatility framework. We find that their approach leads to the same pricing formula for the discrete variance swap as that derived via the PIDE approach. Unfortunately, the forward characteristic function approach cannot be naturally extended to pricing exotic variance swaps. Rujivan and Zhu (2012), Zheng and Kwok (2014) use the conditional expectation technique to evaluate discrete variance swaps and exotic variance swaps, respectively, under the Heston model with jumps. While it is plausible to extend the conditional expectation approach to price exotic variance swaps, we prefer the PIDE approach since the PIDE approach provides a unified pricing formulation for exotic discrete variance derivatives regardless of the return specification for the discrete realized variance. On the other hand, the conditional expectation approach has to deal with each type of return specification separately.

The rest of this paper is organized as follows. In the next section, we present the model specification of the 3/2-stochastic volatility model with jumps in the dynamics of the asset price process. We then propose the PIDE formulation for pricing contingent claims with payoffs that are dependent on the asset prices monitored at discrete time instants. We illustrate how to use specific integral transform techniques to find the fundamental solutions of the partial integro-differential equations. The pricing formula for the discrete variance swap is derived as a demonstrative example. In Section 3, we consider analytic pricing of a pair of third generation variance derivatives: gamma swaps and corridor variance swaps. In Section 4, we present the results of our numerical tests that were performed to illustrate the effective numerical valuation of the quasi-closed-form pricing formulas of the exotic variance swaps. Also, we present detailed analysis on the pricing properties of the exotic variance swaps with respect to different sets of parameters, like the correlation between asset price and its instantaneous variance, sampling frequency, volatility of variance. Moreover, we examine accuracy and numerical stability issues in the numerical valuation of the pricing formulas. The conclusive remarks and summary of findings are presented in the last section.

2 3/2-stochastic volatility models and pricing formulation of variance derivatives

In this section, we first present the 3/2 stochastic volatility model specification. We then derive the PIDE formulation for pricing a contingent claim on the discrete realized variance. Since each component term in the discrete realized variance is the squared asset return, which involves asset prices $S_{t_{i-1}}$ and S_{t_i} at successive monitoring instants t_{i-1} and t_i, the backward induction procedure for calculating the time-t_0 expected value naturally breaks down into a two-step procedure over the successive time intervals (t_{i-1}, t_i) and $[t_0, t_{i-1})$. Across the monitoring time instant t_{i-1} that separates the two time intervals, an appropriate jump condition on a specified path dependent state variable in the PIDE formulation is applied. As an illustration of the pricing procedure, we show how to find the fair strike of a vanilla variance swap under the 3/2-model with jumps.

2.1 3/2-model with jumps

We assume that the dynamics of the asset price S_t and its instantaneous variance v_t under a risk neutral measure $\mathbb{Q}$ is governed by

$$\begin{aligned}\frac{\mathrm{d}S_t}{S_t} &= (r - d - \lambda m)\,\mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t^1 + (e^J - 1)\,\mathrm{d}N_t,\\ \mathrm{d}v_t &= v_t[p(t) - qv_t]\,\mathrm{d}t + \epsilon v_t^{3/2}\,\mathrm{d}W_t^2,\end{aligned} \tag{2.1}$$

where r is the riskfree interest rate, d is the dividend yield, W_t^1 and W_t^2 are two correlated standard Brownian motions with $\mathrm{d}W_t^1\,\mathrm{d}W_t^2 = \rho\,\mathrm{d}t$. Also, N_t is a Poisson process with constant arrival rate λ, assumed to be independent of W_t^1 and W_t^2. The random jump size of the log asset price is denoted by J, which has a normal distribution with mean ν and variance ζ^2. Also, J is assumed to be independent of the two Brownian motions and the Poisson process N_t. The compensator parameter m is given by $\mathbb{E}_{\mathbb{Q}}[e^J - 1] = e^{\nu+\zeta^2/2} - 1$. The drift term q and the correlation coefficient ρ are assumed to be constant. To allow for flexibility in model calibration, we may generalize the level parameter $p(t)$ in the dynamics of v_t to be a deterministic continuous function of time.

The 3/2-dynamics of the variance process exhibits the mean-reverting feature. The mean-reverting rate depends on the current variance level, thus establishing a more volatile volatility structure than that of the Heston model. We let w_t be the reciprocal of the variance v_t. By applying Ito's lemma, w_t is seen to follow the time-inhomogeneous CIR process:

$$\mathrm{d}w_t = [q + \epsilon^2 - p(t)w_t]\,\mathrm{d}t - \epsilon\sqrt{w_t}\,\mathrm{d}W_t^2. \tag{2.2}$$

Certain technical conditions are required in order to avoid anomalies in the 3/2-process. For a CIR process (like the variance process in the Heston model and w_t defined above), it is well known that the coefficients have to satisfy the Feller boundary condition in order to avoid hitting the zero value, causing v_t in the 3/2-process to explode with nonzero probability. In our model, this condition is expressed as $q \geq -\epsilon^2/2$. When pricing options under the share measure, Lewis (2000) shows that v_t has zero probability of explosion if and only if $\rho < \epsilon/2$. By imposing the two constraints: $q > 0$ and $\rho < 0$ in our model, it becomes sufficient to avoid unboundedness in the variance process. Note that $\rho < 0$ is not that restrictive since negative correlation between S_t and v_t confirms quite well with the leverage effect commonly observed in the market: volatility goes up as asset price drops.

For the purpose of option pricing, the analytical tractability of the 3/2-model relies on the availability of the closed-form joint conditional Fourier-Laplace transform of the terminal log asset price and the quadratic variation. By assuming S_t in eq. (2.1) to be free of jump, and letting $\ln S_t e^{(r-d)(T-t)}$ be denoted by X_t, Carr and Sun (2007) obtain

$$\mathbb{E}\Big[e^{ikX_T-\mu\int_t^T v_s\,\mathrm{d}s}\Big|X_t, v_t\Big] = e^{ikX_t}\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\Big[\frac{2}{\epsilon^2 y(v_t,t)}\Big]^{\alpha} M\Big(\alpha,\gamma,-\frac{2}{\epsilon^2 y(v_t,t)}\Big), \tag{2.3}$$

where

$$\begin{aligned} y(v_t,t) &= v_t\int_t^T e^{\int_t^u p(s)\,\mathrm{d}s}\,\mathrm{d}u, \\ \alpha &= -\Big(\frac{1}{2}-\frac{\tilde{q}_k}{\epsilon^2}\Big)+\sqrt{\Big(\frac{1}{2}-\frac{\tilde{q}_k}{\epsilon^2}\Big)^2+\frac{2c_k}{\epsilon^2}}, \\ \gamma &= 2\Big(\alpha+1-\frac{\tilde{q}_k}{\epsilon^2}\Big),\ \tilde{q}_k=\rho\epsilon ik-q,\ c_k=\mu+\frac{k^2+ik}{2}. \end{aligned}$$

The confluent hypergeometric function $M(\alpha,\gamma,z)$ is defined as

$$M(\alpha,\gamma,z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{z^n}{n!},$$

where $(\alpha)_n=(\alpha)(\alpha-1)\cdots(\alpha+n-1)$ is the Pochhammer symbol. Baldeaux and Badran (2014) extend the above result by adding a compound Poisson jump component to the dynamics of S_t in the case where $p(t)$ is a constant.

However, the joint density of the log asset price and the instantaneous variance is required in pricing exotic variance swaps, like the discrete gamma swaps and corridor variance swaps. To the best of our knowledge, the joint density of the pair (X_t, v_t) under the 3/2-model derived in this paper is new in the literature. In Section 3, we present the derivation of the joint density of (X_t, v_t) and show how to use it to derive the pricing formulas of exotic variance swaps.

2.2 PIDE pricing formulation

Let $\{t_i|i=0,1,\cdots,N\}$ be an increasing sequence of monitoring dates within $[0,T]$ such that $0=t_0<t_1<\cdots<t_N=T$. For any fixed $i\neq 0$, we try to evaluate the European contingent claim whose terminal payoff at maturity t_i is given by the integrable bivariate function $F_i(S_{t_i}, I_{t_i})$, where I_{t_i} is the recorded asset price at an earlier time t_{i-1}.

To establish the PIDE formulation, we introduce the state variable I_t (Little and Pant, 2001) to capture the realization of the asset price at t_{i-1}, where

$$I_t=\int_0^t\delta(u-t_{i-1})S_u\,\mathrm{d}u=\begin{cases} S_{t_{i-1}} & t_{i-1}\le t \\ 0 & 0\le t<t_{i-1}, \end{cases} \tag{2.4}$$

where $\delta(\cdot)$ is the Dirac delta function. Since I_t changes value only at t_{i-1} and remains constant over $[0,t_{i-1})$ and $[t_{i-1},t_i]$, the governing equation of the price function can be solved in a two-step backward procedure. This is done by solving the PIDE backward in time from t_i to t_{i-1}, applying an appropriate jump condition at t_{i-1}, and then solving PIDE backward again from t_{i-1} to t_0. Subsequently, we write S_i as S_{t_i}, $i=0,1,\cdots,N$, for notational convenience. Also, similar notations are used for v_{t_i} and I_{t_i}.

