Equity-credit modeling under affine jump-diffusion models with jump-to-default

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ABSTRACT

This article considers the stochastic models for pricing credit-sensitive financial derivatives using the joint equity-credit modeling approach. The modeling of credit risk is embedded into a stochastic asset dynamics model by adding the jump-to-default feature. We discuss the class of stochastic affine jump-diffusion models with jump-to-default and apply the models to price defaultable European options and credit default swaps. Numerical studies of the equity-credit models are also considered. The impact on the pricing behavior of derivative products with the added jump-to-default feature is examined.

1 Introduction

The affine jump-diffusion (AJD) models have been widely used in continuous time modeling of stochastic evolution of asset prices, bond yields and credit spreads. Some of the well known examples include the stochastic volatility (SV) model of Heston (1993), stochastic volatility jump-diffusion models (SVJ) of Bates (1996) and Bakshi *et al.* (1997), and stochastic volatility coherent jump model (SVCJ) of Duffie *et al.* (2000). The AJD models possess flexibility to capture the dynamics of market prices in various asset classes, while also admit nice analytical tractability. The affine term structure models, which fall into the family of AJD models, have been frequently used to study the dynamics of bond yields and credit spreads (Duffie and Singleton, 1999).

A number of studies have addressed the importance of including jump dynamics to valuation and hedging of derivatives. In the modeling of equity derivatives, Bakshi *et al.* (1997) illustrate that the stochastic volatility model augmented with the jump-diffusion feature produces a parsimonious fit to stock option prices for both short-term and long-term maturities. Empirical studies reported by Bates (1996), Pan (2002) and Erakar (2004) show that the inclusion of jumps in the modeling of stock price is necessary to reconcile the time series behavior of the underlying with the cross-sectional pattern of option prices. In particular, Erakar (2004) concludes from his empirical studies that simultaneous jumps in stock price and return variance are important in catering for different volatility regimes.

While the AJD models have been successfully applied in valuation of both equity and credit derivatives, the joint modeling of equity and credit derivatives have not been fully addressed in the literature. Recently, a growing literature has highlighted such an interaction between equity risk (stock return and its variance) and credit risk (firm default risk). While the risk neutral distribution of stock return is fully conveyed by traded option prices of different strikes and maturities, the information of the arrival rate of default can be extracted from the bond yield spreads or credit default swap spreads. With the growing liquidity of the credit default swap (CDS) markets, the CDS spreads provide more reliable and updated information about the credit risk of firms. Achyara and Johnson (2007) find that the CDS market contains forward looking information on equity return, in particular during times of negative credit outlooks. For equity options, Cremers *et al.* (2008), Zhang *et al.* (2009) and Cao *et al.* (2010) show that the out-of-the-money put options, which depict the negative tail of the underlying risk neutral distribution, are closely linked to yield spreads and CDS spreads of the reference firm.

Several innovative equity-credit models have been proposed in the literature. Carr and Linetsky (2006) propose an equity-credit hybrid model in which the stock price is sent to a cemetery state upon the arrival of default of the reference company. Carr and Wu (2009) introduce another equity-credit hybrid model which incorporates jump-to-default in which the equity price drops to zero given the default arrival. Carr and Madan (2010) consider a local volatility model enhanced by jump-to-default. Mendoza-Arriaga *et al.* (2010) and Bayraktar and Yang (2011), respectively, propose a flexible modeling framework to unify the valuation of equity and credit derivatives using the time-changed Markov process and multiscale stochastic volatility. Cheridito and Wugalter (2012) propose a general framework under affine models with possibility of default for the simultaneous modeling of equity, government bonds, corporate bonds and derivatives.

In this article, we propose an equity-credit model under the general affine jump-diffusion framework in the presence of jump-to-default (JtD-AJD model). We illustrate how to use the proposed JtD-AJD model for pricing defaultable European options and credit default swaps. The article is organized as follows. In the next section, we present the mathematical framework of the affine jump-diffusion with jump-to-default. The reduced form approach is adopted, where the default process is modeled as a Cox process with stochastic intensity. We illustrate how to apply the Fourier transform technique to derive the joint characteristic function of the stock price distribution in the JtD-AJD model. In Section 3, we consider pricing of defaultable

European contingent claims and credit default swaps using the JtD-AJD model. We manage to obtain closed form pricing formulas of these two credit-sensitive derivative products as a demonstration for nice analytical tractability of the proposed equity-credit model. In Section 4, we discuss the practical implementation of the jump-to-default feature to several popular option pricing models. We also consider numerical valuation of defaultable European options using various numerical approaches, like the Fast Fourier transform (FFT) techniques and Monte Carlo simulation. The impact of various jump parameters on the pricing behavior of defaultable European options is examined. Conclusive remarks are presented in the last section.

2 Affine jump diffusion with jump-to-default

This section summarizes the mathematical framework of the AJD model (Duffie *et al.*, 2000). Consider the filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, Q)$, the AJD process of the vector stochastic state variable X_t is defined in the state space $\mathcal{D} \subset \mathbb{R}^n$ as follows:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t + dZ_t, \tag{1a}$$

where Q is some appropriate equivalent martingale measure adopted for pricing contingent claims, $\mathcal{G}_t = \sigma\{X_s|\, s < t\}$ is the natural filtration generated by the vector state variable X_t, W_t is an $\mathcal{G}_t$-standard Brownian motion in $\mathbb{R}^n$, $\mu : \mathcal{D} \to \mathbb{R}^n$ is the drift vector, $\sigma : \mathcal{D} \to \mathbb{R}^{n\times n}$ is the diffusion matrix, and Z_t is a pure jump process whose jumps have a fixed probability distribution ν on $\mathbb{R}^n$ and arrive with intensity $\{\lambda(X_t) : t \geq 0\}$, $\lambda : \mathcal{D} \to [0, \infty)$. The interest rate process is specified as $\{r(X_t) : t \geq 0\}$, $r : \mathcal{D} \to [0, \infty)$. Under the AJD model, the parameter functions $\mu(X_t)$, $\sigma(X_t)$, $\lambda(X_t)$ and the risk-free interest rate $r(X_t)$ are specified as follows:

$$\begin{aligned}
&\mu(X) = K_0 + K_1 \cdot X, && K_0 \in \mathbb{R}^n \text{ and } K_1 \in \mathbb{R}^{n\times n};\\
&\{\sigma(X)\sigma(X)^T\}_{ij} = \{H_0\}_{ij} + \{H_1\}_{ij} \cdot X, && H_0 \in \mathbb{R}^{n\times n} \text{ and } H_1 \in \mathbb{R}^{n\times n\times n};\\
&\lambda(X) = l_0 + l_1 \cdot X, && l_0 \in \mathbb{R} \text{ and } l_1 \in \mathbb{R}^n;\\
&r(X) = r_0 + r_1 \cdot X,, && r_0 \in \mathbb{R} \text{ and } r_1 \in \mathbb{R}^n.
\end{aligned}$$

Note that one can choose (l_0, l_1) and (r_0, r_1) appropriately to preclude negative interest rate and jump intensity. Without loss of generality, we let the first component of X_t be the logarithm of the pre-default stock price S_t. Under the AJD framework, stochastic interest rate and stochastic volatility (expressed as some linear combination of the vector state variable X_t) can be incorporated. Also, the correlation structures between the different factors can be introduced by specifying the diffusion matrix $\sigma(X_t)$. It is worth noting that the drift vector

$\mu(X_t)$ is determined by requiring the stock price process to be a martingale under the equivalent martingale measure Q.

