Discussion on "Pricing perpetual fund protection with withdrawal option"

by Hans U. Gerber and Elias S.W. Shiu

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Professors Gerber and Shiu have done a refined job of evaluating the value of the dynamic fund protection option with perpetual life, and relating the fund protection option with the perpetual maximum option and the Russian option (perpetual American lookback option). In particular, they have done a detailed and insightful analysis of the optimal exercise strategy of these options. The two general concepts in arbitrage option pricing theory, smooth paste condition and replicating portfolio, are demonstrated as part of the solution procedure. In this discussion note, we would like to derive the price functions of the dynamic fund protection option and maximum option by relating them to the value of a protection fund with rights to reset to value of another guaranteed fund.

We consider a primary fund with value process S_2^t, which is protected with reference to another (guaranteed) fund with value process S_1^t. The holder of the primary fund has the right to reset the value to that of the guaranteed fund upon exercising the reset right. The number of resets allowed can be finite or infinite. Further, the protection fund is assumed to have perpetual life and withdrawal right. With infinite number of resets, the holder should always exercise the reset right whenever the value of the primary fund falls to the value of the guaranteed fund. This reset strategy is exactly the same as the perpetual dynamic fund protection option considered by Gerber and Shiu. When the holder is allowed to reset only once, the perpetual protection fund is equivalent to the perpetual maximum option. This is because the fund becomes S_2^t upon withdrawal and S_1^t upon reset, corresponding to the payoff that takes the maximum of S_1^t and S_2^t. In this discussion note, we would like to obtain the price formula of such perpetual protection fund with withdrawal right and n reset rights. By setting $n = 1$ and $n \to \infty$, we recover the price functions of the perpetual maximum option and the dynamic fund protection option, respectively. We also discuss the monotonic properties of the value functions and the threshold values at which the holder should withdraw or reset.

With both withdrawal and reset rights, the option pricing model has two-sided thresholds. Let S_2 and S_1 denote the values of the primary fund and guaranteed fund at the current time (taken to be the zero), respectively. Under the Black-Scholes risk neutral valuation framework, the value processes are assumed to follow the lognormal processes

$$\frac{dS_i^t}{S_i^t} = (r - \zeta_i)\, dt + \sigma_i\, dZ_i, \quad i = 1, 2, \tag{1}$$

where r is the riskless interest rate, ζ_i is the dividend yield of fund i, σ_i is the volatility parameter of fund i and dZ_i is the standard Wiener process. We assume $dZ_1\, dZ_2 = \rho\, dt$, where ρ is the correlation coefficient between S_1^t and S_2^t. Let $V_n(S_1, S_2)$ denote the value of the perpetual protection fund with n reset rights and withdrawal right. In our pricing formulation, we take advantage of the linear homogeneity property of $V_n(S_1, S_2)$; and accordingly, we define

$$W_n(x) = \frac{V_n(S_1, S_2)}{S_2}, \quad x = \frac{S_1}{S_2}. \tag{2}$$

This corresponds to the choice of S_2 as the numeraire. In the continuation region, the governing equation for $W_n(x)$ takes the form

$$\frac{\sigma^2}{2}x^2\frac{d^2W_n}{dx^2} + (\zeta_1 - \zeta_2)x\frac{dW_n}{dx} - \zeta_2 W_n = 0, \quad x_n^w < x < x_n^r, \tag{3}$$

where $\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2, x_m^w$ and x_n^r are the threshold values for x at which the holder should optimally withdraw and reset, respectively. The boundary conditions are prescribed as follows:

$$W_n(x_n^w) = 1 \qquad \text{and} \quad W_n'(x_n^w) = 0, \tag{4a}$$

$$W_n(x_n^r) = x_n^r W_{n-1}(1) \quad \text{and} \quad W_n'(x_n^r) = W_{n-1}(1). \tag{4b}$$

Upon withdrawal, $V_n(S_1, S_2)$ becomes S_2 and so we have $W_n(x_n^w) = 1$. When the holder resets at $x = x_n^r$, the option writer has to supply enough funding to increase the number of units of the primary fund such that the new fund value equals S_1. The corresponding number of units equals x_n^r, which is the ratio of the fund values at the reset threshold x_n^r. Subsequently, the protection fund has one reset right less and x becomes 1 since the values of the new "upgraded" fund and guaranteed fund are identical upon reset. Hence, the value of the protection fund at reset threshold becomes $x_n^r W_{n-1}(1)$. When $n = 0$, there is no reset right so we have $W_0(x) = 1$ for all values of x. The derivative conditions at x_n^w and x_n^r represent optimality conditions at the withdrawal and reset thresholds, respectively.

