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# KNOCK-IN AMERICAN OPTIONS

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A knock-in American option under a trigger clause is an option contract in which the option holder receives an American option conditional on the underlying stock price breaching a certain trigger level (also called barrier level). We present analytic valuation formulas for knock-in American options under the Black-Scholes pricing framework. The price formulas possess different analytic representations, depending on the relation between the trigger stock price level and the critical stock price of the underlying American option. We also performed numerical valuation of several knock-in American options to illustrate the efficacy of the price formulas. © 2004 Wiley Periodicals, Inc. *Jrl Fut Mark* 24:179–192, 2004

## INTRODUCTION

The trigger clause in an option contract refers to the feature where the option underlying the contract is triggered to become alive or other embedded features in the contract become activated when certain preset trigger conditions are met. Trigger clauses are commonly found in

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derivative contracts. For example, the issuer of a convertible bond can activate the callable feature only when the underlying stock price exceeds the trigger level consecutively for a number of trading days. Also, a convertible bond may have its conversion price lowered when the stock price recorded on certain dates falls below some threshold level. As another example, we may have a bond whose interests are being accrued only when the dollar/Yen exchange rate stays outside a certain corridor range. In executive warrants issued by companies to their employees, it is common to have the reset feature where the strike price and/or the maturity date of the warrants can be altered, subject to certain preset trigger conditions on the movement of the price of the company stock.

Knock-in options with a trigger clause are closely related to barrier options. Barrier options are common path-dependent options traded in the financial markets. The derivation of the price formula for barrier options was pioneered by Merton (1973) in his seminal paper on option pricing. A list of price formulas for one-asset barrier options and multi-asset barrier options can be found in the articles by Rich (1994) and Wong and Kwok (2003), respectively. Gao et al. (2000) analyzed option contracts with both knock-out barrier and American early exercise features. In this article, we consider knock-in American options that are triggered into existence (knock-in) only when the underlying stock price falls below a certain preset barrier (or threshold) level. Let  $S$  denote the stock price and  $H$  be the barrier level. The holder of the contract is entitled to receive an American option with strike price  $X$  and maturity date  $T$  when  $S$  falls below  $H$  during the life of the option; otherwise, the option contract expires worthless on the maturity date  $T$ . When the underlying knock-in option is a European option, there exists a simple valuation formula where the price of a knock-in European option is given by the difference of the prices of the European vanilla option and the knock-out European barrier option. Unfortunately, such a valuation approach does not apply when the knock-in option is an American option. Haug (2001) presented analytic valuation formulas for knock-in American options. However, his formulas are valid only under the condition  $H \leq X$  (such a restriction has not been explicitly stated in his article). He has neglected the possible interaction of the knock-in region and the exercise region of the underlying American option. Here, we would like to present the analytic valuation formulas for knock-in American options that are applicable under all possible cases.

This article is organized as follows. In the next section, we present the derivation of the analytic valuation formulas using the Black-Scholes pricing framework for knock-in American options under a trigger clause. The valuation formulas take different analytic forms, depending on the relation between the trigger level  $H$  and the critical stock price at which the American option should be optimally exercised. The different analytic forms reflect the various possibilities of interaction of the knock-in region of the option contract and the underlying exercise region of the American option. We then present numerical results that demonstrate the efficacy of the valuation formulas. Some comments on the implementation of the numerical calculations are given. The article ends with conclusive remarks in the last section.

## DERIVATION OF VALUATION FORMULAS

We consider the valuation of knock-in American call options under the Black-Scholes pricing framework. The stock price  $S$  is assumed to follow the risk neutral process

$$\frac{dS}{S} = (r - q)dt + \sigma dZ \quad (1)$$

where  $r$  and  $q$  are the constant risk-free interest rate and dividend yield, respectively,  $\sigma$  is the volatility, and  $dZ$  is a standard Wiener process. Let  $t$  denote the current time,  $T$  be the maturity date of the knock-in American call option, and write  $\tau = T - t$  as the time to expiry. We assume that the down-in trigger clause entitles the holder to receive an American call option, with maturity date  $T$  and strike price  $X$  when  $S$  falls below the threshold level  $H$ . We let  $C_{di}(S, \tau; X, H)$  denote the price of the down-and-knock-in American call option and  $C(S, \tau; X)$  denote the price of the underlying American option received upon knock-in. The governing equation for  $C_{di}(S, \tau; X, H)$  is given by the usual Black-Scholes equation