Let $U_i(S, v, I_t, t)$ be the time-t price function of the contingent claim with terminal payoff $F_i(S_i, I_i)$ on maturity date t_i. For $t \le t_i$, the risk neutral valuation principle gives

$$U_i(S, v, I, t) = e^{-r(t_i - t)}\mathbb{E}^t_{\mathbb{Q}}[F_i(S_i, I_i)], \tag{2.5}$$

where $\mathbb{E}^t_{\mathbb{Q}}[\cdot]$ denotes the expectation conditional on the information by time t under the risk neutral measure $\mathbb{Q}$. By the Feymann-Kac theorem, the governing PIDE for $U_i(S, v, I, t)$ is given by

$$\frac{\partial U_i}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U_i}{\partial S^2} + \rho\epsilon v^2 S\frac{\partial^2 U_i}{\partial S \partial v} + \frac{\epsilon^2}{2}v^3\frac{\partial^2 U_i}{\partial v^2} + (r - d - \lambda m)S\frac{\partial U_i}{\partial S} + [p(t)v - qv^2]\frac{\partial U_i}{\partial v}$$
$$+ \delta(t - t_{i-1})S\frac{\partial U_i}{\partial I} - rU_i + \lambda\mathbb{E}_{\mathbb{Q}}[U_i(Se^J, v, I, t) - U_i(S, v, I, t)] = 0. \tag{2.6}$$

By the definition of I_t stated in eq. (2.4), there is a jump in the value for I across t_{i-1} while I assumes constant value across $[t_0, t_{i-1})$ and $[t_{i-1}, t_i]$. Therefore, $\frac{\partial U}{\partial I} = 0$ at $t \neq t_{i-1}$. However, there should be no jump in the value of U_i across t_{i-1}. Since $I = 0$ at times prior to t_{i-1} and $I = S$ at times immediately after t_{i-1}, the jump condition across t_{i-1} is stated as

$$\lim_{t \to t^+_{i-1}} U_i(S, v, S, t) = \lim_{t \to t^-_{i-1}} U_i(S, v, 0, t). \tag{2.7}$$

Since I assumes constant value over the two subintervals (t_0, t_{i-1}) and (t_{i-1}, t_i), we may omit the dependency of U_i on I in the PIDE for simplicity. In summary, we have the following two-step backward procedure for solving the PIDE:

(i) For $t \in (t_{i-1}, t_i)$, the governing PIDE reduces to

$$\frac{\partial U_i}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U_i}{\partial S^2} + \rho\epsilon v^2 S\frac{\partial^2 U_i}{\partial S \partial v} + \frac{\epsilon^2}{2}v^3\frac{\partial^2 U_i}{\partial v^2} + (r - d - \lambda m)S\frac{\partial U_i}{\partial S}$$
$$+ [p(t)v - qv^2]\frac{\partial U_i}{\partial v} - rU_i + \lambda\mathbb{E}_{\mathbb{Q}}[U_i(Se^J, v, I, t) - U_i(S, v, I, t)] = 0, \tag{2.8}$$

with terminal condition: $U_i(S, v, t_i) = F_i(S, S_{i-1})$. Here, S_{i-1} is the realized asset price at t_{i-1} as captured by I, which can be regarded as a known parameter since the value of I does not change in (t_{i-1}, t_i).

(ii) For $t \in [t_0, t_{i-1})$, we also solve the same governing PIDE subject to the terminal condition at time t_{i-1} as specified by the jump condition prescribed in eq. (2.7).

A close scrutiny of eq. (2.8) reveals that U_i is explicitly solvable over the time interval (t_{i-1}, t_i) since the terminal payoff function $F_i(S, S_{i-1})$ is independent of v and S_{i-1} is a known parameter. However, since the solution U_i at t^-_{i-1} has dependence on v, the solution of U_i at t_0 cannot be obtained in analytic closed form. We manage to express U_i at t_0 in a quasi-closed form in terms of an integral, where the integrand is the product of the transition density and the known solution U_i at t^-_{i-1}.

For convenience, we introduce $x = \ln S$ and $\tau = t_i - t$ for $t \le t_i$. To solve eq. (2.8) for $U_i(x, v, t)$ over (t_{i-1}, t_i), we take the Fourier transform of the governing equation with respect to x. The Fourier transform of $U_i(x, v, \tau)$ is defined by

$$\mathscr{F}[U_i(x, v, \tau)](k) = \int_{-\infty}^{\infty} e^{-ikx}U_i(x, v, \tau)\,\mathrm{d}x, \tag{2.9}$$

where k is the transform variable. We define $H_i(k, v, \tau)$ to be

$$H_i(k, v, \tau) = \exp(-s(k)\tau)\mathscr{F}[U_i(x, v, \tau)](k),$$

where

$$s(k) = ik(r - d - \lambda m) - (r + \lambda) + \lambda \exp(ik\nu - \frac{\zeta^2}{2}k^2).$$

By taking the Fourier transform on both sides, eq. (2.8) is transformed into an one-dimensional partial differential equation in H_i as follows:

$$\frac{\partial H_i}{\partial \tau} = -c_k v H_i + [p(t) + \tilde{q}_k v^2]\frac{\partial H_i}{\partial v} + \frac{\epsilon^2}{2}v^3\frac{\partial^2 H_i}{\partial v^2}, \tag{2.10}$$

with initial condition: $\mathscr{F}[F_i(e^x, S_{i-1})](k)$, where $c_k = (k^2 + ik)/2$ and $\tilde{q}_k = \rho\epsilon ik - q$.

Proposition 1 *Let $\widehat{H}_i(k, v, \tau)$ denote the fundamental solution to eq. (2.10) with initial condition: $\widehat{H}_i(k, v, 0) = 1$, then*

$$\widehat{H}_i(k, v, \tau) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)}\Big[\frac{2}{\epsilon^2 y(v,t)}\Big]^\alpha M\Big(\alpha, \gamma, -\frac{2}{\epsilon^2 y(v,t)}\Big), \tag{2.11}$$

where $y(v_t, t)$, α and γ are given in eq. (2.3) (with $\mu = 0$ in c_k). Moreover, the solution for $U_i(x, v, t)$ over (t_{i-1}, t_i) is given by

$$U_i(x, v, \tau) = \mathscr{F}^{-1}\Big[\exp(s(k)\tau)\widehat{H}_i(k, v, \tau)\mathscr{F}[F_i(e^x, S_{i-1})](k)\Big]. \tag{2.12}$$

Proof. The proof of Proposition 1 is relegated to Appendix A. ■

To proceed with the solution of $U_i(x, v, \tau)$ over $[t_0, t_{i-1})$, once $U_i(x, v, t^+_{i-1})$ has been obtained using eq. (2.12), we apply the jump condition in eq. (2.7) to obtain $U_i(x, v, t^-_{i-1})$. In general, $U_i(x, v, t^-_{i-1})$ would have functional dependence on both x and v [see eqs. (3.3a, 3.3b) for the gamma swaps]. Consequently, the joint transition density function of (x, v) is required to find the second-step solution of the PIDE. Given a closed-form joint transition density function, the solution $U_i(x, v, \tau)$ over $[t_0, t_{i-1}]$ can be obtained by evaluating its martingale representation as a double integral, by virtue of the Feynman-Kac Theorem. Interestingly, for discrete variance swaps, it can be shown that $U_i(x, v, t^-_{i-1})$ depends on v only [see eqs. (2.15a, 2.15b)]. As a result, only the marginal density function of v is required in the solution procedure.

2.3 Fair strike formulas for discrete variance swaps

Suppose $\{t_i | i = 0, 1, \cdots, N\}$ are the discrete sampling dates for the discrete realized variance over $[0, T]$, where T is the maturity date of the variance swap contract. The discrete realized variance is commonly computed based on either the actual rate of return or log rate of return. Let $V^{(1)}$ be the discrete realized variance over $[0, T]$ based on the actual rate of return as defined by

$$V^{(1)} = \frac{F_A}{N}\sum_{i=1}^{N}\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2, \tag{2.13a}$$

where F_A is the annualized factor. We take $F_A = 252$ for daily monitoring and $F_A = 52$ for weekly monitoring. Alternatively, the discrete realized variance over $[0, T]$ based on the log rate of return is given by

$$V^{(2)} = \frac{F_A}{N}\sum_{i=1}^{N}\Big(\ln\frac{S_i}{S_{i-1}}\Big)^2. \tag{2.13b}$$

We assume the time intervals between successive monitoring dates to be uniform and denote this common time interval as $\triangle t$. For the stochastic volatility model with jumps defined in eq. (2.1), it is known that as $\Delta t \to 0$, we have

$$\lim_{\triangle t\to 0}\sum_{i=1}^{N}\Big(\ln\frac{S_i}{S_{i-1}}\Big)^2=\int_0^T v_t\,\mathrm{d}t+\sum_{k=1}^{N_T}J_k^2,$$

where the last term sums all the squared jumps occurring within $[0,T]$. The fair strike of the vanilla swap on discrete realized variance (either in actual return or log return) is given by

$$K^{(n)}=\mathbb{E}_{\mathbb{Q}}[V^{(n)}],\quad n=1,2.$$

Due to the additivity property of expectation, calculating the fair strike amounts to evaluating individual risk neutral expectation of the squared return: $\mathbb{E}_{\mathbb{Q}}[(\frac{S_i-S_{i-1}}{S_{i-1}})^2]$ or $\mathbb{E}_{\mathbb{Q}}[(\ln\frac{S_i}{S_{i-1}})^2]$.

