Signaling Game Models of Equity Financing under Information Asymmetry and Finite Project Life

Qiuqi Wang and Yue Kuen Kwok*

Department of Mathematics
Hong Kong University of Science and Technology
Hong Kong, China

Abstract

We develop the real option signaling game models of equity financing of a risky project under asymmetric information, where the firm quality is known to the firm management but not outside investors. Unlike the usual assumption of perpetuity of investment, we assume that the time window of the investment opportunity has a finite time horizon. The firm chooses the optimal time to issue equity to raise capital for the investment project. The number of shares of equity issued to fund the project depends on the outside investors' belief on the firm quality. The low-type firm has the incentive to sell overpriced securities through mimicking the investment strategy of the high-type firm in terms of investment timing and number of equity shares. On the other hand, the high-type firm may adopt the separating strategy by imposing mimicking costs on the low-type firm. We examine the incentive compatibility constraints faced by the firm under different quality types and discuss characterization of the separating and pooling equilibriums. We also explore how the separating and pooling equilibriums evolve over the time span of the investment opportunity. The information costs and abnormal returns exhibit interesting time dependent behaviors, in particular, at time close to expiry of the investment opportunity.

Keywords: equity financing, signaling games, separating and pooling equilibriums, real options

JEL classification: G31, G32

1 Introduction

A firm can finance an investment project using three common sources: internal funds, issuance of new debt or equity. The well-known pecking order theory in corporate finance (Myers and Majluf, 1984) postulates that internal financing is first adopted. When internal financing becomes exhausted, the firm may issue debt. Issuance of equity is used only when debt issuance becomes non-viable. Equity financing is less preferred since investors believe that the firm manager may take advantage of the over-valuation of the firm when new shares are issued. In response, investors place a lower value on the new equity issuance since investors have less knowledge of the firm's prospect. Adverse selection problems may arise since outside investors may not distinguish whether the firm is high-type or low-type, and the adverse selection cost of financing increases when investors are faced with a higher

*Correspondence author; E-mail: maykwok@ust.hk.

level of information asymmetry. Since debt is less sensitive to private information, the pecking order theory concludes that debt is preferred over equity under information asymmetry. Besides, the additional shares issued under equity financing would mean diluting ownership of existing shareholders.

Many of the earlier works on financing investment opportunities are static models, and the firm may not have the option of waiting. Indeed, in the classical paper of pecking order theory, Myers and Majluf (1984) assume that the investment opportunity evaporates soon. Some of the findings of the classical pecking order theory are reversed when the static models are extended to the dynamic real option signaling game models. Strebulaev *et al.* (2014) show that the tradeoff between debt and equity relies on asymmetric information regarding the firm's quality and riskiness of the project. In general, a project with a lower success probability is financed by equity, while less risky project is financed by debt. Unlike the static models, dynamic real option investment models (Dixit and Pindyck, 1994) allow option value of waiting, where the firm manager may choose the optimal investment time until the stochastic revenue flow rate reaches some optimal threshold. The real option value of waiting is similar to the option value of an American call option with an embedded optimal stopping right. Grenadier and Wang (2005) argue that the standard real options approach fails to account for the presence of information asymmetry between firm manager and outside investors in financing investment opportunities. The timing of investment and amount of capital raised are dependent on the interaction of option value of waiting and revelation of information on the firm quality. Using four corporate finance examples, Grenadier and Malenko (2011) illustrate that signaling incentives may erode the value of the option of waiting and speed up early exercise in some models, while option exercise may be delayed in other models. Xu and Li (2010) analyze equity financing decisions during financial distress with belief updating under asymmetric information. Bouvard (2014) proposes a signaling model to reveal the value of an investment project via sequential investments, thus initiating a learning phase where outside investors can update their beliefs on the probability of holding a high quality project. Besides capital financing, real options signaling game models have been adopted to analyze various corporate finance issues, like liquidation timing of a distressed firm (Nishihara and Shibata, 2017), strategic investment games of incumbent and entrant firms (Watanabe, 2016), decisions on selling out IPO (Nishihara, 2016), mergers and acquisitions strategies of bidder and target firms (Leung and Kwok, 2018). A comprehensive and general review of signaling theory can be found in Connelly *et al.* (2011).

Morellec and Schürhoff (2011) develop the dynamic real option signaling game models of corporate investment and financing decisions on raising funds to invest in a risky project. In their models, the firm management knows more about the project quality than potential external investors. The investment opportunity is assumed to have perpetual life instead of immediate evaporation. The firm has the discretion over the choice of equity or debt financing, and the investment timing. They show that the firm management can signal its quality type to outside investors using timing of investment and debt-equity mix. With information asymmetry, the low-type firm has the incentive to issue equity or debt securities that mimic those offered by the high-type firm, resulting in overvalued securities for the low-type firm and undervalued securities for the high-type firm. Since asymmetric information raises the cost of external funding for high-type firm, the high-type firm may choose to separate to reveal its true quality. These costs are quantified as information costs. Besides, abnormal returns are resulted upon revelation of firm quality at investment time. On the other hand, in a pooling equilibrium in which firm of either type raises funds in the same form and invest at the same time, asymmetric information reduces the value of the high-type firm and increases the cost of investment, while the low-type firm may benefit from information asymmetry. They show that this would force the high-type firm to speed up investment compared to the perfect information benchmark. The dynamic real option signaling game models of financing and investment with debt-equity mix have been extended by Clausen and Flor (2015) to include abandonment right and assets-in-place. They show that mature firms with larger assets-in-place mainly use debt financing, whereas high-growth firms without assets-in-place use equity financing more frequently and signal their type via early investment.

Most real option models of investment and financing in existing literature assume investment opportunity to be perpetual. The perpetuity assumption is used to enhance analytical tractability in determining the optimal investment thresholds and real option value functions. However, as new technologies become shorter in life span in recent technology environment, the usual perpetuity assumption in real option models may become questionable. Gryglewicz *et al.* (2008) pioneer the study of finite project life in real option investment models, where they assume the project revenue flow to last for a finite time horizon. They discover scenarios where investment behaviors under their finite life models may reverse some classical results under the perpetual models. For example, their finite life models show an acceleration of investment under increased uncertainty. This is particularly true when the project life is short and uncertainty level is low.

In this paper, we extend the perpetual real option signaling game model of Morellec and Schürhoff (2011) to a finite time horizon. Unlike most other earlier papers which consider tradeoff between equity and debt financing, we focus on information asymmetry between the firm manager and outside investors in issuance of equity to fund a risky project, where the investment opportunity has a finite time horizon. We analyze the optimal thresholds under separating and pooling equilibriums of the real option signaling model in equity financing. The quality of the firm (high-type or low-type) issuing new equity to finance an investment project is known only to the firm manager but not outside investors of the new shares. In our finite time model, the optimal investment thresholds are time dependent. The nature of separating and pooling equilibriums, information costs and abnormal returns change as the calendar time approaches the expiration date of the investment opportunity. In most cases, we manage to establish existence and uniqueness of the binding thresholds under various incentive compatibility constraints. We also perform characterization of the optimal investment thresholds under separating and pooling equilibrium. Also, we analyze the asymptotic behaviors of the optimal investment thresholds at the limits of perpetuity and close to expiry. Our finite time real option signaling game model provides comprehensive analysis of the time evolution of the investment strategies of equity financing under information asymmetry.

