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Contagion models with interacting default intensity processes

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Abstract

Credit risk is quantified by the loss distribution due to unexpected changes in the credit quality of the counterparty in a financial contract. Default correlation risk refers to the risk that a bundle of risky obligors may default together. To understand the clustering phenomena in correlated defaults, we consider credit contagion models which describe the propagation of financial distress from one risky obligor to another. We present the contagion model of portfolio credit risk of multiple obligors with interacting default intensity processes where the default of one firm may trigger the increase of default intensity of other related firms. As an application, we consider how correlated default risks between the protection seller and the underlying entity may affect the credit default premium in a credit default swap.

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1. Introduction

Credit risk is the likelihood that a contractual party may not meet its obligations, like payment of coupons or principal in a bond contract, thus causing a financial loss of the counterparty. Broadly speaking, financial loss due to a credit event is quantified by the loss distribution due to expected changes in the credit quality (downgrade or default) of a contractual party. The three basic attributes for quantifying default loss is the probability of default, loss given default and exposure at default.

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There are two major approaches that attempt to describe the default processes of risky obligors, commonly known as the structural models and reduced form (intensity) models. The structural models use the contingent claims approach in option pricing theory where the value of firm's assets is used as the underlying state variable. A firm defaults when the firm assets are insufficient to honor contractual payments. This approach attempts to provide a structural interpretation of default. The reduced form approach assumes that default occurs unpredictably at an exogenous intensity or hazard rate. There is no structural interpretation of default, and the intensity is calibrated from market prices. The dynamics of default are prescribed under a pricing measure in the framework of point processes. Let τ denote the random default time of a risky obligor. The default process is defined by

$$H_t = \mathbf{1}_{\{\tau \leq t\}} = \begin{cases} 1 \text{ if } \tau \leq t \\ 0 \text{ otherwise} \end{cases}. \tag{1.1}$$

Note that H_t is a point process with one jump of size one upon default.

One major concern in the pricing and management of credit risk in an investment portfolio is the occurrence of multiple defaults of different obligors within the portfolio. This correlation risk is directly linked to the inter-dependence between default events. The development of quantitative models for analyzing correlated default risk has recently become a focus of attention for academics, regulators, and practitioners. Inter-dependence between defaults stems from at least two sources. First, the financial health of a firm varies with general macroeconomic factors. Since different firms are affected by common macroeconomic factors, we have inter-dependence between their defaults through these factors. Another inter-dependent default structures are caused by direct links between firms such as business relations, like borrower-lender relationship. The likelihood of default of a commercial bank is likely to increase if some of its major borrowers or counterparties default. For example, the South Korean banking crisis is commonly attributed to non-performing of a primary firm so that the likelihood of default of the secondary firm depends on the credit event of the primary firm.

To introduce default correlation under the reduced form framework, one may set the default intensity dynamics be driven by a common set of macroeconomic factors. Conditioned to the realization of the macroeconomic state variables, the default times are conditionally independent. The contagion models take one step further by introducing additional dependence to account for default clustering, an empirical fact that default times tend to concentrate in certain periods of time. To model the phenomenon that the default of one firm may increase the likelihood of default of other related firms, Jarrow and Yu (2001) and Yu (2007) create the default contagion effect by introducing a positive jump in the default intensity whenever there is an occurrence of default of a counterparty. For example, considering a portfolio of 3 obligors, the inter-dependent default intensities of the 3 obligors under the contagion model with interacting default intensity processes may be formulated as

$$
\begin{aligned}
\lambda_t^A &= a_{10} + a_{12}\mathbf{1}_{\{\tau_B \le t\}} + a_{13}\mathbf{1}_{\{\tau_C \le t\}} + a_{14}\mathbf{1}_{\{\tau_B \le t, \tau_C \le t\}} \\
\lambda_t^B &= a_{20} + a_{21}\mathbf{1}_{\{\tau_A \le t\}} + a_{23}\mathbf{1}_{\{\tau_C \le t\}} + a_{24}\mathbf{1}_{\{\tau_A \le t, \tau_C \le t\}} \\
\lambda_t^C &= a_{30} + a_{31}\mathbf{1}_{\{\tau_A \le t\}} + a_{32}\mathbf{1}_{\{\tau_B \le t\}} + a_{34}\mathbf{1}_{\{\tau_A \le t, \tau_B \le t\}},
\end{aligned}
\tag{1.2}
$$

where λ_t^A is the default intensity of obligor A, etc.

