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OPTIONS WITH MULTIPLE RESET RIGHTS

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The reset right embedded in a derivative refers to the feature that the holder can alter certain terms in the derivative contract according to some preset rules. In this paper, we consider options that allow the holder to reset the strike price with preset number of times at any moment during the life of the option. The determination of the optimal reset policies adopted by the holder leads to a free boundary value problem. We explore how the critical asset value at which the holder should exercise the reset right optimally depends on the number of reset rights remaining, the relative magnitude of the riskless interest rate and dividend yield, the original strike price set at initiation, etc. In particular, we examine the asymptotic behaviors of the optimal reset policies at infinite time to expiry and the existence of threshold time earlier than which the holder should never shout.

Keywords: Reset options; shout feature; optimal reset policies; free boundary value problems.

1. Introduction

The reset right embedded in a derivative instrument refers to the feature that the holder can alter certain terms of the derivative contract according to some pre-set rules or conditions. In option contracts, the terms that can be reset include the strike price, maturity date, etc. The time to reset may be chosen optimally by the holder at any moment (the exercise of reset is normally called shouting). Alternatively, the reset may be automatic upon the satisfaction of certain preset conditions (see the examples analyzed by Gray and Whaley [4]). For example, in the reset strike put option, the strike price is reset to the prevailing stock price upon shouting. The earliest type of option with the reset right is the shout call option, which has the embedded feature that allows its holder to lock in the profit via shouting [5]. The terminal payoff of a shout call option is $\max(S_T - X, S_t - X, 0)$, where X is the strike price, S_T is the terminal stock price and S_t is the prevailing stock price at the shouting instant t. Reset features are also common in bonds and funds. For example, the Canadian segregated funds contain multiple embedded reset rights that allow the holder to reset the guarantee level and the maturity date during the life of the contract [6]. In general, the embedded reset feature is attractive to investors who would like to lock in the gain at the prevailing state and seek potential higher gain.

It is relatively straightforward to price derivatives with the automatic reset feature. Dependent on whether the reset conditions are satisfied or not, the payoff on the reset date is the maximum of the reset payoff and the original payoff. However, the pricing of a derivative with voluntary reset right leads to an optimal stopping problem. In the valuation procedure, it is necessary to investigate the optimal shouting policy, that is, to determine the critical asset price at which it is optimal to shout.

Unlike the early exercise feature in American options where the holder can exercise only once, it is common to allow multiple reset opportunites in reset options. In this paper, we consider the optimal shouting policies of a put option with multiple rights to reset the strike price. Whenever the holder shouts, the strike price of the put is reset to the prevailing stock price. The number of shouts commenced throughout the life of the option is more than once but with an upper limit. The *shout floor* corresponds to the special type of reset put option where the strike price is not set at initiation, rather it is put in place at the first shout [1]. This paper is an extension of an earlier work by the authors [2] that deals with reset put option with single shouting right. As an analogy, this is similar to the extension of pricing of vanilla options to compound European options. We consider the characterization of the optimal shouting boundary, in particular, the monotonicity properties of the price functions and critical asset prices with respect to the number of outstanding reset rights remaining. Our results confirm with the financial intuition that the holder exercises the reset rights at a lower critical asset price (the put is less out-of-the-money) when there are more reset rights outstanding and less time remaining before expiration. Similar to the one-reset case, we show that the relative magnitude of the riskless interest rate r and dividend yield q plays a crucial role in the optimal shouting policy. When $r > q$, it is never optimal to shout when the remaining life of the option is longer than some threshold length of time, and this threshold length increases with more reset rights outstanding. On the other hand, when $r < q$, it is always optimal to shout when the asset price reaches some critical asset value, and this critical value decreases with more reset rights outstanding.

The paper is organized as follows. In the next section, we present the linear complementarity formulation of the free boundary value problem associated with the pricing of the n-reset put option. We also develop the put-call parity relation between the reset put and the shout call. We show that the lookback options are related to the limiting cases of reset put options with infinite number of shouting rights. In Sec. 3, the characterization of the optimal reset policies of the n-reset put options is examined. We start with the exploration of the optimal shouting policies of the reset put options through numerical experiments by computing their option value and shouting boundary under different set of parameter values. We then illustrate how the optimal shouting policies would depend on the monotonicity properties of the price functions of the at-the-money reset put options. We examine and postulate some analytic properties of the optimal shouting boundary of the reset put options. For the n-reset shout floor, we manage to obtain an analytic representation of the price formula and deduce the corresponding optimal reset policies. The paper ends with summaries and conclusive remarks in the last section.

2. Pricing Formulation of Reset Put Options

We consider the pricing models of put options with multiple rights to reset the strike price throughout the life of the option contract. Let n denote the maximum number of resets allowed for the holder and X be the strike price set at initiation. Let t_j denote the time of the jth reset to be chosen optimally by the holder of a n-reset put option, and S_j^* denote the critical asset value at the reset instant t_j. Since the new reset strike price should be higher than the previous strike price, we should have $S_j^* > S_{j-1}^*$ and $S_j^* > X$ are observed in all resets. The terminal payoff of the n-reset put option is given by $\max(S_\ell^* - S_T, 0)$, where S_ℓ^* is the strike set in the last reset by the holder.

We adopt the usual Black-Scholes pricing framework in our model, where the asset price S under the risk neutral measure is assumed to follow the lognormal diffusion process

$$\frac{dS}{S} = (r - q)dt + \sigma dZ\,, \tag{1}$$

where r and q are the constant riskless interest rate and dividend yield, σ is the volatility and dZ is the standard Wiener process. Let $V_n(S, \tau; X)$ denote the value of the n-reset put option with time to expiry τ. Let T denote the expiration date of the option and t be the current time so that time to expiry $\tau = T - t$. Upon the jth reset, the reset put becomes an at-the-money $(j-1)$-reset put where the strike price equals the prevailing asset price at the reset instant. The strike price of the reset put with j reset rights remaining will be S_{j+1}^*; and for notational convenience, we write $S_{n+1}^* = X$. The voluntary reset right leads to the following optimal stopping problem where

$$V_j(S, \tau; S_{j+1}^*) = \sup_{t_j \in \mathcal{T}_{t,T}^{(j)}} E^*[e^{-r(t_j - t)} V_{j-1}(S_{t_j}, T - t_j; S_{t_j}) | S_t = S]\,,$$

$$j = n, n-1, \ldots, 1\,, \tag{2}$$

where E^* denotes the expectation under the risk neutral measure, and $\mathcal{T}_{t,T}^{(j)}$ is the set of stopping times between t and T associated with the jth reset right, $j = 1, \ldots, n$.