The generalized Fourier transform of the terminal payoff $F_i(S,S_{i-1})$ associated with a discrete vanilla variance swap takes the following forms, depending on whether the actual rate of return or log rate of return is used:

(i) Actual rate of return

$$\begin{aligned}\mathscr{F}\big[F_i^{(1)}(S,S_{i-1})\big] &= \mathscr{F}\Big[\Big(\frac{S-S_{i-1}}{S_{i-1}}\Big)^2\Big]=\mathscr{F}\Big[\Big(\frac{e^x}{S_{i-1}}-1\Big)^2\Big]\\ &= 2\pi\Big[\frac{\delta(k+2i)}{S_{i-1}^2}-\frac{2\delta(k+i)}{S_{i-1}}+\delta(k)\Big].\end{aligned}\tag{2.14a}$$

(ii) Log rate of return

$$\begin{aligned}\mathscr{F}\big[F_i^{(2)}(S,S_{i-1})\big] &= \mathscr{F}\Big[\Big(\ln\frac{S}{S_{i-1}}\Big)^2\Big]=\mathscr{F}[(x-\ln S_{i-1})^2]\\ &= 2\pi[-\delta''(k)-2i\,\delta'(k)\ln S_{i-1}+\delta(k)(\ln S_{i-1})^2],\end{aligned}\tag{2.14b}$$

where $\delta(\cdot)$ is the Dirac delta function, and $\delta'(\cdot)$, $\delta''(\cdot)$ denote the first order and second order derivatives of $\delta(\cdot)$, respectively.

Recall $x=\ln S$ and let $U_i^{(n)}(x,v,\tau)$ denote the solution to eq. (2.8) corresponding to the terminal condition: $F_i^{(n)}(e^x,S_{i-1}), n=1,2$. Using eq. (2.12), we derive the solution for $U_i^{(1)}$ and $U_i^{(2)}$ at $t=t_{i-1}$ $(\tau=\Delta t)$ as follows:

$$\begin{aligned}&U_i^{(1)}(x,v,\Delta t)\\ =\;& \mathscr{F}^{-1}\Big[\exp(s(k)\Delta t)\widehat{H}_i(k,v,\triangle t)\mathscr{F}\Big[\Big(\frac{e^x}{S_{i-1}}-1\Big)^2\Big]\Big]\\ =\;& \int_{-\infty}^{\infty}e^{ikx}\exp(s(k)\Delta t)\widehat{H}_i(k,v,\Delta t)\Big[\frac{\delta(k+2i)}{S_{i-1}^2}-\frac{2\delta(k+i)}{S_{i-1}}+\delta(k)\Big]\mathrm{d}k\\ =\;& e^{s(-2i)\Delta t}\widehat{H}_i(-2i,v,\Delta t)-2e^{s(-i)\Delta t}+e^{-r\Delta t}\end{aligned}\tag{2.15a}$$

and

$$\begin{aligned}&U_i^{(2)}(x,v,\Delta t)\\ =\;& \mathscr{F}^{-1}[\exp(s(k)\Delta t)\widehat{H}_i(k,v,\Delta t)\mathscr{F}[(x-\ln S_{i-1})^2]]\\ =\;& \int_{-\infty}^{\infty}e^{ikx}\exp(s(k)\Delta t)\widehat{H}_i(k,v,\Delta t)\big[-\delta''(k)-2i\delta'(k)\ln S_{i-1}+\delta(k)(\ln S_{i-1})^2\big]\mathrm{d}k\\ =\;& -f''(0)+2if'(0)\ln S_{i-1}+f(0)(\ln S_{i-1})^2\\ =\;& -g''(0),\end{aligned}\tag{2.15b}$$

where

$$f(k) = g(k)e^{ikx}, \;\; g(k) = \exp(s(k)\Delta t)\widehat{H}_i(k, v, \Delta t).$$

The above calculation clearly shows that for the discrete variance swap, the intermediate solution $U_i^{(n)}(x, v, \tau)$ depends on v only. This observation greatly simplifies the calculation of the second-step solution, since we only need to evaluate a univariate expectation using the known marginal transition density function. The required transition density of v_t can be obtained by observing that $w_t = 1/v_t$ is a CIR process. In fact, Jeanblanc *et al.* (2009) show that

$$\begin{aligned}
& p_{v_t}(v_t, t|v_0, 0) \\
= \;& \tilde{p}_{w_t}\Big(\frac{1}{v_t}, t\Big|\frac{1}{v_0}, 0\Big)\frac{1}{v_t^2} \\
= \;& \frac{l(0,t)}{2l^*(0,t)v_t^2}\exp\left(\frac{-\frac{1}{v_0} - \frac{1}{v_t}l(0,t)}{2l^*(0,t)}\right)\left(\frac{v_0\, l(0,t)}{v_t}\right)^{\bar{\nu}/2} I_{\bar{\nu}}\left(\frac{\sqrt{l(0,t)}}{l^*(0,t)\sqrt{v_0 v_t}}\right),
\end{aligned} \qquad (2.16)$$

where

$$\bar{\nu} = \frac{2}{\epsilon^2}(q+\epsilon^2) - 1, \;\; l(s,t) = \exp\Big(\int_s^t p(u)\,\mathrm{d}u\Big), \;\; l^*(s,t) = \frac{\epsilon^2}{2}\int_s^t l(s,u)\,\mathrm{d}u.$$

Note that $I_{\bar{\nu}}$ is the modified Bessel function of the first kind of order $\bar{\nu}$ as defined by

$$I_{\bar{\nu}}(z) = \Big(\frac{z}{2}\Big)^{\bar{\nu}}\sum_{k=0}^{\infty}\frac{\big(\frac{z^2}{2}\big)^k}{k!\Gamma(\bar{\nu}+k+1)}.$$

More precisely, the risk neutral expectation of the first term ($i = 1$) can be computed by a one-step backward procedure since S_0 is known. For $i \geq 2$, it is necessary to implement the two-step backward calculation from t_i to t_{i-1}, then t_{i-1} to t_0. Summarizing the above results, we obtain the following pricing formula for discrete variance swaps.

Proposition 2 *The fair strike of a variance swap with discrete sampling on dates:* $0 = t_0 < t_1 < \cdots < t_N = T$ *is given by*

$$K^{(n)} = \frac{F_A}{N}\,e^{r\Delta t}\left[U_1^{(n)}(x_0, v_0, \Delta t) + \sum_{i=2}^{N}\int_0^{\infty} U_i^{(n)}(x, v, \Delta t)p_v(v, t_{i-1}|v_0, 0)\,\mathrm{d}v\right], \;\; n = 1, 2. \qquad (2.17)$$

3 Third generation variance derivatives: Gamma swaps and corridor variance swaps

The exotic discrete variance swaps considered in this section are commonly known as the third generation variance derivatives. They are structured so as to provide more specific exposure to equity variance by adding weights at different monitoring instants in the evaluation of the accumulated discrete realized variance. Embedded with the weight mechanism, they are also called weighted variance swaps. The discrete weighted realized variance assumes the form

$$\frac{F_A}{N}\sum_{i=1}^{N} w_i\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2 \quad \text{or} \quad \frac{F_A}{N}\sum_{i=1}^{N} w_i \ln\Big(\frac{S_i}{S_{i-1}}\Big)^2$$

based on actual rate of return or log rate of return, respectively. In this section, we consider two particular types of weighted variance swaps, namely, the gamma swaps and corridor variance swaps. The respective w_i takes the forms

$$w_i = \frac{S_i}{S_0} \quad \text{and} \quad w_i = \mathbf{1}_{\{L<S_{i-1}\leq U\}} \text{ (or } w_i = \mathbf{1}_{\{L<S_i\leq U\}}).$$

The motivation for these two weighted variance swaps and their uses have been discussed in Lee (2010) and Zheng and Kwok (2014).

Analogous to the variance swap, the fair strike price of the weighted variance swap requires the evaluation of the risk neutral expectation of the weight-adjusted squared returns, which is given by

$$\mathbb{E}_{\mathbb{Q}}\Big[w_i\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2\Big] \text{ or } \mathbb{E}_{\mathbb{Q}}\Big[w_i\Big(\ln\frac{S_i}{S_{i-1}}\Big)^2\Big] \quad i = 1, 2, \cdots, N.$$

For both the gamma swaps and corridor variance swaps, w_i is essentially a function of S_i or S_{i-1}. Therefore, the weight-adjusted squared return defined in the above can still be regarded as a European contingent claim with a bivariate payoff function $E_i(S_i, S_{i-1})$. As a result, the two-step PIDE approach for pricing discrete variance swaps can be applied to pricing gamma swaps and corridor variance swaps.

Due to the existence of w_i, the payoff function E_i is no longer homogeneous in S_i and S_{i-1}. As a result, unlike the discrete vanilla variance swaps, the time-t_{i-1} value of $U_i^{(n)}$ for a weighted variance swap has dependence on both S_{i-1} and v_{i-1}. This poses the technical requirement of finding the joint transition density of $\ln S_t$ and v_t in the risk neutral expectation calculation over $[t_0, t_{i-1})$. We write $x_t = \ln S_t$, and let $p_{x,v}(x_t, v_t, t|x_s, v_s, s)$ denote the joint transition density function of x_t and v_t from time s to t. The analytic expression of $p_{x,v}$ is first derived in Lewis' unpublished monograph under the assumption that the mean reversion parameter $p(t)$ is constant. Based on this analytic result, we can evaluate the risk neutral expectation in the second-step calculation via a double integration with respect to x_{i-1} and v_{i-1}.