Information Structure

We characterize the credit risk model by introducing a collection of Cox processes (also known as doubly stochastic Poisson processes). Let the information structure in the economy with a trading period $[0, T]$ be described by the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^T, P\right)$, where $\mathcal{F} = \mathcal{F}_T$ and P is the risk neutral (equivalent martingale) probability measure. Let I be the number of firms in the economy, and $\mathcal{F}_t^X = \sigma(X_s; 0 \le s \le t)$ denotes the market information generated by the macroeconomic factors X_t. Also, $\mathcal{F}_t^i = \sigma(N_s^i; 0 \le s \le t)$ denotes the default information generated by the default process N_t^i of firm $i \in I$. Therefore, the complete information on the macroeconomic factors and the default processes of all firms up to time t is

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^I,$$

where $\mathcal{F}_t^I = \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^{|I|}$ and $\mathcal{F}^i \vee \mathcal{F}^j$ represents the smallest σ-algebra containing $\mathcal{F}^i$ and $\mathcal{F}^j$. Furthermore, the filtration generated by

$$\mathcal{F}_t^{-i} = \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^{i-1} \vee \mathcal{F}_t^{i+1} \vee \cdots \vee \mathcal{F}_t^{|I|}$$

represents the complete default information of all firms other than that of the i^{th} firm, up to time T. Hence,

$$\mathcal{G}_t^i = \mathcal{F}_t^i \vee \mathcal{F}_T^X \vee \mathcal{F}_T^{-i}$$

contains the complete information on the market but excludes the default information of firm i up to time t.

Default Time

Following the standard reduced form approach to model default risk, we characterize the stopping time (default times) τ^i of the i^{th} firm in the Cox process framework. Specifically, we define τ^i by

$$\tau^i = \inf \left\{ t : \int_0^t \lambda_s^i \, ds \ge E^i \right\} \tag{1.3}$$

and $\{E^i\}_{i\in I}$ is a set of independent unit exponential random variables. The probability space is then enlarged to accommodate $\{E^i\}_{i\in I}$, which are independent of $\mathcal{F}_T^X$ and $\mathcal{F}_T^i$ for each i. Each τ^i is characterized by the non-negative and $\mathcal{F}$-measurable process λ_t^i such that

$$\int_0^t \lambda_s^i \, ds < \infty, \quad P\text{-a.s.}$$

for each $t > 0$ and the process

$$M_t^i = H_t - \int_0^{t\wedge\tau^i} \lambda_s^i \, ds$$

is a P-martingale with respect to $\mathcal{F}_t$. Here, λ_t^i is called the default intensity of τ^i. This provides an intuition behind a formal definition of the Cox process.

Suppose that at current time t, firm i has not yet defaulted so that $\tau^i > t$. With respect to the above characterization, the conditional and unconditional survival probabilities of firm i are given by

$$\begin{aligned} P(\tau^i > T|\mathcal{G}_t^i) &= \mathbf{1}_{\{\tau^i>t\}} \exp\left(-\int_t^T \lambda_s^i \, ds\right), \\ P(\tau^i > T|\mathcal{F}_t^X) &= \mathbf{1}_{\{\tau^i>t\}} E\left(\exp\left(-\int_t^T \lambda_s^i \, ds\right) \Big| \mathcal{F}_t\right). \end{aligned} \tag{1.4}$$

In the next section, we present the Markov chain framework of the contagion model of correlated defaults, extending a similar formulation presented by Frey and Backhaus (2004). Markovian chain approach has also been applied by Avellaneda and Wu (2001) to model the default status of a portfolio of risky obligors. The computation procedure that calculates the joint distribution of default times is exemplified. In Section 3, we apply the contagion model of interacting intensities to analyze the correlated default risks of the protection seller and the underlying reference entity in a credit default swap. The procedure of calibrating the parameter functions in the model formulation is also explained. The paper is ended with conclusive summaries and remarks in the last section.