This paper is organized as follows. In Section 2, we present the finite time real option signaling game model formulation of equity financing of an investment project under information asymmetry between the firm manager and outside investors on the firm quality. The value function of the investment project and optimal investment threshold is formulated as an optimal stopping model, similar to finite-lived American option models. The budget constraint on funding the project is analyzed and the issue of adverse selection costs is explored. Section 3 examines the investment choices of the firm under separating equilibrium. We consider the incentive compatibility constraints of both high-type and low-type firm. The existence and uniqueness of the time dependent binding thresholds are established. We discuss the characterization of the least-cost separating equilibrium and the market value of the firm when the incentive compatibility constraints of both firm types are satisfied. In particular, we examine the separating equilibrium thresholds at time close to expiry. Section 4 discusses the optimal investment strategies of the firm and belief system under pooling equilibrium. The technical conditions for the existence of pooling equilibrium are established. We examine the nature of the pooling equilibrium threshold under perpetuity and at time close to expiry. In Section 5, we present numerical studies of the time evolution of the binding thresholds under various incentive compatibility constraints and optimal investment thresholds under separating and pooling equilibrium. We verify the existence of separating and pooling equilibrium by observing the relative positions of the optimal investment thresholds and their relations with the binding thresholds. We analyze how different model parameters may impact the nature of separating and pooling equilibriums. We examine the information costs and abnormal returns of both types of the firm under separating and pooling equilibriums when the calendar time approaches the expiration date of the investment opportunity. Section 6 contains a summary of the major findings and results of the paper.

2 Model formulation of equity financing

The model setup is an extension of the dynamic real option signaling game model of equity financing by Morellec and Schürhoff (2011) under perpetuity to a finite time horizon. A firm is supposed to commit the financing decision of choosing the optimal time to issue equity to raise capital in order to fund the investment of a risky project. The direct cost of investment is assumed to be the constant value I and the investment is irreversible. The time window of the investment opportunity is limited to a fixed finite time horizon T. The project produces a continuous revenue flow rate, where the revenue level depends on the firm type k. We assume only two types of firm quality, where k can be l (low-type) or h (high-type). We assume that all agents in the financial markets are risk neutral and cash flows are discounted at the riskless interest rate r. The revenue flow rate of firm type k is assumed to be $\lambda_k X_t - f$. The multiplier λ_k may be λ_h or λ_l ($\lambda_h > \lambda_l > 0$), corresponding to the high-type firm or low-type firm, respectively. Also, f is the constant rate of operating expenses of the investment project and X_t is the observable stochastic revenue flow rate that evolves according to the following Geometric Brownian motion:

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}Z_t, \quad X_0 > 0. \tag{2.1}$$

Here, Z_t is the standard Brownian motion, μ is the constant drift rate satisfying $\mu < r$ (no bubble condition) and σ is the constant volatility. Let Λ denote the discrete Bernoulli random variable that assumes the two possible values, $\{\lambda_h, \lambda_l\}$ with $\lambda_h > \lambda_l > 0$. We assume that the firm type is the private information held by the firm. The outside investors have only the probabilistic assessment of the firm type, with $P[\Lambda = \lambda_h] = p$ and $P[\Lambda = \lambda_l] = 1 - p$, where p is deterministic and $p \in (0, 1)$.

To the type-k firm, the present value of the perpetual revenue flow generated by the investment project at time t is given by

$$E_t \left[\int_t^\infty \mathrm{e}^{-r(s-t)} \lambda_k X_s \,\mathrm{d}s \middle| X_t = X \right] = \frac{\lambda_k X}{r - \mu}, \quad k = h \text{ or } l, \tag{2.2a}$$

where E_t denotes the expectation based on the information at time t. We write $\Pi(X) = \frac{X}{r-\mu}$ for notational convenience. Let F denote the present value of the future perpetual stream of operating expenses of the investment project, where

$$F = \int_t^\infty \mathrm{e}^{-r(s-t)} f \,\mathrm{d}s = \frac{f}{r}. \tag{2.2b}$$

Let I denote the constant direct upfront cost of investment. The firm management chooses the optimal timing of investment such that the expectation of the discounted revenue flow generated by the investment project is maximized.

Value functions under complete information

The value function of the investment project for the type-k firm, $k = h$ or l, under complete information prior to investment is given by

$$V_k^c(X,t) = \sup_{u\in[t,T]} E_t\left[e^{-r(u-t)}(\lambda_k\Pi(X_u) - F - I)^+ \middle| X_t = X\right], \quad 0 \le X \le X_k^*(t). \tag{2.3}$$

Here, $I_k^c(X) = \lambda_k\Pi(X) - F - I$, $k = h$ or l, is the intrinsic value of the real option value function $V_k^c(X,t)$. The value function under complete information serves as the benchmark for comparison of the value function under information asymmetry. We expect that the value functions of different firm types under separating and pooling equilibriums are less than the benchmark complete information counterparts. The differences in the value functions under complete and incomplete information are quantified as the information costs. In our finite time investment model, the optimal investment time is taken within the finite time interval $[t,T]$. The value function resembles that of a finite life American call option. The corresponding time dependent optimal threshold is denoted by $X_k^*(t)$, where $k = h, l$. As $\lambda_h > \lambda_l$, we have $I_h^c(X) > I_l^c(X)$, so $V_h^c(X,t) > V_l^c(X,t)$ and $X_h^*(t) < X_l^*(t)$ (see Table 1). We assume that the stochastic state variable X_t starts at a sufficiently low level and the firm has to wait until X_t reaches the optimal investment threshold from below.

Belief systems

The signal sent to the investors is X_{inv}, where X_{inv} is the investment threshold of the revenue flow rate level. The belief on Λ can be categorized into three types:

(i) $\Lambda = \lambda_l$, the true "low" type of the firm is revealed to the investors;

(ii) $\Lambda = \lambda_h$, the true "high" type of the firm is revealed to the investors;

(iii) $\Lambda = \lambda_p = p\lambda_h + (1-p)\lambda_l$, a probabilistic belief on Λ since the signal fails to reveal the type of the firm to the investors.