However, this direct approach does not take into account the special functional form of the first-step solution that admits a separable form of $e^{x_{i-1}}h(v_{i-1})$, where $h(\cdot)$ is a univariate function in v_{i-1}. The exponential form in x_{i-1} inspires us to define the log-price-transformed joint density function as follows

$$\widetilde{G}(\tau; -z, v_t|x_s, v_s) = \int_{-\infty}^{\infty} e^{izy} p_{x,v}(y, v_t, t|x_s, v_s, s)\,\mathrm{d}y. \tag{3.1}$$

To achieve better analytic tractability, henceforth we take $p(t)$ to be a constant. The log-price transformed joint density admits an analytic representation as shown in Proposition 3.

Proposition 3 *The log-price-transformed joint density function of log asset price and instantaneous variance under the 3/2-model with jumps is given by*

$$\widetilde{G}(\tau; -z, v_t|x_s, v_s) = G(\tau; -z, v_t, v_s)e^{izx_s}\exp(h(z)\tau),$$

where

$$\begin{aligned} G(\tau; -z, v_t, v_s) \;&=\; \frac{e^{(1+\mu_z)p\tau}}{e^{p\tau}-1}\exp\Big(-\frac{2p}{(e^{p\tau}-1)v_s\epsilon^2} - \frac{2pe^{p\tau}}{(e^{p\tau}-1)v_t\epsilon^2}\Big) \\ &\quad \Big(\frac{2p}{\epsilon^2 v_t}\Big)\Big(\frac{v_s}{v_t}\Big)^{\mu_z} I_{\nu_z}\Big(\frac{4pe^{p\tau/2}}{(e^{p\tau}-1)\epsilon^2\sqrt{v_s v_t}}\Big), \end{aligned}$$

and

$$\begin{aligned}
h(z) &= i(r-d-\lambda m)z + \lambda[\exp(i\nu z - \zeta^2 z^2/2) - 1],\\
\mu_z &= \frac{1}{2}(1+\hat{\theta}_z),\ \ \hat{\theta}_z = \frac{2(q+iz\rho\epsilon)}{\epsilon^2},\ \ c_z = \frac{(z^2-iz)}{2},\\
\tilde{c}_z &= \frac{2c_z}{\epsilon^2},\ \ \nu_z = 2\sqrt{\mu_z^2+\tilde{c}_z},\ \ \tau = t-s.
\end{aligned}$$

Here, $I_{\nu_z}(\cdot)$ is the modified Bessel function of the first kind of order ν_z.

Proof. The proof is presented in Appendix B. ■

As shown in the following subsection, the analytic formula of $\widetilde{G}$ plays a key role in the second-step solution procedure for pricing exotic discrete variance swaps. In particular, the analytic form of $\widetilde{G}$ helps reduce the dimensionality of integration by one, compared with the pricing formula obtained based on the use of the joint transition density function.

3.1 Fair strike formulas for gamma swaps

The derivation of the fair strike formula for the gamma swap, either in actual rate of return or log rate of return, amounts to the following respective risk neutral expectation calculation:

$$\mathbb{E}_{\mathbb{Q}}\Big[S_i\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2\Big] \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}\Big[S_i\Big(\ln\frac{S_i}{S_{i-1}}\Big)^2\Big].$$

Similar to the derivation of the fair strike formula for the vanilla variance swap, we compute the generalized Fourier transform of the terminal payoff of the gamma swap as follows:

(i) actual rate of return

$$\begin{aligned}
\mathscr{F}[E_i^{(1)}(S,S_{i-1})] &= \mathscr{F}\Big[S\Big(\frac{S-S_{i-1}}{S_{i-1}}\Big)^2\Big]\\
&= 2\pi\Big[\frac{\delta(k+3i)}{S_{i-1}^2} - \frac{2\delta(k+2i)}{S_{i-1}} + \delta(k+i)\Big]; \qquad (3.2a)
\end{aligned}$$

(ii) log rate of return

$$\begin{aligned}
\mathscr{F}[E_i^{(2)}(S,S_{i-1})] &= \mathscr{F}\Big[S\Big(\ln\frac{S}{S_{i-1}}\Big)^2\Big]\\
&= 2\pi[-\delta''(k+i) - 2i\delta'(k+i)\ln S_{i-1} + \delta(k+i)(\ln S_{i-1})^2]. \qquad (3.2b)
\end{aligned}$$

Let $U_i^{(n)}(x,v,\tau)$ denote the solution to eq. (2.8) with initial value $E_i^{(n)}(e^x, S_{i-1}), n=1,2$. Using Proposition 1, we obtain $U_i^{(1)}(x,v,\Delta t)$ and $U_i^{(2)}(x,v,\Delta t)$ as follows:

$$\begin{aligned}
&U_i^{(1)}(x,v,\Delta t)\\
&= \mathscr{F}^{-1}\Big[\exp(s(k)\Delta t)\widehat{H}_i(k,v,\Delta t)\Big[\frac{\delta(k+3i)}{S_{i-1}^2} - \frac{2\delta(k+2i)}{S_{i-1}} + \delta(k+i)\Big]\Big]\\
&= e^x[e^{s(-3i)\Delta t}\widehat{H}_i(-3i,v,\Delta) - 2e^{s(-2i)\Delta t}\widehat{H}_i(-2i,v,\Delta t) + e^{s(-i)\Delta t}], \qquad (3.3a)
\end{aligned}$$

and

$$\begin{aligned}
&U_i^{(2)}(x,v,\Delta t)\\
&= \mathscr{F}^{-1}\big[\exp(s(k)\Delta t)\widehat{H}_i(k,v,\Delta t)[-\delta^{(2)}(k+i) - 2i\delta^{(1)}(k+i)\ln S_{i-1} + \delta(k+i)(\ln S_{i-1})^2]\big]\\
&= -f''(-i) + 2if'(-i)\ln S_{i-1} + f(-i)(\ln S_{i-1})^2\\
&= e^x[-g''(-i)], \qquad (3.3b)
\end{aligned}$$

where functions f and g are defined in eq. (2.15b). As mentioned earlier, $U_i(x, v, t_{i-1}^-)$ of the gamma swap is seen to have dependence on both x and v, and in a separable form of $e^x f(v)$.

In a similar manner as in the computation of the fair strike of the vanilla variance swap, the risk neutral expectation calculation of the first term in the discrete realized variance [corresponding to $i = 1$ in the discrete variance formulas shown in eqs. (2.13a) and (2.13b)] does not require the two-step procedure since S_0 is known. For $i \geq 2$, it is necessary to perform the following double integration that evaluates the risk neutral expectation from t_{i-1} to t_0 in the second step of the backward procedure:

$$\mathbb{E}\big[e^{r\Delta t}U_i^{(n)}(x, v, \Delta t)\big] = e^{r\Delta t}\int_0^\infty \int_{-\infty}^\infty U_i^{(n)}(x, v, \Delta t)p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0)\,\mathrm{d}x\,\mathrm{d}v.$$

For $i \geq 2$, we denote the inner integral by $\Psi_i^{(n)}(v, \Delta t|x_0, v_0)$ and obtain

$$\begin{aligned}
&\Psi_i^{(1)}(v, \Delta t|x_0, v_0)\\
=&\int_{-\infty}^\infty U_i^{(1)}(x, v, \Delta t)p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0)\,\mathrm{d}x\\
=&\int_{-\infty}^\infty \Big[e^x\big(e^{s(-3i)\Delta t}\widehat{H}_i(-3i, v, \Delta t) - 2e^{s(-2i)\Delta t}\widehat{H}_i(-2i, v, \Delta t) + e^{s(-i)\Delta t}\big)\Big]\\
&\qquad p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0)\,\mathrm{d}x\\
=&\Big[e^{s(-3i)\Delta t}\widehat{H}_i(-3i, v, \Delta t) - 2e^{s(-2i)\Delta t}\widehat{H}_i(-2i, v, \Delta t) + e^{s(-i)\Delta t}\Big]\widetilde{G}(t_{i-1}; i, v|x_0, v_0),
\end{aligned} \tag{3.4a}$$

and

$$\begin{aligned}
&\Psi_i^{(2)}(v, \Delta t|x_0, v_0)\\
=&\int_{-\infty}^\infty U_i^{(2)}(x, v, \Delta t)p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0)\,\mathrm{d}x\\
=&\int_{-\infty}^\infty \Big[e^x\big(-g''(-i)\big)\Big]p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0)\,\mathrm{d}x\\
=&-g''(-i)\widetilde{G}(t_{i-1}; i, v|x_0, v_0).
\end{aligned} \tag{3.4b}$$

The formula for the fair strike of the gamma swap priced under the 3/2-model with jumps is presented in the following proposition.