2. Markov chain framework

Considering a portfolio of N firms, we associate a random default time τ_i with firm i in the portfolio. The default status of the portfolio is given by the default process

$$\boldsymbol{H}_t = (H_t^1, H_t^2, \cdots, H_t^N) \in \{0,1\}^N = S, \tag{2.1}$$

where $H_t^i = \mathbf{1}_{\{\tau_i \le t\}}$ for $i = 1, 2, \cdots, N$. Here, $\boldsymbol{H}$ is visualized as a finite state Markov chain and S is the state space of $\boldsymbol{H}$. The macroeconomic variables are described by the d-dimensional stochastic process $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_t)_{t\in[0,T]}$ with state space $D \subseteq \mathbb{R}^d$. Let $\boldsymbol{y} \in S$, where $\boldsymbol{y}$ is a vector of default indicators of the risky obligors in the portfolio. For notational convenience, we define the flipped state $\boldsymbol{y}^i \in S$ by

$$\boldsymbol{y}^i(i) = 1 - \boldsymbol{y}(i) \quad \text{and} \quad \boldsymbol{y}^i(j) = \boldsymbol{y}(j), \quad j \in \{1, 2, \cdots, N\} - \{i\}. \quad (2.2)$$

In other words, to obtain $\boldsymbol{y}^i$ from $\boldsymbol{y}$, only the i^{th} component of $\boldsymbol{y}$ is flipped from 1 to 0 or 0 to 1 while all other components remain the same value.

Let $\mathcal{D}([0,\infty), E)$ denote the space of right continuous functions with left limit from $[0,\infty)$ into the Polish space E. We define a measurable space $(\Omega, \mathcal{F})$ in the following manner:

$$\Omega = \Omega_1 \times \Omega_2 \quad \text{where} \quad \Omega_1 = \mathcal{D}([0,\infty), D) \quad \text{and} \quad \Omega_2 = \mathcal{D}([0,\infty), S)$$

and

$$\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \quad \text{where} \quad \mathcal{F}_i \text{ is the Borel } \sigma\text{-field of } \Omega_i, i = 1, 2.$$

For each $\omega \in \Omega$, we write $\omega = (\omega_1, \omega_2)$ where $\omega_i \in \Omega_i, i = 1, 2$. We model $\boldsymbol{\Gamma}$ on $(\Omega, \mathcal{F})$ as follows:

$$\boldsymbol{\Gamma} : [0,\infty) \times \Omega \to D \times S$$

with

$$\boldsymbol{\Gamma}_t(\omega) = (\boldsymbol{\Psi}_t(\omega_1), \boldsymbol{H}_t(\omega_2)) = (\omega_1(t), \omega_2(t)).$$

Suppose the information available to the investor in the market at time t include the history of macroeconomic variables and default status of the portfolio up to time t. Mathematically, the filtration $(\mathcal{F}_t)_{t\ge0}$ on $(\Omega, \mathcal{F})$ is given by

$$\mathcal{F}_t = \mathcal{F}_t^{\boldsymbol{\Psi}} \vee \mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \cdots \vee \mathcal{F}_t^N$$

where

$$\begin{aligned} \mathcal{F}_t^{\boldsymbol{\Psi}} &= \sigma(\boldsymbol{\Psi}_s : 0 \le s \le t) \\ \mathcal{F}_t^i &= \sigma(H_s^i : 0 \le s \le t), \quad i = 1, 2, \cdots, N. \end{aligned}$$

For each $\boldsymbol{\gamma} = (\boldsymbol{\psi}, \boldsymbol{y}) \in D \times S$, we define a family of probability measure $P_{\boldsymbol{\gamma}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0})$ as

$$P_{\boldsymbol{\gamma}} = \mu_{\boldsymbol{\psi}} \times \kappa_{\boldsymbol{y}}(\omega_1, d\omega_2).$$