In particular, at the commencement of the last reset right, the option reduces to an at-the-money vanilla put option.

It can be shown that V_j observes linear homogeneity in S and X so that $V_j(S,\tau;X) = XV_j(S/X,\tau;1)$. When the reset put is at-the-money, that is, $S/X = 1$, we have $V_j(S,\tau;S) = SV_j(1,\tau;1)$. We define $P_j(\tau) = V_{j-1}(1,\tau;1), j = 1,2,\ldots,n$ so that the value of the reset put option at the instant of the jth reset is equal to $SP_j(\tau)$.

2.1. *Linear complementarity formulation*

Let $S_n^*(\tau)$ denote the critical asset price at which it is optimal for the holder to shout the reset put with n reset rights outstanding. The stopping region and continuation region correspond to $S \geq S_n^*(\tau)$ and $S < S_n^*(\tau)$, respectively. In the stopping region, $V_n(S,\tau) = SP_n(\tau)$; while in the continuation region, $V_n(S,\tau)$ satisfies the Black-Scholes equation. The linear complementarity formulation of the free boundary value problem associated with the n-reset put option can be expressed as

$$\begin{aligned}
&\frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V_n}{\partial S^2} - (r-q)S\frac{\partial V_n}{\partial S} + rV_n \geq 0\,, \quad V_n(S,\tau) \geq SP_n(\tau)\,,\\
&\left[\frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V_n}{\partial S^2} - (r-q)S\frac{\partial V_n}{\partial S} + rV_n\right][V_n - SP_n(\tau)] = 0\,,\\
&V_n(S,0) = \max(X-S,0)\,.
\end{aligned} \tag{3}$$

For the n-reset shout floor, the strike price is not prescribed at initiation, but rather being set at the first reset moment. This corresponds to the choice of zero value for the strike price X.

An alternative representation of the governing equation for $V_n(S,\tau)$, valid for $S \in (0,\infty)$ and $\tau \in (0,T]$, is given by

$$\frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V_n}{\partial S^2} - (r-q)S\frac{\partial V_n}{\partial S} + rV_n = \begin{cases} 0\,, & 0<S<S_n^*(\tau)\,,\\ Se^{-q\tau}\frac{d}{d\tau}[e^{q\tau}P_n(\tau)]\,, & S \geq S_n^*(\tau)\,, \end{cases} \tag{4}$$

where the non-homogeneous term in the stopping region is obtained by substituting $V_n(S,\tau) = SP_n(\tau)$ into the Black-Scholes equation. The non-homogeneous term can be interpreted as the rate of delayed compensation required to compensate the holder for not shouting in the stopping region, and it has the following financial interpretation. The reset payoff at the shout moment u is $P_n(u)$ units of asset whose price is $S_u, t \leq u \leq T$, and its no-arbitrage value at time t is $S_te^{q(u-t)}P_n(u)$. Suppose the holder delays his shouting by Δu time periods, he should be compensated with cash amount $S_t[e^{q(u-t)}P_n(u) - e^{q(u+\Delta u-t)}P_n(u+\Delta u)]$. The rate of delayed compensation is given by

$$-\frac{d}{du}[S_te^{q(u-t)}P_n(u)] = -S_te^{-qt}\frac{d}{du}[e^{qu}P_n(u)] = -e^{-qu}S_u\frac{d}{du}[e^{qu}P_n(u)]\,. \tag{5}$$

We then deduce that a necessary condition for optimal shouting is given by the positivity of the quantity $\frac{d}{d\tau}[e^{q\tau}P_n(\tau)]$.

2.2. *Put-call parity relation between reset put and shout call*

The most original class of options with the reset feature resemble the ladder options, which were first called "shout options" by Thomas [5]. Consider the shout option with the call payoff, its terminal payoff is given by $\max(S_T - X, S_u - X)$ if shouting has occurred at time u, and stays at $\max(S_T - X, 0)$ if no shout has commenced throughout the option's life. Here, the "effective ladder" in a shout call option is S_u, but the ladder is chosen by the holder.

Suppose there are n shouting rights allowed in a shout call, and let S_ℓ^* denote the asset price at the last shouting chosen by the holder, $0 \leq \ell \leq n$. The terminal payoff of the n-shout call option will be $\max(S_T - X, S_\ell^* - X)$. We write $U_n(S,\tau)$ as the price of the n-shout call option. Consider the portfolio of holding an n-shout call option and shorting a forward contract with the delivery price same as the strike price of the shout call option. Both derivatives are assumed to have the same initiation date and maturity date T. The terminal payoff of this portfolio is seen to be (i) $\max(S_T - X, 0) - (S_T - X) = \max(X - S_T, 0)$ if there is no shout throughout the whole life of the contracts, and (ii) $\max(S_T - X, S_\ell^* - X) - (S_T - X) = \max(S_\ell^* - S_T, 0)$ if the holder last shouts at asset price S_ℓ^* prior to maturity. The terminal payoff structure is seen to resemble that of the n-reset put option. Since the n-shout call option can be replicated by the combination of the n-reset put option and the corresponding forward contract, both the shout call and reset put should share the same optimal shouting policy. The put-call parity relation between their prices is given by

$$U_n(S,\tau) = V_n(S,\tau) + Se^{-q\tau} - Xe^{-r\tau}\,. \tag{6}$$

2.3. *Limiting case of infinite reset rights*

When the number of allowable reset rights n tends to infinity, the shout call becomes the fixed strike lookback call with terminal payoff: $\max(M_T - X, 0)$, where M_T is the realized maximum asset value. This is because the holder should always shout whenever a new maxima for the asset value is realized. Another justification of such shooting policy is presented in Sec. 3.3. By the put-call parity relation [see Eq. (6)], the price of an infinite-reset put is given by the difference of a fixed strike lookback call and a forward contract; or equivalently, it becomes a lookback option with terminal payoff: $\max(M_T - S_T, X - S_T)$. In particular, when $X = 0$ (corresponds to the shout floor), the terminal payoff becomes that of a floating strike lookback call option.