$$\frac{\partial C_{di}}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 C_{di}}{\partial S^2} - (r - q)S \frac{\partial C_{di}}{\partial S} + rC_{di} = 0$$

for  $S > H$  and  $\tau > 0$  (2)

with left fixed boundary  $H$ . The auxiliary conditions are

$$C_{di}(S, 0) = 0 \quad S > H \quad \text{and} \quad C_{di}(H, \tau) = C(H, \tau; X) \quad (3)$$

where the knock-in American option is modeled as the rebate payment when  $S = H$ . On the other hand, the governing equation for  $C(S, \tau; X)$  is given by

$$\frac{\partial C}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} - (r - q)S \frac{\partial C}{\partial S} + rC = 0$$

for  $S < S^*(\tau)$  and  $\tau > 0$  (4)

where the free boundary  $S^*(\tau)$  is the critical stock price at which the American option should be exercised optimally. The associated auxiliary conditions are

$$C(S^*(\tau), \tau) = S^*(\tau) - X, \quad \frac{\partial C}{\partial S}(S^*(\tau), \tau) = 1 \quad \text{and}$$

$$C(S, 0) = (S - X)^+ \tag{5}$$

It is known that  $S^*(\tau)$  is monotonically increasing with respect to  $\tau$  with

$$S^*(0^+) = \max\left(\frac{r}{q}X, X\right) \quad \text{and} \quad S^*(\infty) = \frac{\mu_+}{\mu_+ - 1}X$$

where

$$\mu_+ = \frac{-(r - q - \frac{\sigma^2}{2}) + \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} \tag{6}$$

(Kwok, 1998). The solution to Equations (2) and (3) can be formally represented by

$$C_{di}(S, \tau; X, H) = \int_0^\tau e^{-r\xi} C(H, \tau - \xi; X) Q(\xi; S) d\xi \tag{7a}$$

where  $\xi$  is the time lapsed from the current time and

$$Q(\xi; S) = \frac{\ln \frac{S}{H}}{\sqrt{2\pi\sigma}} \frac{\exp(-[\ln \frac{S}{H} + (r - q - \frac{\sigma^2}{2})\xi]^2 / (2\sigma^2\xi))}{\xi^{3/2}} \tag{7b}$$

is the density function of the first passage time that the stock price moves from  $S$  to the barrier level  $H$ . Unfortunately, the direct analytic evaluation of the integral is, in general, formidably tedious.

Haug (2001) postulated that  $C_{di}(S, \tau; X, H)$  and  $C(S, \tau; X)$  are related by

$$C_{di}(S, \tau; X, H) = \left(\frac{S}{H}\right)^{1 - [2(r-q)/\sigma^2]} C\left(\frac{H^2}{S}, \tau; H\right) \tag{8}$$

by virtue of the reflection principle. However, for knock-in American call options, the above formula is valid only for  $H \leq X$ . Due to the possible interaction of the knock-in region:  $S \leq H$  and the exercise region:  $S \geq S^*(\tau)$ , the price formula for  $C_{di}(S, \tau; H, X)$  takes different analytic forms under the following cases

- (i)  $H \leq S^*(0^+)$ ,
- (ii)  $H \geq S^*(\infty)$  and
- (iii)  $S^*(0^+) < H < S^*(\infty)$

First, we consider the case

$$S^*(0^+) = H \leq S^*(0^+) \max\left(X, \frac{r}{q}X\right)$$

This corresponds to the scenario where the knock-in region lies completely inside the continuation region of the American option. When

$$S > H, \text{ we have } \frac{H^2}{S} < H < \max\left(X, \frac{r}{q}X\right)$$

so that the point

$$\left(\frac{H^2}{S}, \tau\right)$$

in the  $S - \tau$  plane lies in the continuation region of the American option. Let  $V(S, \tau)$  be defined by

$$V(S, \tau) = \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} C\left(\frac{H^2}{S}, \tau; X\right) \quad \text{for } S > H \quad (9)$$

it can be shown that  $V(S, \tau)$  satisfies the Black-Scholes equation. In addition, we observe

$$V(H, \tau) = C(H, \tau) \quad \text{and} \quad V(S, 0) = \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} \left(\frac{H^2}{S} - X\right)^+ \quad (10)$$