Here, $\mu_{\boldsymbol{\psi}}$ is a probability measure on Ω_1 which gives the law of $\boldsymbol{\Psi}$, $\kappa_{\boldsymbol{y}}$ is a transition kernel from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$, which models the conditional distribution of $\boldsymbol{H}$ for a given trajectory of $\boldsymbol{\Psi}$. Let $|S|$ denote the number of states in S. For $\boldsymbol{y}_i, \boldsymbol{y}_j \in S$, the infinitesimal generator $\Lambda_{[\boldsymbol{\Psi}]}(t) = (\Lambda_{ij}(t|\omega_1))_{|S|\times|S|}$ for $\boldsymbol{H}$ given the path of $\boldsymbol{\Psi}$ is defined as follows.

(a) For $i \neq j$

$$\Lambda_{ij}(t|\omega_1) = \begin{cases} [1 - \boldsymbol{y}_i(k)]\lambda_k(\boldsymbol{\Psi}_t(\omega_1), \boldsymbol{y}_i), & \text{if } \boldsymbol{y}_j = \boldsymbol{y}_i^k \text{ for some } k \\ 0 & \text{else} \end{cases}. \tag{2.3a}$$

The transition rate Λ_{ij} equals $\lambda_k(\boldsymbol{\Psi}_t(\omega_1), \boldsymbol{y}_i)$ when $\boldsymbol{y}_j$ can be obtained from $\boldsymbol{y}_i$ by flipping its k^{th} element from 0 to 1, indicating default of the k^{th} obligor in the portfolio. The factor $1 - \boldsymbol{y}_i(k)$ is included since $\boldsymbol{y}_i(k) = 1$ is an absorbing state.

(b) For $i = j$

$$\Lambda_{ii}(t|\omega_1) = -\sum_{j\neq i} \Lambda_{ij}(t|\omega_1) = -\sum_{k=1}^{N} [1 - \boldsymbol{y}_i(k)]\lambda_k(\boldsymbol{\Psi}_t(\omega_1), \boldsymbol{y}_i). \tag{2.3b}$$

Here, $\lambda_i(\boldsymbol{\Psi}_t, \boldsymbol{H}_t)$ is a strictly positive $\mathcal{F}$-progressively measurable process. Precisely, $\lambda_i(\boldsymbol{\Psi}_t, \boldsymbol{H}_t)$ is the martingale default intensity of firm i, that is, $H_t^i - \int_0^{t\wedge\tau_i} \lambda_i(\boldsymbol{\Psi}_s, \boldsymbol{H}_s)\, ds$ is a $\{\mathcal{F}_t\}$-martingale.

By convention, we order the default indicator vectors according to the ordering of the obligors inside the portfolio. The first state $\boldsymbol{y}_1$ corresponds to no default of any obligor, the second state corresponds to default of the first obligor only, the third state corresponds to default of the second obligor only, etc., the last state $\boldsymbol{y}_{|S|}$ corresponds to default of all obligors.

Conditional transition probabilities

Note that $\boldsymbol{H}$ can be visualized as a conditional time-inhomogeneous Markov chain. For $0 < t \leq s < \infty$, we denote the transition density matrix conditional on the path of $\boldsymbol{\Psi}$ by

$$P(t, s|\omega_1) = (p_{ij}(t, s|\omega_1))_{|S|\times|S|}. \tag{2.4}$$

The transition density matrix $P(t, s|\omega_1)$ can be obtained by solving the corresponding Kolmogorov equations. The backward Kolmogorov equation takes the form

$$\frac{dP(t, s|\omega_1)}{dt} = -\Lambda_{[\boldsymbol{\Psi}]}(t)P(t, s|\omega_1), \quad P(s, s|\omega_1) = I. \tag{2.5}$$