Proposition 4 *The fair strike of a gamma swap with discrete sampling on dates:* $0 = t_0 < t_1 < \cdots < t_N = T$ *is given by*

$$K^{(n)} = \frac{e^{r\Delta t}}{S_0}\frac{F_A}{N}\Big[U_1^{(n)}(x_0, v_0, \Delta t) + \sum_{i=2}^N \int_0^\infty \Psi_i^{(n)}(v, \Delta t|x_0, v_0)\,\mathrm{d}v\Big],\ \ n = 1, 2. \tag{3.5}$$

3.2 Fair strike formulas for corridor variance swaps

In a corridor variance swap, the realized squared return monitored at time t_i is added to the accumulated variance only if S_{i-1} falls within the corridor $(L, U]$. The calculation of the fair strike amounts to evaluating the following risk neutral expectation:

$$\mathbb{E}_\mathbb{Q}\Big[\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2\mathbf{1}_{\{L<S_{i-1}\leq U\}}\Big] \quad \text{and} \quad \mathbb{E}_\mathbb{Q}\Big[\Big(\ln\frac{S_i}{S_{i-1}}\Big)^2\mathbf{1}_{\{L<S_{i-1}\leq U\}}\Big].$$

Alternatively, one may use S_i rather than S_{i-1} as the monitoring asset price of the corridor feature on the i^{th} sampling date for the calculation of the accumulated realized variance. The corresponding risk neutral expectation terms under the two return specifications become

$$\mathbb{E}_{\mathbb{Q}}\Big[\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2 \mathbf{1}_{\{L<S_i\le U\}}\Big] \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}\Big[\Big(\ln\frac{S_i}{S_{i-1}}\Big)^2 \mathbf{1}_{\{L<S_i\le U\}}\Big].$$

Carr and Lewis (2004) develop an approximate pricing and hedging method for the discrete corridor variance swaps by using a portfolio of European options with a continuum of strikes. Zheng and Kwok (2014) present closed form formula for the fair strike of the downside variance swap[2] under the Heston stochastic volatility model with jumps. They argue that a corridor variance swap can be replicated by a downside variance swap and a vanilla variance swap. Therefore, it suffices to compute the respective downside conditional expectation as follows:

$$\mathbb{E}_{\mathbb{Q}}\Big[\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2 \mathbf{1}_{\{S_{i-1}\le U\}}\Big] \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}\Big[\ln\Big(\frac{S_i}{S_{i-1}}\Big)^2 \mathbf{1}_{\{S_{i-1}\le U\}}\Big].$$

Notice that the indicator function is a function of S_{i-1} which is a known quantity by time t_{i-1}, and hence it can be treated as a constant factor in the first-step backward calculation over (t_{i-1}, t_i). Therefore, the first-step solution to eq. (2.8) in the case of downside variance swap is $\mathbf{1}_{\{x\le u\}}U_i^{(n)}$, where $U_i^{(n)}$ is given by eqs. (2.15a) and (2.15b) and $u = \ln U$. For $i \ge 2$, we adopt the same technique as in Zheng and Kwok (2014) to transform the indicator function into the following Fourier integral, so that the state variable x enters into the exponent:

$$\mathbf{1}_{\{S_{i-1}\le U\}} = \mathbf{1}_{\{x_{i-1}\le u\}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iu\omega}}{i\omega} e^{-i\omega x_{i-1}} \mathrm{d}\omega_r, \tag{3.6}$$

where $\omega = \omega_r + i\omega_i$ and $\omega_i \in (-\infty, 0)$. By changing the order of integration of x with ω_r, the expectation calculation coupled with the Fourier transform representation of the indicator function can be expressed as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{x\le u\}} e^{r\Delta t} U_i^{(n)}(x, v, \Delta t)\big] \\
&= e^{r\Delta t}\int_0^{\infty}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}\frac{1}{2\pi}\frac{e^{iu\omega}}{i\omega}e^{-i\omega x}U_i^{(n)}(x, v, \Delta t) p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0)\,\mathrm{d}\omega_r\Big]\mathrm{d}x\,\mathrm{d}v \\
&= \frac{e^{r\Delta t}}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{e^{iu\omega}}{i\omega}U_i^{(n)}(x, v, \Delta t)\widetilde{G}(t_{i-1}; \omega, v|x_0, v_0)\,\mathrm{d}\omega_r\,\mathrm{d}v. \qquad (3.7)
\end{aligned}$$

Furthermore, we define

$$\begin{aligned}
\Phi_i^{(1)}(\omega, v, \Delta t|x_0, v_0) &= U_i^{(1)}(x, v, \Delta t)\widetilde{G}(t_{i-1}; \omega, v|x_0, v_0) \\
&= \Big[e^{s(-2i)\Delta t}\widehat{H}_i(-2i, v, \Delta t) - 2e^{s(-i)\Delta t} + e^{r\Delta t}\Big]\widetilde{G}(t_{i-1}; \omega, v|x_0, v_0), \qquad (3.8\text{a}) \\
\Phi_i^{(2)}(\omega, v, \Delta t|x_0, v_0) &= U_i^{(2)}(x, v, \Delta t)\widetilde{G}(t_{i-1}; \omega, v|x_0, v_0) = -g''(0)\widetilde{G}(t_{i-1}; \omega, v|x_0, v_0). \qquad (3.8\text{b})
\end{aligned}$$

Combining the above results, we obtain the analytic formula for the fair strike of the downside variance swap in the form of a sum of double integrals as summarized in Proposition 5.

[2] The downside variance swap is a special type of corridor variance swap with upper barrier U only.

Proposition 5 *For a discretely sampled downside variance swap with an upper barrier U on dates: $0 = t_0 < t_1 < \cdots < t_N = T$, with S_{i-1} as the monitoring asset price on the i^{th} monitoring date, the fair strike is given by*

$$K^{(n)} = \frac{F_A}{N}\, e^{r\Delta t}\Bigg[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iu\omega}}{i\omega}\Big(\int_0^{\infty} \sum_{i=2}^{N} \Phi_i^{(n)}(\omega, v, \Delta t|x_0, v_0)\,\mathrm{d}v\Big)\mathrm{d}\omega_r \\ + \mathbf{1}_{\{x_0 \le u\}} U_1^{(n)}(x_0, v_0, \Delta t)\Bigg], \quad n = 1, 2, \qquad (3.9)$$

where $u = \ln U$ and $U_1^{(n)}(x_0, v_0, \Delta t)$ are given by eqs. (2.15a) and (2.15b).

By following a similar approach, we can derive the analytic fair strike formula for the downside variance swap with S_i as the monitoring asset price on the i^{th} monitoring date. By adopting a similar Fourier integral representation to eq. (3.6), it is clear that one has to evaluate the expectation of the squared return multiplied by $e^{-i\omega x_i}$. For this new terminal payoff function, the first-step risk neutral expectation calculation of our PIDE approach can be performed as follows:

(i) actual rate of return

$$\begin{aligned} &U_i^{(3)}(\omega, x, v, \Delta t) \\ =&\mathscr{F}^{-1}\Big[\exp(s(k)\Delta t)\widehat{H}_i(k, v, \Delta t)\mathscr{F}\Big[e^{-i\omega x}\Big(\frac{e^x}{I} - 1\Big)^2\Big]\Big] \\ =&e^{-i\omega x}\Big[e^{s(-2i-\omega)\Delta t}\widehat{H}_i(-2i - \omega, v, \Delta t) - 2e^{s(-i-\omega)\Delta t}\widehat{H}_i(-i - \omega, v, \Delta t) \\ &+ e^{s(-\omega)\Delta t}\widetilde{H}_i(-\omega, v, \Delta t)\Big]; \end{aligned} \qquad (3.10a)$$

(ii) log rate of return

$$\begin{aligned} &U_i^{(4)}(\omega, x, v, \Delta t) \\ =&\mathscr{F}^{-1}[\exp(s(k)\Delta t)\widehat{H}_i(k, v, \Delta t)\mathscr{F}[e^{-i\omega x}(x - \ln I)^2]] \\ =&e^{-i\omega x}\big[- g''(-\omega)\big]. \end{aligned} \qquad (3.10b)$$

The second-step risk neutral expectation calculation leads to the following solutions:

$$\begin{aligned} &\Phi_i^{(3)}(\omega, v, \Delta t|x_0, v_0) \\ =&\int_{-\infty}^{\infty} U_i^{(3)}(\omega, x, v, \Delta t)\, p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0)\,\mathrm{d}x \\ =&\Big[e^{s(-2i-\omega)\Delta t}\widehat{H}_i(-2i - \omega, v, \Delta t) - 2e^{s(-i-\omega)\Delta t}\widehat{H}_i(-i - \omega, v, \Delta t) \\ &+ e^{s(-\omega)\Delta t}\widehat{H}_i(-\omega, v, \Delta t)\Big]\widetilde{G}(t_{i-1}; \omega, v|x_0, v_0), \end{aligned} \qquad (3.11a)$$

and

$$\begin{aligned} &\Phi_i^{(4)}(\omega, v, \Delta t|x_0, v_0) \\ =&\int_{-\infty}^{\infty} U_i^{(4)}(\omega, x, v, \Delta t)\, p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0)\,\mathrm{d}x \\ =&- g''(-\omega)\widetilde{G}(t_{i-1}; \omega, v|x_0, v_0). \end{aligned} \qquad (3.11b)$$

Combining the above results, we obtain the analytic fair strike formula of the downside variance swap under the alternative definition of corridor as summarized in Proposition 6.