Both  $V(S, \tau)$  and  $C_{di}(S, \tau; X, H)$  share the same auxiliary condition along  $S = H$ , and they both satisfy the Black-Scholes equation for  $S > H$  and  $\tau > 0$ . Suppose we define  $W(S, \tau)$  where

$$W(S, \tau) = V(S, \tau) - C_{di}(S, \tau; X, H) \quad (11)$$

then  $W(S, \tau)$  satisfies the Black-Scholes equation and observes homogeneous boundary condition along  $S = H$ . The initial condition for  $W(S, \tau)$  is given by

$$W(S, 0) = \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} \left(\frac{H^2}{S} - X\right)^+ \quad \text{for } S > H \quad (12)$$

Let  $c(S, \tau; X)$  denote the price function of the vanilla European call option counterpart. The above initial condition  $W(S, 0)$  matches with

$$\left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} c(S, 0; X)$$

Let  $c_{di}(S, \tau; X, H)$  denote the price function of the European barrier call option with down-and-in barrier  $H$  and strike price  $X$ . The sum of  $W(S, \tau)$  and  $c_{di}(S, \tau; X, H)$  is equal to

$$\left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} c\left(\frac{H^2}{S}, \tau; X\right)$$

so that  $W(S, \tau)$  can be expressed as the difference of price functions of European vanilla and barrier options. Indeed, we have

$$W(S, \tau) = \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} c\left(\frac{H^2}{S}, \tau; X\right) - c_{di}(S, \tau; X, H) \quad S > H, \tau > 0 \quad (13)$$

One can easily check that the above solution to  $W(S, \tau)$  satisfies the Black-Scholes equation, together with homogeneous boundary conditions and initial conditions as specified in Equation (12). Combining the results, we then have

$$\begin{aligned} & C_{di}(S, \tau; X, H) \\ &= \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} \left[ C\left(\frac{H^2}{S}, \tau; X\right) - c\left(\frac{H^2}{S}, \tau; X\right) \right] + c_{di}(S, \tau; X, H) \end{aligned} \quad (14)$$

which is valid for

$$H \leq \max\left(X, \frac{r}{q}X\right)$$

One observes that the price of a knock-in American option can be decomposed into the prices of options of simpler form. The first term in the above price formula represents the early exercise premium associated

with the knock-in American call option, which is obtained by applying the reflection principle to the early exercise premium of the usual American call option. In particular, when  $H \leq X$ , we observe

$$c_{di}(S, \tau; X, H) = \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} c\left(\frac{H^2}{S}, \tau; X\right)$$

so that the price formula (14) reduces to the simpler form as given by Haug [see Equation (8)].

Second, we consider the case  $H \geq S^*(\infty)$ , that is, the trigger level  $S = H$  lies completely inside the exercise region of the American option. Upon the receipt of the American option when the trigger level  $S = H$  is reached, the American option should be exercised at once. Hence, the price formula as depicted in Equation (7a) can be simplified to become

$$\begin{aligned} C_{di}(S, \tau; H, X) &= \int_0^\tau e^{-r\xi}(H - X)Q(\xi; S) d\xi \\ &= (H - X) \left[ \left(\frac{S}{H}\right)^{\alpha-\mu} N(e_1) + \left(\frac{S}{H}\right)^{\alpha+\mu} N(e_2) \right] \end{aligned} \quad (15a)$$

where

$$\mu = \frac{\sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2r\sigma^2}}{\sigma^2}, \quad \alpha = \frac{1}{2} - \frac{r - q}{\sigma^2} \quad (15b)$$

$$e_1 = \frac{\ln \frac{H}{S} + \mu\sigma^2\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad e_2 = \frac{\ln \frac{H}{S} - \mu\sigma^2\tau}{\sigma\sqrt{\tau}} \quad (15c)$$