3. Properties of the Price Functions and Optimal Reset Policies

In this section, we examine the characterization of the optimal shouting boundary $S_n^*(\tau)$ of the n-reset put. We start with the examination of the properties on the price functions $V_n(S, \tau)$ and shouting boundary $S_n^*(\tau)$ from numerical experiments under the following separate cases: (i) $r < q$ and (ii) $r > q$.

3.1. *Numerical results on price functions and optimal shouting boundaries*

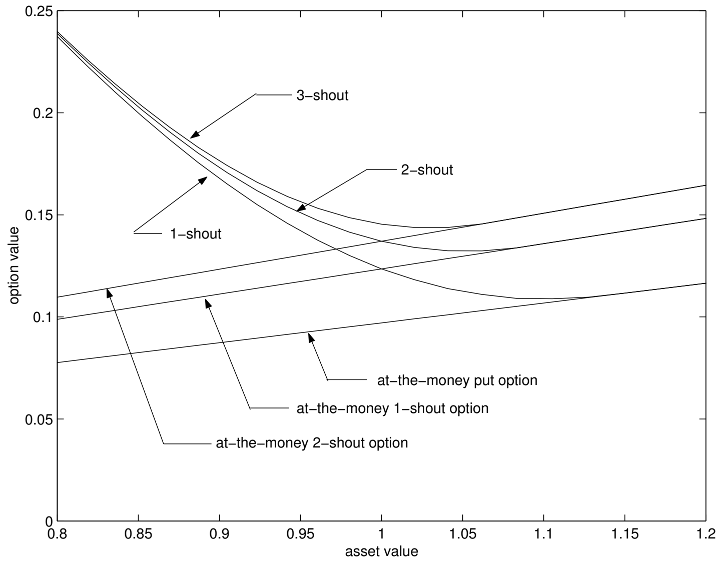
We applied the binomial method to compute the option values and critical asset prices of one-shout, two-shout and three-shout reset put options. To incorporate the reset feature, we adopted the usual dynamic programming procedure of comparing the continuation value and reward value upon reset at each binomial node. The strike price and volatility are taken to be $X = 1.0$ and $\sigma = 0.2$ in all calculations. For $r < q$, we take $r = 0.02$ and $q = 0.06$; while for $r > q$, we take $r = 0.06$ and $q = 0.02$.

In Fig. 1(a), we plot $V_1(S, \tau), V_2(S, \tau)$ and $V_3(S, \tau)$ against S at $\tau = 1$, given that $r < q$. We observe the monotonic property $V_1(S, \tau) < V_2(S, \tau) < V_3(S, \tau)$, which agrees with the intuition that put option with more reset rights should have higher value. At the critical asset price, each of the price curves touches tangentially the line representing the value of the corresponding at-the-money put option. The price function of the at-the-money $(n - 1)$-reset put option is given by $SP_n(\tau)$, which is linear in S. The critical asset prices, $S_1^*(\tau), S_2^*(\tau)$ and $S_3^*(\tau)$, corresponding to the one-shout, two-shout and three-shout reset put options, observe the monotonic property: $S_1^*(\tau) > S_2^*(\tau) > S_3^*(\tau)$.

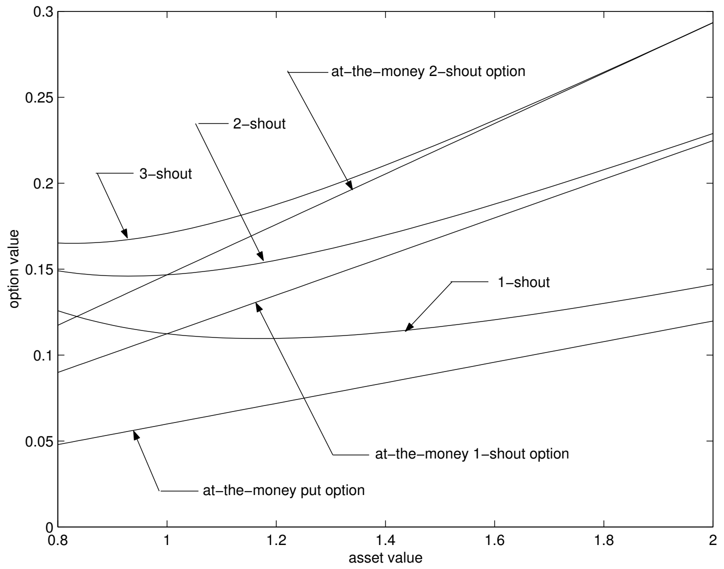
In Fig. 1(b), we plot $V_1(S, \tau), V_2(S, \tau)$ and $V_3(S, \tau)$ against S at $\tau = 12$, given that $r > q$. All of the monotonic properties stated in the last paragraph remain valid. At $\tau = 12$, only the price curve $V_3(S, \tau)$ touches the corresponding at-the-money put option value line. The price curves $V_1(S, \tau)$ and $V_2(S, \tau)$ always stay above the corresponding at-the-money put option value lines, implying that it is never optimal to shout at any asset price level.

Figures 2(a) and 2(b) show the plots of $S_n^*(\tau)$ against $\tau, n = 1, 2, 3$ for $r < q$ and $r > q$, respectively. When $r < q$, we observe that $S_n^*(\tau)$ is defined for $\tau \in (0, \infty)$ and $S_{n+1}^*(\tau) < S_n^*(\tau), n = 1, 2$. Also, $S_n^*(\tau)$ tends to a finite asymptotic value as $\tau \to \infty, n = 1, 2, 3$. From Fig. 2(a), these asymptotic values are estimated to be $1.5, 1.31$ and 1.23, for the one-shout, two-shout and three-shout reset put options, respectively. On the other hand, when $r > q$, the shouting boundaries in Fig. 2(b) reveal that $S_n^*(\tau)$ is defined only for $\tau \in (0, \tau_n^*), n = 1, 2, 3$. These critical values are estimated to be $\tau_1^* \approx 5.71, \tau_2^* \approx 9.55$ and $\tau_3^* \approx 13.0$ for the one-shout, two-shout and three-shout reset put options, respectively. Since $\tau_1^* < \tau_2^* < 12 < \tau_3^*$, the holder of the one-shout option or two-shout option will not shout at any asset price level. This explains why the two price curves $V_1(S, \tau)$ and $V_2(S, \tau)$ stay above the corresponding at-the-money put option value lines in Fig. 1(b).