The individual transition probability $p_{ij}(t,s|\omega_1)$ satisfies the following system of ODE:

$$\begin{cases} \dfrac{dp_{ij}(t,s|\omega_1)}{dt} = -\sum_{k=1}^{|S|}\Lambda_{ik}(t|\omega_1)p_{kj}(t,s|\omega_1) \\ p_{ij}(s,s,\boldsymbol{y}_i,\boldsymbol{y}_j|\omega_1) = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases} \end{cases}. \tag{2.6}$$

Alternatively, the forward Kolmogorov equation takes the form

$$\frac{dP(t,s|\omega_1)}{ds} = P(t,s|\omega_1)\Lambda_{[\boldsymbol{\Psi}]}(s), \quad P(t,t|\omega_1) = I. \tag{2.7}$$

The solution of $P(t,s|\omega_1)$ can be obtained by solving either Eq. (2.5) or Eq. (2.7), and $P(t,s|\omega_1)$ is deterministic for a given path of $(\boldsymbol{\Psi}_t)_{t\geq 0}$ [that is, conditional on $\boldsymbol{\Psi} = \omega_1$].

Marginal distribution of the default time

Once the conditional transition density matrix $P(t,s|\omega_1)$ has been found, it can be used to derive the marginal distribution of τ_i, $i = 1,2,\cdots,N$. The marginal distribution function of the default time τ_i of Obligor i is defined by

$$F_i(t_i) = P_r[\tau_i \leq t_i], \quad i = 1,2,\cdots,N. \tag{2.8}$$

Let $\mu_\psi(\omega_1)$ be the probability measure which gives the law of $\boldsymbol{\Psi}$. To obtain $F_i(t_i)$, we sum over all states j with default of the i^{th} obligor [observing the requirement that $\boldsymbol{y}_j(i) = 1$] of all transition probabilities moving from state 1 (none of the obligors defaults) to state j, and subsequently integrate over the distribution of $\mu_\psi(\omega_1)$. This gives

$$F_i(t_i) = \int \sum_{\boldsymbol{y}_j(i)=1} p_{1j}(0,t_i|\omega_1)\, d\mu_\psi(\omega_1). \tag{2.9}$$

Joint distribution of the default times

The joint distribution of the default times is defined as

$$F(t_1,t_2,\cdots,t_N) = P_r[\tau_1 \leq t_1,\cdots,\tau_N \leq t_N]. \tag{2.10}$$

To express $F(t_1,t_2,\cdots,t_N)$ in terms of $p_{ij}(t_k,k_{k+1}|\omega_1)$, we consider the decomposition of the event $\{\tau_1 \leq t_1,\cdots,\tau_N \leq t_N\}$ into the union of the following mutually exclusive sub-events. Without loss of generality, we assume $t_1 \leq t_2 \leq \cdots \leq t_N$. The first sub-event is the default of all obligors within $[0,t_1]$, whose probability is given by $p_{1M}(0,t_1|\omega_1)$. The second sub-event corresponds to the default of all obligors within $(0,t_2]$, while Obligor 1 but not all obligors have defaulted by t_1. Similarly, in the third sub-event, all obligors have defaulted by t_3. However, Obligor 1 must default within $(0, t_1]$, Obligor 2 must default within $(0, t_2]$ while not all obligors have defaulted by t_2. In the last sub-event, Obligor k must default within $(0, t_k]$, $k = 1, 2, \cdots, N-1$, while not all obligors have defaulted by t_{N-1}. In addition to the above requirements, we also require that once an obligor has defaulted, the default state is an absorbing state.

For notational convenience, we define

$$\mathcal{S}(n) = \{\boldsymbol{y} \in S : \boldsymbol{y}(i) = 1 \text{ for } 1 \leq i \leq n, \boldsymbol{y}(k) = 0 \text{ at least for some } k > n\}, \quad n = 1, 2, \cdots, N-1. \tag{2.11}$$