Budget constraints

For simplicity, we let the firm have one share of common equity before issuance of the new shares of equity for financing the project. We assume that investors would choose to break even in expectation, and the number of shares $n(X_t;\Lambda)$ issued would be based on the belief of the investors on Λ. The number of shares outstanding after financing becomes $1 + n(X_t;\Lambda)$, which indicates dilution of ownership of the incumbent shareholders. The number of shares to be issued at the time of investment is dictated by the budget constraint, where the value of new shares equals the direct investment cost I. At $X_t = X$, the budget constraint gives

$$n(X;\Lambda)\frac{\Lambda\Pi(X) - F}{1 + n(X;\Lambda)} = I,$$

so that the number of new shares to be issued based on the belief on Λ is given by

$$n(X;\Lambda) = \frac{I}{\Lambda\Pi(X) - F - I}. \tag{2.4}$$

As $n(X;\Lambda)$ should be non-negative, $\Lambda\Pi(X) - F - I \ge 0$. This is consistent with the intuition that no share will be issued if the perceived net present value of the project $\Lambda\Pi(X) - F - I$ is negative. This non-negativity condition dictates that the investment threshold under the belief system $\Lambda = \lambda_k$ must observe the lower bound $\hat{X}_k^0$, which is given by

$$\frac{\lambda_k\hat{X}_k^0}{r-\mu} - F - I = 0 \quad \text{or} \quad \hat{X}_k^0 = \frac{(F+I)(r-\mu)}{\lambda_k}, \quad k = h,\ l \text{ or } p. \tag{2.5}$$

Information costs

We would like to characterize the information costs in equity financing under information asymmetry. Since $n(X;\lambda_h) < n(X;\lambda_l)$, the incumbent shareholders of the low-type firm can benefit from a lower ownership dilution when Λ is perceived as λ_h instead of the true type λ_l. This shows why the firm management of the low-type firm has the incentive for pooling. On the other hand, the high-type firm suffers a loss in option value of waiting when it chooses to speed up investment by lowering the investment threshold in a separating equilibrium. When the high-type firm invests under the least-cost separating strategy, the true type is revealed to the outside investors. This leads to a jump in the firm's stock price, termed as abnormal return (Morellec and Schürhoff, 2011). In Section 5, we provide formal definitions of information costs and abnormal returns.

Since the low-type firm has the incentive to sell overpriced securities through mimicking the investment strategy of the high-type firm in terms of investment timing and number of equity shares, this corresponds to a pooling equilibrium where both high-type firm and low-type firm choose the same investment strategy. This shows how asymmetric information imposes information costs on the high-type firm.

3 Separating equilibrium of equity financing

The high-type firm may adopt the separating strategy to reveal its true quality. We would like to examine the characterization of the separating equilibrium of equity financing in which the two types of firm adopt different investment thresholds and issue fairly priced equity. The investigation of a separating equilibrium is related to finding the investment threshold below which the high-type firm remains to be profitable to adopt equity financing for investment while the low-type firm is non-profitable to mimic the high-type firm through equity financing. We explore the time dependence of the separating investment thresholds of two types of the firm, in particular, their behaviors at time close to expiry.

3.1 Incentive compatibility constraint of the low-type firm

First, we consider the incentive compatibility constraint (ICC) of the low-type firm under separating equilibrium. When deciding on whether to mimic or not, the low-type firm balances the gain on the overpricing of the equity shares and loss in investment value of the project due to lowering of investment threshold (speeding up investment time). Suppose the low-type firm chooses to mimic the investment decision of the high-type firm at $X_t = X$, the number of new shares issued equals $n(X;\lambda_h)$. The value $H_l(X)$ of the low-type firm held by the incumbent shareholders right after investment under the belief $\Lambda = \lambda_h$ is given by the diluted ownership of the low-type firm value, where

$$H_l(X) = \frac{\lambda_l\Pi(X) - F}{1 + n(X;\lambda_h)} = \frac{\lambda_l\Pi(X) - F}{\lambda_h\Pi(X) - F}\left[\lambda_h\Pi(X) - F - I\right]. \tag{3.1}$$

To characterize the domain of definition of $H_l(X)$, it is necessary to observe non-negativity of both $n(X;\lambda_h)$ and $H_l(X)$. To ensure non-negativity of $n(X;\lambda_h)$, we require $X \geq \hat{X}_h^0$ [see eq. (2.5)]. On the other hand, $\lambda_l\Pi(X) - F \geq 0 \iff X \geq \frac{F(r-\mu)}{\lambda_l}$. Combining the results, the lower bound of the domain of definition is $\max\left(\hat{X}_h^0, \frac{F(r-\mu)}{\lambda_l}\right)$; and we write $\hat{\hat{X}}_l^0 = \max\left(\hat{X}_h^0, \frac{F(r-\mu)}{\lambda_l}\right)$ for notational convenience. Note that $H_l(X)$ always assumes zero value at $\hat{\hat{X}}_l^0$ and $H_l(X) > 0$ when $X > \hat{\hat{X}}_l^0$. Also, as mimicking requires speeding up investment, the investment threshold X under mimicking would not go beyond the optimal threshold $X_l^*(t)$ of the low-type firm under complete information. Hence, the upper bound of the domain of definition of $H_l(X)$ is $X_l^*(t)$. In summary, the domain of definition of $H_l(X)$ is seen to be $[\hat{\hat{X}}_l^0, X_l^*(t)]$.

Recall that the low-type firm may follow its first-best strategy under complete information and chooses equity financing for investment at the trigger threshold $X_l^*(t)$. To observe the incentive compatibility constraint, the low-type firm may prefer mimicking the high-type firm only if the low-type firm's value after investment $H_l(X)$ is greater than $V_l^c(X,t)$ [defined in eq. (2.3)]. In other words, the low-type firm may prefer mimicking the high-type and invests at the threshold X below $X_l^*(t)$ only if the following incentive compatibility constraint (ICC) of the low-type firm

$$G_l^s(X,t) = H_l(X) - V_l^c(X,t) > 0, \quad \hat{\hat{X}}_l^0 \le X \le X_l^*(t) \tag{3.2}$$

is observed.

In order that the firm of high-type can separate from low-type, the high-type firm should not delay its investment to allow X_t to reach above the low-type firm's binding threshold $\overline{X}_l^s(t)$, where $\overline{X}_l^s(t)$ solves $G_l^s(X,t) = 0$. When $X \le \overline{X}_l^s(t)$, we have $H_l(X) \le V_l^c(X,t)$, so the low-type firm is non-profitable to mimic as high-type since the ICC of the low-type firm is violated [see ineq. (3.2)]. We manage to establish existence and uniqueness of the threshold $\overline{X}_l^s(t)$ such that the binding condition for separating based on the ICC of the low-type firm is satisfied when $X_t \le \overline{X}_l^s(t)$, the details of which are summarized in Lemma 1.

Lemma 1. *There exists a unique threshold value $\overline{X}_l^s(t)$, where $\overline{X}_l^s(t) \in [\hat{\hat{X}}_l^0, X_l^*(t)]$, such that the firm of low-type has no incentive to mimic high-type through equity financing when $X \le \overline{X}_l^s(t)$. That is, the following binding condition for separating equilibrium based on the ICC of the low-type firm holds, where*

$$H_l(X) \le V_l^c(X,t) \tag{3.3}$$

when $X \le \overline{X}_l^s(t)$.

The proof of Lemma 1 is presented in Appendix A. As a remark, though there is no closed form solution for $V_l^c(X,t)$, the time dependent threshold $\overline{X}_l^s(t)$ can be found using numerical methods. In Figure 1, we illustrate the relative positions of the binding threshold and optimal investment threshold, and indicate their significance in characterizing the incentive compatibility constraint and first-best investment strategy of the low-type firm.