Summary of pricing properties. When $r < q$, the optimal shouting boundary exists at all times, that is, $S_n^*(\tau)$ is defined for $\tau \in (0, \infty)$. For a given value of τ, one observes $S_{n+1}^*(\tau) < S_n^*(\tau), n = 1, 2, \ldots$. This can be deduced from the

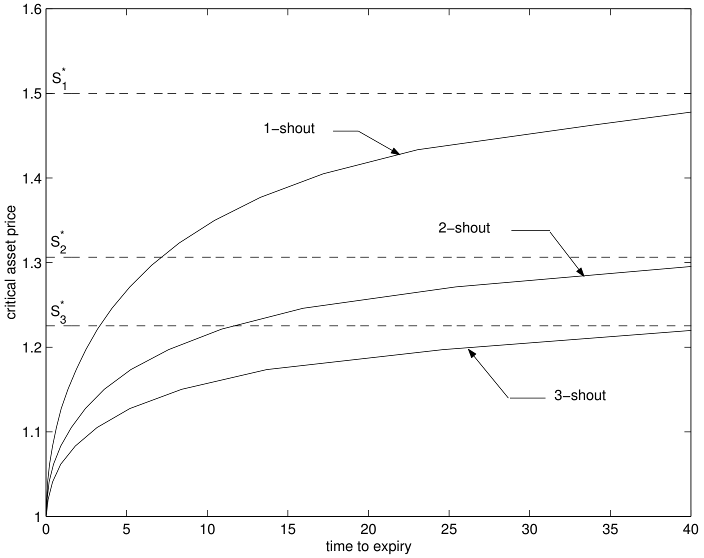


(a)

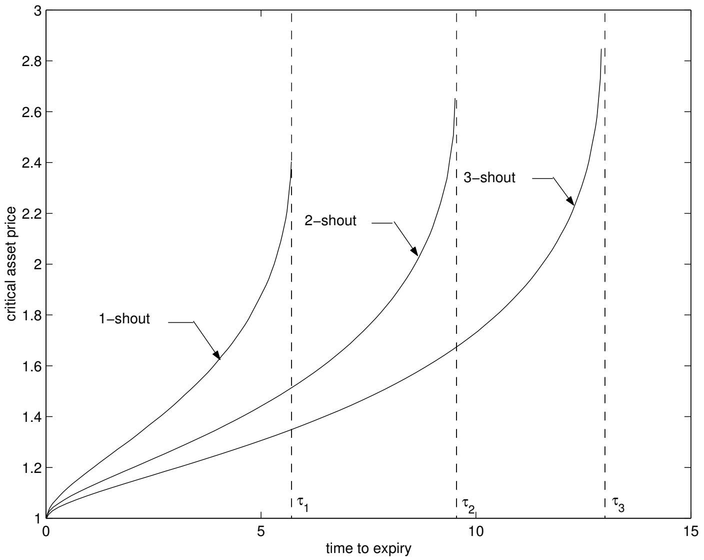


(b)

Fig. 1. (a) Plot of the price function $V_n(S, \tau)$, $n = 1, 2, 3$, against the asset value S at $\tau = 1$, given that $r < q$. The parameter values used in the calculations are: $r = 0.02$, $q = 0.06$, $\sigma = 0.2$ and $X = 1.0$. All price curves touch tangentially the lines representing the values of the corresponding at-the-money put options. (b) Plot of the price function $V_n(S, \tau)$, $n = 1, 2, 3$, against the asset value S at $\tau = 12$, given that $r > q$. The parameter values used in the calculations are: $r = 0.06$, $q = 0.02$, $\sigma = 0.2$ and $X = 1.0$. The price curve of the three-shout put option touches the value line of the corresponding at-the-money put option, while the price curves of the one-shout and two-shout put options always stay above the corresponding at-the-money put option value lines.



(a)



(b)

Fig. 2. (a) Plot of the shouting boundary as a function of time to expiry for the reset put option with right of one, two or three shouts, given that $r < q$. The parameter values used in the calculations are: $r = 0.02$, $q = 0.06$, $\sigma = 0.2$ and $X = 1.0$. The asymptotic values of the critical asset price at infinite time to expiry are found to be 1.5, 1.31 and 1.23, respectively, according to Eqs. (10a) and (10b). (b) Plot of the shouting boundary as a function of time to expiry for the reset put option with right of one, two or three shouts, given that $r > q$. The parameter values used in the calculations are: $r = 0.06$, $q = 0.02$, $\sigma = 0.2$ and $X = 1.0$. The critical values of the time to expiry beyond which it is never optimal to shout are estimated to be 5.71, 9.55 and 13.0, respectively.

financial intuition that the holder should choose to shout at a higher critical asset price with less allowable shouts remaining. The shouting boundary starts at X, that is, $S_n^*(0^+) = X$, and $S_n^*(\tau)$ is an increasing function of τ with a finite asymptotic value at $\tau \to \infty$. Furthermore, from the monotonic property on n, we have

$$\lim_{\tau\to\infty} S_{n+1}^*(\tau) < \lim_{\tau\to\infty} S_n^*(\tau)\,, \quad n = 1, 2, \ldots\,. \tag{7}$$

In the special case $r = q$, we expect $S_n^*(\tau) \to \infty$ as $\tau \to \infty, n = 1, 2, \ldots$.

When $r > q$, $S_n^*(\tau)$ retains the monotonic properties in both n and τ and $S_n^*(\tau)$ also starts at $S_n^*(0^+) = X$. However, $S_n^*(\tau)$ is defined only for $\tau \in [0, \tau_n^*)$, where τ_n^* is the critical value for τ such that it is never optimal for the holder to shout the n-reset put whenever $\tau > \tau_n^*$. With less number of shouts remaining, the holder would become more conservative on the use of the shouting rights. For a given τ, it may occur that it would be optimal to shout a n-reset put at sufficiently high asset price level but never shout for its $(n-1)$-reset counterpart. Hence, we expect $\tau_{n+1}^* > \tau_n^*, n = 1, 2, \ldots$.

3.2. *Properties of $P_n(\tau)$*

The functions $P_n, (\tau), n = 1, 2, \ldots$ play an important role in determining the optimal shouting policies of the n-shout reset put options. The function $P_n(\tau)$ is expected to observe the following properties.