Specification of the default process

Next, we extend the AJD framework by incoporating the jump-to-default feature of the stock price process. Upon the arrival of default, the stock price jumps to some constant level called the cemetery state (Carr and Linetsky, 2006). In principle, the state can be a prior known level or a level that is arbitrarily close to zero. Following the reduced form framework, we assume the default process to be generated by a Cox process that is defined in the same state space $\mathcal{D}$. Formally, we define the first jump time

$$\tau_d = \inf\left\{ t \geq 0 : \int_0^t h(X_s)\, ds \geq e \right\}$$

as the random time of default arrival. Here, e is an independent standard exponential random variable and the intensity process of default arrival (hazard rate) is assumed to be a function of the state variable as represented by $\{h(X_t) : t \geq 0\}$, $h : \mathcal{D} \to [0, \infty)$, and adapted to $\mathcal{G}_t$. The filtration $\mathcal{H}_t = \sigma\left(1_{\{\tau_d < s\}} \middle| s \leq t\right)$ contains the information of whether there has been a default by time t. We define $\mathcal{F}_t = \mathcal{H}_t \vee \mathcal{G}_t$ to be the information set that contains the knowledge of the evolution of the vector state variable X_t and history of default by time t. In order to have the risk adjusted interest rate to be affine, the hazard rate process $h(X_t)$ is specified as

$$h(X) = h_0 + h_1 \cdot X, \qquad h_0 \in \mathbb{R} \text{ and } h_1 \in \mathbb{R}^n. \tag{1b}$$

Similarly, one can choose (h_0, h_1) appropriately to guarantee a positive hazard rate. The above specification falls into the affine term structure framework commonly used in the modeling of interest rate and credit derivatives (Lando, 1998; Duffie and Singleton, 1999).

2.1 Transform analysis

The discounted expectation of a contingent claim that pays $F(X_T)$ when there is no default prior expiration is given by

$$\begin{aligned} & \mathbb{E}^Q\left[\exp\left(-\int_t^T r(X_s)\, ds\right) F(X_T)\, \mathbf{1}_{\{\tau_d > T\}} \middle| \mathcal{F}_t\right] \\ = \; & \mathbf{1}_{\{\tau_d > t\}} \mathbb{E}^Q\left[\exp\left(-\int_t^T R(X_s)\, ds\right) F(X_T) \middle| \mathcal{G}_t\right], \end{aligned}$$

where $R(X_s) = r(X_s) + h(X_s)$ is the risk adjusted discount rate at time s and the payoff $F(X_T)$ is $\mathcal{G}_T$-measurable requiring no knowledge of the default process. The time-t price of the contingent claim conditional on $X_t = X$ is defined by

$$V(X,t) = \mathbb{E}^Q\left[\exp\left(-\int_t^T R(X_s)\,ds\right) F(X_T)\bigg|\,\mathcal{G}_t\right]. \tag{2}$$

By the Feynman-Kac representation formula, we deduce that $V(X,t)$ satisfies the following partial integro-differential equation (PIDE):

$$\frac{\partial V(X,t)}{\partial t} + \mathcal{L}V(X,t) = 0. \tag{3a}$$

Here, $\mathcal{L}$ is the infinitesimal generator as defined by

$$\begin{aligned}\mathcal{L}V(X,t) = {} & \mu(X)\,V_X(X,t) + \frac{1}{2}tr\left[\sigma(X)\,\sigma(X)^T V_{XX}(X,t)\right] \\ & + \lambda(X)\int_{\mathbb{R}^n}\left[V(X+z,t) - V(X,t)\right] d\nu(z) - R(X)\,V(X,t).\end{aligned} \tag{3b}$$

The solution to the above PIDE subject to an exponentially affine form of the terminal condition is stated in Theorem 1.

Theorem 1. *When the terminal payoff function is exponentially affine, where*

$$F(X_T) = \exp(u \cdot X_T),$$

for $u \in \mathbb{C}^n$, the conditional expectation

$$F_0(u, X_t, t; T) = \mathbb{E}^Q\left[\exp\left(-\int_t^T R(X_s)\,ds\right) e^{u\cdot X_T}\bigg|\,\mathcal{G}_t\right] \tag{4}$$

has the solution of the form

$$F_0(u, X_t, t; T) = \exp\left(\alpha(\tau) + \beta(\tau)\cdot X_t\right), \quad \tau = T - t,$$

where $\alpha(\tau)$ and $\beta(\tau)$ satisfy the following system of complex-valued ordinary differential equations (ODEs):

$$\begin{aligned}\alpha'(\tau) &= -R_0 + K_0\beta(\tau) + \frac{1}{2}\beta(\tau)^T H_0\beta(\tau) + l_0\left[\Lambda(\beta(\tau)) - 1\right], \quad \tau > 0, \\ \beta'(\tau) &= -R_1 + K_1^T\beta(\tau) + \frac{1}{2}\beta(\tau)^T H_1\beta(\tau) + l_1\left[\Lambda(\beta(\tau)) - 1\right], \quad \tau > 0,\end{aligned} \tag{5}$$

with the initial conditions: $\alpha(0) = 0,\ \beta(0) = u$. Here, $\beta(\tau)^T H_1\beta(\tau)$ is a vector whose k^{th}

component is given by $\sum_i \sum_j \beta_i \{H_1\}_{ijk} \beta_j$ *and* $\Lambda(c)$ *is the jump transform as defined by* $\Lambda(c) = \int_{\mathbb{R}^n} \exp(c \cdot z)\, d\nu(z)$ *for some* $c \in \mathbb{C}^n$*.*

The proof of Theorem 1 can be found in Duffie *et al.* (2000).

Defaultable bond with fixed recovery

We consider a defaultable zero-coupon bond which pays one dollar when there is no default prior to maturity, otherwise a recovery payment R_p is paid at maturity. Hence, the terminal payoff can be formulated as $\mathbf{1}_{\{\tau_d > T\}} + R_p \times \mathbf{1}_{\{\tau_d \le T\}}$. The non-default component of the zero-coupon bond is given by

$$\begin{aligned} B_0(t, T; X_t) &= \mathbb{E}^Q \left[\exp\left(-\int_t^T r(X_s)\, ds \right) \mathbf{1}_{\{\tau_d > T\}} \middle| \mathcal{F}_t \right] \\ &= \mathbf{1}_{\{\tau_d > t\}} \mathbb{E}^Q \left[\exp\left(-\int_t^T R(X_s)\, ds \right) \middle| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau_d > t\}} \exp\left(\alpha(T-t) + \beta(T-t) \cdot X_t \right), \end{aligned} \tag{6}$$

by virtue of the transform analysis stated in Theorem 1. Here, $\alpha(\tau)$ and $\beta(\tau)$ satisfy the same system of ODEs as depicted in eq. (5), with the corresponding initial conditions specified as $\alpha(0) = 0,\ \beta(0) = (0, 0, ..., 0)^T$. Similarly, the risk neutral discounted expectation of the recovery payment R_p contingent upon default is given by

$$\begin{aligned} B_R(t, T; X_t) &= R_p\, \mathbb{E}^Q \left[\exp\left(-\int_t^T r(X_s)\, ds \right) \mathbf{1}_{\{\tau_d \le T\}} \middle| \mathcal{F}_t \right] \\ &= R_p \left\{ \mathbb{E}^Q \left[\exp\left(-\int_t^T r(X_s)\, ds \right) \middle| \mathcal{G}_t \right] \right. \\ &\qquad \left. - \mathbf{1}_{\{\tau_d > t\}} \mathbb{E}^Q \left[\exp\left(-\int_t^T R(X_s)\, ds \right) \middle| \mathcal{G}_t \right] \right\}. \end{aligned} \tag{7a}$$