Last, we consider the case  $S^*(0^+) < H < S^*(\infty)$ , corresponding to the scenario where the knock-in region  $S \leq H$  is partly inside and partly outside the continuation region of the American option (see Figure 1). Let  $\tau_H$  be the solution to the algebraic equation  $S^*(\tau) = H$ . For  $\tau \leq \tau_H$ , the American option received upon reaching the trigger level should be exercised at once. This is because for  $\tau \leq \tau_H$ , we have  $H \geq S^*(\tau)$  so that the point  $(H, \tau)$  in the  $S - \tau$  plane lies inside the exercise region. Similar to the first case, we define the same set of functions

$$V(S, \tau) = \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} C\left(\frac{H^2}{S}, \tau; X\right) \quad \text{for } S > H \quad (16a)$$

and

$$W(S, \tau) = V(S, \tau) - C_{di}(S, \tau; X, H) \quad \text{for } S > H \quad (16b)$$

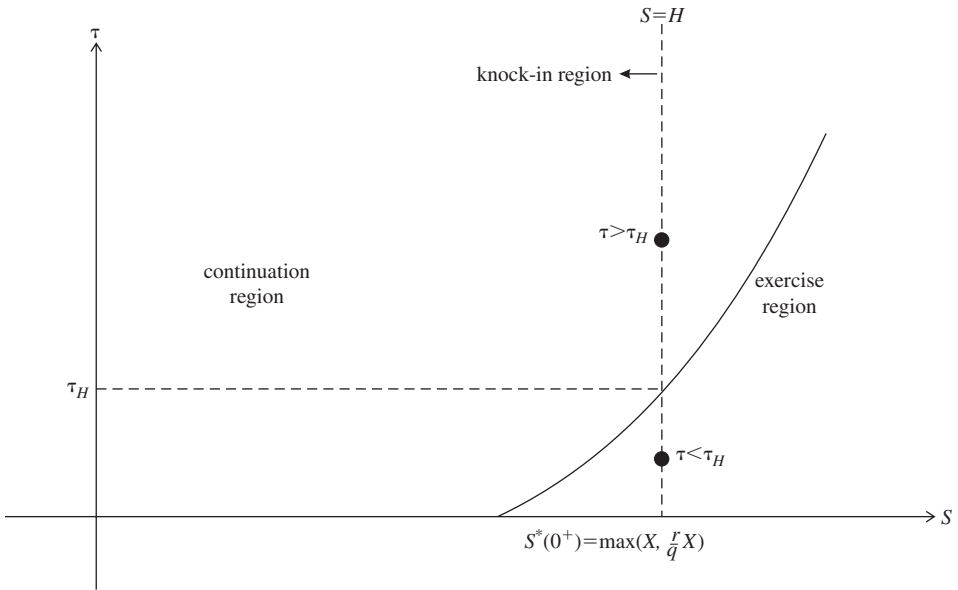


FIGURE 1

When the trigger level  $H$  satisfies  $S(0^+) < H < S^*(\infty)$ , there exists unique value  $\tau_H$  such that  $S^*(\tau_H) = H$ . When  $\tau < \tau_H$ , the American option received upon knock-in should be exercised at once, because  $(H, \tau)$  lies inside the exercise region.

For  $S > H$  and  $\tau > \tau_H$ , the point

$$\left(\frac{H^2}{S}, \tau\right)$$

lies inside the continuation region of the American option. Hence, both  $V(S, \tau)$  and  $W(S, \tau)$  satisfy the Black-Scholes equation, when  $S > H$  and  $\tau > \tau_H$ . Along the barrier  $S = H$ ,  $W(S, \tau)$  observes the boundary condition  $W(H, \tau) = 0, \tau > \tau_H$ . Over the time interval  $\tau \leq \tau_H$ ,  $C_{di}(S, \tau; X, H)$  is given by formula (15a) because  $(S, \tau)$  lies in the exercise region of the American option for  $S > H$  and  $\tau \leq \tau_H$ . In particular, at  $\tau = \tau_H$

$$W(S, \tau_H) = \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} C\left(\frac{H^2}{S}, \tau_H; X\right) - \int_0^{\tau_H} e^{-r\xi} (H - X) Q(\xi; S) d\xi \tag{17}$$

