# Game options analysis of the information role of call policies in convertible bonds


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*Abstract*

In debt financing, existence of information asymmetry on the firm quality between the firm management and bond investors may lead to significant adverse selection costs. We develop the two-stage sequential dynamic two-person game option models to analyze the market signaling role of the callable feature in convertible bonds. We show that firms with positive private information on earning potential may signal their type to investors via the callable feature in a convertible bond. We present the variational inequalities formulation with respect to various equilibrium strategies in the two-person game option models via characterization of the optimal stopping rules adopted by the bond issuer and bondholders. The bondholders' belief system on the firm quality may be revealed with the passage of time when the issuer follows his optimal strategy of declaring call or bankruptcy. Under separating equilibrium, the quality status of the firm is revealed so the information asymmetry game becomes a new game under complete information. To analyze pooling equilibrium, the corresponding incentive compatibility constraint is derived. We manage to deduce the sufficient conditions for the existence of signaling equilibrium of our game option model under information asymmetry. We analyze how the callable feature may lower the adverse selection costs in convertible bond financing. We show how low quality firm may benefit from information asymmetry and vice versa, underpricing of the value of debt issued by a high quality firm.



Keywords: convertible bonds, call provision, signaling equilibrium, game options



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# 1 Introduction

There have been various theories in corporate finance that explain why some firms may be motivated to issue convertibles. These theories can be briefly categorized as agency costs models and asymmetric information models. One aspect of agency costs is related to asset substitution in debts, where managers may tend to take on risky projects after issuing straight debts since the debt holders bear the risks of loss while the managers hold the growth benefit of the firm. Green (1984) postulates that there exists better alignment between investors' conversion value held in convertibles and managers' incentive to improve firm performance, so agency costs in convertible bonds are less compared to those in straight debts. Adverse selection in corporate finance may lead to the phenomenon that capital is raised mainly by low quality firms in the financial markets. Under information asymmetry on the firm's quality, investors may not be able to differentiate between high and low quality firms and tend to undervalue securities issued by a high quality firm. In one of the earlier academic studies on debt financing, Myers and Majluf (1984) argue that a high quality firm may prefer to forego a valuable investment opportunity since the management is unwilling to sell undervalued securities to finance the investment, leading to a socially inefficient outcome. The information signaling approach in the academic literature attempts to explain why issuance of convertibles may help mitigate adverse selection costs arising from information asymmetry on firm's perceived potential earning between bond issuers and investors. Harris and Raviv (1985) pioneer the study of signaling effect of call policies in convertible bonds. Using a discrete time model, they predict that under asymmetric information, the use of forced conversion conveys a negative signal to the market.

There are later studies that examine the optimal contract design in a convertible bond in mitigating adverse selection costs arising from information asymmerty. Kim (1990) shows that the conversion ratio of a convertible bond is a credible parameter in signaling the firms quality to the market, say, issuing a more debt-like convertible (with smaller conversion ratio) conveys more positive information to the market. Stein (1992) considers a two-period convertible bond model in which information asymmetry is assumed to be perfectly resolved at the intermediate time by some external shock. When the financial distress cost is sufficiently large, he shows that the callable-convertible bond may be used by a medium firm (intermediate level of future earning) to signal its type to the market. Consequently, the adverse selection problem is reduced. A bad firm would choose to issue convertible only if the overpricing amount (resulted from mimicking a medium firm) exceeds the expected distress cost; otherwise, a bad firm should choose equity financing. If a good firm chooses to mimic a bad firm or a medium firm, it can only sell what it sees as an underpriced security with no compensation benefit. Since a good firm bears a lower expected distress cost, Stein argues that the good firm would issue debt as an optimal choice of financing. Chakraborty and Yilmaz (2011) consider a two-period model which is similar to that of Stein. Suppose information asymmetry is perfectly resolved at the intermediate time, they show that the callable-convertible bond can be designed such that its market price is independent of market information. The firm (either good or bad) can achieve its first-best efficient outcome by issuing this callable-convertible bond. On the other hand, even when information asymmetry cannot be resolved perfectly, the firm can still achieve the first-best efficient outcome by issuing the callable-convertible bond with a restrictive call provision. Korkeamaki and Moore (2004) examine the role of call provisions in convertible bond design and capital investment. They show that firms with investment options expected to expire sooner will offer weaker call protection. From their empirical studies on the data of convertible bonds and straight debt issues, Krishnaswami and Yaman (2008) show that all three types of contracting costs (moral hazard, adverse selection and financial distress) are important in determining whether a firm chooses to issue convertibles. The likelihood of issuing convertibles is greater for high growth firms and also firms with high level of information asymmetry. Once it has decided to issue a convertible, the bond structure is mainly influenced by financial distress considerations. They also conclude that firms with higher level of firm-specific information asymmetry would issue more debt-like convertibles in order to mitigate the higher adverse selection costs in equity-like securities.

Under the assumption of perfect capital markets, finance theory predicts that a firm should call a convertible bond as soon as its conversion value exceeds its call price (Ingersoll, 1977). However, in contrast to this optimal call policy, various empirical studies have documented that many convertible bonds were called only after the conversion value has exceeded the call price by a wide margin, so called the "delayed call phenomenon". On the other hand, Cowan *et al.* (1993) study calls in which the conversion is out-of-the-money. Various theories have been proposed to explain both in-the-money calls and out-of-the-money calls. A more recent survey of these theories can be found in Sarkar (2003). The empirical tests performed by Sarkar show that late calls are commonly associated with high call premium, dividend yield, tax rate and interest rate, and low coupon and volatility. Also, common stock returns are significantly negative around the announcement of late (in-the-money) calls. On the other hand, positive average stock price reaction is commonly observed for out-of-the-money calls (Cowan *et al.*, 1993), which serves as evidence that supports positive signaling effects. A possible explanation is that corporate managers are willing to pay the premium to convertible bondholders when they receive favorable private information about the firm's quality. The early call can separate from the pool of call announcements that imply bad news.

The interaction between optimal calling and conversion policies in convertibles with an underlying stochastic state variable (firm value or stock price) leads to a two-person stochastic differential game model. Under the assumption of zero bankruptcy cost and tax benefit, the game option model reduces to a zero-sum game. Sirbu and Shreve (2006) develop the variational inequalities formulation of the optimal stopping problem associated with the callable-convertible bond model and provide a full characterization of the optimal calling and conversion timing of the two counterparties. Hennessy and Tserlukevich (2008) extend the two-person game option model by incorporating tax benefit and agency conflicts (risk-shifting), which leads to a non-zero sum stochastic differential game between the equity holder and bondholder. They show that the equilibrium prices are dependent on the Markov perfect equilibriums of an infinite sequence of such games. In a related work, Chen *et al.* (2013) analyze the interaction between issuer's optimal calling and bankruptcy and bondholder's optimal conversion under the presence of credit risk and tax benefit. Using the concept of super solution, they provide a full mathematical characterization of the various optimal stopping times. They also manage to derive a unique Nash equilibrium of the game and the corresponding explicit pricing formulas of the callable-convertible bond. Egami (2010) uses the game options approach to analyze the optimal strategies involved in an investment option using convertible debt financing.

The above two-person game models are limited to complete information. To investigate the information role of issuer's call, it is necessary to incorporate the concept of information updating and signaling game into the system of variational inequalities in the two-person dynamic game models. In this paper, we follow a similar analysis of signaling from investment timing using the concept of "separating equilibrium" and "pooling equilibrium" in Morellec and Schürhoff (2011). Using a two-stage sequential two-person dynamic game model, we analyze the signaling effect from the bankruptcy rule and call policy in our convertible bond model. In the first stage of the sequential games, information asymmetry on firm's quality exists between the two counterparties. The bond investors' belief system may be updated through the optimal decision actions of call or bankruptcy declared by the issuer. Once the quality status is revealed to the bond investors, the second stage is entered and the game is transformed to a new game under complete information. We investigate how the inclusion of call provision may signal credibly the firm type (good firm or bad firm) and compare the adverse selection cost associated with convertible bond financing with and without callable feature. Our model produces the following result that is consistent with empirical findings. A low-quality firm may choose to call an out-of-the-money convertible early under pooling equilibrium provided that the benefit extracted from information asymmetry is higher than the premium paid to knock out an out-of the-money conversion option.

In our model, equity holder chooses the optimal stopping time of calling or declaring bankruptcy so as to maximize the firm value while the bondholder chooses the optimal stopping time of conversion to maximize the sum of expected cashflows. In the process of the competing game, the bondholder can update its information set based on the strategies adopted by the equity holder whose quality type is not known. We show how to determine the optimal strategies under the Perfect Bayesian Equilibrium (PBE) of the game. The type of equilibrium to be analyzed is different from the classical Nash equilibrium since the bondholder's belief changes during the process of competing game through the observation of the optimal strategies adopted by the equity holder. As a result, the bondholder's strategies change in response to the updated belief on the quality type of the equity holder. In order to construct the PBE of the competing game, we adopt the two-stage approach with information asymmetry in the first stage and complete information in the second stage. The corresponding variational inequalities are constructed in each stage. In order to verify the optimality of the strategies, the incentive constraint conditions are imposed into the various stopping sets in the variational inequalities formulation.

This paper is organized as follows. In the next section, we present the model formulation of the convertible bond with reference to the issuer's rights of bankruptcy and call and bondholder's right of voluntary conversion. We derive the expectation of cash flows to the equity holder and bondholder in terms of the belief (information) system of the bondholder.

We construct the belief system on the firm's quality held by the bondholder and introduce the relevant concept of separating and pooling equilibriums. In Section 3, we develop the system of variational inequalities associated with the coupled optimal stopping problems that model the strategic interaction between the two counterparties in a convertible bond. To complete the system, we impose the incentive constraint, which are the necessary conditions required to ensure that the firm has no incentive to deviate from the strategy prescribed in a stopping set. In Section 4, we perform the mathematical characterization of the various optimal stopping sets and derive the corresponding pricing formulas of the convertible bond. We determine the respective continuation region corresponding to different firm types (good or bad) and examine the information role of call. We also study the dependence of adverse selection costs (with and without call provision) on various model parameters. Conclusive remarks of the paper are presented in the last section.

# 2 Callable-convertible bonds under information asymmetry

We consider a firm that has issued a perpetual convertible bond with call provision to finance an investment project. The quality of the investment project can be "high" ($H$-type) or "low" ($L$-type). To model information asymmetry in our callable-convertible bond model, we assume that the firm management (referred as the equity holder henceforth) has complete information about the quality of the project while the bond investors (referred as the single-person bondholder henceforth) have only partial information on the quality. As known to the bondholder at time zero, the stochastic revenue flow rate generated from the investment project can be either $\pi_H x_t$ ($H$-type) or $\pi_L x_t$ ($L$-type), with constant probability of H-type and L-type being $p$ and $1-p$, respectively, where $p \in (0,1)$ and $\pi_H > \pi_L$. The equity holder knows exactly the quality type of the project at time zero. Both $\pi_H$ and $\pi_L$ are assumed to be fixed constants while the stochastic state variable $x_t$ is governed by

$$dx_t = \mu x_t \, dt + \sigma x_t \, dZ_t. \tag{2.1}$$

Here, $\mu$ is the constant drift rate, $\sigma$ is the constant volatility parameter and $Z_t$ is the standard Wiener process. Throughout the paper, we assume that all agents in the financial markets are risk neutral and cash flows are discounted at the riskfree interest rate $r$. As usual in real options investment models, we assume $\mu < r$ in order to satisfy the "no-bubble" condition. The bondholder's information (belief system) on the quality can be described by a vector-valued function $B$ defined by

$$B(x_t, t | x_0 = x) = (p_t, 1 - p_t),$$

where the first and second component in the vector give the time-$t$ updated belief on H-type or L-type, respectively. At time zero, the sequential games model is in the first stage. As stated in the above, the initial belief at time zero is $B(x_0, 0 | x_0 = x) = (p, 1-p)$. The bondholder is able to update his belief on the quality of the project by observing the evolution of the state variable and also the optimal stopping strategies adopted by the equity holder. When the quality type is revealed, the second stage is entered where the belief system becomes that of complete information. The corresponding value of $p_t$ becomes 1 or 0 if the firm quality is $H$-type or $L$-type, respectively. In summary, $p_t$ can only assume three discrete values, namely, 0, $p$ and 1.

To complete the formulation of the callable-convertible bond model, it is necessary to define the relevant model parameters associated with the convertible bond contract. We let $c$ be the constant perpetual coupon rate and $K$ be the call price. With regard to the call provision, the equity holder has the right to call back the bond at any time after issuance of the convertible bond (no call protection period). Upon call by the equity holder, the bondholder can choose whether to receive the cash amount $K$ or convert the bond into equity. We let $\alpha \in [0, 1]$ be the conversion ratio; that is, the bondholder's payoff upon forced conversion in the first stage would be $\alpha$ times the expected value of the revenue flow generated from the project based on the current belief system at the call time. To model financial distress associated with bond issuance, we assume that the firm has the right to declare bankruptcy so as to avoid the equity value to fall below zero. This bankruptcy right relieves the burden of servicing the coupons when the expected value of the revenue flow is low due to low value of $x_t$. Upon declaration of bankruptcy, the bondholder receives $1 - \gamma$ times the equity value after reduction of the liquidation cost, where $\gamma \in [0, 1]$. For the bondholder, he has the voluntary right to convert the bond into equity at any time with the same conversion factor $\alpha$. The interaction of optimal call and bankruptcy by equity holder and optimal conversion by bondholder under the stochastic fundamental $x_t$ leads to a two-person stochastic game option model. In the remaining part of this section, we focus our discussion on the formulation of the first stage of the sequential games where the quality status has not been revealed.

**Expected value of cash flows received by the bondholder and equity holder**

Let $\tau_{con}$ denote the optimal stopping time of bondholder's conversion (referred as conversion time henceforth). Since the quality of the project as seen by the bondholder can be either H-type or L-type, we let $\tau_{b,i}$ denote the optimal stopping time of bankruptcy declared by the equity holder for project quality of type $i$ (Firm $i$), $i = H$ or $L$; and similarly, let $\tau_{c,i}$ denote the optimal stopping time of call announced by the equity holder. Henceforth, $\tau_{b,i}$ and $\tau_{c,i}$ are referred as the bankruptcy time and call time, respectively, for project of type $i$. The bond is terminated at the minimum among all these stopping time $\tau_i = \min(\tau_{con}, \tau_{b,i}, \tau_{c,i})$, with respect to the project quality type, $i = H$ or $L$.

Under the risk neutral valuation framework, all cash flows are discounted at the riskfree interest rate $r$. Provided $\mu < r$, the expected value of the perpetual revenue flow at the level $x_\tau$ of the stochastic state variable at time $\tau$ is given by

$$\mathbb{E}\left[\int_\tau^\infty e^{-ru}\pi_i x_u\, du | x_\tau\right] = \frac{\pi_i x_\tau}{r - \mu}, \quad \pi_i = \pi_H \text{ or } \pi_L.$$

Here, $\mathbb{E}[\cdot]$ is the expectation taken with respect to the randomness of the stochastic fundamental $x_t$. The expected cash flows received by the bondholder and equity holder are functions of $x$, with dependence on the stopping times and belief $B$ on the project quality. When the bondholder makes assessment on the expected value of the perpetual revenue flow of Firm $i$, he cannot use the certain value $\pi_i$ for the cash flow multiplier due to information asymmetry. Let $\tilde{\pi}_i$ denote the cash flow multiplier adopted by the bondholder based on his belief system. Upon receiving the signal from the equity holder via call, the bondholder set $\tilde{\pi}_i$ to be $\pi_i$ if the status of Firm $i$ is revealed and $\pi_i = \bar{\pi} = p\pi_H + (1-p)\pi_L$ if the status of Firm $i$ is not revealed.