The first term can be visualized as the risk-free discount factor as defined by

$$\begin{aligned} B_f(t, T; X_t) &= \mathbb{E}^Q \left[\exp\left(-\int_t^T r(X_s)\, ds \right) \middle| \mathcal{G}_t \right] \\ &= \exp\left(\widetilde{\alpha}(T-t) + \widetilde{\beta}(T-t) \cdot X_t \right), \end{aligned} \tag{7b}$$

where $\widetilde{\alpha}(\tau)$ and $\widetilde{\beta}(\tau)$ satisfy a similar system of ODEs as depicted in eq. (5), except that the risk-free rate (r_0, r_1) replace the role of the risk adjusted discount rate (R_0, R_1). The corresponding initial conditions are specified as: $\widetilde{\alpha}(0) = 0,\ \widetilde{\beta}(0) = (0, 0, ..., 0)^T$. Adding the two components yields the defaultable bond price with fixed recovery as follows:

$$\begin{aligned}
B\left(t,T;X_t\right) &= B_0\left(t,T;X_t\right) + B_R\left(t,T;X_t\right) \\
&= \mathbf{1}_{\{\tau_d>t\}}\left(1-R_p\right)\exp\left[\alpha\left(T-t\right)+\beta\left(T-t\right)\cdot X_t\right] \\
&\quad + \ R_p \exp\left[\widetilde{\alpha}\left(T-t\right)+\widetilde{\beta}\left(T-t\right)\cdot X_t\right]. \qquad (8)
\end{aligned}$$

2.2 Joint characteristic function

The pre-default discounted characteristic function, which provides information on the evolution of the stock price dynamics prior to default, is defined by the following conditional expectation:

$$\begin{aligned}
\Psi\left(\omega,T;X_t,t\right) &= \mathbb{E}^Q\left[\exp\left(-\int_t^T r\left(X_s\right)ds\right)\exp\left(i\omega\cdot X_T\right)\mathbf{1}_{\{\tau_d>T\}}\middle|\,\mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau_d>t\}}\mathbb{E}^Q\left[\exp\left(-\int_t^T R\left(X_s\right)ds\right)\exp\left(i\omega\cdot X_T\right)\middle|\,\mathcal{G}_t\right] \\
&= \mathbf{1}_{\{\tau_d>t\}}\exp\left(\alpha\left(T-t\right)+\beta\left(T-t\right)\cdot X_t\right]), \qquad (9)
\end{aligned}$$

where $\omega=\left(\omega_1,\omega_2,...,\omega_n\right)^T\in\mathbb{R}^n$. Again, $\alpha\left(\tau\right)$ and $\beta\left(\tau\right)$ satisfy the same system of ODEs as depicted in eq. (5), while the corresponding initial conditions are specified as $\alpha\left(0\right)=0,\ \beta\left(0\right)=\left(i\omega_1,i\omega_2,...,i\omega_n\right)^T$. The marginal characteristic function of the first component of X_t, which is the logarithm of the stock price, can be obtained by setting $\omega=\left(\omega,0,...,0\right)^T$. For notational convenience, we write the first component of X_t as x_t in our subsequent discussion.

3 Pricing of defaultable European options and credit default swaps

In this section, we illustrate the pricing of defaultable European contingent claims and credit default swaps under the JtD-AJD model. Provided that analytic solution to the associated system of ODEs is available, we are able to obtain closed form pricing formulas of these credit-sensitive derivatives. Fortunately, nice analytic tractability of the Ricatti system of ODEs is feasible for a wide range of stochastic stock price dynamics models, which include the SV model (Heston, 1993), SVJ model (Bates, 1996; Bakshi *et al.*, 1997), SVCJ model (Duffie *et al.*, 2000; Erakar, 2004), and Carr-Wu's model (Carr and Wu, 2009). Moreover, the exponential affine structure is preserved in the characteristic function of the stock price distribution, which proves to be useful in the computation of risk sensitivity of derivatives.

3.1 Defaultable European contingent claims

Consider a defaultable European contingent claim which pays $P(X_T)$ when no default occurs before maturity and zero payoff upon default (zero recovery). Given that the payoff depends only on the terminal stock price $S_t = \exp(x_t)$, the time-t value of the contingent claim is given by

$$\begin{aligned} P(X_t, t) &= \mathbb{E}^Q \left[\exp\left(-\int_t^T r(X_s)\, ds \right) P(x_T)\, \mathbf{1}_{\{\tau_d > T\}} \middle| \mathcal{F}_t \right] \\ &= \mathbf{1}_{\{\tau_d > t\}} \mathbb{E}^Q \left[\exp\left(-\int_t^T R(X_s)\, ds \right) P(x_T) \middle| \mathcal{G}_t \right], \end{aligned} \tag{10}$$

where $R(X_s) = r(X_s) + h(X_s)$ is the risk-adjusted discount rate at time s. Let $\widetilde{P}(\omega)$ denote the Fourier transform of the terminal payoff with respect to x_T, where

$$\widetilde{P}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x_T} P(x_T)\, dx_T,$$

the terminal payoff can be expressed in the following representation as a generalized Fourier transform integral:

$$P(x_T) = \frac{1}{2\pi} \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} e^{-i\omega x_T} \widetilde{P}(\omega)\, d\omega.$$

Here, the parameter $\varepsilon = \operatorname{Im} \omega$ denotes the imaginary part of ω which falls into some regularity strip, $\varepsilon \in (a, b)$, such that the generalized Fourier transform exists (Lord and Kahl, 2007). By virtue of Fubini's theorem, we obtain the following integral representation of the time-t value of the contingent claim

$$P(X_t, t) = \frac{1}{2\pi} \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \Psi(-\omega)\, \widetilde{P}(\omega)\, d\omega, \tag{11}$$

where

$$\Psi(\omega) = \mathbf{1}_{\{\tau_d > t\}} \mathbb{E}^Q \left[\exp\left(-\int_t^T R(X_s)\, ds \right) \exp(i\omega x_T) \middle| \mathcal{G}_t \right].$$

This is precisely the pre-default discounted first-component marginal characteristic function [see eq. (9)]. If there is a fixed recovery payment R_p to be paid on the maturity date upon earlier default, the present value of this recovery payment is given by

$$\begin{aligned} P_R(X_t, t) = R_p \Bigg\{ &\mathbb{E}^Q \left[\exp\left(-\int_t^T r(X_s)\, ds \right) \middle| \mathcal{G}_t \right] \\ &- \mathbf{1}_{\{\tau_d > t\}} \mathbb{E}^Q \left[\exp\left(-\int_t^T R(X_s)\, ds \right) \middle| \mathcal{G}_t \right] \Bigg\}. \end{aligned} \tag{12}$$

Recall that the jump transform is defined by

$$\Lambda(c) = \int_{\mathbb{R}^n} \exp(c \cdot z) \; d\nu(z) \quad \text{for some} \;\; c \in \mathbb{C}^n.$$

It is worth noting that different terminal payoff functions may impose different restrictions on the regularity strip $\varepsilon \in (a, b)$, inside which the generalized Fourier transform exists. Indeed, one has to choose a particular regularity strip such that both the jump transform and the Fourier transform of the terminal payoff exist. Taking the double exponential jump model as an example, the jump transform exists for $-\eta_1 < \text{Im } \omega < \eta_2$, where $\eta_1 > 1$ and $\eta_2 > 0$ are the parameters defining the sizes of the upward and downward jumps. Usually, there exist certain restrictions on the regularity strip with respect to some standard option payoffs. In case when the regularity strip does not conform with the transformed option terminal payoff function, one may use the put-call parity relation or other relation derived from an appropriate replication portfolio to compute the desired option price.

European call option

Consider a call option which pays $(S_T - K)^+$ at maturity when there is no default prior to the maturity date T and zero otherwise, so the terminal payoff function is given by

$$(S_T - K)^+ \mathbf{1}_{\{\tau_d > T\}} = (e^{x_T} - K)^+ \mathbf{1}_{\{\tau_d > T\}},$$

where $x_T = \ln S_T$. The Fourier transform of the above terminal payoff function is

$$\widetilde{C}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x_T} (e^{x_T} - K)^+ \, dx_T = -\frac{Ke^{i\omega \ln K}}{\omega^2 - i\omega},$$

for $\varepsilon = \text{Im } \omega \in (1, \varepsilon_{\max})$. The upper bound of ε, as denoted by $\varepsilon_{\max}$, can be determined by the non-explosive moment condition $\Psi(-i\varepsilon) < \infty$ (Carr and Madan, 1999). The call option price has the following Fourier integral representation

$$\begin{aligned} C(X_t, t) &= \frac{1}{2\pi} \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \Psi(-\omega) \left[-\frac{Ke^{i\omega \ln K}}{\omega^2 - i\omega} \right] d\omega \\ &= \frac{K}{\pi} \int_0^{\infty} \text{Re} \left\{ \frac{e^{i(\zeta+i\varepsilon)\ln K} \Psi(-(\zeta + i\varepsilon))}{i(\zeta + i\varepsilon) - (\zeta + i\varepsilon)^2} \right\} d\zeta, \;\; \omega = \zeta + i\varepsilon. \end{aligned} \tag{13a}$$

It is worth noting that along the contour $\omega = a + ib$ for $b \in (1, \varepsilon_{\max})$, there is no singularity in the integrand and one can perform numerical integration without much difficulty.