Note that the solution to  $W(S, \tau_H)$  in Equation (17) differs from the earlier formula for  $W(S, \tau)$  in Equation (13) evaluated at  $\tau_H$ . The difference represents the premium associated with the early exercise right of the transformed American price function  $W(S, \tau)$  over the period  $\tau < \tau_H$ . To solve for  $C_{di}(S, \tau; X, H)$  when  $\tau > \tau_H$ , one has to solve for  $W(S, \tau)$



with  $\tau > \tau_H$  based on the “terminal” payoff prescribed at  $\tau = \tau_H$ . The function  $W(S, \tau)$  with  $\tau > \tau_H$  essentially gives the price function of a European down-and-out barrier option with knock-out barrier  $H$  and “terminal” payoff function at  $\tau = \tau_H$  [given by Equation (17)]. Once  $W(S, \tau)$  for  $\tau > \tau_H$  is obtained,  $C_{di}(S, \tau; X, H)$  for  $\tau > \tau_H$  is then given by the difference of  $V(S, \tau)$  and  $W(S, \tau)$  [see Equation (16a)].

## IN-OUT BARRIER PARITY RELATION

For European barrier options, the sum of the prices of down-and-in barrier option and down-and-out barrier option is equal to the price of the European vanilla option. However, such in-out barrier parity relation is not observed for American barrier options. Suppose we let  $C_{do}(S, \tau; X, H)$  denote the American down-and-out barrier call option and write

$$U(S, \tau; X, H) \text{ as the sum of } C_{di}(S, \tau; X, H) \text{ and } C_{do}(S, \tau; X, H)$$

The sum function  $U(S, \tau)$  satisfies the following linear complementarity formulation

$$\begin{aligned} \frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} - (r - q)S \frac{\partial U}{\partial S} + rU &\geq 0, \quad U \geq C_{di}(S, \tau) + (S - X)^+, \\ \left[ \frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} - (r - q)S \frac{\partial U}{\partial S} + rU \right] \{U - [C_{di}(S, \tau) + (S - X)^+]\} &= 0 \\ &\text{for } S > H \\ U(H, \tau) &= C(H, \tau) \\ U(S, 0) &= (S - X)^+ \quad \text{for } S > H \end{aligned} \quad (18)$$

The obstacle function for  $U(S, \tau)$  is  $C_{di}(S, \tau) + (S - X)^+$ , which is always greater than the obstacle function  $(S - X)^+$  for  $C(S, \tau)$ . Because both  $U(S, \tau)$  and  $C(S, \tau)$  share the same boundary and initial conditions, so  $U(S, \tau)$  is guaranteed to be greater than  $C(S, \tau)$ .

The financial intuition of the above result is quite obvious. It suffices to show that a portfolio consisting of an American down-and-in call  $C_{di}(S, \tau; X, H)$  and an American down-and-out call  $C_{do}(S, \tau; X, H)$  always dominates the American nonbarrier call. Suppose the holder of the portfolio follows an exercise policy for the down-and-out call identical to that of the nonbarrier call (though this is suboptimal for the down-and-out call). The exercise payoff of the portfolio is always higher than that of the American nonbarrier call because the portfolio has the extra down-and-in call. During the life of the options, when  $S$  hits the barrier  $H$ , the portfolio becomes the nonbarrier call because one call is knocked out and

the other call is knocked in. At expiry, both the portfolio and the nonbarrier call then have the same value. In all scenarios, the portfolio is at least worth as much as the nonbarrier call; hence, the result.