The sources of cash flows to the bondholder come from the coupon flow, payoff upon voluntary conversion or forced conversion and residual equity value upon bankruptcy. In the first stage of the sequential games, the sum of expected cash flows to the bondholder under information asymmetry at time zero is given by

$$\begin{aligned} &D(x; \tau_{con}, \tau_{b,L}, \tau_{c,L}, \tau_{b,H}, \tau_{c,H}, B) \\ &= \mathbb{E}_p \left[ \mathbb{E} \left[ \int_0^{\tau_i} ce^{-rs}\, ds + \mathbf{1}_{\{\tau_i = \tau_{con}\}} \frac{\alpha \pi_i x_{\tau_{con}}}{r-\mu} e^{-r\tau_{con}} + \mathbf{1}_{\{\tau_i = \tau_{b,i}\}} (1-\gamma) \frac{\pi_i x_{\tau_{b,i}}}{r-\mu} e^{-r\tau_{b,i}} \right.\right. \\ &\quad + \mathbf{1}_{\{\tau_i = \tau_{c,i}\}} e^{-r\tau_{c,i}} \left( \mathbf{1}_{\left\{K \geq \frac{\alpha \tilde{\pi}_i x_{\tau_{c,i}}}{r-\mu}\right\}} K + \mathbf{1}_{\left\{K < \frac{\alpha \tilde{\pi}_i x_{\tau_{c,i}}}{r-\mu}\right\}} \frac{\alpha \pi_i x_{\tau_{c,i}}}{r-\mu} \right) \\ &\quad \left.\left. \Bigg| x_0 = x, B(x,0) = \ (p, 1-p) \right]\right]. \end{aligned} \tag{2.2}$$

The bondholder chooses to convert into shares or receive cash amount $K$, depending on whether the expected value of the perpetual revenue flow based on the uncertain cash flow multiplier $\tilde{\pi}_i$ is higher than $K$ or otherwise; or equivalently, $K < \dfrac{\alpha \tilde{\pi}_i x_{c,i}}{r-\mu}$ or otherwise [see the last term in eq. (2.2)]. Here, the expectation operator $\mathbb{E}_p$ signifies that one has to take the weighted average with respect to the belief on the probabilities of occurrence of the two project types. The debt value has dependence on the optimal bankruptcy times and optimal call times of both types of firms since the status of the equity holder has not been revealed.

In a similar manner, the sum of expected cash flows to the equity holder at time zero of project type $i$ is given by

$$\begin{aligned} &E_i(x; \tau_{con}, \tau_{b,i}, \tau_{c,i}, B) \\ &= \mathbb{E} \Bigg[ \int_0^{\tau_i} (\pi_i x_s - c) e^{-rs}\, ds + \mathbf{1}_{\{\tau_i = \tau_{con}\}} \frac{(1-\alpha)\pi_i x_{\tau_{con}}}{r-\mu} e^{-r\tau_{con}} \\ &\quad + \mathbf{1}_{\{\tau_i = \tau_{c,i}\}} e^{-r\tau_{c,i}} \left( \frac{\pi_i x_{\tau_{c,i}}}{r-\mu} - \mathbf{1}_{\{K \geq \frac{\alpha \tilde{\pi}_i x_{\tau_{c,i}}}{r-\mu}\}} K - \mathbf{1}_{\{K < \frac{\alpha \tilde{\pi}_i x_{\tau_{c,i}}}{r-\mu}\}} \frac{\alpha \pi_i x_{\tau_{c,i}}}{r-\mu} \right) \\ &\quad \Bigg| x_0 = x, B(x,0) = (p, 1-p) \Bigg], \quad i = H \text{ or } L. \end{aligned} \tag{2.3}$$

Though the equity holder has complete information on the project type, the equity holder chooses his optimal decisions on bankruptcy time and call time based on the strategic profile of the bondholder, so the equity value also has dependence on the bondholder's belief.

**Signaling game: separating and pooling equilibrium**

The equity holder may be considered as the sender of signal. In the present context, the sender knows his type (either $H$-type or $L$-type) and the signal is the issuer's call. The bondholder only observes the signal but not the type. In separating equilibrium, each type of sender finds that it is optimal to take a different action. On the other hand, pooling equilibrium occurs under which each type takes the same action, so there is no revelation of the firm's type.

Firstly, we would like to deduce the incentive compatibility constraint for the existence of pooling equilibrium in our dynamic callable-convertible model. Suppose the equity holder of type $i$ would declare call at the level $x$, pooling equilibrium exists if the other type (type $j$) also declares call at the same level. Since both types would send the signal at the same level $x$, the bondholder cannot infer the true quality type so the belief system cannot be updated. On the other hand, if the equity holder chooses not to pool, then the game enters in the second stage of complete information.

*Necessary condition for the existence of pooling equilibrium (incentive compatibility constraint)*

Let $V_{e,j}(x)$ denote the value function of the equity holder of type $j$ evaluated based on the first-best equilibrium under complete information. Since the bondholder cannot distinguish the project quality, he adopts the optimal decision of either receiving the cash amount $K$ or converting into equity based on $K \geq \frac{\alpha\bar{\pi}x}{r-\mu}$ or $K < \frac{\alpha\bar{\pi}x}{r-\mu}$, respectively, where the converted equity value is calculated based on the weighted cash flow multiplier $\bar{\pi} = p\pi_H + (1-p)\pi_L$. The resulting equity value function of type $j$ under pooling is given by the difference of the expected value of revenue flow $\frac{\pi_j x}{r-\mu}$ and the expected value of payoff to the bondholder: $\mathbf{1}_{\{K\geq\frac{\alpha\bar{\pi}x}{r-\mu}\}}K + \mathbf{1}_{\{K<\frac{\alpha\bar{\pi}x}{r-\mu}\}}\frac{\alpha\pi_j x}{r-\mu}$. Pooling equilibrium exists only if the resulting value function of pooling is greater than the value function based on the first-best equilibrium under complete information upon call; that is,

$$\frac{\pi_j x}{r-\mu} - \left(\mathbf{1}_{\{K\geq\frac{\alpha\bar{\pi}x}{r-\mu}\}}K + \mathbf{1}_{\{K<\frac{\alpha\bar{\pi}x}{r-\mu}\}}\frac{\alpha\pi_j x}{r-\mu}\right) \geq V_{e,j}(x). \tag{2.4}$$

As an illustration, $H$-type firm would have the incentive to pool with type $L$ when $\frac{K(r-\mu)}{\alpha\pi_H} < x < \frac{K(r-\mu)}{\alpha\bar{\pi}}$. To verify the claim, from ineq. (2.4) and observing the range of $x$ specified above, we obtain

$$\frac{\pi_H x}{r-\mu} - \left(\mathbf{1}_{\{K\geq\frac{\alpha\bar{\pi}x}{r-\mu}\}}K + \mathbf{1}_{\{K<\frac{\alpha\bar{\pi}x}{r-\mu}\}}\frac{\alpha\pi_H x}{r-\mu}\right) = \frac{\pi_H x}{r-\mu} - K > \frac{(1-\alpha)\pi_H x}{r-\mu},$$

where the last quantity is the value function of $H$-type under complete information upon call. As a result, the corresponding necessary condition for the existence of pooling equilibrium is satisfied. In a similar manner, we can also show that $L$-type firm would have the incentive to pool with $H$-type firm when $\frac{K(r-\mu)}{\alpha\bar{\pi}} < x < \frac{K(r-\mu)}{\alpha\pi_L}$.

*Necessary condition for the existence of separating equilibrium*

On the other hand, type $i$ can be separated from type $j$ only if type $j$ has no incentive to mimic type $i$ by declaring call at the same level (it is preferable to take different signaling actions). Suppose type $j$ chooses to mimic type $i$, the bondholder treats type $j$ as type $i$, so the payoff to bondholder equals $K$ when $K \geq \dfrac{\alpha\pi_i x}{r-\mu}$ and $\dfrac{\alpha\pi_j x}{r-\mu}$ when $K < \dfrac{\alpha\pi_i x}{r-\mu}$. The actual expected value of revenue flow of type $j$ equals $\dfrac{\pi_j x}{r-\mu}$. We conclude that separating equilibrium exists only if

$$\frac{\pi_j x}{r-\mu} - \left(\mathbf{1}_{\{K \geq \frac{\alpha\pi_i x}{r-\mu}\}} K + \mathbf{1}_{\{K < \frac{\alpha\pi_i x}{r-\mu}\}} \frac{\alpha\pi_j x}{r-\mu}\right) \leq V_{e,j}(x). \tag{2.5}$$

**Payoff functions of equity holder and bondholder upon call**

Before we derive the variational inequalities formulation of our dynamic two-stage two-person game option model, it is necessary to prescribe the payoff to different types of equity holder upon call. In the first stage of incomplete information, upon call, the bondholder chooses to receive $K$ or convert into equity depending on the relative magnitude of $K$ and $\dfrac{\alpha\tilde{\pi}_i x}{r-\mu}$, at the level $x$ at which the equity holder declares call. The payoff function of the equity holder of type $i$ at the time of call is given by

$$h_{c,i}^{(p)}(x) = \begin{cases} \frac{\pi_i x}{r-\mu} - K & \text{if } x < \frac{K(r-\mu)}{\alpha\tilde{\pi}_i} \\ \frac{(1-\alpha)\pi_i x}{r-\mu} & \text{if otherwise} \end{cases}. \tag{2.6}$$

Here, we adopt the convention where the superscript "$(p)$" signifies that the payoff function to the equity holder is defined under information asymmetry.

In developing the variational inequalities formulation of our model, it is important to specify the *exact* value for the multiplier $\tilde{\pi}_i$ so that the obstacle function can be obtained. As mentioned in the earlier part of this section, the value of $\tilde{\pi}_i$ depends on whether the status of Firm $i$ can be revealed to the bondholder. More precisely, $\tilde{\pi}_i$ takes the value $\pi_i$ if the separating equilibrium criterion [see ineq. (2.5)] holds. As a result, the status of Firm $i$ can then be revealed. Otherwise, $\tilde{\pi}_i$ takes the expected value $\bar{\pi} = p\pi_H + (1-p)\pi_L$ if the pooling equilibrium criterion [see ineq. (2.4)] holds. Under such scenario, the status of Firm $i$ cannot be revealed.

It may occur that neither separating equilibrium criterion nor pooling equilibrium criterion hold such that $\tilde{\pi}_i$ cannot be determined. One can impose a technical assumption to rule out such ambiguity. More precisely, suppose the call price $K$ observes

$$K < \min(K_1, K_2), \tag{2.7}$$

where $K_1$ is given by Chen *et al.* (2013) (see Lemma D.2) and $K_2$ is explicitly given by

$$K_2 = \frac{c}{r}\frac{\beta^+(1-\beta^-) + (\beta^+ - 1)\beta^- z^{\beta^+-\beta^-} - (\beta^+ - \beta^-)z^{-\beta^-}}{\beta^+(1-\beta^-) + (\beta^+ - 1)\beta^- z^{\beta^+-\beta^-}}.$$

Here, $\beta^+$ and $\beta^-$ are positive root and negative root of the equation $\frac{\sigma^2}{2}\beta(\beta-1)+\mu\beta-r=0$ respectively and $z$ is the unique solution of the following algebraic equation

$$\frac{-\beta^+\beta^-(1-z^{\beta^+-\beta^-})}{\beta^+(1-\beta^-)+(\beta^+-1)\beta^- z^{\beta^+-\beta^-}-(\beta^+-\beta^-)z^{-\beta^-}}=\frac{\pi_L}{\alpha\pi_H}.$$

Under this assumption, it can be shown that exactly one of ineqs. (2.4) and (2.5) hold so that $\tilde{\pi}_i$ can be uniquely determined (see Appendix A). The exact value for $\tilde{\pi}_i$ is found to be

$$\tilde{\pi}_i=\begin{cases}\pi_i & \text{if } x<x_{c,H}\\ \bar{\pi} & \text{if } x\geq x_{c,H}\end{cases},\quad i=L,H. \tag{2.8}$$

Here, $x_{c,H}$ refers to the optimal call threshold of Firm $H$ under complete information. The derivation of $x_{c,H}$ is discussed in later section. Given that $x_{c,H}<\frac{K(r-\mu)}{\alpha\pi_H}$, the explicit expression of $h_{c,i}^{(p)}$ is found to be

$$h_{c,i}^{(p)}(x)=\begin{cases}\frac{\pi_i x}{r-\mu}-K & \text{if } x<\frac{K(r-\mu)}{\alpha\bar{\pi}}\\ \frac{(1-\alpha)\pi_i x}{r-\mu} & \text{if otherwise}\end{cases}. \tag{2.9}$$

When the call issued by the equity holder is optimal with reference to the maximization of the expected value of cash flow based on his own quality type, it is said to be "proactive". On the other hand, the call is said to be "mimicking" if the equity holder chooses to mimic call with respect to equity holder of the other quality type, provided that the necessary condition for the incentive of pooling is satisfied. Note that the payoff to the equity holder is independent of the incentive type of calling (proactive or mimicking).

In the second stage where the quality type has been revealed, the corresponding payoff function of the equity holder of type $i$ upon call is given by

$$\begin{aligned}h_{c,i}(x)&=\begin{cases}\frac{\pi_i x}{r-\mu}-K & \text{if } x<\frac{K(r-\mu)}{\alpha\pi_i}\\ \frac{(1-\alpha)\pi_i x}{r-\mu} & \text{if otherwise}\end{cases}\\ &=\frac{\pi_i x}{r-\mu}-g_{c,i}(x),\end{aligned} \tag{2.10a}$$

where

$$g_{c,i}(x)=\max\left(K,\frac{\alpha\pi_i x}{r-\mu}\right) \tag{2.10$b$}$$

is the payoff to the bondholder under complete information upon call when the quality of type $i$ is revealed.

The specification of the payoff function to the bondholder under information asymmetry upon call [denoted by $g_{c,i}^{(p)}(x)$] depends on how the call is initiated (proactive or mimicking), the details of which will be presented in Section 3.

# 3 Optimal stopping rules and variational inequalities formulation

There are two sequential stages of the two-person options game in our callable-convertible bond model, namely, under information asymmetry and complete information. Let $V_{e,i}^{(p)}(x)$ and $V_d^{(p)}(x)$ be the respective value function of the equity holder of type $i$, $i = H$ or $L$, and the bondholder (quality type not specified) under information asymmetry. Similarly, we define $V_{d,i}(x)$ to be the value function of the bondholder under complete information when the equity holder is type $i$, $i = H$ or $L$. The coupled system of optimal stopping problems is governed by a set of variational inequalities derived from the optimal stopping strategies of the two counterparties in the convertible bond. The two-stage sequential two-person game is solved using backward induction; that is, we first solve for the value functions and optimal stopping regions under complete information (second stage). It is necessary to specify the mechanism under which the game moves from the first stage to the second stage; that is, from information asymmetry to complete information.

**Variational inequalities formulation under complete information**

To present the variational inequalities formulation with reference to the equilibrium strategies of the equity holder and bondholder, it is necessary to prescribe the obstacle functions and the optimal stopping rules of the two counterparties.