Consider the limiting case $n \to \infty$, we have $W_\infty(x_\infty^r) = x_\infty^r W_\infty(1)$. The equation is seen to be satisfied by $x_\infty^r = 1$ [for a more rigorous justification, see Eq. (13)]. This represents immediate reset whenever S_2^t falls to S_1^t, given that the holder has infinite number of reset rights. Furthermore, the corresponding derivative condition becomes $W_\infty'(1) = W_\infty(1)$, which is equivalent to Eq. (3.14) in Gerber-Shiu's paper [they argue that the value of $V(S_1, S_2)$ is unaffected by marginal changes in S_2 when S_2 is "close" to S_1]. We would like to obtain closed form solutions for x_n^r, x_n^w and $W_n(x)$. Since the protection fund with more outstanding reset rights should be more expensive, we have the obvious monotonic properties on the price functions:

$$W_1(x) < W_2(x) < \cdots < W_\infty(x), \tag{5a}$$

from which we deduce the following monotonic properties of the threshold values:

$$x_1^w > x_2^w > \cdots > x_\infty^w, \tag{5b}$$

$$x_1^r > x_2^r > \cdots > x_\infty^r = 1. \tag{5c}$$

When $n = 1$, Gerber and Shiu obtained the solution for $W_1(x)$ in terms of the function

$$g(z) = \frac{\theta_2 z^{\theta_1} - \theta_1 z^{\theta_2}}{\theta_2 - \theta_1}, \quad z > 0, \theta_1 \leq 0 \text{ and } \theta_2 \geq 1, \tag{6}$$

where θ_1 and θ_2 are the two roots of the auxiliary equation:

$$\frac{\sigma^2}{2}\theta(\theta-1)+(\zeta_2-\zeta_1)\theta-\zeta_2=0. \tag{7}$$

The function $g(z)$ satisfies the governing differential equation (3) and the boundary conditions: $g(1)=1$ and $g'(1)=0$. Suppose we set $W_1(x)=g\left(\frac{x}{x_1^w}\right)$ so that $W_1(x_1^w)=1$ and $W_1'(x_1^w)=0$ are automatically satisfied. The other two boundary conditions, $W_1(x_1^r)=x_1^r$ and $W_1'(x_1^r)=1$, lead to the following pair of equations for the determination of x_1^w and x_1^r:

$$\frac{1}{\theta_2-\theta_1}\left[\theta_2\left(\frac{x_1^r}{x_1^w}\right)^{\theta_1}-\theta_1\left(\frac{x_1^r}{x_1^w}\right)^{\theta_2}\right]=x_1^r, \tag{8a}$$

$$\frac{\theta_1\theta_2}{\theta_2-\theta_1}\left[\left(\frac{x_1^r}{x_1^w}\right)^{\theta_1}-\left(\frac{x_1^r}{x_1^w}\right)^{\theta_2}\right]=x_1^r. \tag{8b}$$

Solving the above equations, we obtain

$$x_1^w=\left(\frac{-\theta_1}{1-\theta_1}\right)^{(1-\theta_1)/(\theta_2-\theta_1)}\left(\frac{\theta_2}{\theta_2-1}\right)^{(\theta_2-1)/(\theta_2-\theta_1)}, \tag{9a}$$

$$x_1^r=\left(\frac{-\theta_1}{1-\theta_1}\right)^{-\theta_1/(\theta_2-\theta_1)}\left(\frac{\theta_2}{\theta_2-1}\right)^{\theta_2/(\theta_2-\theta_1)}. \tag{9b}$$