Proposition 6 *For a discretely sampled downside variance swap with an upper barrier U on dates: $0 = t_0 < t_1 < \cdots < t_N = T$ with S_i as the monitoring asset price on the i^{th} monitoring date, the fair strike is given by*

$$K^{(n)} = \frac{F_A}{N}\left[\frac{e^{r\Delta t}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iu\omega}}{i\omega}\Big(U_1^{(n)}(\omega, x_0, v_0, \Delta t) + \int_0^{\infty}\sum_{i=2}^{N}\Phi_i^{(n)}(\omega, v, \Delta t|x_0, v_0)\,\mathrm{d}v\Big)\,\mathrm{d}\omega_r\right],\ n = 3, 4. \tag{3.12}$$

4 Numerical results

In this section, we present the numerical results that were preformed for demonstrating efficiency and accuracy in the numerical evaluation of the fair strike formulas. We also investigate the impact of different parameters on the fair strikes of the various types of weighted variance swaps. We show the comparison of the values obtained in our method with the benchmark results obtained from Monte Carlo simulation for assessing the level of accuracy of our analytic formulas. We also show the relation between the sampling frequency N and the fair strikes for different types of variance swaps. The impact of various model parameters, including correlation coefficient ρ and volatility of variance ϵ, on the fair strikes of the different types of variance swaps are examined.

In our numerical calculations, we adopted the same set of model parameter values from Drimus (2012). These parameter values are obtained through simultaneous calibration of the 3-month and 6-month S&P 500 implied volatilities on July 31, 2009 for the 3/2-model under the assumption of zero jump in the asset price process (see Table 1). We assume $d = 0$, $S_0 = 1$ and $T = 1$. Besides, we take the interest rate to be flat at $r = 0.48\%$ with reference to the Treasury 1-Year Yield Curve on the calibrated date. Furthermore, we assume $U = 1$ for the upper barrier in the corridor of the downside variance swaps and there are 252 trading days in one year.

Parameter	v_0	p	q	ϵ	ρ
Value	0.060025	4.9790	22.84	8.56	-0.99

Table 1: Model parameters of the 3/2-model.

Assessment of numerical accuracy in evaluation of analytic formulas

To assess the accuracy of the newly derived analytic pricing formulas, we compare the numerical fair strike prices of discrete weighted variance swaps to the benchmark results produced by Monte Carlo simulation. The Euler discretization scheme is adopted for performing the Monte Carlo simulations. As pointed out by Drimus (2012), the 3/2-model exhibits more erratic behaviors of the volatility dynamics compared to the Heston model. When calibrated to the same set of option data, the 3/2-variance process is seen to generate more volatile sample paths than the Heston model. We also notice that exceedingly small time step is required in order to minimize numerical instabilities in the Monte Carlo simulation. Also, in order to achieve 4 significant figures accuracy, we may need to perform 1 million paths in Monte Carlo simulation (see Table 2).

Our numerical experiments suggest that the range of v_{i-1} for integration should be taken to be $[0, 10]$ and that of ω_r to be $[-100, 100]$ while $\omega_i = -0.02$ would be sufficient for convergence of the generalized Fourier transform. The calculations were performed on an

Intel i7 PC. By taking advantage of the multi-cores CPU in parallel computing, we have designed parallel codes for the Monte Carlo simulation and numerical integration.

In Table 2, we list the numerical fair strike prices for $N = 52$ produced by the Monte Carlo simulation and numerical integration of the analytic pricing formulas, respectively. The CPU times are also recorded for comparison of computational efficiency. It is obvious from the table that the analytic formulas produce highly accurate results with significantly less computation time. It is worth noting that longer computational time is required in the numerical evaluation of the analytic pricing formulas to produce fair strike prices of the discrete weighted variance swaps based on the log return specification than their actual return counterparts. This is because more time is needed to compute the second derivatives in the formulas. The computation of the fair strike prices of downside variance swaps requires more effort since evaluation of double integrals in the complex domain is involved.

swap type	variance swap		gamma swap		downside corridor	
return convention	actual	log	actual	log	actual	log
analytic	0.080939	0.083874	0.071909	0.073409	0.054224	0.056786
CPU time (s)	(0.742)	(5.742)	(1.341)	(4.232)	(37.50)	(247.89)
MC (0.1M)	0.080853	0.083827	0.071852	0.073355	0.054247	0.056797
CPU time (s)	(211.571)	(211.571)	(211.418)	(211.418)	(220.02)	(220.02)
SE	0.000142	0.000179	0.000074	0.000084	0.000174	0.000207
MC (1M)	0.080937	0.083891	0.071890	0.073394	0.054318	0.056856
CPU time (s)	(2859.911)	(2859.911)	(2839.121)	(2839.121)	(4384.84)	(4384.84)
SE	0.000045	0.000056	0.000023	0.000027	0.000055	0.000064

Table 2: We show the comparison of the fair strike prices of the variance swap, gamma swap and downside corridor swap obtained from the numerical integration of the analytic strike formulas and Monte Carlo simulation. The computational times are measured in units of second. Here, "analytic" stands for numerical evaluation of the analytic formulas via numerical integration, MC (0.1M) and MC (1M) for Monte Carlo simulation using 0.1 million paths and 1 million paths, respectively, and SE for the standard error in the simulation. For Monte Carlo simulation, computational times quoted are the same for both actual return and log return since the discrete realized variance under both conventions can be computed by using the same set of simulation paths.

To illustrate robustness and effectiveness of our pricing method, we performed more detailed numerical tests on pricing variance swaps under different sampling frequencies. In Table 3, we report the numerical comparison of the fair strike prices obtained from numerical valuation of the pricing formulas with the benchmark Monte Carlo simulation results for varying sampling frequencies. Good agreement of numerical results is observed, implying that our pricing method is reliable for any reasonable choice of sampling frequency.

Pricing properties of discrete weighted variance swaps

We would like to investigate pricing properties of these exotic discrete variance swaps under the 3/2-model. First of all, convergence of the fair strike prices as $\Delta t \to 0$ is of particular interest. Under the Heston model, Zhu and Lian (2011) show that the fair strike prices of discrete variance swap typically decrease with the sampling frequency, whereas the fair strike prices of discrete weighted variance swaps are not guaranteed to be a decreasing function (Zheng and Kwok, 2014). In Figure 1, we show the plots of the fair strike price of various

swap type		variance swap		gamma swap		downside corridor	
return convention		actual	log	actual	log	actual	log
N=12	analytic	0.077464	0.086275	0.067708	0.071866	0.050810	0.057640
	MC	0.077680	0.086639	0.067843	0.072056	0.051011	0.058150
N=26	analytic	0.079642	0.084734	0.070318	0.072857	0.052837	0.057163
	MC	0.079443	0.084472	0.070201	0.072708	0.052755	0.056984
N=78	analytic	0.081458	0.083541	0.072546	0.073620	0.054804	0.056640
	MC	0.081754	0.083927	0.072683	0.073786	0.055016	0.056931
N=104	analytic	0.081740	0.083362	0.072893	0.073733	0.055109	0.056524
	MC	0.081624	0.083230	0.072847	0.073676	0.054989	0.056405

Table 3: We show the comparison of the fair strike prices of the variance swap, gamma swap and downside corridor swap obtained from the numerical integration of the analytic strike formulas and Monte Carlo simulation under different sampling frequencies. The computational times are measured in units of second. Here, "analytic" stands for numerical evaluation of the analytic formulas via numerical integration, MC for Monte Carlo simulation using 0.1 million paths.

types of discrete weighted variance swaps against the sampling frequency. We find that the fair strike prices of the vanilla variance swaps and gamma swaps based on the log return specification are always greater than those based on the actual return. A similar pattern has been observed by Zhu and Lian (2011) for discrete variance swaps under the Heston model. For the discrete variance swap based on log return, the fair strike price decreases when the sampling frequency increases. However, when the actual return is used, the fair strike price increases steadily when N increases.[3] Interestingly, the gamma swaps based on both actual return and log return exhibit an increasing trend as the sampling frequency increases. As for the less frequently sampled downside variance swaps, the difference in the fair strike prices can be substantial under the two different corridor monitoring conventions. This indicates that the corridor monitoring convention is more influential in determining the fair strikes than the return specification. Despite the inconclusive monotonicity in the sampling frequency, we do observe that the fair strike price of each type of discrete weighted variance swap eventually converges to a steady level that corresponds to the fair strike price of their continuous counterpart regardless of the return specification and monitoring convention.