*Obstacle functions*

Since the bondholder can always exercise the voluntary conversion right and the corresponding conversion value is $\frac{\alpha\pi_i x}{r-\mu}$ if the equity holder is of type $i$, so the debt value function of Firm $i$ has the following obstacle condition:

$$V_{d,i}(x) \geq \frac{\alpha\pi_i x}{r-\mu}, \quad i = H \text{ or } L. \tag{3.1$a$}$$

On the other hand, the equity holder of type $i$ can always declare bankruptcy to avoid the equity value to fall below zero. Also, the equity holder can choose optimally to call the bond resulting in payoff $h_{c,i}(x)$ [see eq. (2.10a)]. We then have the following obstacle condition for the equity value function of Firm $i$:

$$V_{e,i}(x) \geq \max(h_{c,i}(x), 0), \quad i = H \text{ or } L. \tag{3.1$b$}$$

*Optimal stopping regions*

The optimal stopping region $S_{D,i}$ of optimal voluntary conversion chosen by the bondholder is characterized by

$$S_{D,i} = cl\left(\left\{x \in [0,\infty) : V_{d,i}(x) = \frac{\alpha\pi_i x}{r-\mu}, \frac{\sigma^2}{2}x^2\frac{d^2V_{d,i}}{dx^2} + \mu x\frac{dV_{d,i}}{dx} - rV_{d,i} + c < 0\right\}\right), \tag{3.2$a$}$$

$i = H$ or $L$, where $cl(A)$ denotes the closure of the set $A$. The above inequality condition ensures that $V_{d,i}(x)$ assumes the value of the obstacle function under voluntary conversion rather than forced conversion. When $x \in S_{D,i}$, the payoff to the equity holder of type $i$ becomes $V_{d,i} = \dfrac{(1-\gamma)\pi_i x}{r-\mu}$. When $x \notin S_{D,i}$, the equity holder has the right to terminate the contract (by declaring call or bankruptcy) or chooses to continue, so $V_{e,i}(x)$ satisfies the following linear complementarity relation:

$$\min\{V_{e,i}(x) - \max(h_{c,i}(x), 0), -\frac{\sigma^2}{2}x^2\frac{d^2V_{e,i}}{dx^2} - \mu x\frac{dV_{e,i}}{dx} + rV_{e,i} - \pi_i x + c\} = 0, \ i = H \text{ or } L. \quad (3.2b)$$

On the other hand, the optimal stopping region $S_{E,i}$ of optimal call or bankruptcy adopted by the equity holder of type $i$ is characterized by

$$S_{E,i} = cl\left(\left\{x \in [0,\infty) : V_{e,i}(x) = \max(h_{c,i}(x), 0), \right.\right.$$
$$\left.\left.\frac{\sigma^2}{2}x^2\frac{d^2V_{e,i}}{dx^2} + \mu x\frac{dV_{e,i}}{dx} - rV_{e,i} + \pi_i x - c < 0\right\}\right). \quad (3.3a)$$

Suppose $x \in S_{E,i}$, we have $V_{d,i} = \dfrac{(1-\gamma)\pi_i x}{r-\mu}$ when $V_{e,i}(x) = 0$ (optimal bankruptcy); and $V_{d,i}(x) = \max\left(K, \dfrac{\alpha\pi_i x}{r-\mu}\right)$ when $V_{e,i}(x) = h_{c,i}(x)$ (optimal call). When $x \notin S_{E,i}$, then $V_{d,i}(x)$ satisfies the following linear complementarity relation:

$$\min\{V_{d,i}(x) - \frac{\alpha\pi_i x}{r-\mu}, -\frac{\sigma^2}{2}x^2\frac{d^2V_{d,i}}{dx^2} - \mu x\frac{dV_{d,i}}{dx} + rV_{d,i} - c\} = 0. \quad (3.3b)$$

One may be concerned with potential overlapping of the two stopping regions, $S_{D,i}$ and $S_{E,i}$ (though this is almost ruled out in real convertibles). By imposing certain technical condition on the call price $K$ (to be discussed later), the two optimal stopping regions would become disjoint so that ambiguity of the optimal stopping rules does not occur.

**Variational inequalities formulation under information asymmetry**

Under information asymmetry (first stage of the sequential game), the optimal call policy of the equity holder becomes more complicated due to the possibilities of proactive call and mimicking call (pooling is advantageous). Since mimicking call occurs only if the other type would initiate proactive call, the optimal stopping region corresponding to mimicking call would be a subset of the optimal stopping region corresponding to proactive call of the other type. When the incentive constraint condition is not satisfied (pooling equilibrium does not exist), the status of the quality type of the equity holder is revealed to the bondholder and the game enters into the second stage (complete information).

*Obstacle functions*

Under information asymmetry, the expected equity value envisioned by the bondholder upon voluntary conversion is $\dfrac{\alpha\bar{\pi}x}{r-\mu}$, where $\bar{\pi} = p\pi_H + (1-p)\pi_L$. Therefore, the value function

$V_d^{(p)}(x)$ is bounded below by $\dfrac{\alpha\bar{\pi}x}{r-\mu}$. This gives

$$V_d^{(p)}(x) \geq \frac{\alpha\bar{\pi}x}{r-\mu}. \tag{3.4$a$}$$

Similar to the formulation under complete information, the lower bound for the value function $V_{e,i}^{(p)}(x)$ is given by

$$V_{e,i}^{(p)}(x) \geq \max(h_{c,i}^{(p)}(x), 0), \tag{3.4$b$}$$

where $h_{c,i}^{(p)}(x)$ is specified in eq. (2.9).

*Optimal stopping regions*

Similar to the formulation under complete information, the optimal stopping region $S_D^{(p)}$ of optimal voluntary conversion in the first stage (information asymmetry) is characterized by

$$S_D^{(p)} = cl\left(\left\{x \in [0,\infty) : V_d^{(p)}(x) = \frac{\alpha\bar{\pi}x}{r-\mu}, \quad \frac{\sigma^2}{2}x^2\frac{d^2V_d^{(p)}}{dx^2} + \mu x\frac{dV_d^{(p)}}{dx} - rV_d^{(p)} + c < 0\right\}\right). \tag{3.5$a$}$$

When $x \in S_D^{(p)}$, the payoff to the equity holder of type $i$ becomes $V_{e,i}^{(p)}(x) = \dfrac{(1-\alpha)\pi_i x}{r-\mu}$. When $x \notin S_D^{(p)}$, $V_{e,i}^{(p)}(x)$ satisfies the following linear complementarity relation:

$$\min\{V_{e,i}^{(p)}(x) - \max(h_{c,i}^{(p)}(x), 0), -\frac{\sigma^2}{2}x^2\frac{d^2V_{e,i}^{(p)}}{dx^2} - \mu x\frac{dV_{e,i}^{(p)}}{dx} + rV_{e,i}^{(p)} - \pi_i x + c\} = 0. \tag{3.5$b$}$$

Under information asymmetry, the equity holder may issue proactive or mimicking call. Hence, the optimal stopping region $S_{E,i}^{(p)}$ for the equity holder of type $i$ is the union of $S_{pro,i}^{(p)}$, $S_{mim,i}^{(p)}$ and $S_{b,i}^{(p)}$, corresponding to the respective optimal decision of proactive call, mimicking call and bankruptcy declared by the equity holder. The decisions of proactive call and bankruptcy are similar to those under complete information, so the optimal stopping region $S_{pro,i}^{(p)} \cup S_{b,i}^{(p)}$ can be defined by:

$$\begin{aligned} S_{pro,i}^{(p)} \cup S_{b,i}^{(p)} = cl\Bigg(\Bigg\{x \in [0,\infty) : & V_{e,i}^{(p)}(x) = \max(h_{c,i}^{(p)}(x), 0), \\ & \frac{\sigma^2}{2}x^2\frac{d^2V_{e,i}^{(p)}}{dx^2} + \mu x\frac{dV_{e,i}^{(p)}}{dx} - rV_{e,i}^{(p)} + \pi_i x - c < 0\Bigg\}\Bigg). \end{aligned} \tag{3.6}$$

The characterization of the optimal stopping region $S_{mim,i}^{(p)}$ is more complicated. First, we define $S_{pool,i}^{(p)}$ to be the set of points where the incentive compatibility constraint condition for mimicking call by type $i$ is satisfied. Based on the constraint condition (2.4), we have

$$S_{pool,i}^{(p)} = cl\left(\left\{x \in [0,\infty) : \frac{\pi_j x}{r-\mu} - \left(\mathbf{1}_{\{K \geq \frac{\alpha\bar{\pi}x}{r-\mu}\}}K + \mathbf{1}_{\{K < \frac{\alpha\bar{\pi}x}{r-\mu}\}}\frac{\alpha\pi_j x}{r-\mu}\right) \geq V_{e,j}(x)\right\}\right). \tag{3.7}$$

Recall that pooling occurs only if the other type also chooses to call optimally at the same level $x$. Conditional on $x \in S_{pro,j}^{(p)}$, we have the following three possibilities:

(i) $x$ also lies inside $S^{(p)}_{pro,i}$, then proactive call prevails;

(ii) $x \notin S^{(p)}_{pro,i}$ but $x \in S^{(p)}_{pool,i}$, then mimicking call prevails;

(iii) $x \notin S^{(p)}_{pro,i} \cup S^{(p)}_{pool,i}$, no call signal is initiated by the equity holder. His status of quality type is revealed, and as a result, the game enters into the second stage of complete information.

We can conclude that $S^{(p)}_{mim,i}$ is characterized by

$$S^{(p)}_{mim,i} = (S^{(p)}_{pro,j} \setminus S^{(p)}_{pro,i}) \cap S^{(p)}_{pool,i},$$

and the optimal stopping region for the equity holder of type $i$ is given by

$$S^{(p)}_{E,i} = S^{(p)}_{pro,i} \cup S^{(p)}_{b,i} \cup S^{(p)}_{mim,i}.$$

Similar to the formulation under complete information, suppose $x \notin S^{(p)}_{E,i} \cup S^{(p)}_{E,j}$, then $V^{(p)}_d(x)$ satisfies the following linear complementarity relation:

$$\min\{V^{(p)}_d(x) - \frac{\alpha\bar{\pi}x}{r-\mu}, -\frac{\sigma^2}{2}x^2\frac{d^2V^{(p)}_d}{dx^2} - \mu x\frac{dV^{(p)}_d}{dx} + rV^{(p)}_d - c\} = 0. \tag{3.8}$$

Note that when $x \in S^{(p)}_{E,i}$ and the equity holder may be type $i$ or type $j$ (the other type), the payoff to the bondholder and the equity holder are dependent on how the game is terminated or whether it enters into the second stage of the game. The schematic representation of the evolution of the two-stage sequential stochastic game is summarized in Figure 1. The discussion on the payoff function under various scenarios is presented below.

1. *Termination of the game due to bankruptcy or call*
   This occurs when $x \in S^{(p)}_{E,i}$ and the equity holder is type $i$. We first determine the payoff of the equity holder (Firm $i$). When the optimal stopping decision is due to optimal bankruptcy, the payoff to the equity holder is simply $V^{(p)}_{e,i}(x) = 0$. When the optimal stopping decision is due to optimal call, the payoff to the equity holder is $V^{(p)}_{e,i}(x) = h^{(p)}_{c,i}(x)$. The determination of payoff to the bondholder is complicated by the fact that the bondholder does not know in advance the exact type of the equity holder upon receiving call/bankruptcy signal when $x$ falls within $S^{(p)}_{E,i}$. The exact payoff depends on whether $x$ falls into the stopping region of another type of firm. We let $p_i$, $i = L$ or $H$, be the probability that the firm is of type $i$. Firstly, we consider the case when $x \notin S^{(p)}_{E,j}$, the bondholder expects that call/bankruptcy signal is received with probability $p_i$ and no signal is received with probability $1 - p_i$. If the call/bankruptcy signal is received, the bondholder conjectures that the firm is of type $i$ and makes its optimal decision accordingly (receiving $K$ or conversion when the call signal is received). The corresponding payoff is seen to be

$$f_i(x) = \begin{cases} 0 & \text{if } x \in S^{(p)}_{b,i} \\ \max\left(K, \frac{\alpha\pi_i x}{r-\mu}\right) & \text{if } x \in S^{(p)}_{c,i} \end{cases}. \tag{3.9}$$

If no signal is received, the bondholder conjectures that the firm is of type $j$ and the game enters into the second stage. The corresponding payoff is seen to be $V_{d,j}(x)$. Summing up, the expected payoff to the bondholder is given by

$$V_d^{(p)}(x) = p_i f_i(x) + (1 - p_i)V_{d,j}(x). \tag{3.10a}$$

We consider another case where $x \in S_{E,j}^{(p)}$. If the two types of firm adopt different strategies, the bondholder can still update its belief on the firm's type by observing the signal received. The expected payoff to the bondholder is again the weighted average of payoff with respect to different signals received. We then have

$$V_d^{(p)}(x) = \begin{cases} (1 - p_i) \max\left(K, \frac{\alpha\pi_j x}{r-\mu}\right) & \text{if } x \in S_{b,i}^{(p)} \cap S_{c,j}^{(p)} \\ p_i \max\left(K, \frac{\alpha\pi_i x}{r-\mu}\right) & \text{if } x \in S_{c,i}^{(p)} \cap S_{b,j}^{(p)} \end{cases}. \tag{3.10b}$$

If the two types of firm adopt the same strategies (bankruptcy or call), the bondholder cannot update its belief using the signal received and make his optimal decision using the original belief (when the call signal is received). Thus, the payoff to the bondholder in this case is seen to be

$$V_d^{(p)}(x) = \begin{cases} 0 & \text{if } x \in S_{b,i}^{(p)} \cap S_{b,j}^{(p)} \\ \max\left(K, \frac{\alpha\bar{\pi} x}{r-\mu}\right) & \text{if } x \in S_{c,i}^{(p)} \cap S_{c,j}^{(p)} \end{cases}. \tag{3.10c}$$

2. *Quality status is revealed and the game enters into the second stage*
This occurs when the equity holder is type $j$ while $x \in S_{pro,i}^{(p)} \cup S_{b,i}^{(p)} \setminus S_{E,j}^{(p)}$. Under this scenario, the status is revealed since the equity holder of the other type (type $i$) would have called or declared bankruptcy optimally. However, since $x \notin S_{E,j}^{(p)}$, type $j$ would not call or declare bankruptcy. The game enters into the second stage of complete information, so the payoff to the equity holder is simply

$$V_{e,j}^{(p)}(x) = V_{e,j}(x). \tag{3.11}$$

On the other hand, the bondholder does not know in advance whether the bankruptcy/ call signal is sent when $x$ hits the above stopping sets. If a signal is received, the bondholder conjectures that the firm must be of type $i$. If no signal is received, the bondholder conjectures that the firm must be of type $j$ and the game enters into second stage. Therefore, the payoff to the bondholder is seen to be

$$V_d^{(p)}(x) = \begin{cases} (1 - p_i)V_{d,j}(x) & \text{if } x \in S_{b,i}^{(p)} \\ p_i \max\left(K, \frac{\alpha\pi_i x}{r-\mu}\right) + (1 - p_i)V_{d,j}(x) & \text{if } x \in S_{c,i}^{(p)} \end{cases}. \tag{3.12}$$

*Optimal stopping time and incentive constraint*
Based on the variational inequalities developed above, the optimal stopping times adopted

by Firm $i$ and bondholder should be defined as the first hitting time to its corresponding stopping set, respectively. Unlike the case under complete information (Chen *et al.*, 2013), the bondholder's belief may be updated during the game and the optimal stopping times adopted by the two counterparties change accordingly. This happens when the state variable hits the boundary of the stopping regions of the equity holder in which the bondholder may receive some signal from the equity holder.

We let $\tau^*_{b,i}$, $\tau^*_{c,i}$ $i = L$ or $H$ and $\tau^*_{con}$ be the optimal bankruptcy time of Firm $i$, the optimal call time of Firm $i$ and optimal conversion time of bondholder, respectively. During the first stage of the game under information asymmetry, the optimal stopping times $\tau^*_{b,i}$, $\tau^*_{c,i}$ and $\tau^*_{con}$ should be taken as the first hitting time to the optimal stopping sets $S^{(p)}_{b,i}$, $S^{(p)}_{c,i}$ and $S^{(p)}_{D}$, respectively. On the other hand, if the state variable $x_t$ hits the optimal stopping set of Firm $j$, where $S^{(p)}_{E,j} = S^{(p)}_{b,j} \cup S^{(p)}_{c,j}$, $j \neq i$ before hitting the stopping sets $S^{(p)}_{E,i}$ or $S^{(p)}_{D}$, the bondholder's belief changes and the game enters into second stage of complete information. The optimal stopping times $\tau^*_{b,i}$, $\tau^*_{c,i}$ and $\tau^*_{con}$ become the first hitting time of the optimal stopping sets $S_{b,i}$, $S_{c,i}$ and $S_D$, respectively.