It can be shown by replacing $k = \ln K$, $\zeta = -v$ and $\varepsilon = \alpha + 1$ that the expression in eq. (13a) is equivalent to the pricing formulation in Carr and Madan (1999), where

$$C\left(X_t, t\right) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \operatorname{Re}\left\{ \frac{e^{-ivk}\Psi\left(v - i\left(\alpha+1\right)\right)}{-\left(v - i\alpha\right)\left[v - i\left(\alpha+1\right)\right]} \right\} dv. \tag{13b}$$

In other words, one would obtain the same analytic expression of the Fourier integral representation no matter whether one considers the transform with respect to the log-stock price or log-strike price.

Lord and Kahl (2007) propose two approaches to compute the above Fourier integral. The first approach is the direct numerical integration using an adaptive numerical quadrature, such as the Gauss-Kronrod quadrature (for example, the "quadgk" subroutine in Matlab). The adaptive quadrature can achieve high order of accuracy by choosing appropriately different optimal damping factors α for options at different strikes. The second approach is to employ the Fast Fourier transform (FFT) technique to invert the Fourier integral to obtain option prices on a uniform grid of log-strikes. Since option prices at discrete strikes are obtained on a uniform grid of log-strikes, one needs to perform interpolation to obtain the option price at an arbitrary strike price (Carr and Madan, 1999). For short-maturity options, the interpolation errors can be quite substantial.

European put option

Suppose a put option pays at maturity the following payoff: $\left(K - S_T\right)^+ \mathbf{1}_{\{\tau_d > T\}} = \left(K - e^{x_T}\right)^+ \mathbf{1}_{\{\tau_d > T\}}$ when there is no default prior to maturity, and a recovery payment $R_P \mathbf{1}_{\{\tau_d \leq T\}}$ to be paid at maturity when default occurs during the contractual period. The Fourier transform of the payoff of the non-default component of the put option is given by

$$\widetilde{P}_0\left(\omega\right) = \int_{-\infty}^{\infty} e^{i\omega x_T}\left(K - e^{x_T}\right)^+ dx_T = -\frac{Ke^{i\omega \ln K}}{\omega^2 - i\omega},$$

for $\varepsilon = \operatorname{Im} \omega \in \left(-\varepsilon_{\max}, 0\right)$. Inside the regularity strip where the above Fourier transform is well defined, the non-default component has the following integral representation

$$\begin{aligned} P_0\left(X_t, t\right) &= \frac{1}{2\pi} \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \Psi\left(-\omega\right)\left(-\frac{Ke^{i\omega \ln K}}{\omega^2 - i\omega}\right) d\omega \\ &= \frac{K}{\pi} \int_0^\infty \operatorname{Re}\left\{ \frac{e^{i(\zeta + i\varepsilon)\ln K}\Psi\left(-\left(\zeta + i\varepsilon\right)\right)}{i\left(\zeta + i\varepsilon\right) - \left(\zeta + i\varepsilon\right)^2} \right\} d\zeta,\ \omega = \zeta + i\varepsilon. \end{aligned} \tag{14}$$

It is interesting to find that the Fourier transform of the terminal payoff function for the put option and the call option counterpart both have the same integral representation, though subject to different constraints on the regularity strip. Note that $\operatorname{Im} \omega \in \left(1, \varepsilon_{\max}\right)$ for the call option and Im $\omega \in (-\varepsilon_{\max}, 0)$ for the put option.

As shown earlier, the recovery payment can be obtained similar to $P_R(X_t, t)$ in eq. (12). The defaultable European put option price is then given by

$$P(X_t, t) = P_0(X_t, t) + P_R(X_t, t). \tag{15}$$

Remark
For the Merton jump-diffusion model, there is no additional restriction on the regularity strip. For the Kou double exponential jump model, the jump transform exists only for Im $\omega \in (-\eta_1, \eta_2)$, where $\eta_1 > 1$ and $\eta_2 > 0$. In this case, it is more convenient to implement the put option formula [which requires $\varepsilon = \text{Im } \omega \in (-\infty, 0)$] and obtain the call option price using the put-call parity relation (to be discussed next).

Put-call parity relation under jump-to-default

Take the recovery payment to be $R_p = K$. In the presence of jump-to-default, a portfolio of a long call and a short put has the terminal payoff

$$\begin{aligned} & (S_T - K)^+ \mathbf{1}_{\{\tau_d > T\}} - \left[(K - S_T)^+ \mathbf{1}_{\{\tau_d > T\}} + K\mathbf{1}_{\{\tau_d \le T\}}\right] \\ = \; & (S_T - K)\, \mathbf{1}_{\{\tau_d > T\}} - K\mathbf{1}_{\{\tau_d \le T\}}. \end{aligned}$$

Hence, the difference of defaultable European call and put prices is given by

$$\begin{aligned} C(X_t, t) - P(X_t, t) &= \mathbf{1}_{\{\tau_d > t\}} \mathbb{E}^Q \left[\exp\left(-\int_t^T R(X_s)\, ds\right) (S - K) \middle| \mathcal{G}_t \right] \\ &\quad - \mathbb{E}^Q \left[K \exp\left(-\int_t^T r(X_s)\, ds\right) \middle| \mathcal{G}_t \right] \\ &\quad + \mathbf{1}_{\{\tau_d > t\}} \mathbb{E}^Q \left[K \exp\left(-\int_t^T R(X_s)\, ds\right) \middle| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau_d > t\}} S_t - K B_f(t, T), \end{aligned} \tag{16}$$

where $B_f(t, T)$ is defined in eq. (7b). The put-call parity relation in the presence of jump-to-default is seen to be the same as the standard relation. This is consistent with the model-free property of the put-call parity relation.

3.2 Credit default swap

A credit default swap (CDS) is an over-the-counter (OTC) credit protection contract in which a protection buyer pays a stream of fixed premium (CDS spread) to a protection seller and in return entitles the protection buyer to receive a contingent payment upon the occurrence of a pre-defined credit event. The CDS spread is set at initiation of the contract in such a way that the expected present value of the premium leg received by the protection seller equals that of the protection leg received by the protection buyer. We would like to determine the fair CDS spread, assuming a fixed and known recovery rate of the underlying risky bond. In practice, the recovery rate is commonly set to be $30-40\%$ for corporate bonds in the US and Japanese markets.

Based on the proposed JtD-AJD model, the hazard rate of default arrival is assumed to be affine [see eqs. (1a,b)], where

$$\begin{aligned} h\left(X_t\right) &= h_0 + h_1 \cdot X, \text{ for } h_0 \in \mathbb{R} \text{ and } h_1 \in \mathbb{R}^n; \\ dX_t &= \mu\left(X_t\right) dt + \sigma\left(X_t\right) dW_t + dZ_t. \end{aligned}$$

The survival probability is given by

$$S\left(t,T\right) = \mathbb{E}^Q \left[\exp\left(-\int_t^T h\left(X_u\right) du \right) \middle| \mathcal{G}_t \right].$$

For simplicity, we assume the hazard rate to be independent of the interest rate process. Hence, the price of a defaultable zero-coupon bond with zero recovery is given by

$$D\left(t,T\right) = \mathbb{E}^Q \left[\exp\left(-\int_t^T r\left(X_u\right) + h\left(X_u\right) du \right) \middle| \mathcal{G}_t \right] = B_f\left(t,T\right) S\left(t,T\right).$$

Both $S(t,T)$ and $D(t,T)$ can be readily obtained using the transform analysis.