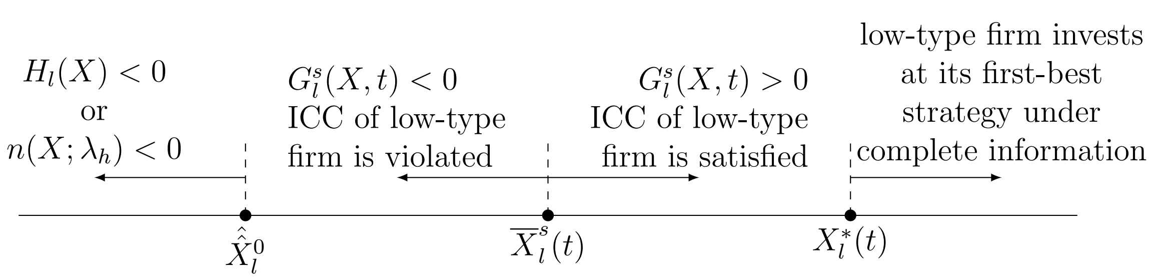


Figure 1: Financial significance of the relative positions of the binding threshold and optimal investment threshold on the incentive compatibility constraint and the investment strategy of the low-type firm.

3.2 Incentive compatibility constraint of the high-type firm

Next, we consider the incentive compatibility constraint (ICC) from the perspective of the high-type firm under separating equilibrium. The ICC of the high-type firm is dictated by the value of separating from the low-type at investment threshold X to be greater than or equal to the value of not separating. Under the belief $\Lambda = \lambda_l$, which dictates that the high-type firm cannot separate as it is perceived as low-type, the value of the incumbent shareholders of the high-type firm is given by

$$\frac{\lambda_h \Pi(X) - F}{1 + n(X;\lambda_l)} = \frac{\lambda_h \Pi(X) - F}{\lambda_l \Pi(X) - F}[\lambda_l \Pi(X) - F - I].$$

This is because the number of shares that has to be issued is $n(X;\lambda_l)$ in order to meet the investment cost I when the high-type firm is perceived as low-type. Since $n(X;\lambda_l) > n(X;\lambda_h)$, the firm value is more diluted due to unfavorable belief on the firm type. Let $V_h^m(X,t)$ denote the value of the high-type firm that follows this strategy of not separating from the low-type firm and $X_h^{*m}(t)$ be the corresponding time dependent optimal threshold of investment under $\Lambda = \lambda_l$. The optimal stopping rule dictates that

$$V_h^m(X,t) = \sup_{u\in[t,T]} E_t\left[\mathrm{e}^{-r(u-t)}\frac{\lambda_h \Pi(X_u) - F}{\lambda_l \Pi(X_u) - F}[\lambda_l \Pi(X_u) - F - I]\middle| X_t = X\right]. \tag{3.4}$$

For convenience, we denote the intrinsic value of the high-type firm perceived as low-type by

$$H_h(X) = \frac{\lambda_h \Pi(X) - F}{\lambda_l \Pi(X) - F}[\lambda_l \Pi(X) - F - I].$$

We can deduce that $X_h^*(t) < X_h^{*m}(t) < X_l^*(t)$ since $I_h^c(X) > H_h(X) > I_l^c(X)$. In Table 1, we summarize the real option value functions with differing intrinsic values, and deduce their relative magnitudes and relative positions of their optimal thresholds.

Table 1: Real option value functions and optimal thresholds with differing intrinsic values.

real option value functions	intrinsic values	optimal threshold
$V_h^c(X,t)$	$I_h^c(X) = \frac{\lambda_h X}{r-\mu} - F - I$	$X_h^*(t)$
$V_h^m(X,t)$	$H_h(X) = \frac{\frac{\lambda_h X}{r-\mu}-F}{\frac{\lambda_l X}{r-\mu}-F}\left(\frac{\lambda_l X}{r-\mu} - F - I\right)$	$X_h^{*m}(t)$
$V_l^c(X,t)$	$I_l^c(X) = \frac{\lambda_l X}{r-\mu} - F - I$	$X_l^*(t)$
$V_h^c > V_h^m > V_l^c$	$I_h^c(X) > H_h(X) > I_l^c(X)$	$X_h^*(t) < X_h^{*m}(t) < X_l^*(t)$

At a given level X, the value upon immediate investment of the high-type firm is $\lambda_h \Pi(X) - F - I$. A necessary condition for the high-type firm to separate from low-type is given by

$$G_h^s(X,t) = \lambda_h \Pi(X) - F - I - V_h^m(X,t) \geq 0, \quad \hat{X}_h^0 \leq X \leq X_h^{*m}(t). \tag{3.5}$$

The above condition is characterized as the ICC of the high-type firm. The domain of definition of $G_h^s(X,t)$ is the intersection of the domains of definition of the two constituent functions: $\lambda_h \Pi(X) - F - I$ and $V_h^m(X,t)$.

The next step is to analyze the mathematical properties of $G_h^s(X,t)$. We manage to show that there exists a unique root $\overline{X}_h^s(t)$ of $G_h^s(X,t)$ within $(\hat{X}_h^0, X_h^{*m}(t))$, where $G_h^s(X,t) \geq 0$ when $X \geq \overline{X}_h^s(t)$. These results are summarized in Lemma 2, the proof of which is presented in Appendix B.

Lemma 2. *There exists a unique root of $G_h^s(X,t)$ that lies within $(\hat{X}_h^0, X_h^{*m}(t))$. Also, the ICC of the high-type firm of separating is satisfied where*

$$\lambda_h \Pi(X) - F - I \geq V_h^m(X,t) \tag{3.6}$$

when $X \geq \overline{X}_h^s(t)$. In other words, the high-type firm prefers not to separate from low-type when X has not reached the level $\overline{X}_h^s(t)$.

Based on the results in Lemmas 1 and 2, in order to satisfy the ICC of both firm types, the relative magnitudes of the two binding thresholds are required to observe $\overline{X}_h^s(t) \leq \overline{X}_l^s(t)$, where $\overline{X}_h^s(t)$ and $\overline{X}_l^s(t)$ are the respective root of $G_h^s(X,t)$ and $G_l^s(X,t)$. When $X \in [\overline{X}_h^s(t), \overline{X}_l^s(t)]$, the high-type firm remains profitable to separate while the low-type firm has no incentive to mimic high-type (see Figure 2).

By virtue of uniqueness of root of $G_h^s(X,t)$ and $G_h^s(X,t) \geq 0$ when $X \geq \overline{X}_h^s(t)$ (see Figure B2), we can deduce that $\overline{X}_h^s(t) \leq \overline{X}_l^s(t)$ if and only if $G_h^s(\overline{X}_l^s(t),t) \geq 0$. In subsequent discussion, we assume $G_h^s(\overline{X}_l^s(t),t) \geq 0$ in our analysis of separating equilibrium; if otherwise, separating equilibrium does not prevail since there does not exist X such that the ICC of both firm types are satisfied. By examining the definitions of $G_h^s(X,t)$ and $G_l^s(X,t)$, it is straightforward to establish (i) $\overline{X}_h^s(t)$ is increasing with respect to λ_l; (ii) $\overline{X}_l^s(t) - \overline{X}_h^s(t)$ is sufficiently large when $\lambda_l \ll \lambda_h$, and (iii) $\overline{X}_l^s(t) = \overline{X}_h^s(t)$ when $\lambda_l = \lambda_h$. These mathematical properties are sufficient to allow us to postulate that $\overline{X}_h^s(t) \leq \overline{X}_l^s(t)$. The rigorous mathematical proof is not straightforward to establish since $G_h^s(X,t)$ and $G_l^s(X,t)$ involve real option value functions with embedded optimal stopping right. On the other hand, depending on the relative magnitudes of λ_h and λ_l, it is possible to have either $X_h^*(t) < \overline{X}_l^s(t)$ or $X_h^*(t) \geq \overline{X}_l^s(t)$. The financial significance related to the relative positions of the thresholds, $\overline{X}_h^s(t)$, $\overline{X}_l^s(t)$ and $X_h^*(t)$, is illustrated in the numerical plots presented in Section 5.