We define the stopping times $t^{(p)}_{b,i}$, $t^{(p)}_{c,i}$ and $t^{(p)}_{D}$ to be the respective first hitting time of the stopping sets $S^{(p)}_{b,i}$, $S^{(p)}_{c,i}$ and $S^{(p)}_{D}$, respectively. The optimal stopping times $\tau^*_{b,i}$, $\tau^*_{c,i}$ ($i = L$ or $H$) and $\tau^*_{con}$ can be expressed as

$$\tau^*_{b,i} = \begin{cases} t^{(p)}_{b,i} & \text{if } t^{(p)}_{b,i} \leq \min\{t^{(p)}_{b,j}, t^{(p)}_{c,j}\} \\ \inf\{t \geq \min\{t^{(p)}_{b,j}, t^{(p)}_{c,j}\} : x_t \in S_{b,i}\} & \text{if } t^{(p)}_{b,i} > \min\{t^{(p)}_{b,j}, t^{(p)}_{c,j}\} \end{cases}, \tag{3.13$a$}$$

$$\tau^*_{c,i} = \begin{cases} t^{(p)}_{c,i} & \text{if } t^{(p)}_{c,i} \leq \min\{t^{(p)}_{b,j}, t^{(p)}_{c,j}\} \\ \inf\{t \geq \min\{t^{(p)}_{b,j}, t^{(p)}_{c,j}\} : x_t \in S_{c,i}\} & \text{if } t^{(p)}_{c,i} > \min\{t^{(p)}_{b,j}, t^{(p)}_{c,j}\} \end{cases}, \tag{3.13$b$}$$

$$\tau^*_{con} = \begin{cases} t^{(p)}_{D} & \text{if } t^{(p)}_{D} \leq \min\{t^{(p)}_{b,j}, t^{(p)}_{c,j}\} \\ \inf\{t \geq t^{(p)}_{E,j} : x_t \in S_{D,i}\} & \text{if } t^{(p)}_{D} > \min\{t^{(p)}_{b,j}, t^{(p)}_{c,j}\} \end{cases}. \tag{3.13$c$}$$

**Perfect Bayesian Equilibrium**

To ensure that the optimal stopping times defined above constitute the so called Perfect Bayesian Equilibrium[2] (PBE), it is essential to impose some extra conditions on the stopping sets $S^{(p)}_{b,i}$ and $S^{(p)}_{c,i}$, $i = L$ or $H$. These conditions are often called the *incentive constraints*. They are used to ensure that each type of firm has no strict incentive to deviate from the strategy prescribed in the stopping sets. Furthermore, they also guarantee that the belief system is consistent with respect to the strategic profile of the equity holder of either type and that of the bondholder.

[2]In general, the Perfect Bayesian equilibrium is a strategic profile $(\tau^*_{b,L}, \tau^*_{b,H}, \tau^*_{c,L}, \tau^*_{c,H}, \tau^*_{con})$ associated with a belief system $B$ endowed with the following two properties: (i) The value function of each party (equity holder of type $i$ or bondholder) is maximized given that the other parties have adopted the strategies described in the strategic profile (property of sequentially rationality). (ii) The belief system $B(S)$ is determined as much as possible based on the strategic profile and Bayes' rule (property of consistency).

*Separation*

For $x \in S^{(p)}_{E,i} \setminus S^{(p)}_{E,j}$, Firm $i$ either declares call (if $x \in S^{(p)}_{c,i}$) or declare bankruptcy (if $x \in S^{(p)}_{b,i}$) while Firm $j$ does nothing. On the other hand, the bondholder's belief conjectures that the firm's type must be type $j$ if no signal (either call or bankruptcy) is received at that time.

To ensure that the strategies described above constitute the desired PBE, we require that

(1) Firm $j$ has no strict incentive to *mimic* Firm $i$ and adopt its optimal strategy under complete information. This requires

$$\begin{cases} V_{e,j}(x) > 0 & \text{if } x \in S^{(p)}_{b,i} \\ V_{e,j}(x) > \frac{\pi_j x}{r-\mu} - K\mathbf{1}_{\left\{K > \frac{\alpha\pi_i x}{r-\mu}\right\}} - \frac{\alpha\pi_j x}{r-\mu}\mathbf{1}_{\left\{K \leq \frac{\alpha\pi_i x}{r-\mu}\right\}} & \text{if } x \in S^{(p)}_{pro,i} \end{cases}. \tag{3.14a}$$

(2) Firm $i$ has no strict incentive to delay its exercising decision when $x_t$ hits the stopping sets $S^{(p)}_{E,i} \setminus S^{(p)}_{E,j}$. The corresponding sufficient condition is given by

$$\begin{cases} V_{e,i}(x) \leq 0 & \text{if } x \in S^{(p)}_{b,i} \\ V_{e,j}(x) \leq \frac{\pi_i x}{r-\mu} - K\mathbf{1}_{\left\{K > \frac{\alpha\pi_i x}{r-\mu}\right\}} - \frac{\alpha\pi_i x}{r-\mu}\mathbf{1}_{\left\{K \leq \frac{\alpha\pi_i x}{r-\mu}\right\}} & \text{if } x \in S^{(p)}_{pro,i} \end{cases}. \tag{3.14b}$$

For $x \in S^{(p)}_{b,i} \cap S^{(p)}_{pro,j}$, Firm $i$ declares bankruptcy and Firm $j$ declares call at the same time. We state the following required conditions for ensuring that this strategic profile constitutes PBE.

(1) Neither Firm $i$ nor Firm $j$ has strict incentive to mimic another type of firm. This requires

$$\begin{cases} \frac{\pi_i x}{r-\mu} - K\mathbf{1}_{\left\{K > \frac{\alpha\pi_j x}{r-\mu}\right\}} - \frac{\alpha\pi_i x}{r-\mu}\mathbf{1}_{\left\{K \leq \frac{\alpha\pi_j x}{r-\mu}\right\}} \leq 0 \\ \frac{\pi_j x}{r-\mu} - K\mathbf{1}_{\left\{K > \frac{\alpha\pi_j x}{r-\mu}\right\}} - \frac{\alpha\pi_j x}{r-\mu}\mathbf{1}_{\left\{K \leq \frac{\alpha\pi_j x}{r-\mu}\right\}} \geq 0 \end{cases}. \tag{3.15a}$$

(2) Both types of firm have no strict incentive to delay its exercising decision given a set of off-the-equilibrium belief. This requires

$$\begin{cases} 0 \geq V_{e,i}(x) \\ \frac{\pi_j x}{r-\mu} - K\mathbf{1}_{\left\{K > \frac{\alpha\pi_i x}{r-\mu}\right\}} - \frac{\alpha\pi_j x}{r-\mu}\mathbf{1}_{\left\{K \leq \frac{\alpha\pi_i x}{r-\mu}\right\}} \geq V_{e,j}(x) \end{cases}. \tag{3.15b}$$

*Pooling*

For $x \in S^{(p)}_{b,i} \cap S^{(p)}_{b,j}$ or $x \in S^{(p)}_{c,i} \cap S^{(p)}_{c,j}$, both types of firm either issue call or declare bankruptcy at the same time. To ensure that this strategic pair constitute the PBE, we require that both firms have no strict incentive to delay their exercising decision. In other words, both types of firms have incentive to pool with the firm of another type instead of having its type being revealed to the bondholder. Mathematically, we require that both $V_{e,i}(x)$ and $V_{e,j}(x)$ are non-positive for $x \in S^{(p)}_{b,i} \cap S^{(p)}_{b,j}$ and

$$\begin{cases} \frac{\pi_i x}{r-\mu} - K\mathbf{1}_{\left\{K > \frac{\alpha\bar{\pi} x}{r-\mu}\right\}} - \frac{\alpha\pi_i x}{r-\mu}\mathbf{1}_{\left\{K \leq \frac{\alpha\bar{\pi} x}{r-\mu}\right\}} \geq V_{e,i}(x) \\ \frac{\pi_j x}{r-\mu} - K\mathbf{1}_{\left\{K > \frac{\alpha\bar{\pi} x}{r-\mu}\right\}} - \frac{\alpha\pi_j x}{r-\mu}\mathbf{1}_{\left\{K \leq \frac{\alpha\bar{\pi} x}{r-\mu}\right\}} \geq V_{e,j}(x) \end{cases} \quad \text{for } x \in S^{(p)}_{c,i} \cap S^{(p)}_{c,j}. \tag{3.16}$$

Provided that the stopping sets $S_{b,i}^{(p)}$ and $S_{c,i}^{(p)}$ observe all of the above properties [see eqs. (3.14a,b)-(3.16)], one can show that the stopping times defined in eqs. (3.13a,b,c) constitute the PBE associated with the objective functions (2.2) and (2.3). Furthermore, the value functions $V_{e,i}^{(p)}(x)$ and $V_d^{(p)}(x)$ obtained from the variational inequalities formulation satisfy

$$V_{e,i}^{(p)}(x) = E_i(x; \tau_{con}^*, \tau_{b,i}^*, \tau_{c,i}^*, B), \quad i = L \text{ or } H, \tag{3.17$a$}$$

$$V_d^{(p)}(x) = D(x; \tau_{con}^*, \tau_{b,L}^*, \tau_{b,H}^*, \tau_{c,L}^*, \tau_{c,H}^*, B). \tag{3.17$b$}$$

# 4 Characterization of stopping regions and determination of value functions

Next, we proceed to obtain the optimal stopping policies and the corresponding value functions by solving the variational inequalities developed in the last section. Note that the game consists of two stages in which the first stage is under information asymmetry while the second stage is under complete information. One should solve the two-stage game option model backwards by first solving for the value functions in the second stage, then solve for the value functions in the first stage. To determine the value functions of each stage, one has to determine the various optimal stopping regions (bankruptcy, call and conversion). Once the stopping regions are determined, the value function of each counterparty can be solved using the linear complementarity formulation [see eqs. (3.2a,b)-(3.3a,b)].

**Stopping regions and value functions under complete information (second stage)**

The complete information case (second stage) can be done using a similar analysis as in Chen *et al.* (2013) (see Theorem 4.2). We summarize the result in the following proposition and refer the readers to Chen *et al.* (2013) for the detailed derivation of Proposition 1.

**Proposition 1** *There are two critical thresholds $x_{b,i}$ and $x_{c,i}$ (with $x_{b,i} < x_{c,i}$) such that the optimal stopping regions can be expressed as*

$$S_{b,i} = [0, x_{b,i}], \quad S_{c,i} = \left[x_{c,i}, \frac{c}{\alpha\pi_i}\right], \quad S_d = \left[\frac{c}{\alpha\pi_i}, \infty\right), \quad i = L \text{ or } H. \tag{4.1}$$

*Here, the thresholds $x_{b,i}$ and $x_{c,i}$ are called the optimal bankruptcy threshold and optimal call threshold of the equity holder. These two thresholds can be solved using the following smooth-pasting conditions:*

$$\begin{cases} \frac{dV_{e,i}^{(p)}}{dx}\Big|_{x=x_{b,i}} = 0 \\ \frac{dV_{e,i}^{(p)}}{dx}\Big|_{x=x_{c,i}} = \frac{\pi_i}{r-\mu} \end{cases}. \tag{4.2}$$

*The corresponding value functions $V_{e,i}(x)$ and $V_d(x)$ are given by*

$$V_{e,i}(x) = \begin{cases} 0 & \text{if } x \le x_{b,i} \\ \frac{\pi_i x}{r-\mu} - \frac{c}{r} + \left(\frac{c}{r} - \frac{\pi_i x_{b,i}}{r-\mu}\right) A(x; x_{b,i} < x_{c,i}) & \\ + \left(\frac{c}{r} - K\right) A(x; x_{c,i} < x_{b,i}) & \text{if } x_{b,i} < x < x_{c,i} \\ \frac{\pi_i x}{r-\mu} - \max\left\{K, \frac{\alpha \pi_i x}{r-\mu}\right\} & \text{if } x \ge x_{c,i} \end{cases}, \tag{4.3a}$$

$$V_{d,i}(x) = \begin{cases} \frac{(1-\gamma)\pi_i x}{r-\mu} & \text{if } x \le x_{b,i} \\ \frac{c}{r} + \left(\frac{(1-\gamma)\pi_i x}{r-\mu} - \frac{c}{r}\right) A(x; x_{b,i} < x_{c,i}) & \\ + \left(K - \frac{c}{r}\right) A(x; x_{c,i} < x_{b,i}) & \text{if } x_{b,i} < x < x_{c,i} \\ \max\left\{K, \frac{\alpha \pi_i x}{r-\mu}\right\} & \text{if } x \ge x_{c,i} \end{cases}. \tag{4.3b}$$

*Here, $A(x; a < b)$ is the value of Arrow security corresponding to the event where the state variable $x_t$ hits $a$ before $b$, the explicit form of which is given by*

$$A(x; a < b) = \frac{b^{\beta^+ - \beta^-} - x^{\beta^+ - \beta^-}}{b^{\beta^+ - \beta^-} - a^{\beta^+ - \beta^-}} \left(\frac{x}{a}\right)^{\beta^-}.$$

**Stopping regions and value functions under incomplete information (first stage)**

Next, we proceed to find the value functions in the first case (incomplete information). Similar to that for the second stage, we first characterize the various optimal stopping regions. The analysis is complicated by the possibility that the optimal bankruptcy region of one type of firm and optimal call region of another type of firm may overlap. We need to derive a necessary and sufficient condition under which the above phenomenon does not happen. In fact, it can be shown that the call region and bankruptcy region do not overlap if and only if the following condition holds:

$$x_{b,L} < x_{c,H} \Leftrightarrow \frac{\pi_L}{\pi_H} > z, \tag{4.4}$$

where $z \in (0, 1)$ is the unique root of the algebraic equation

$$K = \frac{c}{r} \frac{\beta^+(1-\beta^-) + (\beta^+ - 1)\beta^- z^{\beta^+ - \beta^-} - (\beta^+ - \beta^-) z^{-\beta^-}}{\beta^+(1-\beta^-) + (\beta^+ - 1)\beta^- z^{\beta^+ - \beta^-}}.$$

We first consider the case when $\frac{\pi_L}{\pi_H} > z$. As there will be no overlapping between the optimal bankruptcy region and optimal call/conversion region, so one can perform the characterization of optimal bankruptcy region and optimal call/conversion region separately.

**Proposition 2** *Under the assumption that $\frac{\pi_L}{\pi_H} > z$, there are two critical thresholds $x^*_{b,L}$ and $x^*_{c,H}$ satisfying $x_{b,H} < x^*_{b,L} < x^*_{c,H} < x_{c,L}$ such that the optimal stopping regions are given by*

$$S^{(p)}_{b,H} = [0, x_{b,H}], \quad S^{(p)}_{b,L} = [0, x^*_{b,L}], \quad S^{(p)}_{c,H} = \left[x^*_{c,H}, \frac{c}{\alpha \pi_H}\right], S^{(p)}_{c,L} = \left[x_{c,L}, \frac{c}{\alpha \pi_L}\right),$$

$$S_D^{(p)} = \left[\frac{c}{\alpha\bar{\pi}}, \infty\right).$$

*Here, the threshold $x^*_{b,L}$ and $x^*_{c,H}$ are called the optimal bankruptcy threshold of Firm L and optimal call threshold of Firm H, respectively. These threshold values can be obtained using the following smooth-pasting conditions:*

$$\begin{cases} \frac{dV_{e,L}^{(p)}(x)}{dx}\Big|_{x=x^*_{b,L}} = 0 \\ \frac{dV_{e,H}^{(p)}(x)}{dx}\Big|_{x=x^*_{c,H}} = \frac{\pi_H}{r-\mu} \end{cases}. \tag{4.5}$$

The corresponding value functions $V_{e,L}(x)$, $V_{e,H}(x)$ and $V_d(x)$ are given by

$$V_{e,H}^{(p)}(x) = \begin{cases} 0 & \text{if } x \le x_{b,H} \\ V_{e,H}(x) & \text{if } x_{b,H} < x \le x^*_{b,L} \\ \frac{\pi_H x}{r-\mu} - \frac{c}{r} + \left[V_{e,H}(x^*_{b,L}) + \frac{c}{r} - \frac{\pi_H x^*_{b,L}}{r-\mu}\right] A(x; x^*_{b,L} < x^*_{c,H}) & \\ \qquad + \left(\frac{c}{r} - K\right) A(x; x^*_{c,H} < x^*_{b,L}) & \text{if } x^*_{b,L} < x < x^*_{c,H} \\ \frac{\pi_H x}{r-\mu} - K\mathbf{1}_{\left\{K \ge \frac{\alpha\bar{\pi}x}{r-\mu}\right\}} - \frac{\alpha\pi_H x}{r-\mu}\mathbf{1}_{\left\{K < \frac{\alpha\bar{\pi}x}{r-\mu}\right\}} & \text{if } x > x^*_{c,H} \end{cases}. \tag{4.6a}$$

$$V_{e,L}^{(p)}(x) = \begin{cases} 0 & \text{if } x \le x^*_{b,L} \\ \frac{\pi_L x}{r-\mu} - \frac{c}{r} + \left(\frac{c}{r} - \frac{\pi_L x^*_{b,L}}{r-\mu}\right) A(x; x^*_{b,L} < x^*_{c,H}) & \\ \qquad + \left[V_{e,L}(x^*_{c,H}) + \frac{c}{r} - \frac{\pi_L x^*_{c,H}}{r-\mu}\right) A(x; x^*_{c,H} < x^*_{b,L}) & \text{if } x^*_{b,L} < x < x^*_{c,H} \\ V_{e,L}(x) & \text{if } x^*_{c,H} \le x < x_{c,L} \\ \frac{\pi_L x}{r-\mu} - K\mathbf{1}_{\left\{K \ge \frac{\alpha\bar{\pi}x}{r-\mu}\right\}} - \frac{\alpha\pi_L x}{r-\mu}\mathbf{1}_{\left\{K < \frac{\alpha\bar{\pi}x}{r-\mu}\right\}} & \text{if } x > x_{c,L} \end{cases}. \tag{4.6b}$$

$$V_d^{(p)}(x) = \begin{cases} 0 & \text{if } x \le x_{b,H} \\ pV_{d,H}(x) & \text{if } x_{b,H} < x < x^*_{b,L} \\ \frac{c}{r} + [pV_{d,H}(x^*_{b,L}) + (1-p)\frac{(1-\gamma)\pi_L x^*_{b,L}}{r-\mu} - \frac{c}{r}]A(x; x^*_{b,L} < x^*_{c,H}) & \\ \qquad + [pK + (1-p)V_{d,L}(x^*_{c,H}) - \frac{c}{r}]A(x; x^*_{c,H} < x^*_{b,L}) & \text{if } x^*_{b,L} < x < x^*_{c,H} \\ pK + (1-p)V_{d,L}(x) & \text{if } x^*_{c,H} < x < x_{c,L} \\ \max\left(K, \frac{\alpha\bar{\pi}x}{r-\mu}\right) & \text{if } x > x_{c,L} \end{cases}. \tag{4.6c}$$

The detailed derivation of the analytic results in Proposition 2 is presented in Appendix B. The schematic plots of the various stopping regions are shown in Figure 2. One can observe from Proposition 2 that each type of firm adopts different bankruptcy strategy and call strategy under information asymmetry so that the signaling time (whenever bankruptcy or call) is a credible signal for bondholders to identify the firm's quality (High or Low). In particular, when $x$ falls within $(x_{b,H}, x^*_{b,L})$ and the firm does not declare bankruptcy, then its $H$-type status is revealed to the bondholder. On the other hand, when $x$ falls within $(x^*_{c,H}, x_{c,L})$ and no call is issued by the firm, then $L$-type status is revealed to the bondholder.