The default indicator vector $\boldsymbol{y}_{j_n}$ at time t_n must be chosen from $\mathcal{S}(n)$ since the first n obligors have defaulted within $(0, t_n]$ but not all of the obligors have defaulted by t_n. Assuming $t_1 \leq t_2 \leq \cdots \leq t_N$, the joint distribution function can be expressed as

$$\begin{aligned} &F(t_1, t_2, \cdots, t_N) \\ &= \int_{\Omega_1} \Bigg[p_{1M}(0, t_1|\omega_1) + \sum_{\boldsymbol{y}_{j_1} \in \mathcal{S}(1)} p_{1j_1}(0, t_1|\omega_1) p_{j_1 M}(t_1, t_2|\omega_1) + \\ &\sum_{\substack{\boldsymbol{y}_{j_1} \in \mathcal{S}(1) \\ \boldsymbol{y}_{j_2} \in \mathcal{S}(2)}} p_{1j_1}(0, t_1|\omega_1) p_{j_1 j_2}(t_1, t_2|\omega_1) p_{j_2 M}(t_2, t_3|\omega_1) + \cdots + \\ &\sum_{\substack{\boldsymbol{y}_{j_1} \in \mathcal{S}(1) \\ \vdots \\ \boldsymbol{y}_{j_{N-1}} \in \mathcal{S}(N-1)}} p_{1j_1}(0, t_1|\omega_1) p_{j_1 j_2}(t_1, t_2|\omega_1) \cdots p_{j_{N-1} M}(t_{N-1}, t_N|\omega_1) \Bigg] \\ &d\mu_{\boldsymbol{\psi}}(\omega_1), \end{aligned} \tag{2.12}$$

where $\boldsymbol{y}_{j_n}$ observes the property: $\boldsymbol{y}_{j_n}(\ell) \geq \boldsymbol{y}_{j_{n-1}}(\ell)$ for $\ell = 1, 2, \cdots, N$, $n = 2, \cdots, N-1$, This condition is dictated by "default is an absorbing state". That is, once $\boldsymbol{y}_{j_{n-1}}(\ell)$ becomes one then $\boldsymbol{y}_{j_n}(\ell)$ cannot be zero.

3. Counterparty risk of credit default swaps

In a vanilla credit default swap (CDS), the protection buyer pays periodic premium to the protection seller. In return, the buyer is entitled to receive compensation from the seller on finanical loss upon default of the underlying reference entity. There have been several papers (Kim and Kim, 2003; Leung and Kwok, 2005; Walker, 2005) which discuss specifically on the counterparty risk in credit default swaps. In this section, we would like to propose a simple default contagion model that examines how correlated default risks between the protection seller and the underlying reference entity may affect the credit default swap premium. In particular, our model assumes that the default intensity of the protection seller and reference entity are subject to a positive jump in value upon the occurrence of an external shock event. To put into real life perspective of our model, we may consider a credit default swap on a risky Korean bond whose protection seller is a Korean financial institution. Though the Korean financial institution may offer protection on the Korean bond at a lower credit default swap premium, we may query whether the reduction in swap premium would be sufficient to compensate for the higher counterparty risk. This is because the Korean protection seller may share higher level of correlated risk with the Korean reference entity upon the arrival of a country wide shock (like the 1997 economic meltdown in Korea).

Model formulation

Let τ_C and τ_R denote the random default time of the counterparty and reference asset, respectively, and τ_S be the random time of arrival of the external shock S. The arrival of the shock is modeled as a Poisson event with constant mean intensity λ_S. Prior to the arrival of the shock, the default intensities λ_t^C and λ_t^R are assumed to be $a_C(t)$ and $a_R(t)$, where $a_C(t)$ and $a_R(t)$ are deterministic functions of t. Upon arrival of S, λ_t^C jumps from $a_C(t)$ to $\alpha_R a_C(t)$, and similarly, λ_t^R jumps from $a_R(t)$ to $\alpha_R a_R(t)$. Here, the proportional factors α_C and α_R are assumed to be positive constants, with $\alpha_C > 1$ and $\alpha_R > 1$. In summary, the default intensities of the three events are given by

$$\begin{aligned}\lambda_t^R &= a_R(t)[(\alpha_R - 1)\mathbf{1}_{\{\tau_S \le t\}} + 1] \\ \lambda_t^C &= a_C(t)[(\alpha_C - 1)\mathbf{1}_{\{\tau_S \le t\}} + 1] \\ \lambda_t^S &= \lambda_S. \end{aligned} \tag{3.1}$$