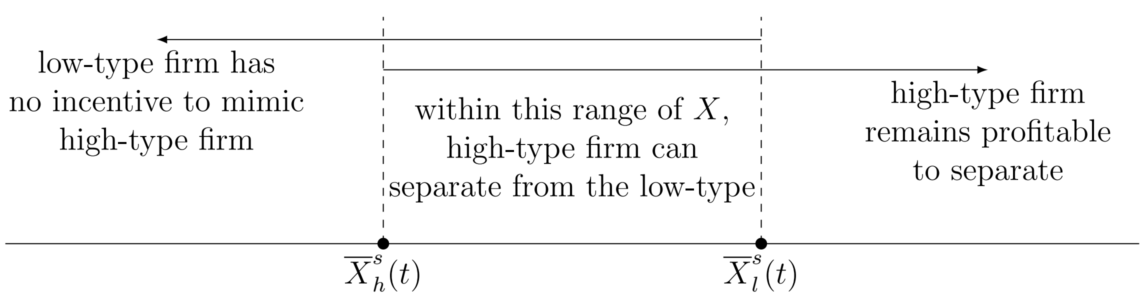


Figure 2: Financial significance related to the relative positions of the binding thresholds $\overline{X}_l^s(t)$ and $\overline{X}_h^s(t)$ derived from the incentive compatibility constraints of both types of the firm.

3.3 Least-cost separating equilibrium

To examine the nature of the least-cost separating equilibrium, we start with the high-type firm and consider the two scenarios: $(i) X_h^*(t) < \overline{X}_l^s(t)$, $(ii) X_h^*(t) \geq \overline{X}_l^s(t)$. In scenario (i), the high-type firm invests optimally at the first-best optimal threshold $X_h^*(t)$ since its real option value function is maximized at $X_h^*(t)$. Given that $X_h^*(t) < \overline{X}_l^s(t)$, the low-type firm has no incentive to mimic the high-type firm to invest at $X_h^*(t)$ since the ICC of the low-type firm is violated. Therefore, the high-type firm can separate from low-type and invests at $X_h^*(t)$. For the second scenario where $X_h^*(t) \geq \overline{X}_l^s(t)$, though the ICC for the high-type firm is satisfied when $X \in [\overline{X}_h^s(t), \overline{X}_l^s(t)]$, the high-type firm would not invest at $X < \overline{X}_l^s(t)$ since the cost of separating is lower at $\overline{X}_l^s(t)$. This is because the strategy of investing at a lower threshold means higher dilution due to more shares to be issued. On the other hand, in order that the separating strategy constitutes a Perfect Bayesian Equilibrium (PBE), the high-type firm should not invest at a threshold that is above $\overline{X}_l^s(t)$. The proof of the strategy of investing at $\overline{X}_l^s(t)$ that constitutes a PBE is presented in Appendix C.

The signal sent by the firm management to outside investors is the threshold X_{inv} at which investment takes place. Combining these results, since $X_h^*(t)$ and $\overline{X}_l^s(t)$ are unique, we conclude that the unique least-cost separating equilibrium of the high-type firm under $G_h^s(\overline{X}_l^s(t), t) \geq 0$ is to invest at $X_{inv} = \min(X_h^*(t), \overline{X}_l^s(t))$. The separating equilibrium prevails under the belief system that the firm is perceived as low-type when it invests at X_{inv} higher than $\min(X_h^*(t), \overline{X}_l^s(t))$.

Once the separating equilibrium strategy is identified, the real option value $V_h^s(X,t)$ of the high-type firm under separating can be computed by respecting the corresponding strategy of investing at $\min(X_h^*(t), \overline{X}_l^s(t))$ at all times. In other words, the real option value function $V_h^s(X,t)$ is not dictated by the optimal stopping rule. Indeed, $V_h^s(X,t)$ resembles the up-barrier call option defined on the domain of definition $[0, \min(X_h^*(t), \overline{X}_l^s(t))]$, where both boundaries $X_h^*(t)$ and $\overline{X}_l^s(t)$ can be found earlier in separate calculations. The rebate and terminal payoff of $V_h^s(X,t)$ are both equal to $\lambda_h \Pi(X) - F - I$. The full formulation of $V_h^s(X,t)$ is presented in Section 5 [see eq. (5.2)].

Under the assumption of $G_h^s(\overline{X}_l^s(t), t) \geq 0$, the low-type firm invests optimally at $X_l^*(t)$ as its first-best strategy. As a result, the real option value of the low-type firm before investment equals $V_l^c(X,t)$ for $X < X_l^*(t)$, where $V_l^c(X,t)$ is defined in eq. (2.3).

For $X < \min(X_h^*(t), \overline{X}_l^s(t))$, the firm type has not been identified, so the market value of the firm is the expectation of the real option value under the two possible types according to the belief system: $\mathrm{P}[\Lambda = \lambda_h] = p$ and $\mathrm{P}[\Lambda = \lambda_l] = 1 - p$. Given that $\overline{X}_h^s(t) \leq \overline{X}_l^s(t)$, the characterization of the separating equilibrium, belief system and the market value of the firm before investment are summarized in Proposition 3.

Proposition 3. *For $X \in [\overline{X}_h^s(t), \overline{X}_l^s(t)]$, the incentive compatibility constraint for the high-type firm is satisfied, but that of the low-type firm is not satisfied. Suppose the firm is of high-type, then it invests more aggressively than its first-best threshold $X_h^*(t)$ when $\overline{X}_l^s(t) < X_h^*(t)$; otherwise, the high-type firm chooses to invest optimally at $X_h^*(t)$ at zero separating cost. There exists a unique least-cost separating equilibrium in which the high-type firm invests at $\min(X_h^*(t), \overline{X}_l^s(t))$. The low-type firm invests optimally at $X_l^*(t)$ as its first-best strategy. These equilibrium strategies of the high-type firm and low-type firm can be sustained under the belief system:*

$$\Lambda(X_{inv}) = \begin{cases} \lambda_l & \text{if } X_{inv} > \min(\overline{X}_l^s(t), X_h^*(t)), \\ \lambda_h & \text{otherwise}, \end{cases} \tag{3.7a}$$

where X_{inv} is the investment threshold. The market value of the firm before investment is given by

$$V_m^s(X,t) = pV_h^s(X,t) + (1-p)V_l^c(X,t), \tag{3.7b}$$

where $p = \mathrm{P}[\Lambda = \lambda_h]$.