It is not surprising to see that Firm $H$ (Firm $L$) tends to declare bankruptcy at a lower (higher) threshold since $x_{b,H} < x^*_{b,L}$ since Firm $H$ has relatively larger revenue flow. On the other hand, the result indicates that Firm $H$ tends to declare out-of-the-money call at a lower threshold (since $x^*_{c,H} < x_{c,L}$) since Firm $H$ has stronger revenue flow and it is willing to pay the premium at an earlier time. This implies that an earlier out-of-the-money call can deliver some favorable information about the firm type to the investors and the corresponding stock price moves positively. This agrees with the empirical study by Cowan *et al.*, (1993) stating that positive average stock price reaction is commonly observed for out-of-the-money call.

It is also instructive to investigate the value functions of the equity holder (type $L$ and type $H$) and the bondholder under this scenario. In Figures 3(a-c), we show the various plots of the equity value functions of each type and debt value functions against $x$. The common set of parameter values used in the numerical calculations for plotting the value functions are chosen to be: $r = 0.05$, $\mu = 0.02$, $\sigma = 0.3$, $\pi_L = 1$, $\pi_H = 1.2$, $p = 0.4$, $c = 1$, $\alpha = 0.3$, $K = 12$, $\gamma = 0.2$.

In Figure 3(a), we show the plot of the equity value function $V^{(p)}_{e,H}(x)$ of type $H$. When $x \leq x_{b,H}$, the equity holder of type $H$ optimally chooses to declare bankruptcy, so $V^{(p)}_{e,H}(x) = 0$. When $x_{b,H} < x \leq x^*_{b,L}$, the quality type is revealed to the bondholder since the equity holder does not declare bankruptcy (see Figure 2). In this case, $V^{(p)}_{e,H}(x)$ is simply the equity value under complete information, that is, $V^{(p)}_{e,H}(x) = V_{e,H}(x)$. When $x^*_{b,L} < x < x^*_{c,H}$, it is the continuation region for both firm types and the game remains to be in the first stage. The equity holder of type $H$ chooses to declare call (out-of-the-money call) when $x_t$ hits $x^*_{c,H}$ from below. On the other hand, its firm type will be revealed to bondholder if $x_t$ hits $x^*_{b,L}$ from above, thus the value function $V^{(p)}_{e,H}(x)$ is simply given by (4.6a). When $x^*_{c,H} \leq x < \frac{K(r-\mu)}{\alpha\bar{\pi}}$, the equity holder of type $H$ optimally chooses to call. Since $x < \frac{K(r-\mu)}{\alpha\bar{\pi}}$, the bondholder chooses to receive the cash amount $K$ [see eq. (2.6)]. This gives $V^{(p)}_{e,H}(x) = \frac{\pi_H x}{r-\mu} - K$. When $x \geq \frac{K(r-\mu)}{\alpha\bar{\pi}}$, either the equity holder of type $H$ issues call (forced conversion) or the bondholder voluntarily converts into shares, so we have $V^{(p)}_{e,H}(x) = \frac{(1-\alpha)\pi_H x}{r-\mu}$. It is worthwhile to point out that there is a downward jump in the equity value at the conversion threshold $\frac{K(r-\mu)}{\alpha\bar{\pi}}$. This is because the equity holder of both types issue the call near the conversion threshold so that the firm's type cannot be revealed to the bondholder. The bondholder has to make its decision (receiving cash or convert into shares) based on the relative magnitude of $K$ and the *expected conversion value* $\frac{\alpha\bar{\pi}x}{r-\mu}$. On the other hand, the equity holder knows its type and its equity value is calculated based on the *actual* value of $\pi_i$, namely, $\pi_i = \pi_H$. This generates a jump at the conversion threshold since the bondholder changes its optimal decision at this critical threshold. Figure 3(b) shows the plot of equity value function of type $L$ $[(V^{(p)}_{eL}(x)]$ and similar set of patterns as discussed in the above are found in the corresponding plot for the $L$-type firm.

In Figure 3(c), we plot the debt value function $V^{(p)}_d(x)$ against $x$. The value function $V^{(p)}_d(x)$ is given by eq. (4.6c). When $x \leq x^*_{b,H}$, the firm (regardless of firm type) is sure to declare bankruptcy and the firm quality is not revealed. Hence, the debt value is the weighted average of the salvage value (after the reduction of bankruptcy cost), where $V_d^{(p)}(x) = \frac{(1-\gamma)\bar{\pi}x}{r-\mu}$. When $x \in (x_{b,H}, x^*_{b,L}]$, the probability that the firm declares bankruptcy is $1-p$ since its quality status is $L$-type. On the other hand, the probability that the firm takes no action is $p$. Under such scenario, the status of $H$-type is then revealed. Summarizing the above results, the value function is given by

$$V_d^{(p)}(x) = pV_{d,H}(x) + (1-p)\frac{(1-\gamma)\pi_L x}{r-\mu}.$$

When $x^*_{b,L} < x < x^*_{c,H}$, no action is taken by the equity holder as $x$ lies in the continuation region. The game remains at the first stage. When the state variable $x_t$ either hits $x^*_{b,L}$ from above or hits $x^*_{c,H}$ from below, an appropriate optimal decision action will occur (subject to certain probability). When $x^*_{c,H} \le x < x_{c,L}$, there exists probability $p$ that the firm declares call since its quality is $H$-type and the bondholder chooses to receive $K$ (since $K > \frac{\alpha\pi_H x}{r-\mu}$). Likewise, there exists probability $1-p$ that the firm takes no action and the status of $L$-type is revealed. Summing the results together, the debt value function is seen to be

$$V_d^{(p)}(x) = pK + (1-p)V_{d,L}(x).$$

When $x \ge x_{c,L}$, either the firm is sure to declare call and the status of quality type is not revealed to bondholder or the bondholder converts the bond voluntarily into shares. In the event of call, the bondholder chooses to receive $K$ when $x$ is less than $\frac{K(r-\mu)}{\alpha\bar{\pi}}$ and chooses to receive $\frac{\alpha\bar{\pi}x}{r-\mu}$ (force conversion) if otherwise. The resulting debt value function is given by

$$V_d^{(p)}(x) = \begin{cases} K & \text{if } x_{c,L} \le x < \frac{K(r-\mu)}{\alpha\bar{\pi}} \\ \frac{\alpha\bar{\pi}x}{r-\mu} & \text{if } x \ge \frac{K(r-\mu)}{\alpha\bar{\pi}} \end{cases}.$$

On the other hand, we consider the other case where $\dfrac{\pi_L}{\pi_H} \le z$. An overlapping of the bankruptcy region of Firm $L$ and the call region of Firm $H$ occurs. In order to characterize the stopping regions, one has to determine this overlapping region and it can be shown to be a bounded interval. The remaining part (outside this region) will be divided into two sub-intervals where the left interval is the bankruptcy region while the right interval is the call / conversion region. One can then follow a similar analysis as above to determine the remaining region. We summarize the results in Proposition 3.

**Proposition 3** *Under the assumption that* $\frac{\pi_L}{\pi_H} \le z$, *the optimal stopping regions can be characterized as*

$$S_{b,H}^{(p)} = [0, x_{b,H}], \quad S_{b,L}^{(p)} = [0, x_{b,L}], \quad S_{c,H}^{(p)} = \left[x_{c,H}, \frac{c}{\alpha\pi_H}\right], \quad S_{c,L}^{(p)} = \left[x_{c,L}, \frac{c}{\alpha\pi_L}\right]$$

$$S_D = \left[\frac{c}{\alpha\bar{\pi}}, \infty\right).$$

*Here, $x_{b,L} > x_{c,H}$ so that there is an overlapping of the stopping regions $S^{(p)}_{b,L}$ and $S^{(p)}_{c,H}$. The corresponding value functions are found to be*

$$V^{(p)}_{e,H}(x) = \begin{cases} 0 & \text{if } x \le x_{b,H} \\ V_{e,H}(x) & \text{if } x_{b,H} < x \le x_{c,H} \\ \frac{\pi_H x}{r-\mu} - K\mathbf{1}_{\left\{K \ge \frac{\alpha\bar{\pi}x}{r-\mu}\right\}} - \frac{\alpha\pi_H x}{r-\mu}\mathbf{1}_{\left\{K < \frac{\alpha\bar{\pi}x}{r-\mu}\right\}} & \text{if } x > x_{c,H} \end{cases} . \tag{4.7a}$$

$$V^{(p)}_{e,L}(x) = \begin{cases} 0 & \text{if } x \le x_{b,L} \\ V_{e,L}(x) & \text{if } x_{b,L} \le x < x_{c,L} \\ \frac{\pi_L x}{r-\mu} - K\mathbf{1}_{\left\{K \ge \frac{\alpha\bar{\pi}x}{r-\mu}\right\}} - \frac{\alpha\pi_L x}{r-\mu}\mathbf{1}_{\left\{K < \frac{\alpha\bar{\pi}x}{r-\mu}\right\}} & \text{if } x > x_{c,L} \end{cases} . \tag{4.7b}$$

$$V^{(p)}_{d}(x) = \begin{cases} 0 & \text{if } x \le x_{b,H} \\ pV_{d,H}(x) & \text{if } x_{b,H} < x < x_{c,H} \\ p\max\left(K, \frac{\alpha\pi_H x}{r-\mu}\right) & \text{if } x_{c,H} < x < x_{b,L} \\ p\max\left(K, \frac{\alpha\pi_H x}{r-\mu}\right) + (1-p)V_{d,L}(x) & \text{if } x_{b,L} < x < x_{c,L} \\ \max\left(K, \frac{\alpha\bar{\pi}x}{r-\mu}\right) & \text{if } x > x_{c,L} \end{cases} . \tag{4.7c}$$

We observe from Proposition 3 that every point in the interval $[0, \infty)$ must lie in either the stopping region of equity holder or the stopping region of bondholder. Either the game is terminated immediately or the game enters into second stage immediately after the issuance of convertible bond. Indeed, this represents an uninteresting degenerate case.

# 5 Adverse selection cost

Under incomplete information, $H$-type firm suffers from a higher adverse selection cost since the equity component of the convertible bond is often underpriced in the market. It is more apparent for non-callable convertible bond since the resulting bond value at both the upper barrier (voluntarily conversion) and lower barrier (bankruptcy) depend on the equity value. In this section, we shall investigate how the inclusion of the callable provision in the convertible bond reduces the adverse selection cost. Intuitively, we define the *adverse selection cost* of a security to be the relative price change of the security under the existence of incomplete information:

$$\text{adverse selection cost } = \frac{V_{d,H}(x) - V^{(p)}_d(x)}{V_{d,H}(x)},$$

where $x$ should be chosen in the continuation region so that no information is revealed to the bondholder and the game remains at the first stage. In order to investigate the role of the call provision in affecting the adverse selection cost, we shall compare the adverse selection cost of *non-callable convertible bond* and *callable convertible bond*. In Figures 4(a) and 4(b), we present the plots of the respective adverse selection cost of the two convertible bonds against probability $p$ and the ratio of profit flow constants $\frac{\pi_H}{\pi_L}$, respectively. The common set of parameter values used in the numerical calculations are chosen to be: $r = 0.05$, $\mu = 0.02$, $\sigma = 0.3$, $c = 1$, $\alpha = 0.3$, $K = 12$, $\gamma = 0.2$.

In plotting Figure 4(a), the parameter values of $\pi_H$ and $\pi_L$ are chosen to be 2 and 1, respectively. The level of the state variable $x$ is chosen to be 0.45. We observe that the adverse selection costs of the two bonds are larger when the bondholder's belief is biased low. This cost decreases to zero as $p$ increases to 1 since the market perception is getting close to the actual firm's quality as $p$ tends to 1. On the other hand, we observe that the adverse selection cost of the callable convertible bond is about 50% lower then that of the non-callable counterpart. Since the call price $K$ of the callable convertible bond is relatively low so that the equity holder chooses to declare the out-of-the-money call before the voluntary conversion by the bondholder, the upper barrier of the callable-convertible bond is equal to the cash amount $K$ instead of equity. Recall that only the equity component of a security is being underpriced in the market under incomplete information. We conclude that the inclusion of the callable feature can greatly reduce the adverse selection cost.

In plotting Figure 4(b), the values of $p$ and $\pi_L$ are chosen to be 0.4 and 1, respectively. The level of the state variable $x$ is chosen to be 0.31. We observe that the adverse selection costs of two types of convertible bonds are both increasing with respect to the ratio of $\frac{\pi_H}{\pi_L}$. Also, we observe that the adverse selection cost of the non-callable convertible bond increases faster than that of the callable counterpart. This is because the non-callable convertible bond are more equity-like (as payments at the upper barrier and lower barrier are both settled by equity), the mispricing problem is more apparent. Therefore, the price difference in the two types of convertible bonds is more significantly under incomplete information.

# 6 Conclusion

A two-stage sequential two-person stochastic game option model with information asymmetry is proposed to analyze the information role of the optimal call policies in a callable-convertible bond. Under information asymmetry, the equity holder may choose to call due to either regular optimal stopping rule or pooling with the other type in order to extract the benefit of information asymmetry. When pooling equilibrium prevails, the low quality firm may choose to call earlier in order to pool the signal of the quality type to the bondholder. In our analysis of the Perfect Bayesian Equilibrium of the optimal strategies adopted by the bondholder and equity holder, we establish the incentive compatibility constraint for pooling equilibrium. The true status of firm quality may be revealed either by declaration of call or bankruptcy, or no action taken when the value of the stochastic fundamental falls in the stopping region of firm of other type. Under separating equilibrium, the quality status of the firm is revealed. As a result, the two-stage game enters into the second stage of complete information.

Mathematically, we provide the full characterization of the optimal stopping rules of call, conversion and bankruptcy adopted by the bondholder and equity holder. We also present the variational inequalities formulation with respect to various equilibrium strategies in the two-person sequential game option model of the callable-convertible bond. We discuss the required conditions for ensuring that the optimal stopping strategies constitute the Perfect Bayesian Equilibrium. Once the stopping rules are fully characterized, we manage to determine the equity value functions of both the high quality firm and low quality firm, and the debt value function. We also show that the inclusion of the callable feature in a convertible bond and lower level of information asymmetry help reduce adverse selection cost.