Next, we investigate the pricing sensitivity of these discrete weighted variance swaps with respect to the parameter value of correlation coefficient and volatility of variance, respectively. For convenience, we only present the plots of the fair strike prices based on the log return specification. Figure 2 exhibits the sensitivity of the fair strike price against correlation coefficient, ρ. The fair strike price of the vanilla variance swap is less sensitive to ρ than that of the gamma swap. The fair strike of the variance swap is almost flat with varying ρ, whereas the gamma swap exhibits a moderate increasing trend as ρ increases. This is expected, since the fair strike of the continuously sampled variance swap is independent of ρ. On the other hand, the weight process of the gamma swap has the function of mitigating extreme variance spikes if $\rho < 0$ and intensifying extreme variance if $\rho > 0$, which explains the observed increasing trend. The fair strike price of the downside variance swap is seen to be the most sensitive to the change of ρ and it is a decreasing function of ρ. This is

[3]As discussed in Bernard and Cui (2014), one needs to be cautious in making any conclusion on the monotonicity out of these numerical experiments, since it is possible that the observed pattern is only true for a particular range of model parameter values.

consistent with the intuition that the correlation would play a decisive role through the barrier feature. The downside variance swaps are worth more when the asset and volatility innovations are negatively correlated. A higher value of the volatility of variance ϵ normally indicates more volatile behavior of the volatility dynamics in the Heston model, and hence makes volatility derivatives more valuable. However, the corresponding pricing behavior is quite different under the 3/2-model. Figure 3 demonstrates the relation of the fair strike prices with varying values of ϵ. As ϵ increases, the fair strike prices of all discrete weighted variance swaps decrease in a convex manner.[4]

5 Conclusion

In this paper, we propose a two-step partial integro-differential equation approach to derive analytic price formulas for various discrete weighted variance swaps, including variance swaps, gamma swaps and corridor variance swaps. Quasi-closed-form pricing formulas are obtained by virtue of the closed-form log-price-transformed joint transition density of the log asset price and instantaneous variance. Though alternative methods for deriving these pricing formulas may exist, we find that the proposed method is more robust and flexible. Our approach provides a unified pricing framework for all types of discrete weighted variance swaps based on either type of return specification. We remark that the proposed method can be applied to price other path dependent derivatives under the 3/2-model as well, such as the moment swaps. We performed comprehensive numerical tests with the analytic pricing formulas and made comparison on the numerical results with the benchmark Monte Carlo simulation. The pricing of exotic variance swaps using numerical integration of analytic price formulas compete favorably well over the Monte Carlo simulation method with regard to run time efficiency. Some interesting and unique pricing behaviors of these discrete weighted variance swaps under the 3/2-model are revealed. Despite its small impact on the value of fair strike price, the return specification may affect the convergence behavior of the fair strike price significantly. The corridor convention can cause a significant difference in the fair strike price of a downside variance swap, in particular when the sampling frequency is small. While the pricing properties of these discrete weighted variance swaps with respect to the correlation coefficient exhibit similar behavior as that under the Heston model, their sensitivity with respect to varying value of the volatility of variance is completely different. In fact, the fair strike price of each type of discrete weighted variance swaps decreases as the volatility of variance increases.

[4]As an attempt to provide an intuitive explanation of this pricing behavior, we consider the continuous monitoring variance swap since the fair strike prices of the continuous and discrete monitoring variance swaps should exhibit a similar behavior when ϵ changes. The reciprocal w_t of the 3/2 variance process is a CIR process with mean equals $(q+\epsilon^2)/p$ [see eq. (2.3)]. The expected value of w_t changes with this long term mean, and so does ϵ^2 in the numerator. When ϵ^2 increases, $\mathbb{E}[w_t]$ is increasing with leading order of ϵ^2. We expand $\mathbb{E}[1/w_t]$ by Taylor series as: $\mathbb{E}[1/w_t] \approx 1/\mathbb{E}[w_t] + \mathrm{var}(w_t)/(\mathbb{E}[w_t])^3$. Since $\mathbb{E}[w_t]$ increases with leading order of ϵ^2, we can estimate that $\mathrm{var}(w_t) = O(\epsilon^4)$ as $\mathrm{var}(w_t)$ involves squared term of w_t. Since $1/\mathbb{E}[w_t]$ and $\mathrm{var}(w_t)/(\mathbb{E}[w_t])^3$ are decreasing functions of ϵ^2, so $\mathbb{E}[1/w_t]$ would be a decreasing function of ϵ^2 as well. For the continuous variance swap, the expected value $\mathbb{E}[\int_0^T v_s \, \mathrm{d}s] = \mathbb{E}[\int_0^T 1/w_t \, \mathrm{d}s]$ is then a decreasing function of ϵ^2.

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Appendix A. Proof of Proposition 1

At $\tau = 0$, we have

$$U_i(x, v, 0) = \mathscr{F}^{-1}\big[\exp(s(k)0)\widehat{H}_i(x, v, 0)\mathscr{F}[F_i(e^x, S_{i-1})]\big] = F_i(e^x, S_{i-1}),$$

so the initial condition is easily seen to be satisfied. Next, we take

$$H_i(k, v, \tau) = \widehat{H}_i(k, v, \tau)\mathscr{F}[F_i(e^x, S_{i-1})],$$

and substitute into the two sides of eq. (2.10). By canceling out the term $\mathscr{F}[F_i(e^x, S_{i-1})]$ on both sides, H_i is seen to satisfy eq. (2.10). The Fourier transform of U_i is obtained by the relation:

$$\widetilde{U}_i(k, v, \tau) = \exp(s(k)\tau)H_i(k, v, \tau) = \exp(s(k)\tau)\widehat{H}_i(x, v, \tau)\mathscr{F}[F_i(e^x, S_{i-1})]\big].$$

Subsequently, U_i is obtained by taking the inverse Fourier transform of $\widetilde{U}_i$. In fact, the fundamental solution is closely related to the marginal Fourier-Laplace transform of x given in eq. (2.3) with $\mu = 0$. One can show this relationship by using the Parseval identity.

We decompose the governing stochastic differential equation of x into the continuous part x^C and the jump part x^J as follow:

$$x^C = \int_s^t (r - d - \lambda m - \frac{1}{2}v)\,\mathrm{d}u + \int_s^t \sqrt{v_u}\,\mathrm{d}W_u^1 \quad \text{and} \quad x^J = \sum_{i=N_s+1}^{N_t} J_i.$$

Let $p_{x_i}(x_i, t_i|x, v, t)$ denote the transition density of x_i at time t_i given v and x at time t. Suppose we treat $e^{ikx}\widehat{H}_i(k, v, \tau)$ to be the marginal characteristic function given in eq. (2.3) with $\mu = 0$, then

$$\widehat{H}_i(k, v, \tau) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)}\Big[\frac{2}{\epsilon^2 y(v, t)}\Big]^\alpha M\Big(\alpha, \gamma, -\frac{2}{\epsilon^2 y(v, t)}\Big),$$

where

$$\begin{aligned}
y(v, t) &= v\int_t^{t_i} e^{\int_t^u p(s)\,\mathrm{d}s}\,\mathrm{d}u,\\
\alpha &= -\Big(\frac{1}{2} - \frac{\tilde{q}_k}{\epsilon^2}\Big) + \sqrt{\Big(\frac{1}{2} - \frac{\tilde{q}_k}{\epsilon^2}\Big)^2 + \frac{2c_k}{\epsilon^2}},\\
\gamma &= 2\Big(\alpha + 1 - \frac{\tilde{q}_k}{\epsilon^2}\Big), \;\; \tilde{q}_k = \rho\epsilon ik - q, \;\; c_k = \frac{k^2 + ik}{2}.
\end{aligned}$$

By evaluating the risk neutral expectation of the discounted payoff, we obtain

$$\begin{aligned}
U_i(x, v, \tau) &= \int_{-\infty}^{\infty} e^{-r\tau}F_i(e^{x_i}, S_{i-1})p_{x_i}(x_i, t_i|x, v, t)\,\mathrm{d}x_i\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\tau}\mathscr{F}[F_i(e^{x_i}, S_{i-1})]\overline{\mathscr{F}[p_{x_i}(x_i, t_i|x, v, t)]}\,\mathrm{d}k\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\tau}\mathscr{F}[F_i(e^{x_i}, S_{i-1})]\int_{-\infty}^{\infty} e^{ikx_i}p_{x_i}(x_i, t_i|x, v, t)\,\mathrm{d}x_i\,\mathrm{d}k\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\tau}\mathscr{F}[F_i(e^{x_i}, S_{i-1})]\mathbb{E}[e^{ikx + ikx^C + ikx^J}|x, v]\,\mathrm{d}k\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{F}[F_i(e^{x_i}, S_{i-1})]e^{ikx}\widehat{H}_i(k, v, \tau)\exp(s(k)\tau)\,\mathrm{d}k\\
&= \mathscr{F}^{-1}\big[\mathscr{F}[F_i(e^{x_i}, S_{i-1})]\widehat{H}_i(k, v, \tau)\exp(s(k)\tau)\big].
\end{aligned}$$

Appendix B. Deviation of Proposition 3

Let $p_{x,v}(x_t, v_t, t|x_s, v_s, s)$ denote the joint density of x_t and v_t conditional on x_s and v_s at time $s \leq t$. By observing

$$x_t = x_s + \int_s^t \Big(r - d - \lambda m - \frac{1}{2}v)\,\mathrm{d}u + \int_s^t \sqrt{v_u}\,\mathrm{d}W_u^1 + \sum_{i=N_s+1}^{N_t} J_i,$$

we obtain the following translation invariant relation

$$p_{x,v}(x_t, v_t, t|x_s, v_s, s) = p_{x,v}(x_t - x_s, v_t, t|0, v_s, s).$$

By virtue of the above relation, we may assume $x_s = 0$ for simplicity. Though an explicit formula for the joint transition density is not available, the x_t-transformed joint density $\widetilde{G}(\tau; -z, v_t|x_s, v_s)$ of $p_{x,v}(x_t, v_t, t|x_s, v_s, s)$ can be obtained in closed form, where $\tau = t - s$ and z is the transform variable.