(i) If $r \leq q$, then

$$\frac{d}{d\tau}[e^{q\tau} P_n(\tau)] > 0 \quad \text{for} \quad \tau \in (0, \infty)\,. \tag{8}$$

(ii) If $r > q$, there exists a unique critical value $\tau_n^* \in (0, \infty)$ such that

$$\frac{d}{d\tau}[e^{q\tau} P_n(\tau)]|_{\tau=\tau_n^*} = 0\,, \tag{9a}$$

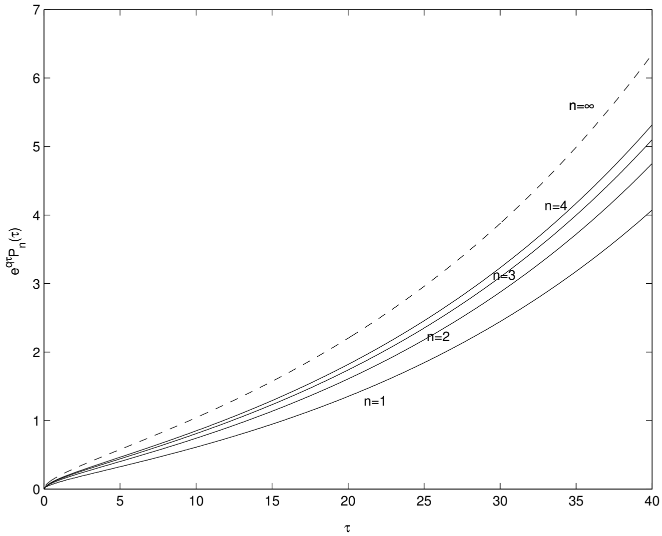
and

$$\frac{d}{d\tau}[e^{q\tau} P_n(\tau)] > 0 \quad \text{for} \quad \tau \in (0, \tau_n^*)\,, \tag{9b}$$

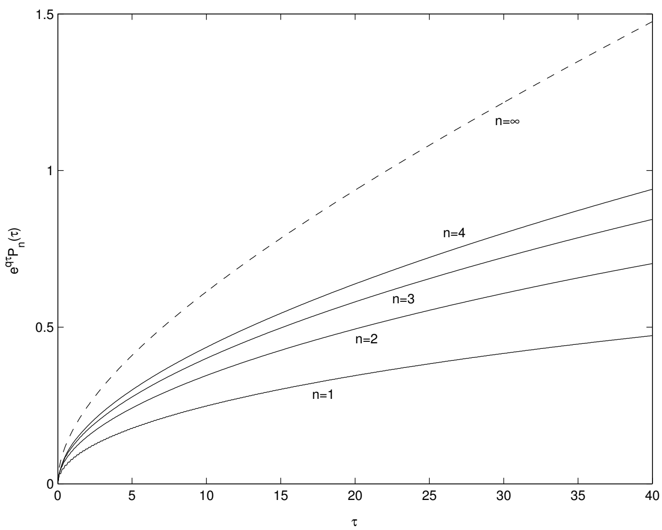
$$\frac{d}{d\tau}[e^{q\tau} P_n(\tau)] < 0 \quad \text{for} \quad \tau \in (\tau_n^*, \infty)\,. \tag{9c}$$

In addition, we have $\tau_n^* < \tau_{n+1}^*$ and $\lim_{n\to\infty} \tau_n^* = \infty$.

We prove the validity of Eq. (8) in Appendix A. The results in Eqs. (9a), (9b) and (9c) are stated as conjecture. Due to the lack of the analytic formulas for $P_n(\tau), n > 1$, the rigorous proof of Eqs. (9a), (9b) and (9c) is not available. We illustrate the validity of the above results through numerical verification (see Figs. 3(a)–(c)). The following observations verify the validity of Eq. (9a). When $r > q$ (see Fig. 3(c)), the functions $e^{q\tau} P_n(\tau), n = 1, 2, 3$, attain their respective absolute maxima at $\tau_1^* \approx 5.71 \tau_2^* \approx 9.55$ and $\tau_3^* \approx 13.0$. These threshold time values agree well with those obtained in Fig. 2(b). In each of these figures, the curve for

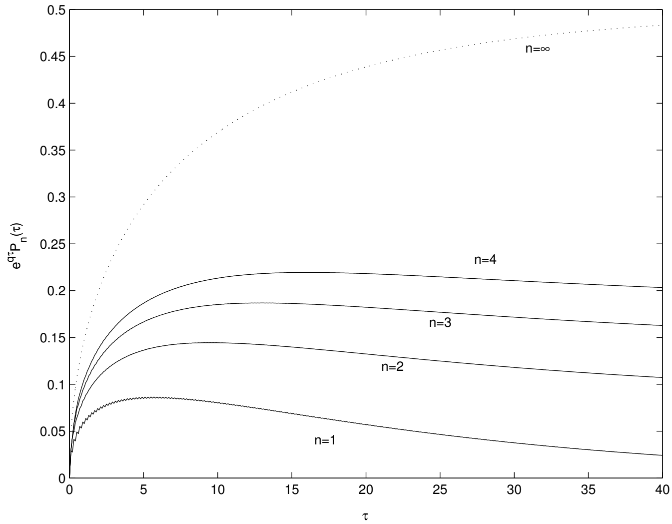


(a)



(b)

Fig. 3. (a)The curves show the plot of $e^{q\tau}P_n(\tau),\ n = 1, \ldots, 4, \infty$, against τ, given that $r < q$. The parameter values used in the calculations are: $r = 0.02$, $q = 0.06$, $X = 1.0$ and $\sigma = 0.2$. The curves are monotonically increasing for all values of τ. The price function $SP_\infty(\tau)$ corresponds to the value of shout floor with infinite number of reset rights. (b) The curves show the plot of $e^{q\tau}P_n(\tau),\ n = 1, \ldots, 4, \infty$, against τ, given that $r = q$. The parameter values used in the calculations are: $r = q = 0.02$, $X = 1.0$ and $\sigma = 0.2$. The curves are monotonically increasing and concave downward for all values of τ. (c) The curves show the plot of $e^{q\tau}P_n(\tau),\ n = 1, \cdots, 4, \infty$, against τ, given that $r > q$. The parameter values used in the calculations are: $r = 0.06$, $q = 0.02$, $X = 1$ and $\sigma = 0.2$. The maximum values of $P_n(\tau), n = 1, 2, 3$, are attained roughly at $\tau_1^* = 5.71$, $\tau_2^* = 9.55$ and $\tau_3^* = 13.0$, respectively. These estimated values for τ_1^*, τ_2^* and τ_3^* agree well with those obtained in Fig. 2(b).