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**Appendix A - Proof of eq. (2.7)**

We would like to show that exactly one of ineqs. (2.4) and (2.5) holds when $K < \min(K_1, K_2)$. This assumption on $K$ would guarantee that the optimal call threshold $x_{c,L}$ of Firm $L$ under complete information exists and satisfies $x_{c,L} < \dfrac{K(r-\mu)}{\alpha\pi_H}$. Without loss of generality, we consider the case for $i = H$ and $j = L$.

For $x < x_{c,L}$, we have $x < \frac{K(r-\mu)}{\alpha\pi_H}$ so that

$$\frac{\pi_L x}{r-\mu} - \mathbf{1}_{\left\{K > \frac{\alpha\pi_H x}{r-\mu}\right\}} K - \mathbf{1}_{\left\{K \le \frac{\alpha\pi_H x}{r-\mu}\right\}} \frac{\alpha\pi_L x}{r-\mu} = \frac{\pi_L x}{r-\mu} - K < V_{e,L}(x).$$

The last inequality follows from the result in Chen *et al.* (2013). Hence, ineq. (2.5) is satisfied. On the other hand, it can be shown that

$$\frac{\pi_L x}{r-\mu} - \mathbf{1}_{\left\{K > \frac{\alpha\bar{\pi} x}{r-\mu}\right\}} K - \mathbf{1}_{\left\{K \le \frac{\alpha\bar{\pi} x}{r-\mu}\right\}} \frac{\alpha\pi_L x}{r-\mu} = \frac{\pi_L x}{r-\mu} - K < V_{e,L}(x).$$

This shows that ineq. (2.4) is violated and the exact value of $\tilde{\pi}_i$ is given by $\pi_i$.

For $x \ge x_{c,L}$, by following similar procedures in Chen *et al.* (2013), one can show that

$$\begin{aligned} &\frac{\pi_L x}{r-\mu} - \mathbf{1}_{\left\{K > \frac{\alpha\pi_H x}{r-\mu}\right\}} K - \mathbf{1}_{\left\{K \le \frac{\alpha\pi_H x}{r-\mu}\right\}} \frac{\alpha\pi_L x}{r-\mu} \\ > &\frac{\pi_L x}{r-\mu} - \mathbf{1}_{\left\{K > \frac{\alpha\pi_L x}{r-\mu}\right\}} K - \mathbf{1}_{\left\{K \le \frac{\alpha\pi_L x}{r-\mu}\right\}} \frac{\alpha\pi_L x}{r-\mu} = V_{e,L}(x). \end{aligned}$$

The above step reveals that ineq. (2.5) is violated. On the other hand, one can show easily that

$$\begin{aligned} &\frac{\pi_L x}{r-\mu} - \mathbf{1}_{\left\{K > \frac{\alpha\bar{\pi} x}{r-\mu}\right\}} K - \mathbf{1}_{\left\{K \le \frac{\alpha\bar{\pi} x}{r-\mu}\right\}} \frac{\alpha\pi_L x}{r-\mu} \\ > &\frac{\pi_L x}{r-\mu} - \mathbf{1}_{\left\{K > \frac{\alpha\pi_L x}{r-\mu}\right\}} K - \mathbf{1}_{\left\{K \le \frac{\alpha\pi_L x}{r-\mu}\right\}} \frac{\alpha\pi_L x}{r-\mu} = V_{e,L}(x). \end{aligned}$$

This shows ineq. (2.4) is satisfied and the exact value of $\tilde{\pi}_i$ is given by $\bar{\pi}$.

The case for $i = L$ and $j = H$ can be proven in a similar fashion.

## Appendix B - Proof of Proposition 2

Under the assumption that $\frac{\pi_L}{\pi_H} > z$, the optimal bankruptcy region and optimal call region are disjoint so that one can characterize these two stopping regions separately.

*Intervals of various optimal stopping regions*

Firstly, we state the following lemma regarding the intervals of various optimal stopping regions which is useful in the characterization of stopping regions later. The result can be established easily using the definition of stopping sets [see eqs. (3.5a) and (3.6a)] and following the argument adopted by Chen *et. al.* (2013) [see Lemma C1 of Chen *et. al.* (2013)].

**Lemma B1**

Given that the values functions $V_{e,i}^{(p)}(x)$ $i = L, H$, and $V_d^{(p)}(x)$ satisfy the variational inequalities formulation described in Section 3, the intervals of various optimal stopping regions are given by

$$S_{b,i}^{(p)} \subseteq \left[0, \min\left(\frac{K(r-\mu)}{\pi_i}, \frac{c}{\pi_i}\right)\right), \quad S_{pro,i}^{(p)} \subseteq \left[\frac{K(r-\mu)}{\pi_i}, \frac{c}{\alpha\pi_i}\right], \quad S_D^{(p)} \subseteq \left[\frac{c}{\alpha\bar{\pi}}, \infty\right). \quad (B.1)$$

*Characterization of the optimal bankruptcy region*

Since $V_{e,H}^{(p)}(x) \geq V_{e,L}^{(p)}(x) \geq 0$ for $\pi_H > \pi_L$, then we have

$$S_{b,H}^{(p)} \subseteq S_{b,L}^{(p)}.$$

Based on this simple fact, we can characterize the bankruptcy region by (i) first characterizing the stopping set $S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$ and (ii) then characterize the stopping set $S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$.

*(i) Characterization of $S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$*

We shall establish that $S_{b,L}^{(p)} \cap S_{b,H}^{(p)} = [0, x_{b,H}]$, where $x_{b,H}$ is the optimal call threshold of Firm $H$ under complete information. This can be done in two steps. Firstly, we establish that $S_{b,L}^{(p)} \cap S_{b,H}^{(p)} = [0, x_0]$ if $x_0 \in S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$. Secondly, we show the upper bound of $\sup\{S_{b,L}^{(p)} \cap S_{b,H}^{(p)}\} = x_{b,H}$.

We argue that $[0, x_0] \subseteq S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$ for $x_0 \in S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$. It is obvious that both types of firm should declare bankruptcy when there is no revenue flow ($x = 0$). It then follows that $0 \in S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$. Suppose $(0, x_0) \setminus (S_{b,L}^{(p)} \cap S_{b,H}^{(p)}) \neq \phi$, one can use the fact that $S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$ is closed to show that $(0, x_0) \setminus (S_{b,L}^{(p)} \cap S_{b,H}^{(p)})$ is open and can be expressed as the disjoint union of open intervals:

$$(0, x_0) \setminus (S_{b,L}^{(p)} \cap S_{b,H}^{(p)}) = \bigcup_{n \in \mathbb{N}} (\underline{x}_n, \overline{x}_n).$$

For any $x \in (\underline{x}_n, \overline{x}_n)$ for some $n \in \mathbb{N}$, we must have $x \notin S_{b,H}^{(p)}$. On the other hand, we note that $x < x_0 < \frac{K(r-\mu)}{\pi_H}$ from eq. (B.1). We also have $x \notin S_{c,H}^{(p)}$. Since $K < K_1$, we can deduce from Chen *et al.* (2013) that the threshold $x_{c,H}$ exists and $x_{c,H} < \frac{c_H}{\alpha\pi_H}$. Together with the fact that $S_D^{(p)} \subseteq \left[\frac{c_H}{\alpha\bar{\pi}}, \infty\right)$, we deduce that $x \notin S_D^{(p)}$. From eq. (3.5b), the value function $V_{e,L}^{(p)}(x)$ satisfies the following linear complementarity relation:

$$\begin{cases} \min\left(V_{e,L}^{(p)}(x) - 0, -\frac{\sigma^2}{2}x^2\frac{d^2V_{e,L}^{(p)}}{dx^2} - \mu x\frac{dV_{e,L}^{(p)}}{dx} + rV_{e,L}^{(p)} - \pi_L x + c\right) = 0, \quad x \in (\underline{x}_n, \overline{x}_n); \\ V_{e,L}^{(p)}(\underline{x}_n) = 0, \quad V_{e,L}^{(p)}(\overline{x}_n) = 0. \end{cases} \tag{B.2}$$

From Lemma B1, we have $x < x_0 < \frac{c}{\pi_H}$, it is seen that

$$-\frac{\sigma^2}{2}x^2\frac{d^2 0}{dx^2} - \mu x\frac{d0}{dx} + r(0) - \pi_L x + c = \pi_L x + c > \pi_L\left(\frac{c}{\pi_H}\right) + c > 0.$$

This implies that 0 is the supersolution of the eq. (B.2) so that $V_{e,L}^{(p)}(x) \leq 0$. Together with the obstacle condition (3.1b), we deduce that $V_{e,L}^{(p)}(x) = 0$ for all $x \in (\underline{x}_n, \overline{x}_n)$. Therefore, $(\underline{x}_n, \overline{x}_n) \subseteq S_{b,L}^{(p)}$. This implies that $V_{e,H}^{(p)}(x) = V_{e,H}(x)$ for $x \in (\underline{x}_n, \overline{x}_n)$ from eq. (3.11). Using the incentive constraint conditions embedded in $S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$ and eq. (4.3a), we have $V_{e,H}^{(p)}(x_0) = 0$ and $x_0 < x_{b,H}$. Since $x < x_0$, it follows from eq. (4.3a) that $V_{e,H}^{(p)}(x) = V_{e,H}(x) = 0$ for all $x \in (\underline{x}_n, \overline{x}_n)$. So we also have $(\underline{x}_n, \overline{x}_n) \subseteq S_{b,H}^{(p)}$. This contradiction implies that $[0, x_0] \subseteq S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$.

Next, we show that $x_{b,H} = \sup(S_{b,L}^{(p)} \cap S_{b,H}^{(p)})$, where $x_{b,H}$ is the optimal bankruptcy threshold of Firm $H$ under complete information. Using the incentive constraint eq. (3.10a) embedding in $S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$ and the obstacle condition eq. (3.1b), we have $V_{e,H}(x) = 0$ for all $x \in S_{b,L}^{(p)} \cap S_{b,H}^{(p)}$ so that $x \leq x_{b,H}$ from eq. (4.3a). This implies $\sup(S_{b,L}^{(p)} \cap S_{b,H}^{(p)}) \leq x_{b,H}$.

Suppose that $\sup(S_{b,L}^{(p)} \cap S_{b,H}^{(p)}) < x_{b,H}$, this leads to a contradiction by showing both types of firm has incentive to declare bankruptcy at some threshold between $\sup\left(S_{b,L}^{(p)} \cap S_{b,H}^{(p)}\right) < x_{b,H}$. We first note that $S_{c,H}^{(p)}$ is non-empty in $[0, \frac{K(r-\mu)}{\alpha\pi_H}]$ under condition (2.7). For any $x \in (\sup(S_{b,L}^{(p)} \cap S_{b,H}^{(p)}), \inf S_{c,H}^{(p)})$, we have $x \notin S_{e,H}^{(p)}(x) \cup S_D^{(p)}$ and the value function $V_{e,L}^{(p)}$ satisfies the following linear complementarity relation:

$$\begin{cases} \min\left(V_{e,L}^{(p)}(x) - \max(0, h_{c,i}^{(p)}(x)) - \frac{\sigma^2}{2}x^2\frac{d^2V_{e,L}^{(p)}}{dx^2} - \mu x\frac{dV_{e,L}^{(p)}}{dx} + rV_{e,L}^{(p)} - \pi_L x + c\right) = 0, \\ \qquad\qquad x \in (\sup S_{b,H}^{(p)}, \inf S_{c,H}^{(p)}); \\ V_{e,L}^{(p)}(\sup(S_{b,L}^{(p)} \cap S_{b,H}^{(p)})) = 0, \quad V_{e,L}^{(p)}(\inf S_{c,H}^{(p)}) = V_{e,L}(\inf S_{c,H}^{(p)}) \end{cases} . \tag{B.3}$$

Using the condition that $K < K_1$, one can establish that $[0, \frac{K(r-\mu)}{\alpha\pi_H}] \cap S_{D,L} = \phi$ [see Chen *et al.* (2013)]. For any $x \in (\sup S_{b,H}^{(p)}, \inf S_{c,H}^{(p)}) \subseteq [0, \frac{K(r-\mu)}{\alpha\pi_H}]$, the value function $V_{e,L}(x)$ satisfies the linear complementarity relation shown in eq. (3.2b). On the other hand, $h_{c,i}^{(p)}(x) = K$ for $x < \frac{K(r-\mu)}{\alpha\pi_H}$. We observe that $V_{e,L}(x)$ is also the solution of eq. (B.3). By the uniqueness of solution of eq. (B.3), we deduce that $V_{e,L}^{(p)}(x) = V_{e,L}(x)$. Note that $x_{b,H} < x_{b,L}$, we deduce that

$$V_{e,L}^{(p)}(x) = V_{e,L}(x) = 0 \quad \text{for } x \leq x_{b,H}$$

This implies that $V_{e,L}^{(p)}(x) = 0$ for $x \leq x_{b,H}$ and $[0, x_{b,H}] \subseteq S_{b,L}^{(p)}$.

To arrive contradiction, we note that $V_{e,H}^{(p)}(x) = V_{e,H}(x) = 0$ for $x \in (\sup S_{b,H}^{(p)}, x_{b,H}]$ from eq.(3.11) and eq.(4.3a). Hence, we conclude that $(\sup S_{b,H}^{(p)}, x_{b,H}] \subseteq S_{b,H}^{(p)}$. This contradiction implies that $x_{b,H} = \sup(S_{b,L}^{(p)} \cap S_{b,H}^{(p)})$. We finally have

$$S_{b,L}^{(p)} \cap S_{b,H}^{(p)} = [0, x_{b,H}]. \tag{B.4}$$

*(ii) Characterization of* $S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$

We first establish the connectedness property of $S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$ by showing $(x_{b,H}, x_1] \subseteq S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$ for any $x_1 \in S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$. Using the incentive constraint embedded in the stopping sets $S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$ and $S_{c,H}^{(p)}$ [see eqs. (3.14b), (3.15b)] and obstacle condition (3.1b), we have $V_{e,L}(x) = 0$ for $x \in S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$ and $V_{e,H} = \frac{\pi_H x}{r-\mu} - \max\left(K, \frac{\alpha\pi_H x}{r-\mu}\right)$. Using eq. (4.3a), we can show that $S_{b,L}^{(p)} \setminus S_{b,H}^{(p)} \subseteq [0, x_{b,L}]$ and $S_{c,H}^{(p)} \subseteq [x_{c,H}, \infty)$.

For any $x \in (x_{b,H}, x_1)$, we have $x < x_1 < x_{b,L}$. Note that $x_{b,L} < x_{c,H}$ from condition (4.4), this implies that $x \notin S_{e,H}^{(p)}$. Also $x_{c,H} < \frac{K(r-\mu)}{\pi_H} < \frac{c}{\alpha\pi_H}$, we have $x \notin S_D^{(p)}$. Hence, $V_{e,L}^{(p)}(x)$ should satisfy the following linear complementarity relation:

$$\begin{cases} \min\left(V_{e,L}^{(p)}(x) - 0, -\frac{\sigma^2}{2}x^2\frac{d^2V_{e,L}^{(p)}}{dx^2} - \mu x\frac{dV_{e,L}^{(p)}}{dx} + rV_{e,L}^{(p)} - \pi_L x + c\right) = 0, \quad x \in (x_{b,H}, x_1) \\ V_{e,L}^{(p)}(x_{b,H}) = 0, \quad V_{e,L}^{(p)}(x_1) = 0. \end{cases} \tag{B.5}$$

Using the fact that $x < \frac{c}{\pi_L}$ (see Lemma A1), one can show that 0 is the supersolution of the eq. (B.5) so that $V_{e,L}^{(p)}(x) \leq 0$. Together with the obstacle condition (3.1b), we can conclude that $V_{e,L}^{(p)}(x) = 0$ for all $x \in (x_{b,H}, x_1)$ and $(x_{b,H}, x_1] \subseteq S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$. We define the critical threshold $x_{b,L}^*$ to be $x_{b,L}^* = \sup(S_{b,L}^{(p)} \setminus S_{b,H}^{(p)})$. Then we can conclude that

$$(x_{b,H}, x_{b,L}^*] \subseteq S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}. \tag{B.6a}$$

Since the value function $V_{e,L}^{(p)}(x)$ is assumed to be smooth at $x = x_{b,L}^*$, then the critical threshold $x_{b,L}^*$ satisfies the following *smooth pasting* condition:

$$\frac{dV_{e,L}^{(p)}(x)}{dx}\Big|_{x=x_{b,L}^*} = 0. \tag{B.6b}$$

*Characterization of the optimal call region and conversion region*

Since the payoff upon call $h_{c,i}^{(p)}(x)$ assumes two different forms in different regimes [see eq. (2.6)]. We characterize those stopping regions in each of the intervals $[0, \frac{K(r-\mu)}{\alpha\bar{\pi}})$ and $[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty)$ separately.