Our assumed model falls within the framework of a contagion model with interacting intensities. The probabilities of transition between various states of event occurrences can be solved using the Markov chain formulation. Accordingly, we let

$$\boldsymbol{H}_t = (H_t^R \quad H_t^C \quad H_t^S), \tag{3.2}$$

where $H_t^R = \begin{cases} 1 \text{ if } \tau_R \le t \\ 0 \text{ if } \tau_R > t \end{cases}$, and similar definition for H_t^C and H_t^S. There are eight possible states of the default process $\boldsymbol{H}$. The infinitesimal generator $\Lambda(t)$ can be readily found to be

$$\begin{aligned}
&\Lambda_{11} = -[a_R(t) + a_C(t) + \lambda_S], \Lambda_{12}(t) = a_R(t), \Lambda_{13}(t) = a_C(t), \Lambda_{14}(t) = \lambda_S,\\
&\Lambda_{22} = -[a_C(t) + \lambda_S], \Lambda_{25} = a_C(t), \Lambda_{26} = \lambda_S,\\
&\Lambda_{33} = -[a_R(t) + \lambda_S], \Lambda_{35} = a_R(t), \Lambda_{37} = \lambda_S,\\
&\Lambda_{44} = -[\alpha_R a_R(t) + \alpha_C a_C(t)], \Lambda_{46} = \alpha_R a_R(t), \Lambda_{47} = \alpha_C a_C(t),\\
&\Lambda_{55} = -\lambda_S, \Lambda_{58} = \lambda_S, \Lambda_{66} = -\alpha_C a_C(t), \Lambda_{68} = \alpha_C a_C(t),\\
&\Lambda_{77} = -\alpha_R a_R(t), \Lambda_{78} = \alpha_R a_R(t),
\end{aligned}$$

while all other entries are zero. The transition probability matrix P is governed by the forward Kolmogorov equation

$$\frac{dP(t,u)}{du} = P(t,u)\Lambda(u), \quad 0 \le t \le u, \tag{3.3}$$

with $P(t,t) = I$. Since $\Lambda(u)$ is upper triangular, individual transition probability $p_{ij}(t,u)$ can be solved successively in a sequential manner. Some of these probability values are found to be

$$\begin{aligned}
p_{11}(t,T) &= e^{-\int_t^T [a_R(u)+a_C(u)+\lambda_S]\,du}\\
p_{13}(t,T) &= e^{-\int_t^T [a_R(u)+\lambda_S]\,du}[1 - e^{-\int_t^T a_C(u)\,du}]\\
p_{14}(t,T) &= \lambda_S e^{-\int_t^T [a_R(u)+a_C(u)]\,du}\\
&\quad \int_t^T e^{-\int_s^T [(\alpha_R-1)a_R(u)+(\alpha_C-1)a_C(u)]\,du-\lambda_S s}\,ds.
\end{aligned}$$

The marginal distribution for τ_R is given by

$$\begin{aligned}
P_r[\tau_R > T|\mathcal{F}_t] &= p_{11}(t,T) + p_{13}(t,T) + p_{14}(t,T) + p_{17}(t,T)\\
&= e^{-\int_t^T a_R(u)\,du}\Bigg[e^{-\lambda_S(T-t)}\\
&\quad + \lambda_S \int_t^T e^{-\int_s^T (\alpha_R-1)a_R(u)\,du-\lambda_S(s-t)}\,ds\Bigg]. \qquad (3.4)
\end{aligned}$$