3.4 Separating equilibrium strategies at time close to expiry

It is instructive to examine the asymptotic limits of the separating equilibrium strategies of both types of firm at time close to expiry. The real option value function $V_k^c(X,t)$, $k = l$ or h, resembles an American call option on an underlying asset with continuous dividend yield q and strike K. The exercise payoff of the American call option is $S - K$, where S is the price of the underlying asset. Note that q is equivalent to $r - \mu$ and strike K is equivalent to $\frac{(F+I)(r-\mu)}{\lambda_k}$ in $V_k^c(X,t)$. Let $S^*(\tau)$ denote the early exercise boundary of the American call option, where τ is the time to expiry. It is known that (Kwok, 2008)

$$\lim_{\tau \to 0^+} S^*(\tau) = K \max\left(1, \frac{r}{q}\right),$$

where r is the interest rate. Referring back to $V_k^c(t)$ and $X_k^*(t)$, we deduce that the asymptotic limit at $t \to T^-$ of the first-best investment threshold of firm of type k is given by

$$\lim_{t \to T^-} X_k^*(t) = \frac{r(F+I)}{\lambda_k}, \quad k = l \text{ or } h. \tag{3.8}$$

To examine $\overline{X}_l^s(t)$ as $t \to T^-$, we recall that $H_l(X) = 0$ at $X = \hat{\hat{X}}_l^0$, which is the lower bound of the domain of definition of $H_l(X)$. On the other hand, the strike of the low-type firm's option $V_l^c(X,t)$ is $\frac{(F+I)(r-\mu)}{\lambda_l}$, which is seen to be greater than $\hat{\hat{X}}_l^0 = \max\left(\frac{(F+I)(r-\mu)}{\lambda_h}, \frac{F(r-\mu)}{\lambda_l}\right)$. Therefore, $V_l^c(X,T^-)$ is close to zero at $X = \hat{\hat{X}}_l^0$ since it is likely to expire out-of-the-money at T. Hence, the root of $G_l^s(X,T^-)$ tends to $\hat{\hat{X}}_l^0$ at time close to expiry, giving

$$\lim_{t \to T^-} \overline{X}_l^s(t) = \hat{\hat{X}}_l^0. \tag{3.9}$$

Combining the results in eqs. (3.8, 3.9) and inferring from Proposition 3, we deduce that at $t \to T^-$, the high-type firm invests at

$$\min(X_h^*(T^-), \overline{X}_l^s(T^-)) = \min\left(\frac{r(F+I)}{\lambda_h}, \hat{\hat{X}}_l^0\right)$$

and the low-type firm invests at $\frac{r(F+I)}{\lambda_l}$ under the least-cost separating equilibrium.

4 Pooling equilibrium of equity financing

In a pooling equilibrium of equity financing, regardless of its firm type, the firm invests at the same threshold and issues the same number of equity shares to finance the capital outlay. The outside investors are not able to distinguish the firm type, so the belief on Λ is

$$\Lambda = \lambda_p = p\lambda_h + (1-p)\lambda_l. \tag{4.1}$$

The number of shares to be issued is determined by the budget constraint, where

$$n(X;\lambda_p)\frac{\lambda_p \Pi(X) - F}{1 + n(X;\lambda_p)} = I,$$

so that

$$n(X;\lambda_p) = \frac{I}{\lambda_p \Pi(X) - F - I}.$$

To ensure non-negativity of $n(X;\lambda_p)$, we require $X \geq \hat{X}_p^0$, where $\hat{X}_p^0 = \frac{(F+I)(r-\mu)}{\lambda_p}$.

4.1 Incentive compatibility constraints

Similar to the separating equilibrium discussed in Section 3, we consider the ICC of both types of firm. For the low-type firm, in order that pooling equilibrium exists, pooling with high-type should dominate its first-best strategy. Let $H_l^p(X)$ be the value of the low-type firm right after investment under the belief $\Lambda = \lambda_p$. The ICC of the low-type firm under pooling equilibrium is given by

$$H_l^p(X) = \frac{\lambda_l \Pi(X) - F}{1 + n(X; \lambda_p)} = \frac{\lambda_l \Pi(X) - F}{\lambda_p \Pi(X) - F}[\lambda_p \Pi(X) - F - I] > V_l^c(X,t). \tag{4.2}$$

The domain of definition of $H_l^p(X)$ is found to be $[\hat{\hat{X}}_p^0, X_l^*(t)]$, where $\hat{\hat{X}}_p^0 = \max\left(\hat{X}_p^0, \frac{F(r-\mu)}{\lambda_l}\right)$. Let $\overline{X}_l^p$ be the binding threshold of the low-type firm under pooling such that $\overline{X}_l^p(t)$ solves

$$G_l^p(X,t) = H_l^p(X) - V_l^c(X,t) = 0.$$

By following a similar proof as in Lemma 1, we can establish that there exists a unique threshold value $\overline{X}_l^p(t)$, where $\hat{\hat{X}}_p^0 < \overline{X}_l^p(t) < X_l^*(t)$, such that low-type firm has no incentive to pool with high-type through equity financing when $X \leq \overline{X}_l^p(t)$. Furthermore, since $\lambda_p < \lambda_h$, $\overline{X}_l^p(t)$ lies between $\overline{X}_l^s(t)$ and $X_l^*(t)$.

Recall from Proposition 3 that the separating equilibrium strategy of the high-type firm is to invest at $X_{inv} = \min(X_h^*(t), \overline{X}_l^s(t))$. Since the high-type firm cannot do better in pooling with low-type firm than its first-best value $V_h^c(X,t)$, so pooling equilibrium of the high-type firm exists only if $\overline{X}_l^s(t) < X_h^*(t)$. Using the geometric properties of $G_l^s(X,t)$ [see Figures A2(a,b)], we deduce that

$$\overline{X}_l^s(t) < X_h^*(t) \iff G_l^s(X_h^*(t), t) > 0. \tag{4.3}$$

Let $X_h^{*p}(t)$ denote the critical threshold that the high-type firm chooses to pool with low-type and $V_h^p(X,t)$ denote the corresponding value function of the high-type firm under pooling. Since the high-type firm chooses to pool with low-type only if the value at $\overline{X}_l^s(t)$ under pooling is higher than that under separating equilibrium. Mathematically, this technical condition can be expressed as

$$V_h^p(\overline{X}_l^s(t), t) > V_h^s(\overline{X}_l^s(t), t). \tag{4.4}$$

In summary, the conditions specified in eqs. (4.3) and (4.4) represent the necessary conditions for the existence of pooling equilibrium.