(c)

Fig. 3. (*Continued*)

$e^{q\tau}P_\infty(\tau) = e^{q\tau}\lim_{n\to\infty} P_n(\tau)$ is also included for comparison. We compute $P_\infty(\tau)$ using the relation: $SP_\infty(\tau) = c_{\ell b}(S, S, \tau)$, where $c_{\ell b}(S, M, \tau)$ denotes the price function of a floating strike lookback call option with realized maximum asset value M. Such relation can be deduced from the observation that an infinite-reset shout floor is effectively a floating strike lookback call and the realized maxima of asset value at initiation is simply the current asset value S [see Sec. 2.3 and Eq. (11b)].

3.3. *Analytic properties of the optimal shouting boundary*

At those times when the term $\frac{d}{d\tau}[e^{q\tau}P_n(\tau)]$ in the non-homogeneous term in Eq. (4) is negative, the holder should never shout optimally. Hence, we deduce that for $r > q$, the optimal shouting boundary $S_n^*(\tau)$ is defined only for $\tau \in (0, \tau_n^*)$, where τ_n^* is defined in Eq. (9a). This result is verified by the numerical plots shown in Figs. 2(b) and 3(c). When $r < q$, we have $\dfrac{d}{d\tau}[e^{q\tau}P_n(\tau)] > 0$ for all $\tau > 0$; correspondingly, $S_n^*(\tau)$ is defined for all $\tau > 0$. In particular, we manage to derive the asymptotic value for $S_n^*(\tau)$ as $\tau \to \infty$.

Theorem 3.1. *Let* $S_{n,\infty}^*$ *denote* $\lim_{\tau\to\infty} S_n^*(\tau)$. *For* $r < q$, *we have*

$$S_{n,\infty}^* = \left(1 + \frac{1}{\alpha}\right)\frac{X}{\beta_n} \tag{10a}$$

where $\alpha = 2(q-r)/\sigma^2$, $\beta_1 = 1$ *and*

$$\beta_n = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\beta_{n-1}^{1+\alpha}\,. \tag{10b}$$

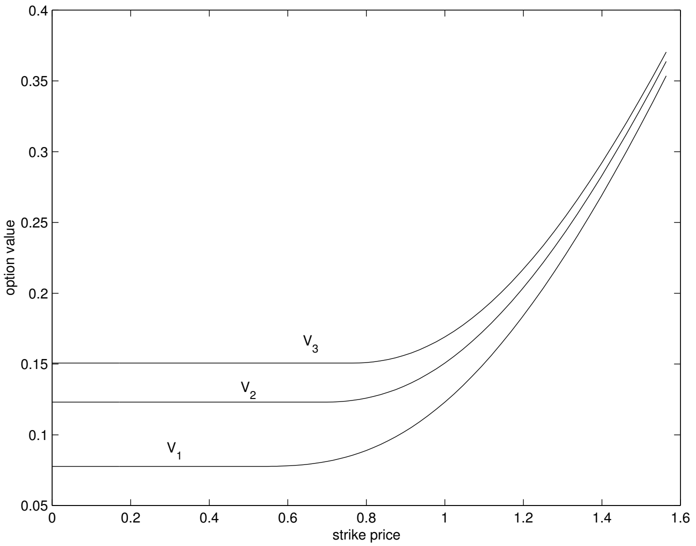


Fig. 4. Plot of the price function $V_n(S, \tau; X), n = 1, 2, 3$, of the n-reset put option against the initial strike price X. The parameter values used in the calculations are: $r = 0.06$, $q = 0.02$, $\sigma = 0.2$, $\tau = 5.0$ and $S = 1.0$. The slopes of the price curves tend to one as $X \to \infty$ and tend to zero as $X \to 0$.

In addition, we have the monotonic property $S^*_{n,\infty} < S^*_{n-1,\infty}, n = 2, 3, \ldots,$ *and* $\lim_{n\to\infty} S^*_{n,\infty} = X$. *If* $r = q$, *then* $S^*_{n,\infty}$ *becomes infinite.*

The proof of Theorem 3.1 is presented in Appendix A.

Infinite-reset put. Let $S^*_\infty(\tau)$ denote the critical asset price at which it is optimal to shout the infinite-reset put option. Since $S^*_\infty(\tau)$ is non-decreasing function of $\tau, S^*_\infty(\tau) \leq \lim_{n\to\infty} S^*_{n,\infty} = X$, for all τ. Together with the constraint that $S^*_\infty(\tau) \geq X$, we deduce that $S^*_\infty(\tau) = X$ for all $\tau \geq 0$. This agrees with the intuition that the holder should shout the infinite-reset put whenever the option becomes in-the-money (see Sec. 2.3). We see that the pricing model of the infinite-reset put is no longer a free boundary value problem since the exercise boundary becomes deterministic.

3.4. *Dependence of price functions on the strike price*

In Fig. 4, we show the plot of the price function of the n-reset put option against the initial strike price X. The parameter values used in the calculations are:

$$r = 0.06,\ q = 0.02,\ \sigma = 0.2,\ \tau = 5.0 \text{ and } S = 1.0\,.$$

The slopes of the price curves observe the properties:

$$\lim_{X\to 0} \frac{\partial V_n}{\partial X} = 0 \text{ and } \lim_{X\to\infty} \frac{\partial V_n}{\partial X} = 1\,.$$

These analytic properties can be understood by the following financial arguments. At exceedingly high value of X, the holder should never reset the strike price optimally, so the reset put is equal to that of the vanilla put. We then have

$$\lim_{X\to\infty} \frac{\partial V_n}{\partial X} = \lim_{X\to\infty} \frac{\partial p_E}{\partial X} = 1\,.$$

At infinitesimally small value of X, the reset put is sufficiently deep-in-the-money so that the put should be shouted immediately to establish a new strike price. This implies $S_n^*(\tau) \to 0$ as $X \to 0$. The put is almost sure to stay in the stopping region so that $V_n(\tau) \approx SP_n(\tau)$; and accordingly,

$$\lim_{X\to 0} \frac{\partial V_n}{\partial X} = 0\,.$$

3.5. *Shout floors*

Unlike the reset put, the shout floor has no preset strike price at the initiation of the contract. We may visualize a shout floor as a special case of a reset put where the strike price X is zero. The value of the n-reset shout floor shares the same governing equation as that of n-reset put option except that the terminal payoff is set to be zero. The price function of a n-reset shout floor takes different forms according to $r \le q$ or $r > q$, whose properties are summarized in Theorem 3.2.