We now consider the determination of the fair CDS spread c. Consider a CDS contract with unit notional to be initiated at time t and expiring at time T, the present value of the premium leg $L_P\left(t,T\right)$ can be formulated as

$$L_P\left(t,T\right) = \mathbb{E}^Q \left[\int_t^T c \exp\left(-\int_t^s r\left(X_u\right) + h\left(X_u\right) du \right) ds \middle| \mathcal{G}_t \right].$$

On the other hand, the corresponding present value of the protection leg $L_R\left(t,T\right)$ is given by

$$L_R\left(t,T\right) = \left(1-w\right) \mathbb{E}^Q \left[\int_t^T h\left(X_s\right) \exp\left(-\int_t^s r\left(X_u\right) + h\left(X_u\right) du \right) ds \middle| \mathcal{G}_t \right],$$

where w is the loss given default of the risky bond (assumed to be fixed and known). Equating the two payment legs yields the CDS spread as follows:

$$c = (1 - w) \frac{\mathbb{E}^Q \left[\int_t^T h(X_s) \exp\left(-\int_t^s r(X_u) + h(X_u)\, du \right) ds \middle| \mathcal{G}_t \right]}{\mathbb{E}^Q \left[\int_t^T \exp\left(-\int_t^s r(X_u) + h(X_u)\, du \right) ds \middle| \mathcal{G}_t \right]}. \tag{17}$$

When the default intensity assumes the constant value λ, we recover the obvious result: $c = (1 - w)\lambda$. By interchanging the order of taking expectation and performing integration, the denominator and numerator in eq. (17) can be simplified as

$$\mathbb{E}^Q \left[\int_t^T \exp\left(-\int_t^s r(X_u) + h(X_u)\, du \right) ds \middle| \mathcal{G}_t \right] = \int_t^T D(t, s)\, ds,$$

and

$$\mathbb{E}^Q \left[\int_t^T h(X_s) \exp\left(-\int_t^s r(X_u) + h(X_u)\, du \right) ds \middle| \mathcal{G}_t \right] = \int_t^T B_f(t, s) \left[-\frac{\partial S}{\partial s}(t, s) \right] ds,$$

respectively. Hence, the CDS spread can be expressed as

$$c = (1 - w) \left[\int_t^T D(t, s)\, ds \right]^{-1} \int_t^T B_f(t, s) \left[-\frac{\partial S}{\partial s}(t, s) \right] ds, \tag{18}$$

which can be computed directly from the risk adjusted discount factor and survival probability.

4 Numerical valuation of defaultable options and impact of jump-to-default feature

In the last two sections, we presented the general mathematical formulation of equity-credit modeling under affine-diffusion models with the jump-to-default feature and demonstrated the pricing of defaultable European options and credit default swaps using the proposed equity-credit formulation. In this section, we discuss the practical implementation of the jump-to-default feature to several popular option pricing models. Our choices of the stochastic stock price dynamics models include Heston's stochastic volatility model (Heston, 1993), Merton's jump-diffusion model (Merton, 1976), and Kou's double exponential jump model (Kou, 2002). First, we briefly review the mathematical formulation of each of these popular stock price dynamics models and show that they are all nested under the stochastic volatility jump-diffusion model (SVJ). We then demonstrate how to find the closed form formula of the characteristic function of the SVJ model with a specified set of parameter functions. Next, we report the numerical experiments that were performed on valuation of the prices of defaultable European options using various numerical approaches, namely,

(i) direct numerical integration of the Fourier integral representation of the price function;

(ii) Fast Fourier transform algorithm of inverting the Fourier transform;

(iii) Monte Carlo simulation of the terminal stock price and computation of the sampled averaged discounted expectation of the terminal payoff.

Besides comparing option prices with varying moneyness (ratio of strike price to stock price) and maturities under different stock price dynamics models, it is also instructive to compare the implied volatility values and examine the nature of implied volatility smile patterns under various moneyness conditions.

4.1 Stochastic dynamics of stochastic volatility and jump-diffusion models with jump-to-default

We present the stochastic differential equations that govern the stock price dynamics under Heston's stochastic volatility model. Also, we derive the moment generating functions of Merton's Gaussian jump-diffusion model and Kou's exponential jump-diffusion model. The stochastic volatility (with no price jumps) and jump-diffusion models (with non-stochastic volatility) are actually nested by the SVJ models since they can be recovered from the SVJ model by switching off the jump component and stochastic volatility component, respectively. We now derive the closed form representation of the characteristic function of the SVJ model with jump-to-default.

Heston's stochastic volatility model

Let $x(t) = \ln S(t)$ be the logarithm of the pre-default stock price $S(t)$. The pre-default dynamics of Heston's stochastic volatility model is specified by

$$\begin{aligned} dx\,(t) &= \left[r(t) - \frac{1}{2}\nu\,(t) + h\,(t) \right] dt + \sqrt{\nu\,(t)}\; dW_s, \quad x(0) = x_0, \\ d\nu\,(t) &= \kappa\,[\theta - \nu\,(t)]\, dt + \sigma_\nu \sqrt{\nu\,(t)}\; dW_\nu, \quad \nu(0) = \nu_0, \end{aligned} \tag{19}$$

where the instantaneous variance $\nu\,(t)$ is assumed to follow a square-root process with a mean-reversion level θ, mean-reversion speed κ, and volatility of volatility σ_ν. The correlation between the price process and instantaneous variance is given by ρ, where $\mathrm{E}[dW_s dW_\nu] = \rho\; dt$. The nice analytical tractability of the stochastic volatility model with jump-to-default prevails provided that the generalization of the hazard rate $h\,(t)$ is state dependent on the instantaneous variance $\nu\,(t)$.

As a remark, we consider the degenerate case where the variance $\nu(t)$ is taken to be constant. This reduces to the standard Black-Scholes formulation, except with the inclusion of jump-to-default. Let σ be the constant volatility, h be the constant hazard rate and r be the constant interest rate. The pre-default dynamics of the stock price is given by

$$dx(t) = \left(r - \frac{\sigma^2}{2} + h\right) dt + \sigma \, dW_s. \tag{20}$$

The call price formula can be obtained as follows:

$$c(S,\tau) = SN(d_1) - Ke^{-(r+h)\tau} N(d_2), \quad \tau = T - t, \tag{21}$$

where K is the strike price, and

$$d_1 = \frac{\ln \frac{S}{K} + \left(r + h + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \; d_2 = d_1 - \sigma\sqrt{\tau}.$$

This call price formula is simply the standard Black-Scholes option price formula with the replacement of the riskfree interest rate r by the default risk adjusted discount rate $r+h$.

Stochastic stock price models with jumps

To incorporate the stock price jumps prior the arrival of jump-to-default, we may modify the stock price dynamics as

$$dx(t) = \left[r(t) - \frac{1}{2}\nu(t) - \lambda m + h(t)\right] dt + \sqrt{\nu(t)} \, dW_s + q \, dN(t), \tag{22}$$

where the jump in stock price is modeled by the counting process $dN(t)$ with intensity λ and jump size q. Here, the compensator $m = \mathbb{E}^Q[\exp(q) - 1]$ is the expected jump size which renders the discounted stock price process to be a martingale under the equivalent martingale measure Q. The two popular distribution specifications of the jump size q are the Gaussian distribution (Merton model) and the double exponential distribution (Kou model).