### AMERICAN UP-AND-IN PUTS

One may apply the above analytic procedures to derive the price formulas for American up-and-in puts. For reference, we quote the price formula for an American up-and-in put corresponding to

$$H \geq \min\left(X, \frac{r}{q}X\right)$$

which has close analogy to the price formula in Equation (14). Let  $p(S, \tau; X)$  and  $P(S, \tau; X)$  denote the price function of a European vanilla put and its American counterpart, respectively, and  $p_{ui}(S, \tau; X, H)$  and  $P_{ui}(S, \tau; X, H)$  denote the price function of a European up-and-in put and its American counterpart, respectively. When

$$H \geq \min\left(X, \frac{r}{q}X\right)$$

we have

$$\begin{aligned} & P_{ui}(S, \tau; X, H) \\ &= \left(\frac{S}{H}\right)^{1-[2(r-q)/\sigma^2]} \left[ P\left(\frac{H^2}{S}, \tau; X\right) - p\left(\frac{H^2}{S}, \tau; X\right) \right] + p_{ui}(S, \tau; X, H) \quad (19) \end{aligned}$$

### NUMERICAL RESULTS

We performed numerical experiments to verify the validity of the analytic price formulas derived in the last section. The price formulas contain the price function of the nonbarrier American option function  $C(S, \tau; X)$ , which has no explicit closed form analytic formula. In the literature, there exists a wide variety of numerical methods and analytic approximation methods for the numerical valuation of  $C(S, \tau; X)$ . It is well known that explicit numerical schemes, like the binomial method, commonly suffer from degradation of accuracy in pricing barrier options when rebates are incorporated into the pricing algorithm through numerical boundary condition (Kwok & Lau, 2001). This is because the numerical rebate value takes finite time to propagate into the interior of the computational domain. In Figure 2, we illustrate the comparison of accuracy of computing a knock-in American call option using two different methods (2) the full binomial method with the American option price function as rebate (1) the use of price formula (14) where

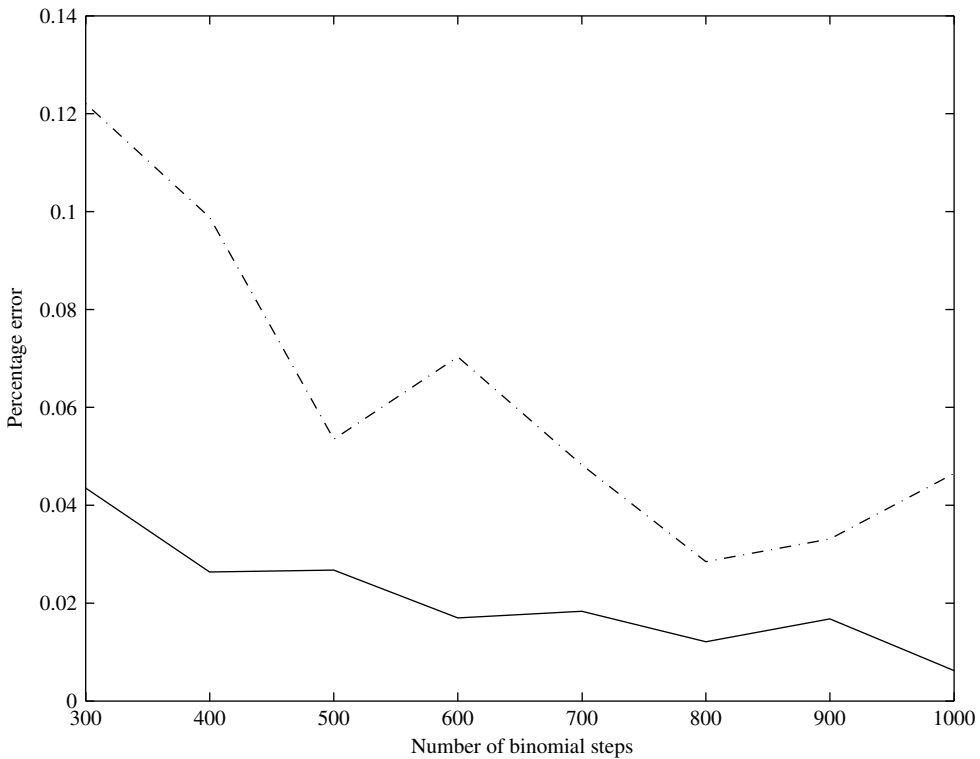


FIGURE 2

The two curves show the plot of percentage error against the number of binomial steps used in computing a knock-in American call option using the full binomial method with American option value as rebate (shown in dashed line) and price formula (14) where  $C(S, \tau; X)$  is obtained from binomial calculations (shown in solid line). The binomial calculations based on price formula (14) are shown to be more accurate and exhibiting less erratic behaviors.