4.2 Determination of threshold and belief system of pooling equilibrium

Under pooling, the firm value upon investment at threshold X for firm type k, $k = l$ or h, is

$$\frac{\lambda_k \Pi(X) - F}{\lambda_p \Pi(X) - F}[\lambda_p \Pi(X) - F - I].$$

The firm of type k can choose the optimal investment threshold X_k^{*p} and the associated real option value $V_k^p(X,t)$ is given by

$$V_k^p(X,t) = \sup_{u \in [t,T]} E_t \left[\mathrm{e}^{-r(u-t)} \frac{\lambda_k \Pi(X_u) - F}{\lambda_p \Pi(X_u) - F}[\lambda_p \Pi(X_u) - F - I] \middle| X_t = X \right], \quad k = h \text{ or } l. \tag{4.5}$$

By comparing the respective investment payoffs, it is seen that

$$X_h^*(t) < X_h^{*p}(t) < X_l^{*p}(t) < X_l^*(t). \tag{4.6}$$

Pooling equilibrium exists only if $\overline{X}_l^s(t) < X_h^*(t)$ and the ICC of high-type firm as specified in eq. (4.4) is satisfied. It is necessary to consider the different scenarios corresponding to different relative positions of $\overline{X}_l^p(t)$, $X_h^{*p}(t)$ and $X_l^{*p}(t)$. Since the ICC of low-type firm has to be satisfied, it is irrelevant to consider the scenario where $\overline{X}_l^p(t) > X_l^{*p}(t)$. On the other hand, while $X_h^{*p}(t) < X_l^{*p}(t)$, $\overline{X}_l^p(t)$ may lie below $X_h^{*p}(t)$ or between $X_h^{*p}(t)$ and $X_l^{*p}(t)$. We explore the existence of pooling equilibrium under two possible belief systems and two cases of relative positions (*i*) $\overline{X}_l^p(t) \leq X_h^{*p}(t)$, (*ii*) $\overline{X}_l^p(t) > X_h^{*p}(t)$. We consider the following belief systems:

1. Belief system I

$$\Lambda(X_{inv}) = \begin{cases} \lambda_h, & \text{if } X_{inv} \leq \overline{X}_l^s(t), \\ \lambda_p, & \text{if } \overline{X}_l^s(t) < X_{inv} \leq X_h^{*p}(t), \\ \lambda_l, & \text{otherwise.} \end{cases}$$

   The outside investors cannot determine the exact type of the firm if it invests at X_{inv}, when $\overline{X}_l^s(t) < X_{inv} < X_h^{*p}(t)$.

   (a) $\overline{X}_l^p(t) \leq X_h^{*p}(t)$
   The high-type firm would choose to invest at the optimal threshold $X_h^{*p}(t)$ under pooling equilibrium. Also, the low-type firm would also choose to invest at $X_h^{*p}(t)$ since the ICC of the low-type firm is satisfied at $X_h^{*p}(t)$ since $\overline{X}_l^p(t) \leq X_h^{*p}(t)$. Under the belief system, the low-type firm would not deviate to the optimal pooling threshold $X_l^{*p}(t)$ since it would be perceived as low-type. Therefore, pooling equilibrium exists where the firm invests at $X_h^{*p}(t)$, regardless of its type.

   (b) $\overline{X}_l^p(t) > X_h^{*p}(t)$
   Since $\overline{X}_l^p(t) > X_h^{*p}(t)$, the ICC of the low-type firm is not satisfied if the low-type firm chooses to invest at $X_h^{*p}(t)$. Also, the low-type firm would not invest at its optimal pooling threshold $X_l^{*p}(t)$, since it would be perceived as low-type according to the belief system. Hence, pooling equilibrium does not exist.

2. Belief system II

$$\Lambda(X_{inv}) = \begin{cases} \lambda_h, & \text{if } X_{inv} \leq \overline{X}_l^s(t), \\ \lambda_p, & \text{if } \overline{X}_l^s(t) < X_{inv} \leq X_l^{*p}(t), \\ \lambda_l, & \text{otherwise.} \end{cases}$$

   We extend the upper bound of the interval of X_{inv} within which $\Lambda(X_{inv}) = \lambda_p$ from $X_h^{*p}(t)$ to $X_l^{*p}(t)$.

   (a) $\overline{X}_l^p(t) \leq X_h^{*p}(t)$
   Under this new belief system where $\Lambda(X_{inv})$ remains to be λ_p until X_{inv} reaches to the higher optimal threshold $X_l^{*p}(t)$, the high-type firm would remain to invest optimally at $X_h^{*p}(t)$ while the low-type firm would deviate from $X_h^{*p}(t)$ to invest at its optimal pooling threshold $X_l^{*p}(t)$. Pooling equilibrium does not exist since the low-type firm benefits from deviation of investing at the lower threshold $X_h^{*p}(t)$.

   (b) $\overline{X}_l^p(t) > X_h^{*p}(t)$
   Pooling equilibrium does not exist since the high-type firm would invest optimally at $X_h^{*p}(t)$ while the low-type firm would not invest at $X_l^{*p}(t)$ when the ICC of the low-type firm is not satisfied.

In summary, provided that the following three conditions: $\overline{X}_l^s(t) < X_h^*(t)$, $V_h^p(\overline{X}_l^s(t),t) > V_h^s(\overline{X}_l^s(t),t)$ and $\overline{X}_l^p(t) \le X_h^{*p}(t)$ are satisfied, pooling equilibrium exists under belief system I. In addition, the following technical conditions on the binding thresholds, first-best threshold and pooling threshold have to be satisfied.

1. Two technical conditions on $\overline{X}_l^s(t)$ and $X_h^*(t)$, namely,
   (i) $\overline{X}_l^s(t) < X_h^*(t)$ since the high-type firm cannot do better in pooling than its first-best value;
   (ii) $V_h^p(\overline{X}_l^s(t),t) > V_h^s(\overline{X}_l^s(t),t)$ since the high-type firm value at $\overline{X}_l^s(t)$ under pooling equilibrium should be higher than that under separating equilibrium.
2. The binding threshold $\overline{X}_l^p(t)$ derived from the ICC of the low-type firm has to be less than or equal to the optimal investment threshold $X_h^{*p}(t)$ of the high-type firm under pooling equilibrium, namely, $\overline{X}_l^p(t) \le X_h^{*p}(t)$.

The characterization of pooling equilibrium and the associated belief system is summarized in Proposition 4.

Proposition 4. *Suppose $\overline{X}_l^s(t) < X_h^*(t)$, $V_h^p(\overline{X}_l^s(t),t) > V_h^s(\overline{X}_l^s(t),t)$ and $\overline{X}_l^p(t) \le X_h^{*p}(t)$, then there exists a Pareto-dominant pooling equilibrium where firm of either type would optimally invest at $X_h^{*p}(t)$. The pooling equilibrium can be sustained under the belief system*

$$\Lambda(X_{inv}) = \begin{cases} \lambda_h, & \text{if } X_{inv} \le \overline{X}_l^s(t), \\ \lambda_p, & \text{if } \overline{X}_l^s(t) < X_{inv} \le X_h^{*p}(t), \\ \lambda_l, & \text{otherwise}, \end{cases}$$

where X_{inv} is the investment threshold. Since $X_h^{*p}(t) < X_l^*(t)$, the low-type firm invests more aggressively at $X_h^{*p}(t)$ than its first-best strategy $X_l^*(t)$ under pooling equilibrium.