Gaussian jump distribution (Merton model)

The Gaussian jump distribution is a normal distribution with mean μ_j and volatility σ_j so that $q \sim N(\mu_j, \sigma_j)$. The moment generating function of the Gaussian jump distribution is given by

$$M_M(\zeta) = \mathbb{E}^Q\left[e^{\zeta q}\right] = \exp\left(\mu_j \zeta + \frac{\sigma_j^2}{2}\zeta^2\right). \tag{23}$$

The corresponding jump compensator is given by

$$m = \exp\left(\mu_j + \frac{\sigma_j^2}{2}\right) - 1.$$

Double exponential jump distribution (Kou model)

The double exponential jump has the asymmetric density function:

$$f(q) = p\eta_1 e^{-\eta_1 q}\mathbf{1}_{\{q>0\}} + (1-p)\,\eta_2 e^{\eta_2 q}\mathbf{1}_{\{q<0\}},$$

where $\eta_1 > 1$, $\eta_2 > 0$ and $0 < p < 1$. The probability of an upward jump and a downward jump are given by p and $1-p$, respectively. Empirically, it is common to have $p < 1/2$ such that the jump is asymmetric with bias to downward jumps. The moment generating function of the double exponential jump distribution is given by

$$M_K(\zeta) = \mathbb{E}^Q\left[e^{\zeta q}\right] = p\,\frac{\eta_1}{\eta_1 - \zeta} + (1-p)\,\frac{\eta_2}{\eta_2 + \zeta}, \tag{24}$$

which is well defined for $-\eta_1 < \zeta < \eta_2$. The corresponding jump compensator is given by

$$m = p\frac{\eta_1}{\eta_1 - 1} + (1-p)\,\frac{\eta_2}{\eta_2 + 1} - 1.$$

In these earlier versions of jump-diffusion models, the volatility is assumed to be constant. One may combine all the features of stochastic volatility, price jumps and jump-to-default in the generalized SVJ model with jump-to-default. We demonstrate how to find the closed form analytic representation of the characteristic function of the SVJ model with jump-to-default, whose dynamics is specified as follows:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t + dZ_t,$$

with the state variables $X_t = (x_t, \nu_t)^T$. The parameter functions are defined by

$$\begin{aligned}
\mu(X_t) &= \begin{pmatrix} r_t - \frac{1}{2}\nu_t - \lambda m + h_t \\ \kappa(\theta - \nu_t) \end{pmatrix}, \\
\sigma(X_t) &= \begin{pmatrix} \sqrt{\nu_t} & 0 \\ \rho\sigma_\nu\sqrt{\nu_t} & \sigma_\nu\sqrt{1-\rho^2}\sqrt{\nu_t} \end{pmatrix}, \\
dZ_t &= \begin{pmatrix} q \\ 0 \end{pmatrix} dN_t.
\end{aligned}$$

4.2 Characteristic functions

We would like to demonstrate how to derive the closed form representation of the characteristic function of the SVJ model with jump-to-default. According to the governing equation of the AJD process as depicted in eqs. 1(a,b), we set the coefficient functions to be

$$
\begin{aligned}
\mu(X) &= \begin{pmatrix} r-\lambda m+h \\ \kappa\theta \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa \end{pmatrix} X \\
\sigma(X)\,\sigma(X)^T &= \begin{pmatrix} \nu & \rho\sigma_\nu\nu \\ \rho\sigma_\nu\nu & \sigma_\nu^2\nu \end{pmatrix}.
\end{aligned}
$$

Taking the interest rate r and hazard rate h to be constant, the characteristic function with $u = (u_0, u_1, u_2)^T$ is given by

$$
\begin{aligned}
\Psi(u, X_t, t; T) &= \mathbb{E}^Q\left[\exp\left(-\int_t^T R(X_s)\,ds\right) e^{u\cdot X_T}\middle|\,\mathcal{G}_t\right] \\
&= \exp\left[\alpha(\tau) + \beta_1(\tau)\,x_t + \beta_2(\tau)\,\nu_t\right], \quad \tau = T-t. \qquad (25)
\end{aligned}
$$

The time dependent coefficient functions, $\alpha(\tau)$ and $\beta(\tau) = (\beta_1(\tau), \beta_2(\tau))^T$, are obtained by solving the following system of ODEs

$$
\begin{aligned}
\frac{d\alpha(\tau)}{d\tau} &= (r-\lambda m+h)\,\beta_1(\tau) + \kappa\theta\beta_2(\tau) + \lambda\left[\Lambda(\beta(\tau)) - 1\right] - h - r, \\
\frac{d\beta_1(\tau)}{d\tau} &= 0, \qquad (26) \\
\frac{d\beta_2(\tau)}{d\tau} &= \frac{1}{2}\sigma_\nu^2\beta_2^2(\tau) + \rho\sigma_\nu\beta_1(\tau)\,\beta_2(\tau) - \kappa\beta_2(\tau) + \frac{1}{2}\beta_1^2(\tau) - \frac{1}{2}\beta_1(\tau),
\end{aligned}
$$

with initial conditions: $\alpha(0) = u_0$ and $\beta(0) = (u_1, u_2)^T$. Since there are only jumps in stock price, the jump transform

$$
\Lambda(\beta(\tau)) = \int_{\mathbb{R}^n} \exp(\beta(\tau)\cdot z)\,d\nu(z) = \int f(q)\exp(\beta_1(\tau)\,q)\,dq
$$

can be explicitly computed as:
(i) $\Lambda(\beta(\tau)) = M_M(\beta_1(\tau))$ under the Gaussian jump-diffusion model;
(ii) $\Lambda(\beta(\tau)) = M_K(\beta_1(\tau))$ under the double exponential jump model.

It is seen that $\beta_1(\tau)$ admits a trivial solution of $\beta_1(\tau) = u_1$. As a result, the governing ODE for $\beta_2(\tau)$ can be recasted as a Riccati equation

$$
\frac{d\beta_2(\tau)}{d\tau} = b_0 + b_1\beta_2(\tau) + b_2\beta_2^2(\tau),
$$

where

$$b_0 = \frac{1}{2}u_1 (u_1 - 1), \; b_1 = \rho\sigma_\nu u_1 - \kappa, \; b_2 = \frac{1}{2}\sigma_\nu^2,$$

with initial condition: $\beta_2(0) = u_2$. The system of ODEs has the following explicit solution:

$$\begin{aligned} \alpha(\tau) &= u_0 + \{(r - \lambda m + h) u_1 + \lambda [\Lambda(u_1) - 1] - h - r\} \tau \\ &\quad + \kappa\theta \left[r_- \tau - \frac{1}{b_2} \ln \left(\frac{1 - g\exp(-\tau d)}{1-g} \right) \right], \\ \beta_1(\tau) &= u_1, \; \beta_2(\tau) = r_- \frac{1 - \widetilde{g}\exp(-\tau d)}{1 - g\exp(-\tau d)}, \end{aligned} \tag{27}$$

where

$$r_\pm = \frac{1}{2b_2}(-b_1 \pm d), \; g = \frac{r_- - u_2}{r_+ - u_2}, \; \widetilde{g} = \frac{r_+}{r_-} g, \; d = \sqrt{b_1^2 - 4b_0 b_2}.$$

Remark

When the ODEs governing $\alpha(\tau)$ and $\beta_j(\tau)$, $j = 1, 2$, have coupled or non-linear terms, we may not have closed form solution. One then has to resort to numerical method such as the fourth-order Runga-Kutta method to solve the system of ODEs. Nevertheless, the characteristic function perserves the simple exponential expression with argument that is a linear combination of the solution of the system of ODEs.

4.3 Numerical valuation of defaultable European options

First, we present numerical calculations on pricing of defaultable European options under the JtD-AJD model using different approaches of evaluating the Fourier integral of the option price function. We also examine the impact of the jump-to-default feature on the values of the defaultable European options and the corresponding implied volatility smile patterns. In our numerical calculations, the model parameters are specified as: $\rho = -0.3, \kappa = 5, \theta = 0.12, \sigma_\nu = 0.2, \nu_0 = 0.09, \lambda = 0.5, h = 0.02$. The model parameters are chosen based on the following assumptions. The stochastic volatility has a mean-reversion speed with a half-life of 0.2 year under a moderately upward sloping term structure. The hazard rate of 0.02 implies an CDS spread of around $100 - 150$ bps. For the stock price jump, we assume the arrival intensity to be 0.5 and the jump sizes are assumed to be either the Merton jump or double exponential jump. The values of the jump parameters are taken to be:

Merton jump: $\mu_j = -0.12, \sigma_j = 0.15$;

Double exponential jump: $p = 0.25, \eta_1 = 8$ and $\eta_2 = 6$.