$C(S, \tau; X)$  is obtained by the binomial method. The parameter values used in the calculations are:  $X = 100$ ,  $r = 0.1$ ,  $q = 0.09$ ,  $\tau = 1.0$ ,  $\sigma = 0.3$ ,  $H = 110$ ,  $S = 110.5$ . The limiting values of the critical stock price of the American call option are found to be  $S^*(0^+) = 111.11$  and  $S^*(\infty) = 162.09$ , so that the trigger level  $H$  observes

$$X < H < \frac{r}{q}X$$

The “exact” option value is obtained by choosing 10,000 time steps in the full binomial scheme. We plot the percentage error of the numerical results against the number of binomial steps used (see Figure 2). The percentage error using the full binomial method (shown in dashed line) is invariably greater than that obtained using price formula (14) (shown in solid line). Also, the convergence behaviors of the full binomial method are shown to be more erratic.

**TABLE I**  
 The Entries in the Table Show the Comparison of the Numerical Accuracy of Computation of Knock-in American Call Option Value Under Different Scenarios of Trigger Level  $H$  and Varying Stock Price Level  $S$

Trigger Level	$(H, S)$	Haug's Formula	Formula (14)	Formula (15a-c)	"Exact" Solution
$H < X$	(99, 99.5)	10.7434	10.7434	—	10.7432
	(99, 110.5)	6.8222	6.8222	—	6.8224
$X < H < \frac{r}{q}X$	(110, 110.5)	17.2081	17.2063	—	17.2062
	(110, 120.5)	12.5767	12.5411	—	12.5409
	(110, 140.5)	6.4314	6.3551	—	6.3553
	(110, 160.5)	3.1450	3.0664	—	3.0667
$\frac{r}{q}X < H < S^*(\infty)$	(130, 130.5)	32.1595	32.1286	—	32.1285
	(130, 140.5)	26.2911	25.6663	—	25.6659
	(130, 150.5)	21.2959	20.1786	—	20.1773
$H > S^*(\infty)$	(170, 170.5)	69.6604	—	69.4759	69.4759
	(170, 180.5)	62.9935	—	59.3868	59.3874

Table I provides more details about our numerical experiments that were performed to verify various price formulas of knock-in American call options under different scenarios of trigger level  $H$  and varying stock price level  $S$ . The parameter values used in the knock-in American call option model are  $X = 100$ ,  $r = 0.1$ ,  $q = 0.09$ ,  $\tau = 1.0$  and  $\sigma = 0.3$ . Correspondingly, we have  $S^*(0^+) = 111.11$  and  $S^*(\infty) = 162.09$ . To obtain the "exact" solution to the option value for a given set of  $H$  and  $S$  values (see the last column), we performed calculations with 10,000 time steps using the binomial scheme. The nonbarrier American option values in the price formulas are obtained using the binomial scheme with the same number of time steps. The entries in the third column reveal the limitation of Haug's formula. His formula provides accurate results only for  $H < X$ . When

$$X < H < \frac{r}{q}X$$

the entries in the fourth column show that our price formula (14) gives very accurate results. Similarly, for  $H > S^*(\infty)$ , price formula (15a-c) also gives superb accuracy (see the fifth column). We also examined whether price formula (14) can serve as an approximation formula when

$$\frac{r}{q}X < H < S^*(\infty)$$

Our numerical results (last three entries in the fourth column) show that price formula (14) indeed can provide reasonably accurate option values