Credit swap premium

Let T be the maturity date of the CDS and assume unit value for the par of the underlying reference asset. We assume that the swap premium payments are made continuously at a constant swap rate $C(T)$. We assume ρ to be the deterministic recovery rate of the reference asset upon default. The contingent compensation payment of $1 - \rho$ is made by the protection seller during $(t, t + dt]$ provided that there has been no default during $(0, t)$ and default of the reference asset occurs during the infinitesimal time interval $(t, t + dt]$. The expected present value of contingent compensation payment over $(t, t + dt]$ is $(1 - \rho)e^{-rt}[p_{11}(0,t)a_R(t) + p_{14}(0,t)\alpha_R a_R(t)]\,dt$. The probability of no default up to time t is given by $p_{11}(0,t) + p_{14}(0,t)$ and the expected present value of the swap premium payment over $(t, t + dt]$ is $C(T)e^{-rt}[p_{11}(0,t)+p_{14}(0,t)]\,dt$. By equating the expected present value of the swap premium payment and contingent compensation payment upon default over the whole period $[0, T]$, we obtain

$$C(T) = \frac{(1-\rho)\int_0^T e^{-rt}[p_{11}(0,t)a_R(t) + p_{14}(0,t)\alpha_R a_R(t)]\,dt}{\int_0^T e^{-rt}[p_{11}(0,t) + p_{14}(0,t)]\,dt}. \tag{3.5}$$

Substituting the known solutions of $p_{11}(t)$ and $p_{14}(t)$, we obtain an analytic expression for $C(T)$. When there is no default risk of the counterparty, we then have $a_C = 0$. In this case, the credit default swap rate without counterparty risky is given by

$$\overline{C}(T) = \frac{(1-\rho)\int_0^T \left[a_R(t)e^{-\int_0^t r+a_R(u)\,du}\left(e^{-\lambda_S t} + \alpha_R\int_0^t \lambda_S e^{-\int_s^t (\alpha_R-1)a_R(u)\,du-\lambda_S s}\,ds\right)\right]dt}{\int_0^T e^{-\int_0^t r+a_R(u)\,du}\left(e^{-\lambda_S t} + \int_0^t \lambda_S e^{-\int^t [(\alpha_R-1)a_R(u)\,du-\lambda_S s}\,ds\right)dt}. \tag{3.6}$$

Calibration of the parameter functions

The parameter function $\alpha_R(t)$ in the intensity α_t^R can be calibrated using the term structure of prices of defaultable bonds issued by the reference entity. Let $B_R(t,T)$ denote the time-t price of the defaultable bond with unit par and zero recovery upon default. Under the risk neutral measure $\mathbb{P}$ and constant riskfree interest rate r, the defaultable bond price $B_R(t,T)$ is given by

$$B_R(t,T) = e^{-r(T-t)}E_{\mathbb{P}}[\mathbf{1}_{\{\tau_R>T\}}|\mathcal{F}_t] = e^{-r(T-t)}P_r[\tau_R > T|\mathcal{F}_t]. \tag{3.7}$$

We can also establish the following relation between the parameter function $\alpha_R(t)$ and the term structure of $B_R(t,T)$:

$$\begin{aligned}\frac{\partial B_R}{\partial T}(t,T) &= -rB_R(t,T) + a_R(T)B_R(t,T) + a_R(T)(\alpha_R - 1)B_R(t,T)\\ &\quad - a_R(T)(\alpha_R-1)e^{-\int_t^T a_R(u)\,du}e^{-\lambda_s(T-t)-r(T-t)}.\end{aligned} \tag{3.8}$$

4. Conclusion

A robust and versatile default correlation models should reflect the following two empirical facts: (i) default of one firm may trigger an increase of the default intensities of other related firms, (ii) default times tend to concentrate in certain periods of time (clusters of default). In this paper, we present the Markov chain framework of modeling default contagion via the interacting intensities approach, and apply the Markov chain techniques in calculating the joint default distribution of the random default times of multiple risky obligors within a portfolio. We develop the three-firm contagion model to analyze the counterparty risk of the protection seller of a credit default swap. To model the correlated risk of defaults, the protection seller and reference entity are subject to a positive jump in default intensity upon the arrival of an external shock. We obtain the credit default swap premium with and without default risk of the protection seller. We also manage to calibrate the parameter functions in the contagion model using market prices of traded bonds.

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