By setting

$$x_1^r=W_{n-1}(1)x_n^r \quad \text{and} \quad x_1^w=W_{n-1}(1)x_n^w \tag{10}$$

in Eqs. (8a,b) and considering the function $g\left(\frac{x}{x_n^w}\right)$, we observe that

$$g\left(\frac{x_n^r}{x_n^w}\right)=x_n^rW_{n-1}(1) \quad \text{and} \quad g'\left(\frac{x_n^r}{x_n^w}\right)=W_{n-1}(1). \tag{11}$$

Hence, we deduce that

$$W_n(x)=g\left(\frac{x}{x_n^w}\right), \quad x_n^w<x<x_n^r. \tag{12}$$

Once we know x_1^w, x_1^r and $W_1(x)$, we can compute $W_1(1)=g\left(\frac{1}{x_1^w}\right)$; then we apply the recursive relations (10) to obtain x_2^w and x_2^r, also $W_2(x)=g\left(\frac{x}{x_2^w}\right)$ and $W_2(1)=g\left(\frac{1}{x_2^w}\right)$, and so forth. From the recursive relations (10) and the monotonic properties on $W_n(x)$ in Eq. (5a), we deduce immediately the monotonic properties on x_n^w and x_n^r [see Eqs. 5(b,c)].

Consider the limiting case where $n\to\infty$, the boundary conditions (4b) become

$$W_\infty(x_\infty^r)=x_\infty^rW_\infty(1) \quad \text{and} \quad W_\infty'(x_\infty^r)=W_\infty(1). \tag{13}$$

By virtue of the monotonic increasing property of $W'_\infty(x)$, the curve of $y = W_\infty(x)$ and the line $y = W_\infty(1)x$ can intersect at only one point, namely, $x = 1$. Hence, the equation $W_\infty(x^r_\infty) = x^r_\infty W_\infty(1)$ can have the unique root, $x^r_\infty = 1$. The other condition becomes $W'_\infty(1) = W_\infty(1)$. Hence, the governing equation for the value of the perpetual fund protection option, which is equal to $W_\infty(x)$, is given by

$$\frac{\sigma^2}{2}x^2\frac{dW^2_\infty}{dx^2} + (\zeta_1 - \zeta_2)x\frac{dW_\infty}{dx} - \zeta_2 W_\infty = 0, \quad x^w_\infty < x < 1, \tag{14}$$

subject to the auxiliary conditions:

$$W_\infty(x^w_\infty) = 1 \quad \text{and} \quad W'_\infty(x^w_\infty) = 0, \tag{15a}$$

$$W'_\infty(1) = W_\infty(1). \tag{15b}$$

The solution to $W_\infty(x)$ is easily seen to be

$$W_\infty(x) = \frac{h(x)}{h(x^w_\infty)}, \quad x^w_\infty < x < 1, \tag{16}$$

where

$$h(x) = (\theta_2 - 1)x^{\theta_1} - (\theta_1 - 1)x^{\theta_2}, \quad x > 0. \tag{17}$$

Note that $h(x)$ satisfies Eq. (14) and the Robin boundary condition (15b). The boundary condition $W_\infty(x^w_\infty) = 1$ is satisfied by the inclusion of the multiplicative factor $1/h(x^w_\infty)$ in $W_\infty(x)$. The optimality condition, $W'_\infty(x^w_\infty) = 0$, gives the following algebraic equation for x^w_∞:

$$h'(x^w_\infty) = \theta_1(\theta_2 - 1)(x^w_\infty)^{\theta_1} - \theta_2(\theta_1 - 1)(x^w_\infty)^{\theta_2} = 0. \tag{18}$$

Alternatively, from the recursive relations (10), we deduce that [see Eq. (4.5) in Gerber-Shiu's paper]

$$\frac{x^w_1}{x^r_1} = \frac{x^w_\infty}{x^r_\infty} = x^w_\infty. \tag{19}$$

Also, from Eqs. (12) and (16), we obtain another relation [see Eqs. (4.11) and (4.17) in Gerber-Shiu's paper]

$$W_\infty(x) = g\left(\frac{x}{x^w_\infty}\right) = \frac{h(x)}{h(x^w_\infty)}, \quad x^w_\infty < x < 1. \tag{20}$$

In Figure 1, we show the plots of the price functions $W_n(x)$ for $n = 1, 2, 3$ and ∞. The values of the price functions increase monotonically with increasing number of reset rights and always stay above 1. We also plot the threshold values, x^w_n and x^r_n, against $1/n$ in Figure 2. The monotonic properties on x^w_n and x^r_n as stated in Eqs. (5b,c) are verified. In particular, we observe that x^r_n tends to 1 as $n \to \infty$.