Theorem 3.2. *Let $R_n(S,\tau)$ denote the price function of the n-reset shout floor.*

(i) *If $r \le q$,*

$$R_n(S,\tau) = SP_n(\tau)\,, \quad \tau \in (0,\infty)\,. \tag{11a}$$

(ii) *If $r > q$,*

$$R_n(S,\tau) = \begin{cases} SP_n(\tau) & \tau \in (0,\tau_n^*] \\ e^{-q(\tau-\tau_n^*)} SP_n(\tau_n^*)\,, & \tau \in (\tau_n^*,\infty)\,, \end{cases} \tag{11b}$$

where τ_n^* is the unique solution to $\frac{d}{d\tau}[e^{q\tau}P_n(\tau)] = 0$.

The proof of Theorem 3.2 is given in Appendix A. The above analytic representation of $R_n(S,\tau)$ is not an explicit analytic formula. Recall that $P_n(\tau) = V_{n-1}(1,\tau;1)$, so one has to find $V_1(S,\tau), V_2(S,\tau), \ldots$, successively in order to obtain $P_n(\tau)$.

Optimal shouting policies of shout floors. First, consider the case $r \le q$. Since we have $R_n(S,\tau) = SP_n(\tau)$ for all values of τ, we deduce that the first shouting right will be utilized at once at any asset price level. Next, when $r > q$, the n-reset shout floor will not be shouted at any asset price when $\tau > \tau_n^*$. However, it will be shouted at once at any asset price level whenever $\tau \le \tau_n^*$. Once the first shouting has occurred, the n-reset shout floor reduces to the at-the-money $(n-1)$-reset put option. The subsequent optimal shouting policies will be governed by those of the multi-reset put option.

4. Conclusion

In this paper, we have analyzed the optimal shouting policies of options with multiple shouting rights. The relations between the shout call option, reset put option and shout floors have been examined. The behaviors of the shouting boundaries depend crucially on the relative magnitude of the riskless interest rate r and dividend yield q. When $r \leq q$, the shouting boundary is defined at all times. This implies that at any time during the life of the option, the holder should choose to shout optimally when the asset value reaches some threshold value. On the other hand, when $r > q$, there exists a threshold time value earlier than which it is never optimal for the holder to shout at any asset value level. The optimal shouting policies of the multi-reset shout floor have some striking properties. When $r \leq q$, the shout floor should be shouted at once at any time and at any asset price level. When $r > q$, there exists a critical time earlier than which it is never optimal for the holder to shout the shout floor. However, the shout floor should be shouted at once at any asset price upon reaching the critical time. When the first shouting has occurred in a multi-reset shout floor, the shout floor becomes the corresponding at-the-money reset put option with one less shouting right.

A number of interesting analytic formulas have been derived in the paper. The analytic representation of the price function of the n-reset shout floor is deduced. In addition, we obtain the asymptotic critical asset prices at infinte time to expiry for the n-reset put options when $r < q$.

Several results on the monotonic properties with regard to the critical asset prices and shouting boundaries are established through theoretical arguments togehter with numerical experiments. Some of these properties are: (1) an option with more reset rights should have higher value compared to its counterpart with less; (2) the holder shouts at a lower critical asset price with more shouting rights outstanding; (3) the holder chooses to shout at a lower critical asset price for a shorter-lived option; (4) the critical value of the time to expiry beyond which it is never optimal to shout increases with more shouting rights outstanding. In particular, when the number of allowable shouts tends to infinity, the reset put options are related to lookback options. All these monotonic properties agree with financial intuitions.

Appendix A

Proof of Eq. (8). We prove by mathematical induction. The proposition holds when $n = 1$ (see [2]). Assume

$$\frac{d}{d\tau}[e^{q\tau}P_n(\tau)] > 0$$

and since $P_{n+1}(\tau) = V_n(1,\tau;1)$, it suffices to show

$$\frac{d}{d\tau}[e^{q\tau}V_n(1,\tau;1)] > 0 \quad \text{for} \quad \tau > 0\,.$$

Let us consider the functions

$$\tilde{V}_n(S,\tau) = e^{q\tau}V_n(S,\tau;1) \quad \text{and} \quad \tilde{V}_n^\delta(S,\tau) = e^{q(\tau+\delta)}V_n(S,\tau+\delta;1) \quad \text{with} \quad \delta > 0\,,$$

we would like to show that

$$\tilde{V}_n^\delta(S,\tau) > \tilde{V}_n(S,\tau) \quad \text{for all} \quad S,\tau > 0\,.$$

It is seen that $\tilde{V}_n(S,\tau)$ and $\tilde{V}_n^\delta(S,\tau)$ satisfy

$$\frac{\partial \tilde{V}_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 \tilde{V}_n}{\partial S^2} - (r-q)S\frac{\partial \tilde{V}_n}{\partial S} + (r-q)\tilde{V}_n \geq 0\,, \quad \tilde{V}_n(S,\tau) \geq Se^{q\tau}P_n,(\tau)\,,$$

$$\left[\frac{\partial \tilde{V}_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 \tilde{V}_n}{\partial S^2} - (r-q)S\frac{\partial \tilde{V}_n}{\partial S} + (r-q)\tilde{V}_n\right][\tilde{V}_n(S,\tau) - Se^{q\tau}P_n(\tau)] = 0\,,$$

$$\tilde{V}_n(S,0) = \max(1-S,0)$$

and

$$\frac{\partial \tilde{V}_n^\delta}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 \tilde{V}_n^\delta}{\partial S^2} - (r-q)S\frac{\partial \tilde{V}_n^\delta}{\partial S} + (r-q)\tilde{V}_n^\delta \geq 0\,, \quad \tilde{V}_n^\delta(S,\tau) \geq Se^{q(\tau+\delta)}P_n(\tau+\delta)\,,$$

$$\left[\frac{\partial \tilde{V}_n^\delta}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 \tilde{V}_n^\delta}{\partial S^2} - (r-q)S\frac{\partial \tilde{V}_n^\delta}{\partial S} + (r-q)\tilde{V}_n^\delta\right][\tilde{V}_n^\delta(S,\tau) - Se^{q(\tau+\delta)}P_n(\tau+\delta)] = 0\,,$$

$$\tilde{V}_n^\delta(S,0) = \tilde{V}_n(S,\delta)\,,$$

respectively. Let $p_E(S,\tau;1)$ denote the price of a European put option with unit strike. It can be checked easily that

$$\frac{\partial}{\partial \tau}e^{q\tau}p_E(S,\tau;1) \geq 0 \ \text{ for } r \leq q\,,$$

so

$$e^{q\delta}p_E(S,\delta) \geq p_E(S,0) \ \text{ for } r \leq q\,.$$

Clearly $V_n(S,\tau;1) > p_E(S,\tau;1)$ for all $S,\tau > 0$, so we have

$$\tilde{V}_n^\delta(S,0) = e^{q\delta}V_n(S,\delta;1) > e^{q\delta}p_E(S,\delta;1) \geq p_E(S,0) = \max(1-S,0) = \tilde{V}_n(S,0)\,.$$