*Characterization of the stopping regions within* $[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty)$

We first characterize the optimal call region and conversion region in the interval $[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty)$. We would like to show that either forced conversion (due to call) or voluntary conversion must occur when $x \in [\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty)$. Using the objective functions in eqs. (2.2) and (2.3) as well as eqs. (3.17a) and (3.17b), we establish that

$$\begin{aligned}
&V_d^{(p)}(x) + pV_{e,H}^{(p)}(x) + (1-p)V_{e,L}^{(p)}(x) \\
&= D(x, \tau_{con}^*, \tau_{b,L}^*, \tau_{c,L}^*, \tau_{b,H}^*, \tau_{c,H}^*, B) + pE_H(x, \tau_{con}^*, \tau_{b,H}^*, \tau_{c,H}^*, B) + (1-p)E_L(x, \tau_{con}^*, \tau_{b,L}^*, \tau_{c,L}^*, B) \\
&\leq \frac{\bar{\pi}x}{r-\mu}.
\end{aligned} \tag{B.7}$$

On the other hand, one can use the obstacle conditions [see eqs. (3.1a) and (3.1b)] and show that

$$V_d^{(p)}(x) + pV_{e,H}^{(p)}(x) + (1-p)V_{e,L}^{(p)}(x) \geq \frac{\bar{\pi}x}{r-\mu}.$$

This implies that $V_d^{(p)}(x) + pV_{e,H}^{(p)}(x) + (1-p)V_{e,L}^{(p)}(x) = \frac{\bar{\pi}x}{r-\mu}$. One can use the obstacle conditions [see eqs. (3.1a) and (3.1b)] and deduce that

$$V_d^{(p)}(x) = \frac{\alpha\bar{\pi}x}{r-\mu}, \qquad V_{e,i}^{(p)}(x) = h_{c,i}^{(p)}(x) = \frac{(1-\alpha)\pi_i x}{r-\mu}.$$

This implies that either forced conversion or voluntary conversion occurs when $x$ lies within in the interval $[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty)$. Therefore, the optimal stopping region in the interval $\left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty\right)$ can be characterized as follows:

(i) If $\frac{K(r-\mu)}{\alpha\bar{\pi}} < \frac{c}{\alpha\pi_H}$, then we have

$$\begin{aligned}
S_{c,H}^{(p)} \cap \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty\right) &= \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \frac{c}{\alpha\pi_H}\right) \\
S_{c,L}^{(p)} \cap \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty\right) &= \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \frac{c}{\alpha\pi_L}\right), \\
S_D^{(p)} \cap \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty\right) &= \left[\frac{c}{\alpha\bar{\pi}}, \infty\right).
\end{aligned} \tag{B.8a}$$

(ii) If $\frac{K(r-\mu)}{\alpha\bar{\pi}} \geq \frac{c}{\alpha\pi_H}$, then we have

$$S_{c,H}^{(p)} \cap \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty\right) = \phi$$

$$S_{c,L}^{(p)} \cap \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty\right) = \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \frac{c}{\alpha\pi_L}\right),$$

$$S_D^{(p)} \cap \left[\frac{K(r-\mu)}{\alpha\bar{\pi}}, \infty\right) = \left[\frac{c}{\alpha\bar{\pi}}, \infty\right). \qquad (B.8b)$$

*Characterization of the stopping regions within* $[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}]$

We then focus on the characterization of optimal call and conversion regions in the interval $\left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]$. Since $\frac{K(r-\mu)}{\alpha\bar{\pi}} < \frac{c}{\alpha\bar{\pi}}$ and $S_D^{(p)} \subseteq [\frac{c}{\alpha\bar{\pi}}, \infty)$, then $S_D^{(p)} \cap [0, \frac{K(r-\mu)}{\alpha\bar{\pi}}] = \phi$. It remains to characterize the optimal call region in the same interval, we first establish that it is optimal for both types of firms to declare call at $x = \frac{K(r-\mu)}{\alpha\bar{\pi}}$. Substituting $x = \frac{K(r-\mu)}{\alpha\bar{\pi}}$ into ineq. (B.7), we have

$$\begin{aligned}
&V_d^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) + pV_{e,H}^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) + (1-p)V_{e,L}^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) \\
&= D(x, \tau_{con}^*, \tau_{b,L}^*, \tau_{c,L}^*, \tau_{b,H}^*, \tau_{c,H}^*, B) + pE_H(x, \tau_{con}^*, \tau_{b,H}^*, \tau_{c,H}^*, B) + (1-p)E_L(x, \tau_{con}^*, \tau_{b,L}^*, \tau_{c,L}^*, B) \\
&\le \frac{\bar{\pi}\frac{K(r-\mu)}{\alpha\bar{\pi}}}{r-\mu} = \frac{K}{\alpha}.
\end{aligned}$$

On the other hand, the obstacle conditions (3.1a) and (3.1b) imply

$$\begin{aligned}
&V_d^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) + pV_{e,H}^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) + (1-p)V_{e,L}^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) \\
&\ge \frac{\alpha\bar{\pi}\frac{K(r-\mu)}{\alpha\bar{\pi}}}{r-\mu} + \frac{\bar{\pi}\frac{K(r-\mu)}{\alpha\bar{\pi}}}{r-\mu} - K = \frac{\alpha\bar{\pi}\frac{K(r-\mu)}{\alpha\bar{\pi}}}{r-\mu} + \frac{\bar{\pi}\frac{K(r-\mu)}{\alpha\bar{\pi}}}{r-\mu} - \frac{\alpha\bar{\pi}x}{r-\mu} = \frac{\bar{\pi}\frac{K(r-\mu)}{\alpha\bar{\pi}}}{r-\mu} = \frac{K}{\alpha}.
\end{aligned}$$

Combining the result with ineq. (B.9) and the obstacle conditions (3.1) and (3.1b), we deduce that

$$V_d^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) + pV_{e,H}^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) + (1-p)V_{e,L}^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) = \frac{K}{\alpha}$$

$$\Rightarrow V_d^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) = K, \quad V_{e,i}^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right) = \frac{\pi_i(\frac{K(r-\mu)}{\alpha\bar{\pi}})}{r-\mu} - K = h_i^{(p)}\left(\frac{K(r-\mu)}{\alpha\bar{\pi}}\right).$$

So we have $\frac{K(r-\mu)}{\alpha\bar{\pi}} \in S_{c,L}^{(p)} \cap S_{c,H}^{(p)}$.

The remaining characterization of the optimal call region is similar to that of the optimal bankruptcy region. Using eqs. (3.17a) and (3.17b) and the objective functions in eqs. (2.2) and (2.3), we can establish that

$$V_{e,H}^{(p)}(x) - \left(\frac{\pi_H x}{r-\mu} - K\right) \le V_{e,L}^{(p)}(x) - \left(\frac{\pi_L x}{r-\mu} - K\right).$$

For any $x \in S_{c,L}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]$, we have

$$V_{e,H}^{(p)}(x) - \left(\frac{\pi_H x}{r-\mu} - K\right) \leq \frac{\pi_L x}{r-\mu} - K - \left(\frac{\pi_L x}{r-\mu} - K\right) = 0$$

$$\Rightarrow V_{e,H}^{(p)}(x) \leq \frac{\pi_H x}{r-\mu} - K.$$

Together with the obstacle condition (3.1b), we conclude that $V_{e,H}^{(p)}(x) = \frac{\pi_H x}{r-\mu} - K$ so that $x \in S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]$. In other words, Firm $H$ chooses to declare the out-of-the-money call if Firm $L$ declares call at the same time. This implies

$$S_{c,L}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right] \subseteq S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right].$$

Using this fact, we first characterize the stopping region $S_{c,L}^{(p)} \cap S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]$ and then $(S_{c,H}^{(p)} \setminus S_{c,L}^{(p)}) \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]$.

*Characterization of* $S_{c,L}^{(p)} \cap S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]$

We argue that $[x_2, \frac{K(r-\mu)}{\alpha\bar{\pi}}] \subseteq \left(S_{c,L}^{(p)} \cap S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]\right)$ for $x_2 \in (S_{c,L}^{(p)} \cap S_{c,H}^{(p)}) \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]$. Suppose that $[x_2, \frac{K(r-\mu)}{\alpha\bar{\pi}}] \setminus \left(S_{c,L}^{(p)} \cap S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]\right) \neq \phi$, the set must be open and it can be expressed as a countable disjoint union of open sets:

$$\left[x_2, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right] \setminus \left(S_{c,L}^{(p)} \cap S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]\right) = \bigcup_{n\in\mathbb{N}} (\underline{x}_n, \overline{x}_n).$$

Take $x \in (\underline{x}_n, \overline{x}_n)$, we have $x \geq x_2 > \frac{K(r-\mu)}{\pi_L}$ from eq. (4.4), so that $x \notin S_{b,L}^{(p)}$ and $x \notin S_{e,L}^{(p)} \cup S_D^{(p)}$ for any $x \in (\underline{x}_n, \overline{x}_n)$, the value function $V_{e,H}^{(p)}(x)$ satisfies the following linear complementarity relation:

$$\begin{cases} \min\left\{V_{e,H}^{(p)}(x) - \left(\frac{\pi_H x}{r-\mu} - K\right), \frac{-\sigma^2 x^2}{2}\frac{d^2V_{e,H}^{(p)}}{dx^2} - \mu x \frac{dV_{e,H}^{(p)}(x)}{dx} + rV_{e,H}^{(p)}(x) - \pi_H x + c\right) = 0, \ x \in (\underline{x}_n, \overline{x}_n); \\ V_{e,H}^{(p)}(\underline{x}_n) = \frac{\pi_H \underline{x}_n}{r-\mu} - K, \quad V_{e,H}^{(p)}(\overline{x}_n) = \frac{\pi_H \overline{x}_n}{r-\mu} - K \end{cases} . \tag{B.8}$$

One can show that $\frac{\pi_H x}{r-\mu} - K$ is the supersolution of eq. (B.8) so that $V_{e,H}^{(p)}(x) \leq \frac{\pi_H x}{r-\mu} - K$. Together with the obstacle condition (3.1b), we deduce that $V_{e,H}^{(p)}(x) = \frac{\pi_H x}{r-\mu} - K$ for all $x \in (\underline{x}_n, \overline{x}_n)$ and conclude that $(\underline{x}_n, \overline{x}_n) \subseteq S_{c,H}^{(p)}$. This implies that $V_{e,L}^{(p)}(x) = V_{e,L}(x)$ for $x \in (\underline{x}_n, \overline{x}_n)$. Using the incentive constraint embedded in $S_{c,L}^{(p)} \cap S_{c,H}^{(p)}$, we get $x > x_2 \geq x_{c,L}$. It follows from eq. (4.3a) that $V_{e,L}^{(p)}(x) = V_{e,L}(x) = \frac{\pi_L x}{r-\mu} - K$. Thus, we deduce that $x \in S_{c,L}^{(p)}$. This contradiction implies that $[x_2, \frac{K(r-\mu)}{\alpha\bar{\pi}}] \subseteq \left(S_{c,L}^{(p)} \cap S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]\right)$.

We proceed to argue that $\inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}) = x_{c,L}$. Using the incentive constraint condition embedded in $S_{c,L}^{(p)} \cap S_{c,H}^{(p)}$ [see eq. (3.16)] and the obstacle condition (3.1b), we have $V_{e,L}(x) = \frac{\pi_L x}{r-\mu} - K$ for any $x \in S_{c,L}^{(p)} \cap S_{c,H}^{(p)}$. From eq. (4.3a), we have $x \geq x_{c,L}$ and $\inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}) \geq x_{c,L}$. Suppose that $\inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}) > x_{c,L}$, one can use a similar technique as in the proof of $\sup\left(S_{b,L}^{(p)} \cap S_{b,H}^{(p)}\right) = x_{b,H}$ and show that both types of firm has strict incentive to declare call at some threshold between $x_{c,L}$ and $\inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)})$.

For any $x \in (\sup S_{b,L}^{(p)}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}))$, we note that $x \notin S_{e,L}^{(p)}(x) \cup S_D^{(p)}$ and the value function $V_{e,H}^{(p)}$ satisfies the following linear complementarity relation:

$$\begin{cases} \min\left(V_{e,H}^{(p)}(x) - \max(0, h_{c,H}^{(p)}(x)) - \frac{\sigma^2}{2}x^2\frac{d^2V_{e,H}^{(p)}}{dx^2} - \mu x\frac{dV_{e,H}^{(p)}}{dx} + rV_{e,H}^{(p)} - \pi_H x + c\right) = 0, \\ \hfill x \in (\sup S_{b,L}^{(p)}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)})); \\ V_{e,H}^{(p)}(\sup S_{b,L}^{(p)}) = V_{e,H}(\sup S_{b,L}^{(p)}), \quad V_{e,H}^{(p)}(\inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)})) = \frac{\pi_H(\inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}))}{r-\mu} - K \end{cases} . \tag{B.11}$$

It can be seen that $V_{e,H}(x)$ is the supersolution of eq. (B.11) and $V_{e,H}^{(p)}(x) \leq V_{e,H}(x)$. On the other hand, $V_{e,H}(x) = \frac{\pi_H x}{r-\mu} - K$ for $x \in (x_{c,H}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}))$ from eq. (4.3a). Together with the obstacle condition (3.1b), we have $V_{e,H}^{(p)}(x) = \frac{\pi_H x}{r-\mu} - K$ for $x \in (x_{c,H}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}))$ and $(x_{c,H}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)})) \subseteq S_{c,H}^{(p)}$. This implies that $V_{e,L}^{(p)}(x) = V_{e,L}(x)$ for $x \in (x_{c,H}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}))$ from eq. (3.11). Since $x_{c,L} \in (x_{c,H}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}))$, we can deduce from eq. (4.3a) that $V_{e,L}^{(p)}(x) = V_{e,L}(x) = \frac{\pi_L x}{r-\mu} - K$ for $x \in (x_{c,L}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}))$ so that $(x_{c,L}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)})) \subseteq S_{c,L}^{(p)}$ and $(x_{c,L}, \inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)})) \subseteq S_{c,L}^{(p)} \cap S_{c,H}^{(p)}$. The contradiction implies $\inf(S_{c,L}^{(p)} \cap S_{c,H}^{(p)}) = x_{c,L}$ and

$$\left[x_{c,L}, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right] = S_{c,L}^{(p)} \cap S_{c,H}^{(p)} \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]. \tag{B.12}$$

*Characterization of* $(S_{c,H}^{(p)} \setminus S_{c,L}^{(p)}) \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right]$

Lastly, we characterize the stopping region $(S_{c,H}^{(p)} \setminus S_{c,L}^{(p)}) \cap [0, \frac{K(r-\mu)}{\alpha\bar{\pi}}]$. The analysis is similar to that of $S_{b,L}^{(p)} \setminus S_{b,H}^{(p)}$ and details are omitted here. Finally, we deduce that

$$\left(S_{c,H}^{(p)} \setminus S_{c,L}^{(p)}\right) \cap \left[0, \frac{K(r-\mu)}{\alpha\bar{\pi}}\right] = \left[x_{c,H}^*, x_{c,L}\right), \tag{B.13a}$$

where $x_{c,H}^* = \inf(S_{c,H}^{(p)} \setminus S_{c,L}^{(p)})$ is the optimal call threshold of Firm $H$ and satisfies the following smooth pasting condition:

$$\frac{dV_{e,H}^{(p)}(x)}{dx}|_{x=x_{c,H}^*} = \frac{d}{dx}\left(\frac{\pi_H x}{r-\mu} - K\right)|_{x=x_{c,H}^*} = \frac{\pi_H}{r-\mu}. \tag{B.13b}$$

*Determination of the value functions*

Once the optimal stopping regions are determined, the value functions can be determined as follows:

(i) If $x$ lies in the stopping regions of the equity holder or bondholder, the corresponding value functions can be determined using eqs. (3.9)-(3.12).