The above parameter values are chosen to be similar to those in Broadie and Kaya (2006). Also, we assume constant interest rate to be 2% and the spot stock price to be $S_0 = \exp(x_0) = 100$

(unless otherwise specified).

Tables 1 and 2 present the comparison of numerical estimates of defaultable European call option prices at varying maturities: 3-month ($t = 0.25$), 6-month ($t = 0.5$), 1-year ($t = 1$) and 2-year ($t = 2$) with the Merton jump and double exponential jump, respectively. For the adaptive numerical integration approach, we use the Gauss-Kronrod quadrature (with relative tolerance of 10^{-8}) to compute the Fourier integral. In the Monte Carlo simulation calculations, we apply the Euler scheme for the numerical simulation of the log stock price and stochastic volatility processes, where a reflecting boundary is imposed for the latter. In order to achieve high accuracy, the simulation is repeated 1,000,000 times and the time step is kept at 0.001. As shown in Tables 1 and 2, the numerical option prices obtained from valuation of the exact characteristic function using adaptive numerical integration are very close to the Monte Carlo estimates (standard deviation of the Monte Carlo simulation is also reported alongside). Since the jump-to-default feature leads to all-or-nothing in the terminal payoff, the Monte Carlo estimates are seen to be less accurate, in particular for long-term options. As an alternative numerical approach, we use the FFT technique to invert the Fourier integral and use linear interpolation to obtain the option value at the desired strike price. We follow Carr and Madan (1999) to choose the number of grids for the discrete Fourier transform to be $N_{FFT} = 4000$ and the grid size to be $\Delta\omega = 0.25$. This implies a truncation at $\alpha = N_{FFT} \times \Delta\omega = 1,000$. For the FFT implementation, the estimates are consistent with the estimates using numerical integration quadrature up to 4 decimal places.

To analyse the impact of the FFT parameters on numerical accuracy of the option value calculations, we vary the number of grids in the Fourier domain (N_{FFT}) as $250, 1000, 4000$ and 16000, and the grid size ($\Delta\omega$) as $0.05, 0.1, 0.25$ and 0.5. The total number of operations for the FFT algorithm is $N_{FFT} \log_2(N_{FFT})$, so the same number of grids in the Fourier domain implies the same operations count. For each level of the computational budget, the different set of FFT parameters produce different degrees of truncation error (controlled by α), discretization error (controlled by $\Delta\omega$), and interpolation error (controlled by $\Delta k = \dfrac{2\pi}{N_{FFT}\Delta\omega}$). Table 3 reports the option price estimates using the same model parameters as in Table 2 but with different FFT parameters. It is interesting to note that accuracy of the estimates depends on the trade-off in minimizing the different sources of errors. Suppose we would like to achieve higher accuracy in computing the Fourier integral by its discrete approximation and set the grid size $\Delta\omega$ to be small (say, $\Delta\omega = 0.05$), then a large number of grids in the Fourier domain is needed to avoid the truncation error. However, such a small grid size in the Fourier domain introduces a wide dispersion of the log-strike grids and leads to larger interpolation error. Table 3 shows that a significant numerical error appears with $\Delta\omega = 0.05$. Conversely, if we attempt to minimize the interpolation error by setting the spacing $\Delta\omega$ to be large (say, $\Delta\omega = 0.5$), the option price estimate suffers from the discretization error in approximating the Fourier integral. Fortunately, one can use an improved numerical integration scheme to minimize the impact of discretization error. Indeed, when a higher order integration scheme is employed (say, Simpson's rule), the discretization is negligible even when the spacing is set at 0.5 or higher. Our results are consistent with the observation in Carr and Madan (1999). As a conclusion, it is desirable to have a coarse grid size in the Fourier domain in order to minimize the truncation and interpolation errors, and adopt a higher order quadrature to approximate the Fourier integral.

Implied volatility smile patterns with jump-to-default

It is more convenient to use put options to examine the impact of jump-to-default. Empirical evidence shows that the out-of-the-money (OTM) put options are more closely linked to the yield spreads and CDS spreads of the underlying firm. Also, the corresponding skew in the implied volatility smile is strongly correlated with the default risk of the firm. Figure 1 shows the implied volatility smile patterns of defaultable European put options under different jump-to-default (JtD) models at varying maturities of 3 months, 6 months, 1 year and 2 years. We consider the following jump-to-default models: (i) Black-Scholes model with JtD (BS-JtD model); (ii) Heston model with JtD (SV-JtD model); and (iii) SVJ model with JtD (SVJ-JtD) model. We also show the flat Black-Scholes implied volatility as benchmark for the implied volatility smile generated by these models.

A higher implied volatility indicates a higher put option price, so a higher premium to buy the downside protection. It can be seen that the presence of jump-to-default significantly increases the implied volatility of the deep OTM put options and produces a strongly skewed implied volatility smile. The jump-to-default feature adds to the implied volatility smile with magnitude varying from 10 to 30 volatility points as the moneyness moves from 0.8 and 0.6. This is consistent with the empirical evidence that the implied volatility of a deep OTM put option is strongly correlated with the default risk of the underlying firm. In particular, it is important to note that for short-term options, the implied volatility generated by different JtD models converge as the moneyness goes deep out-of-the-money, except for the SVJ-JtD model which produces a slightly different pattern. With the possibility of jump-to-default, the price of a short-term deep OTM option is primarily determined by default risk while the diffusion stock price dynamics is of less importance.

Given the possibility of jump-to-default, the skew in the implied volatility is more persistence for long-term options, a feature that cannot be produced by pure stochastic volatility models as their asymptotic implied volatility smile becomes flat as time-to-maturity increases since the variance reverts to its long-run mean. It is also worth to note that the jump-to-default feature introduces a notable shift in the implied volatility smile for the long-term options (by 10 to 15 volatility points for the BS-JtD model). This suggests that the volatility risk premium (net amount that the implied volatility value exceeds the historical volatility value) embedded in the long-term options is largely attributed to the default risk of the underlying firm.

5 Conclusion

The joint modeling of equity risk and credit exposure is important in any state-of-the-art option pricing models of credit-sensitive equity derivatives. Our proposed equity-credit models attempt to perform pricing of equity and credit derivatives under a unified framework. We have demonstrated the robustness of adding the jump-to-default feature in the popular affine jump-diffusion models for pricing defaultable European claims and credit default swaps. By assuming the hazard rate to be affine, analytic tractability in typical affine jump-diffusion models is maintained even with the inclusion of the jump-to-default feature. Once the analytic formula is available for the characteristic function of the joint equity-credit price dynamics, numerical valuation of the derivative prices can be performed easily using a standard numerical integration quadrature or Fast Fourier transform algorithm. Our numerical experiments showed that accuracy of the Fast Fourier transform algorithm may deteriorate for short-maturity options. Also, volatility skew effects may be significant for short-maturity deep-out-of-the-money puts under the joint modeling of equity and credit risks. This may be attributed to the observation that the price of a short-term deep-out-of-the-money put is more sensitive to default risk. As future research works, one may consider pricing of credit-sensitive exotic equity derivatives, like the variance swap products with default cap feature.