The assumption of induction gives $Se^{q(\tau+\delta)}P_n(\tau+\delta) > Se^{q\tau}P_n(\tau)$. Since the obstacle function and the terminal payoff associated with $\tilde{V}_n^\delta(S,\tau)$ are greater than those of $\tilde{V}_n(S,\tau)$, by applying the maximum principle of variational inequality (see [3]), we obtain the desired result. □

Proof of Theorem 3.1. We write $W_n(S,\tau) = e^{r\tau}V_n(S,\tau)$, and let $W_n^\infty(S)$ denote the limit of $W_n(S,\tau)$ as $\tau \to \infty$. The governing equation for $W_n^\infty(S)$ take the from

$$\frac{\sigma^2}{2}S^2\frac{d^2W_n^\infty}{dS^2} + (r-q)S\frac{dW_n^\infty}{dS} = 0\,, \quad 0 < S < S^*_{n,\infty}\,,$$

with auxiliary conditions

$$W_n^\infty(S^*_{n,\infty}) = \beta_n S^*_{n,\infty} \text{ and } \frac{dW_n^\infty}{dS}(S^*_{n,\infty}) = \beta_n\,,$$

where $\beta_n = \lim_{\tau\to\infty} e^{r\tau}P_n(\tau)$. When $n = 1$, it can be shown easily that

$$\beta_1 = \lim_{\tau\to\infty} e^{r\tau}P_E(1,\tau;1) = 1\,.$$

In general, we have

$$\beta_n = \lim_{\tau\to\infty} e^{r\tau}P_n(\tau) = \lim_{\tau\to\infty} W_{n-1}(1,\tau;1) = W_{n-1}^\infty(1;1)\,.$$

Hence, β_n exists provided that $W_{n-1}^\infty(1;1)$ is defined. The existence of β_n can be argued as follows. Given the existence of β_1, we can determine $W_1^\infty(1;1)$. This guarantees the existence of β_2, and from which we can determine $W_2^\infty(1;1)$, and so forth.

The general solution for $W_n^\infty(S)$ is found to be

$$W_n^\infty(S) = X + CS^{1+\alpha}\,,$$

where $\alpha = 2(q-r)/\sigma^2$ and C is an arbitrary constant. Applying the two auxiliary conditions, we obtain

$$C = \frac{1}{1+\alpha}\frac{\beta_n}{S^{*\alpha}_{n,\infty}} = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{\beta_n^{1+\alpha}}{X^\alpha}\,,$$

$$S^*_{n,\infty} = \left(1+\frac{1}{\alpha}\right)\frac{X}{\beta_n}\,.$$

The recurrence relation for β_n is deduced to be

$$\beta_n = W_{n-1}^\infty(1;1) = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\beta_{n-1}^{1+\alpha}\,.$$

The monotonic relation $\beta_n > \beta_{n-1}$ leads to the monotonic property $S^*_{n-1,\infty} > S^*_{n,\infty}$. Taking the limit $n \to \infty$ in the above recurrence relation for β_n gives

$$\lim_{n\to\infty}\beta_n = 1 + \frac{1}{\alpha}\,.$$

Correspondingly, this implies $\lim_{n\to\infty} S^*_{n,\infty} = X$. The first few values of β_n and $S^*_{n,\infty}$ are obtained as follows:

(i) when $n = 1, \beta_1 = 1$ and $S^*_{1,\infty} = (1+\frac{1}{\alpha})X$;

(ii) when $n = 2, \beta_2 = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$ and $S^*_{2,\infty} = \frac{1+\frac{1}{\alpha}}{1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}}$;

(iii) when $n = 3, \beta_3 = 1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}[1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}]$ and $S^*_{3,\infty} = \frac{1+\frac{1}{\alpha}}{1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}[1+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}]}$.

Proof of Theorem 3.2. Since there is no strike price appearing in the linear complementarity formulation, one deduces that $R_n(S,\tau)$ is linearly homogeneous in S so that $R_n(S,\tau) = Sg_n(\tau)$. The set of governing equations for $g_n(\tau)$ are given by

$$\frac{d}{d\tau}[e^{q\tau}g_n(\tau)] \geq 0\,, \quad g_n(\tau) \geq P_n(\tau)\,,$$

$$\left\{\frac{d}{d\tau}[e^{q\tau}g_n(\tau)]\right\}[g_n(\tau) - P_n(\tau)] = 0 \quad \text{and} \quad g_n(0) = 0\,.$$

Within the time interval where $\frac{d}{d\tau}[e^{q\tau}P_n(\tau)] \geq 0$, we observe that the equations are automatically satisfied by $g_n(\tau) = P_n(\tau)$. However, at those times where $\frac{d}{d\tau}[e^{q\tau}P_n(\tau)] < 0, g_n(\tau)$ must satisfy $\frac{d}{d\tau}[e^{q\tau}g_n(\tau)] = 0$. Now, we consider the following two separate cases:

(i) $r \leq q$.
Since $\frac{d}{d\tau}[e^{q\tau}P_n(\tau)]$ is strictly positive for all $\tau > 0$ and $P_n(0) = 0$, we then have $g_n(\tau) = P_n(\tau), \tau \in (0,\infty)$.

(ii) $r > q$.
For $\tau \in (0,\tau_n^*]$, we deduce similarly that $g_n(\tau = P_n(\tau)$. However, when $\tau > \tau_n^*$, we have $e^{q\tau}g_n(\tau) = e^{q\tau_n^*}P_n(\tau_n^*)$, so that $g_n(\tau) = e^{-q(\tau-\tau_n^*)}P_n(\tau_n^*)$ for $\tau \in (\tau_n^*,\infty)$.

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