(ii) If $x$ lies in the continuation region (i.e. $x \in (x^*_{b,L}, x^*_{c,H})$), the value functions $V^{(p)}_{e,L}(x)$, $V^{(p)}_{e,H}(x)$ and $V^{(p)}_{d}(x)$ can be obtained by solving the following differential equations:

$$\begin{cases} \frac{\sigma^2}{2} x^2 \frac{d^2 V^{(p)}_{e,H}}{dx^2} + \mu x \frac{dV^{(p)}_{e,H}}{dx} - rV^{(p)}_{e,H} + \pi_H x - c = 0, \quad x \in (x^*_{b,L}, x^*_{c,H}), \\ V^{(p)}_{e,H}(x^*_{b,L}) = V_{e,H}(x^*_{b,L}), \quad V^{(p)}_{e,H}(x^*_{c,H}) = \frac{\pi_H x^*_{c,H}}{r-\mu} - K \end{cases}, \tag{B.14a}$$

$$\begin{cases} \frac{\sigma^2}{2} x^2 \frac{d^2 V^{(p)}_{e,L}}{dx^2} + \mu x \frac{dV^{(p)}_{e,L}}{dx} - rV^{(p)}_{e,L} + \pi_L x - c = 0, \quad x \in (x^*_{b,L}, x^*_{c,H}), \\ V^{(p)}_{e,L}(x^*_{b,L}) = 0, \quad V^{(p)}_{e,L}(x^*_{c,H}) = V_{e,L}(x^*_{c,H}) \end{cases}, \tag{B.14b}$$

$$\begin{cases} \frac{\sigma^2}{2} x^2 \frac{d^2 V^{(p)}_{d}}{dx^2} + \mu x \frac{dV^{(p)}_{d}}{dx} - rV^{(p)}_{d} + c = 0, \quad x \in (x^*_{b,L}, x^*_{c,H}), \\ V^{(p)}_{d}(x^*_{b,L}) = pV_{d,H}(x^*_{b,L}) + (1-p)\frac{(1-\gamma)\pi_L x^*_{b,L}}{r-\mu}, \\ V^{(p)}_{d}(x^*_{c,H}) = p\left(\frac{\pi_H x^*_{c,H}}{r-\mu} - K\right) + (1-p)V_{d,L}(x^*_{c,H}) \end{cases}. \tag{B.14c}$$

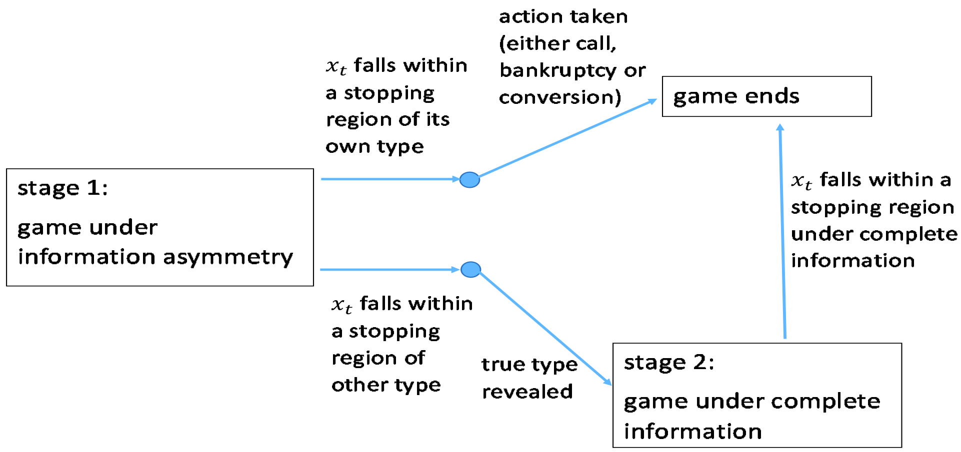


Figure 1: Schematic representation of the two-stage sequential two-person stochastic game model.

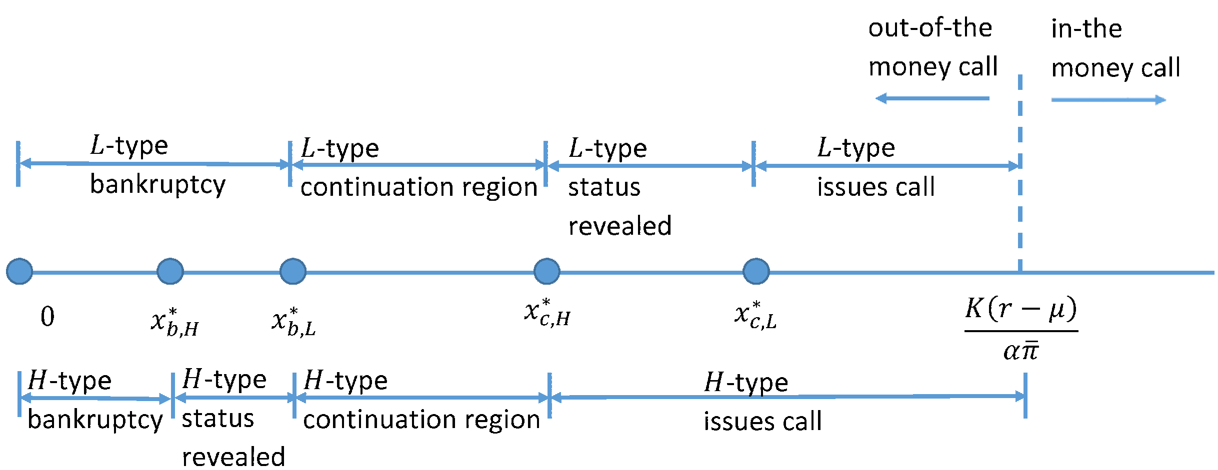


Figure 2: Characterization of the stopping regions under $\frac{\pi_L}{\pi_H} \geq z$.

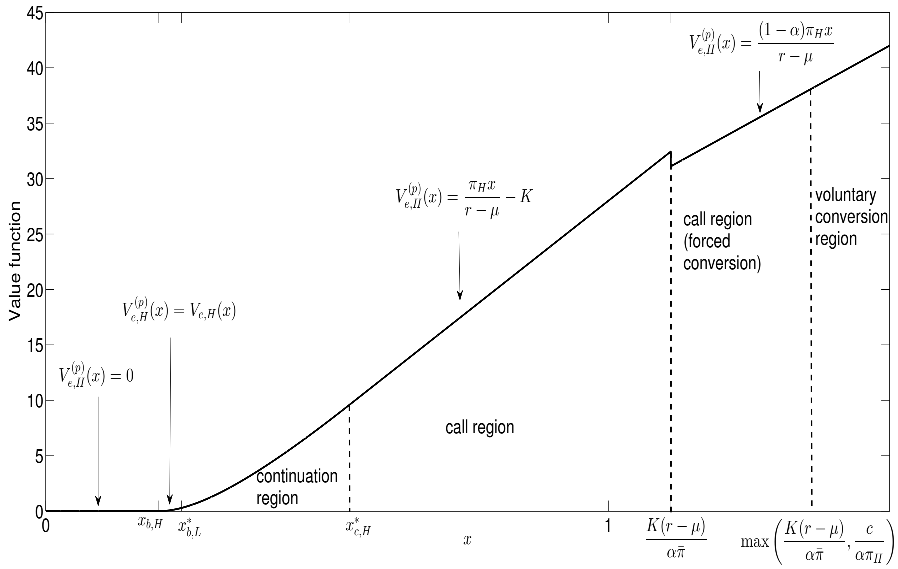


Figure 3a: Plot of the equity value function of type $H$ against $x$. The sequential game is terminated either due to (i) bankruptcy when $x \leq x_{b,H}$, (ii) call when $x^*_{c,H} \leq x < \max\left(\frac{K(r-\mu)}{\alpha\pi_L}, \frac{c}{\alpha\pi_H}\right)$. It may enter into the second stage of game under complete information when $x \in (x_{b,H}, x^*_{b,L})$.

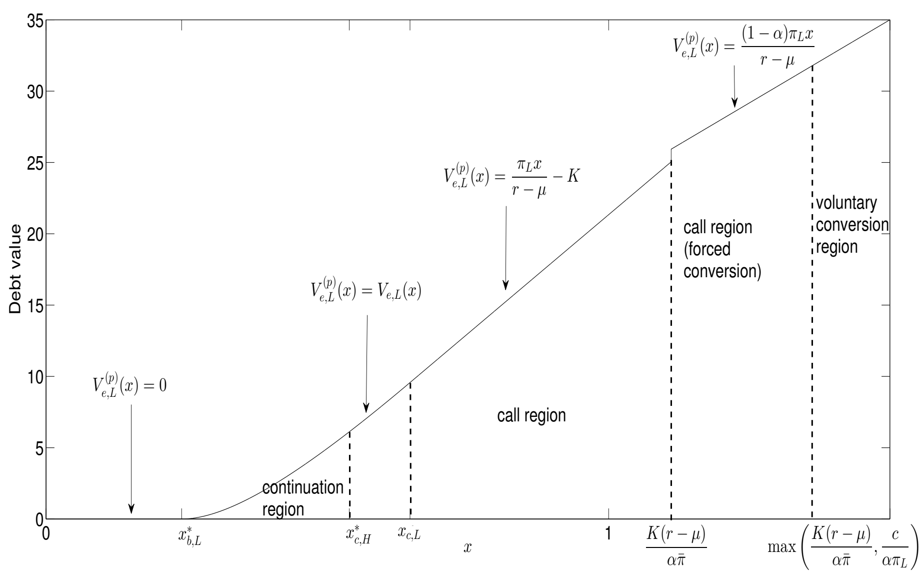


Figure 3b: Plot of the equity value function of type $L$ against $x$. The sequential game is terminated either due to (i) bankruptcy $x \leq x^*_{b,L}$, (ii) call when $x_{c,L} \leq x < \max\left(\frac{K(r-\mu)}{\alpha\pi_L}, \frac{c}{\alpha\pi_L}\right)$. It may enter into the second stage of game under complete information when $x \in (x^*_{c,H}, x_{c,L})$.

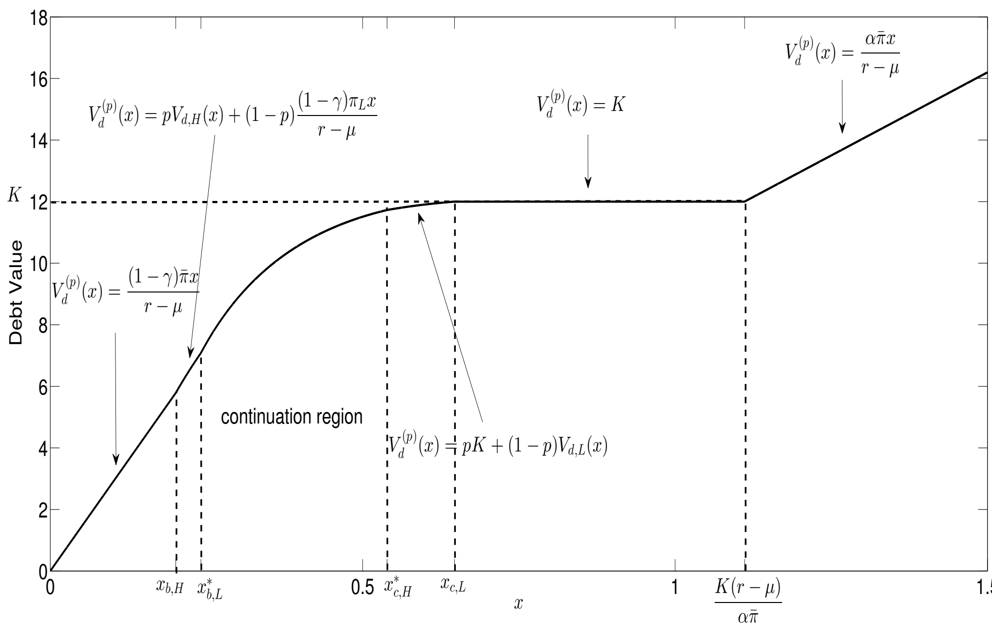


Figure 3c: Plot of the debt value function against $x$. The sequential game is terminated due to (i) bankruptcy when $x \leq x_{b,H}$ or (ii) call when $x \geq x_{c,L}$. It may enter into the second stage of game under complete information when $x \in (x_{b,H}, x_{b,L}^*)$ or $x \in (x_{c,H}^*, x_{c,L})$.

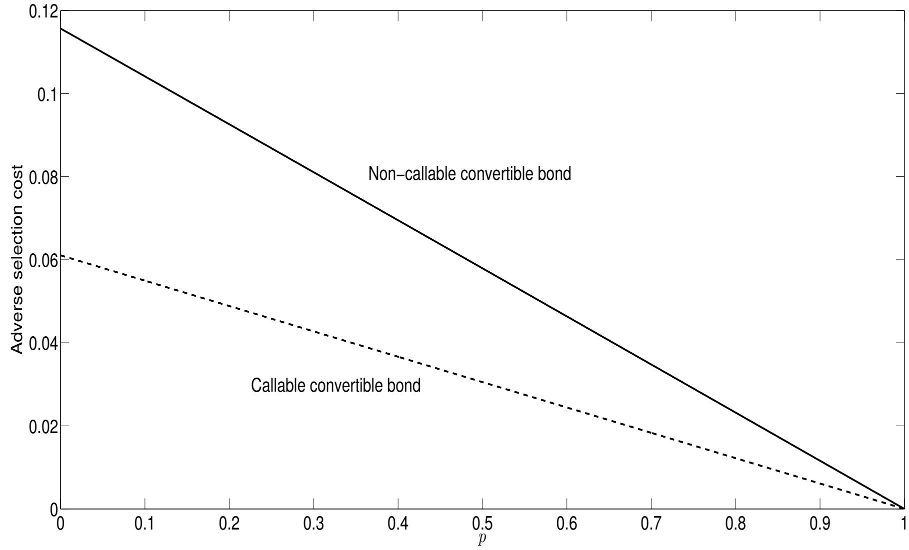


Figure 4a: Plot of the adverse selection cost of the non-callable convertible bond and callable convertible bond against probability $p$. Both adverse selection costs decrease to zero when $p$ increases to 1 (complete information). The adverse selection cost of the callable-convertible bond is always lower than that of the non-callable convertible bond.

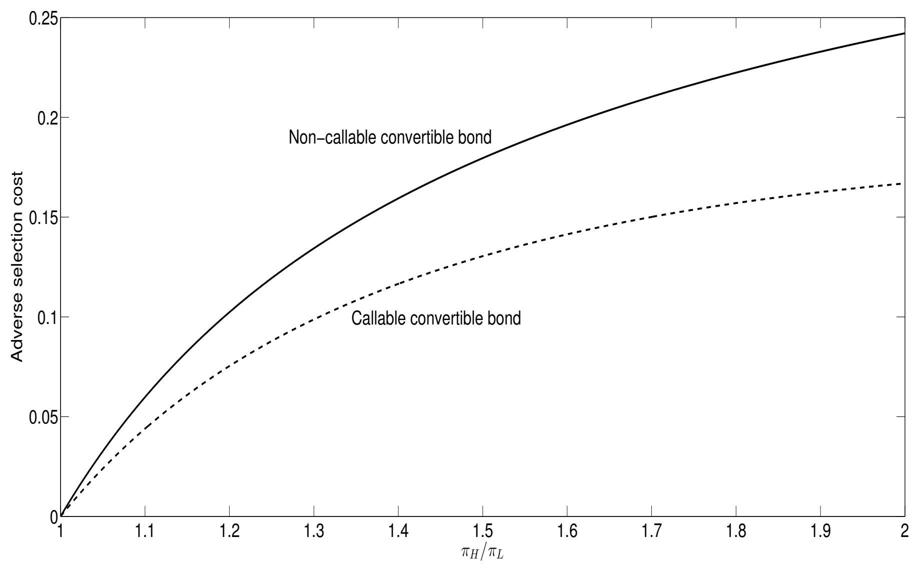


Figure 4b: Plot of the adverse selection cost of the non-callable convertible bond and callable convertible bond against the ratio $\pi_H/\pi_L$. Both adverse selection costs increase when the gap between $\pi_H$ and $\pi_L$ widens. The adverse selection cost of the non-callable convertible bond increases at a faster rate than that of the callable convertible bond.