In summary, we have illustrated that the maximum call and the dynamic protection fund option correspond to the protection fund with rights to reset to a reference guaranteed fund once and infinite number of times, respectively. We obtain the closed form formula for the price function of the protection fund with n reset rights. With finite number of resets, there are two threshold values, an upper threshold for reset and a lower threshold for withdrawal. When infinite number of resets allowed, we prove mathematically that the holder exercises the reset right whenever the value of the protection fund falls to the value of the guaranteed fund.

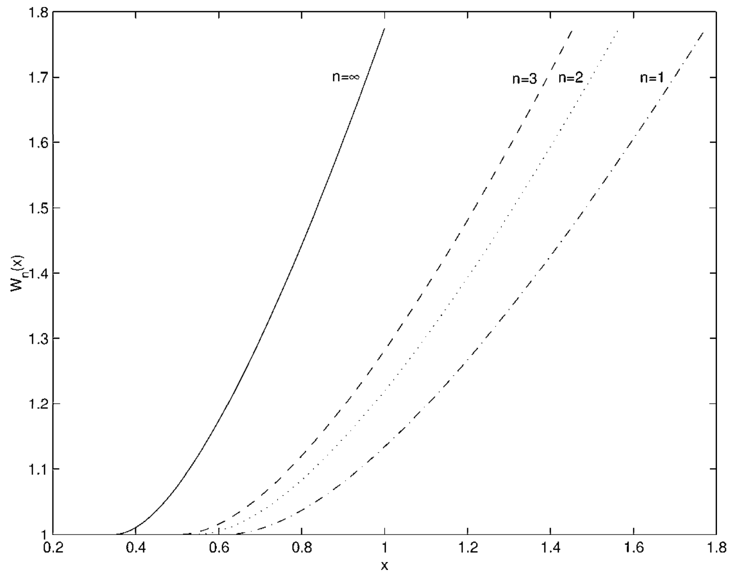

Figure 1 The figure shows the plots of the price functions of the protection fund $W_n(x)$ against x, with varying number of reset rights n.

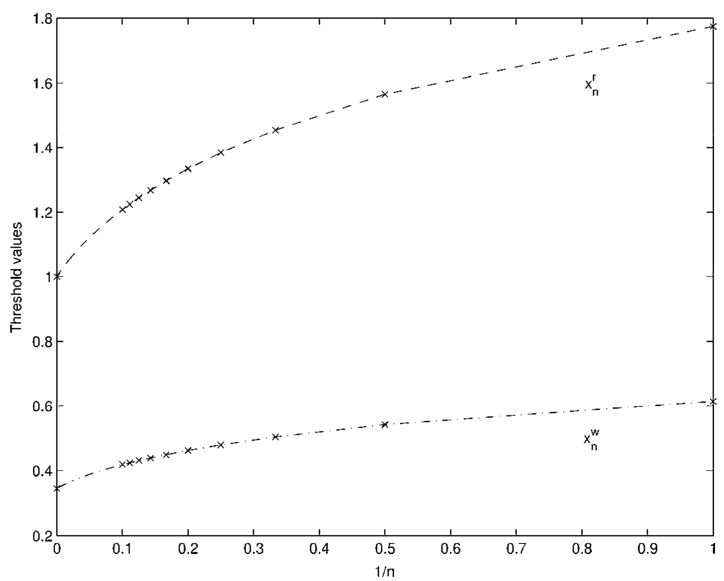

Figure 2 We plot the threshold values, x_n^w and x_n^r, at which the holder of the protection fund should optimally withdraw and reset, respectively, against the reciprocal of the number of reset rights, $1/n$.