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Time-to-Maturity = 0.25

Spot price	G-K quadrature	FFT	Monte Carlo	(standard deviation)
80	0.246	0.246	0.245	(0.002)
90	1.730	1.730	1.729	(0.005)
100	5.961	5.961	5.966	(0.009)
110	13.032	13.032	13.032	(0.012)
120	21.821	21.821	21.821	(0.015)

Time-to-Maturity = 0.5

Spot price	G-K quadrature	FFT	Monte Carlo	(standard deviation)
80	1.338	1.338	1.334	(0.005)
90	4.133	4.133	4.134	(0.009)
100	9.080	9.080	9.080	(0.014)
110	15.937	15.937	15.930	(0.018)
120	24.138	24.138	24.139	(0.022)

Time-to-Maturity = 1

Spot price	G-K quadrature	FFT	Monte Carlo	(standard deviation)
80	4.097	4.097	4.094	(0.011)
90	8.151	8.151	8.158	(0.016)
100	13.724	13.724	13.695	(0.021)
110	20.574	20.574	20.563	(0.027)
120	28.401	28.401	28.398	(0.032)

Time-to-Maturity = 2

Spot price	G-K quadrature	FFT	Monte Carlo	(standard deviation)
80	9.163	9.163	9.164	(0.022)
90	14.341	14.341	14.356	(0.028)
100	20.547	20.547	20.581	(0.035)
110	27.602	27.602	27.615	(0.041)
120	35.336	35.336	35.366	(0.047)

Table 1: Comparison of numerical estimates of European call option prices under the Merton jump model using various numerical approaches.

Time-to-Maturity = 0.25

Spot price	G-K quadrature	FFT	Monte Carlo	(standard deviation)
80	0.312	0.312	0.314	(0.002)
90	1.793	1.793	1.803	(0.005)
100	5.991	5.992	5.994	(0.009)
110	13.055	13.055	13.064	(0.013)
120	21.868	21.868	21.876	(0.016)

Time-to-Maturity = 0.5

Spot price	G-K quadrature	FFT	Monte Carlo	(standard deviation)
80	1.438	1.439	1.440	(0.006)
90	4.229	4.232	4.221	(0.010)
100	9.165	9.168	9.157	(0.014)
110	16.026	16.027	16.024	(0.018)
120	24.243	24.243	24.247	(0.022)

Time-to-Maturity = 1

Spot price	G-K quadrature	FFT	Monte Carlo	(standard deviation)
80	4.254	4.256	4.259	(0.012)
90	8.324	8.326	8.317	(0.017)
100	13.909	13.910	13.874	(0.022)
110	20.769	20.770	20.786	(0.028)
120	28.604	28.605	28.528	(0.032)

Time-to-Maturity = 2

Spot price	G-K quadrature	FFT	Monte Carlo	(standard deviation)
80	9.434	9.435	9.459	(0.023)
90	14.644	14.645	14.650	(0.029)
100	20.871	20.872	20.899	(0.036)
110	27.939	27.939	27.961	(0.042)
120	35.676	35.676	35.624	(0.049)

Table 2: Comparison of numerical estimates of European call option prices under the double exponential jump model using various approaches.

Time-to-Maturity = 0.25

N_{FFT} \ $\Delta\omega$	0.5	0.25	0.1	0.05
250	6.0635	6.1660	7.6496	9.0196
1000	5.9959	6.0053	6.0944	6.4075
4000	5.9912	5.9915	5.9956	6.0205
16000	5.9911	5.9912	5.9916	5.9929
G-K quadrature	5.9912			

Time-to-Maturity = 0.5

N_{FFT} \ $\Delta\omega$	0.5	0.25	0.1	0.05
250	9.2074	9.2693	10.1979	11.5057
1000	9.1677	9.1733	9.2256	9.4128
4000	9.1649	9.1651	9.1675	9.1821
16000	9.1648	9.1650	9.1652	9.1660
G-K quadrature	9.1650			

Time-to-Maturity = 1.0

N_{FFT} \ $\Delta\omega$	0.5	0.25	0.1	0.05
250	13.9319	13.9663	14.4919	15.4494
1000	13.9101	13.9132	13.9420	14.0454
4000	13.9086	13.9088	13.9101	13.9181
16000	13.9086	13.9087	13.9088	13.9093
G-K quadrature	13.9087			

Time-to-Maturity = 2.0

N_{FFT} \ $\Delta\omega$	0.5	0.25	0.1	0.05
250	20.8821	20.8986	21.1579	21.7084
1000	20.8717	20.8733	20.8871	20.9366
4000	20.8710	20.8711	20.8718	20.8756
16000	20.8710	20.8711	20.8711	20.8713
G-K quadrature	20.8711			

Table 3: Comparison of numerical estimates of European call option prices with double exponential jump using the FFT algorithm with varying values of N_{FFT} (number of grids) and $\Delta\omega$ (grid size). The numerical estimates of option prices obtained from the Gauss-Kronrod integration quadrature are used as benchmark values for comparison. The truncation in the Fourier domain is given by $\alpha = N_{FFT}\Delta\omega$. The corresponding spacing in the log-strike determined by the relation: $\Delta k \Delta\omega = 2\pi/N_{FFT}$.

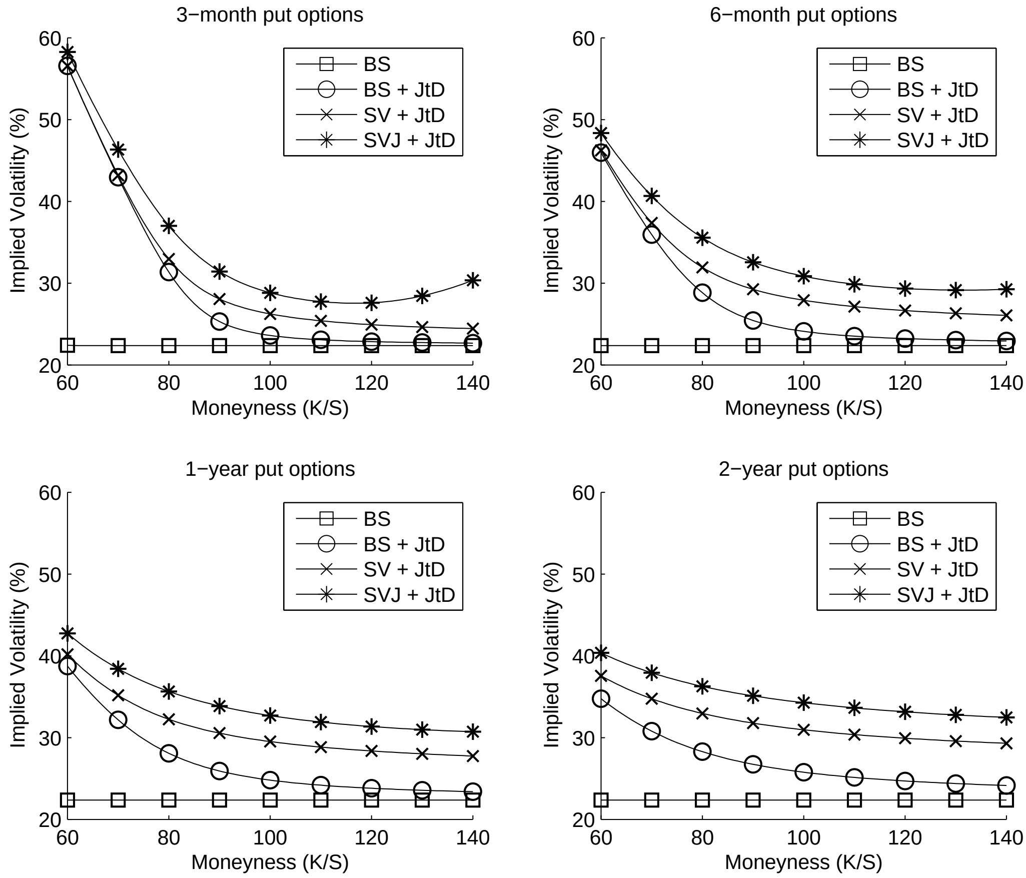

Figure 1: Implied volatility smile patterns of defaultable European put options under various jump-to-default models at varying maturities. The implied volatility values are obtained by inverting the Black-Scholes formula from the model option prices.