As the first step in our analytic derivation, we consider a pure diffusion process x_t by setting the jump component $x^J = 0$. Let $p_d(y, \eta, t|0, v, s)$ be the joint transition density of the pure diffusion process, where y and η are assumed to be known at time t, then p_d satisfies the following Kolmogorov backward equation

$$\frac{\partial p_d}{\partial s} = \frac{1}{2}v\frac{\partial^2 p_d}{\partial y^2} + \rho\epsilon v^2\frac{\partial^2 p_d}{\partial y \partial v} + \frac{\epsilon^2}{2}v^3\frac{\partial^2 p_d}{\partial v^2} + (r - d - \frac{1}{2}v)\frac{\partial p_d}{\partial y} + (pv - qv^2)\frac{\partial p_d}{\partial v},$$

with initial condition: $p_d(y, \eta, t|0, v, t) = \delta(y)\delta(v-\eta)$. The above partial differential equation can be solved analytically by performing the Fourier transform with respect to the variable y. We define the y-transformed density $G(\tau; -z, \eta, v)$ as follows:

$$G(\tau; -z, \eta, v) = \left[\int_{-\infty}^{\infty} e^{izy} p_d(y, \eta, t|0, v, s)\,\mathrm{d}y\right] e^{-i(r-d)z\tau}.$$

The above Kolmogorov backward equation can be simplified into a one-dimensional problem:

$$\frac{\partial G}{\partial \tau} = \frac{\epsilon^2}{2}v^3\frac{\partial^2 G}{\partial v^2} + [pv - (q + iz\rho)v^2]\frac{\partial G}{\partial v} - c_z v G,$$

where

$$c_z = (z^2 - iz)/2,$$

with initial condition: $G(0; z, \eta, v) = \delta(v - \eta)$. We define the following set of transformed variables:

$$\tilde{y} = \frac{\tilde{p}}{v},\ Y = \frac{\tilde{p}}{\eta},\ \tilde{t} = p\tau,\ \tilde{p} = \frac{2p}{\epsilon^2},\ \tilde{c}_z = \frac{2c_z}{\epsilon^2},\ \tilde{q}_z = \frac{2(q + iz\rho\epsilon)}{\epsilon^2},$$

and assume that $G(\tau; z, \eta, v)$ takes the form

$$G(\tau; z, \eta, v) = \frac{Y^2}{\tilde{p}}\Big(\frac{\tilde{y}}{Y}\Big)^{R(z)} e^{[1-R(z)\tilde{t}]} g(\tilde{t}, \tilde{y}, Y, z),$$

where

$$R(z) = -\mu_z - \delta_z,\ \mu_z = \frac{1}{2}(1 + \tilde{q}_z),\ \delta_z = \sqrt{\mu_z^2 + \tilde{c}_z},$$

the partial differential equation for $G(\tau; z, \eta, v)$ can be further transformed to an ordinary differential equation in terms of $g(\tilde{t}, \tilde{y}, Y, z)$ with the independent spatial variable $\tilde{y}$. The solution of this ordinary differential equation is readily available by taking the Laplace transform in the variable $\tilde{y}$. By combining all the above results, we obtain the closed form solution for $G(\tau; -z, \eta, v)$.

The derivation of the Fourier transform of the joint density of x_t and v_t can be extended to the inclusion of jumps in the log asset price process x_t. Recall from eq. (2.1) that x_t under the 3/2-model can be decomposed into the continuous component x^C and the jump component x^J. By applying the above result for the continuous process on x^C and evaluating the compound Poisson process x^J separately, we obtain the log-price-transformed density function $\widetilde{G}(\tau; -z, v_t|x_s, v_s)$ of $p_{x,v}(x_t, v_t, t|x_s, v_s, s)$ as follows:

$$\begin{aligned}
\widetilde{G}(\tau; -z, v_t|x_s, v_s) &= \int_{-\infty}^{\infty} e^{izy} p_{x,v}(y, v_t, t|x_s, v_s, s)\,\mathrm{d}y \\
&= \mathbb{E}\big[e^{izx_t}|x_s, v_t, v_s\big] \\
&= \mathbb{E}\big[e^{izx_s+izx^C+izx^J}|x_s, v_t, v_s\big] \\
&= e^{izx_s}\mathbb{E}\big[e^{izx^C}|x_s = 0, v_t, v_s\big]\mathbb{E}\big[e^{jzx^J}\big] \\
&= e^{izx_s}\left[\int_{-\infty}^{\infty} e^{izx^C} p_d(x^C, v_t, t|0, v_s, s)\,\mathrm{d}x^C\right] e^{[\exp(i\nu z-\zeta^2 z^2/2)-1]\lambda\tau} \\
&= G(\tau; -z, v_t, v_s)e^{i(r-d-\lambda m)\tau}e^{izx_s}e^{[\exp(i\nu z-\zeta^2 z^2/2)-1]\lambda\tau} \\
&= G(\tau; -z, v_t, v_s)e^{izx_s}\exp(h(z)\tau),
\end{aligned}$$

where

$$h(z) = i(r - d - \lambda m)z + \lambda[\exp(i\nu z - \zeta^2 z^2/2) - 1].$$

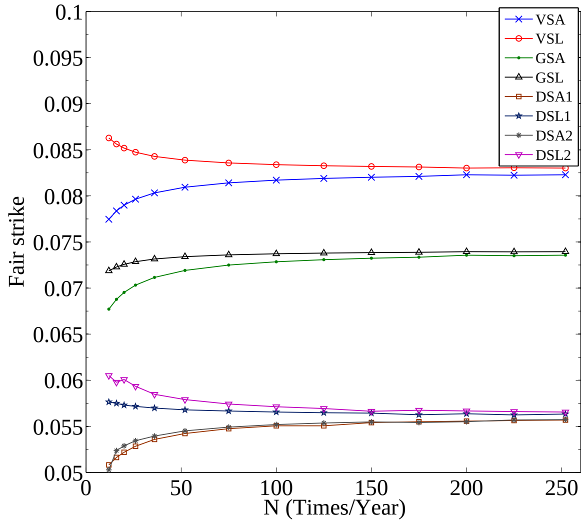

Figure 1: Plots of the fair strike prices of the vanilla variance swaps, gamma swaps and downside variance swaps against sampling frequency, N. The label VSA stands for the vanilla variance swap defined by actual return, VSL for log return. Similarly, GSA stands for the gamma swap defined by actual return, and GSL for log return. Also, DSA1 and DSL1 stand for the downside variance swaps with S_{i-1} as the monitoring asset price on the i^{th} monitoring date based on actual and log return, respectively, while DSA2 and DSL2 for S_i as the monitoring asset price.

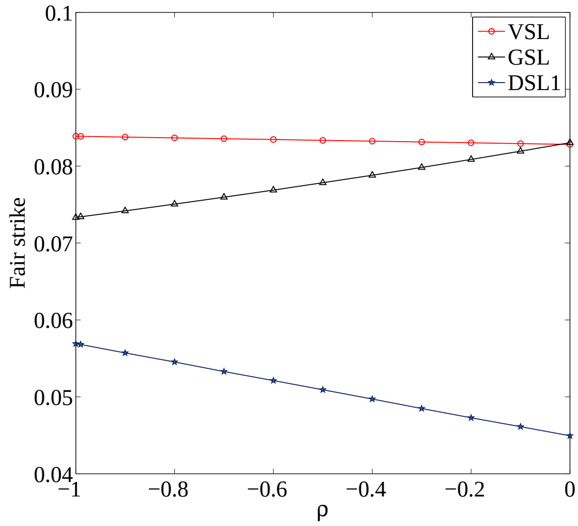

Figure 2: Plots of the fair strike prices of the vanilla variance swap, gamma swap and downside variance swap against correlation coefficient, ρ. The label VSL stands for the vanilla variance swap, GSL for the gamma swap. Also, DSL1 stands for the downside variance swap with S_{i-1} as the monitoring asset price on the i^{th} monitoring date. All discrete variance calculations are based on the convention of log return.

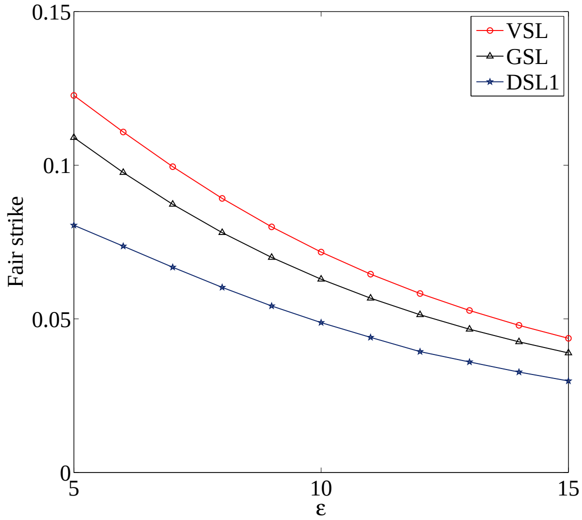

Figure 3: Plots of the fair strike prices of the vanilla variance swap, gamma swap and downside variance swap against volatility of variance, ϵ. The label VSL stands for the vanilla variance swap, GSL for the gamma swap. Also, DSL1 stands for the downside variance swap with S_{i-1} as the monitoring asset price on the i^{th} monitoring date. All discrete variance calculations are based on the convention of log return.