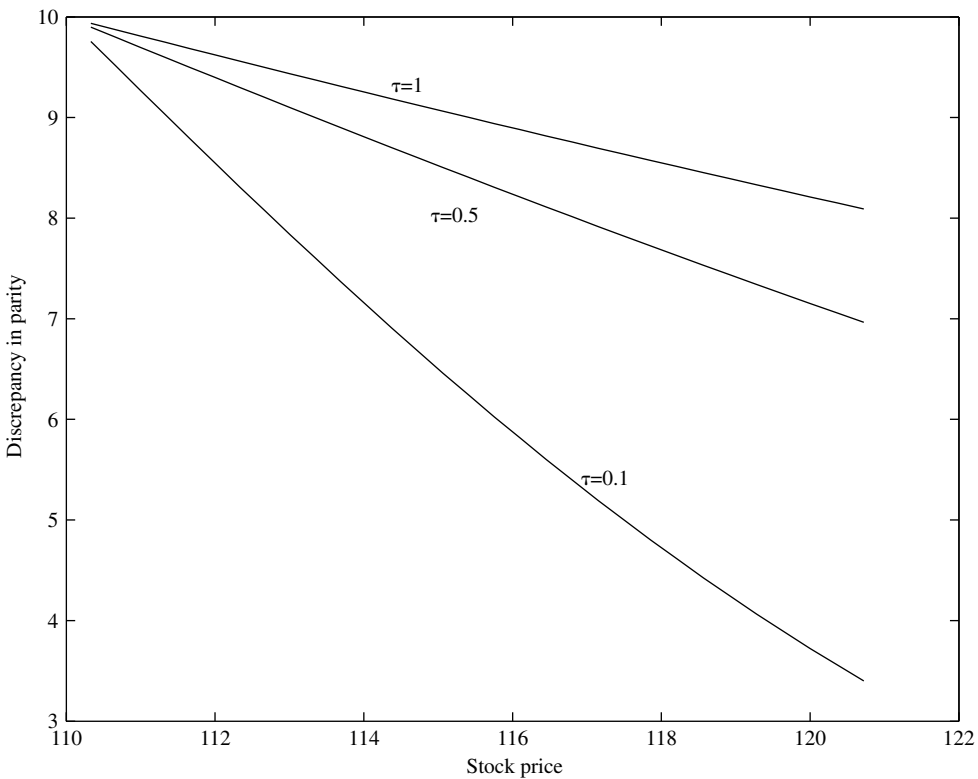
under this scenario. This is attributed to the small difference between  $W(S, \tau_H)$  in Equation (17) and  $W(S, \tau)$  in Equation (13) when both are evaluated at  $\tau_H$ .

Last, we checked the violation of the in-out barrier parity relation by computing the difference

$$C_{di}(S, \tau; X, H) + C_{do}(S, \tau; X, H) - C(S, \tau; X, H)$$

for varying time to expiry  $\tau$  and stock price  $S$ . The above difference is used as a measure of discrepancy in the parity relation. Here, we chose  $H = 110$  and other parameter values for the knock-in American call option were taken to be the same as in previous calculations. The numerical results are obtained using 10,000 time steps in the binomial calculations.

In Figure 3, we plot the discrepancy in parity against stock price  $S$  for varying time to expiry  $\tau$ . The level of discrepancy always stays positive, and it decreases with increasing stock price and decreasing time to expiry.



**FIGURE 3**

The difference  $C_{di}(S, \tau; X, H) + C_{do}(S, \tau; X, H) - C(S, \tau; X)$  is taken as a measure of the discrepancy in the in-out barrier parity relation. The discrepancy always stays positive, and it decreases with increasing stock price and decreasing time to expiry.

## CONCLUSION

We have presented the analytic price formulas for knock-in American options under the Black-Scholes pricing framework. Because the knock-in region and the exercise region of the underlying American option may intersect with each other, the price formulas take different analytic forms depending on the interaction between the knock-in region of the down-in feature of the option contract and the exercise region of the underlying American option. The price function of a knock-in American option can be expressed in terms of the price functions of simple barrier options and American options. Such decomposition facilitates the numerical valuation of knock-in American options. We also showed that the sum of the prices of knock-in and knock-out American options is always greater than the price of the nonbarrier American counterpart. In future work, we may consider the impact of other types of trigger, for example, the holder receives the underlying American option only when the moving average of the stock price over a fixed period falls below some threshold level. Our results and valuation approach may shed light on the analysis of the trigger clauses in other derivative contracts, like the Parisian trigger requirement on the callable feature in convertible bonds.

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