4.3 Pooling equilibrium threshold of the high-type firm at time close expiry and perpetuity

The pooling equilibrium threshold of the high-type firm is given by $X_h^{*p}(t)$, which is the optimal stopping threshold governed by $V_h^p(X,t)$ [see eq. (4.5)]. The exercise payoff of $V_h^p(X,t)$ involves the rational function $\frac{\lambda_h\Pi(X)-F}{\lambda_p\Pi(X)-F}[\lambda_p\Pi(X)-F-I]$, which exhibits non-standard payoff among usual contingent claims. The corresponding asymptotic values of the pooling equilibrium threshold $X_h^*(t)$ at $t \to T^-$ and perpetuity have to be determined with special considerations. We write the exercise payoff of $V_h^p(X,t)$ as

$$H_h^p(X) = \frac{\lambda_h\Pi(X)-F}{\lambda_p\Pi(X)-F}[\lambda_p\Pi(X)-F-I], \quad \hat{X}_h^p \le X < \infty, \tag{4.7}$$

where $\hat{X}_h^p = \max\left(\frac{F(r-\mu)}{\lambda_h}, \hat{X}_p^0\right)$ is the lower bound of the domain of definition of $H_h^p(X)$. The lower bound is imposed in order to ensure non-negativity of $H_h^p(X)$ and $n(X;\lambda_p)$.

Pooling equilibrium threshold at time close to expiry

Suppose the real option of pooling $V_h^p(X,t)$ survives at $t \to T^-$, the value function remains to satisfy the governing equation:

$$\frac{\partial V_h^p}{\partial t} + \frac{\sigma^2}{2}X^2\frac{\partial^2 V_h^p}{\partial X^2} + \mu X\frac{\partial V_h^p}{\partial X} - rV_h^p = 0.$$

By continuity of the option value, we expect that $\lim_{t \to T^-} V_h^p(X,t) = H_h^p(X)$ at $X > \hat{X}_h^p$. We then deduce that

$$\begin{aligned}\left.\frac{\partial V_h^p}{\partial t}\right|_{t=T^-} &= -\frac{\sigma^2}{2}X^2\frac{\partial^2 H_h^p}{\partial X^2} - \mu X\frac{\partial H_h^p}{\partial X} + rH_h^p \\ &= \frac{\sigma^2}{2}X^2[2\lambda_p(\lambda_h - \lambda_p)]\frac{IF(r-\mu)}{[\lambda_p X - F(r-\mu)]^3} \\ &\quad - \mu X\left\{\frac{\lambda_h}{r-\mu} + (\lambda_h - \lambda_p)\frac{IF(r-\mu)}{[\lambda_p X - F(r-\mu)]^2}\right\} \\ &\quad + r\frac{\lambda_h X - F(r-\mu)}{\lambda_p X - F(r-\mu)}\left(\frac{\lambda_p X}{r-\mu} - F - I\right).\end{aligned}$$

According to the theory of optimal stopping in American option models (Kwok, 2008), the optimal stopping threshold at $t \to T^-$ is determined by the threshold at which $\left.\frac{\partial V_h^p}{\partial t}\right|_{t=T^-}$ changes sign. By setting $\left.\frac{\partial V_h^p}{\partial t}\right|_{t=T^-} = 0$, the optimal stopping threshold $X_h^{*p}(T^-)$ at time close to expiry is found by solving the following quartic equation:

$$X^4 + bX^3 + cX^2 + dX + e = 0, \tag{4.8}$$

where

$$\begin{aligned}b &= -\frac{1}{\lambda_h\lambda_p}\{[3F(r-\mu) + rI]\lambda_h + rF\lambda_p\}, \\ c &= \frac{F(r-\mu)}{\lambda_h\lambda_p^2}\{[(\sigma^2 - \mu + 2r)I + 3(r-\mu)F]\lambda_h + [(\mu - \sigma^2 + r)I + 3rF]\lambda_p\}, \\ d &= -\frac{F^2(r-\mu)^2}{\lambda_h\lambda_p^3}\{(r-\mu)(F+I)\lambda_h + [3rF + (\mu + 2r)I]\lambda_p\}, \\ e &= \frac{F^3(r-\mu)^3}{\lambda_h\lambda_p^3}r(F+I).\end{aligned}$$

Pooling equilibrium threshold at perpetuity

We let $V_{\infty,h}^p(X)$ and $X_{\infty,h}^{*p}$ denote the value function and optimal threshold of pooling equilibrium under perpetuity. It is seen that $V_{\infty,h}^p(X)$ is given by (Kwok, 2008)

$$V_{\infty,h}^p(X) = H_h^p(X_{\infty,h}^{*p})\left(\frac{X}{X_{\infty,h}^{*p}}\right)^{\xi_+}, \quad 0 < X \leq X_{\infty,h}^{*p}, \tag{4.9}$$

where ξ_+ is the positive root of the quadratic equation:

$$\frac{\sigma^2}{2}\xi^2 + (\mu - \frac{\sigma^2}{2})\xi - r = 0.$$

Since optimality of the value function $V_{\infty,h}^p(X)$ is observed under the optimal threshold $X_{\infty,h}^{*p}$, we determine $X_{\infty,h}^{*p}$ such that $V_{\infty,h}^p(X)$ is maximized at $X = X_{\infty,h}^{*p}$. Use of calculus shows that the optimal pooling equilibrium threshold $X_{\infty,h}^{*p}$ under perpetuity can be obtained by solving the following cubic equation:

$$X^3 + \tilde{b}X^2 + \tilde{c}X + \tilde{d} = 0, \tag{4.10}$$

where

$$\tilde{b} = -\frac{r-\mu}{(\xi_+-1)\lambda_h\lambda_p}\{\lambda_h[2(\xi_+-1)F+\xi_+I]+\lambda_p\xi_+F\},$$
$$\tilde{c} = \frac{F(r-\mu)^2}{(\xi_+-1)\lambda_h\lambda_p^2}\{\lambda_h(\xi_+-1)(F+I)+\lambda_p[2\xi_+F+(\xi_++1)I]\},$$
$$\tilde{d} = -\frac{\xi_+(F+I)F^2(r-\mu)^3}{(\xi_+-1)\lambda_h\lambda_p^2}.$$

5 Numerical calculations of value functions, thresholds and information costs

In this section, we present numerical calculations of the real option value functions, binding and optimal thresholds of both types of firm under separating and pooling equilibriums. We also analyze the time evolution of the information costs and abnormal returns of the two types of firm. In particular, we discuss the asymptotic behaviors of the binding and optimal thresholds, information costs and abnormal return at time close to expiry of the investment opportunity.

Most of the real option value functions considered in this paper resemble the American call option model, except in some cases where the intrinsic value may involve rational function of the state variable X. Let $V(X,t)$ denote the prototype real option value function with intrinsic value $I(X)$ and $X^*(t)$ be the optimal stopping threshold. The linear complementarity formulation of $V(X,t)$ is given by

$$\left(\frac{\partial V}{\partial t}+\mu X\frac{\partial V}{\partial X}+\frac{\sigma^2}{2}X^2\frac{\partial^2 V}{\partial X^2}-rV\right)[V-I(X)]=0, \quad 0<X<\infty,\, 0<t<T, \tag{5.1}$$

with terminal condition: $V(X,T)=I(X)$ and boundary condition: $V(0,t)=0$. The value matching condition and smooth pasting condition at the optimal stopping threshold $X^*(t)$ are given by

$$V(X^*(t),t)=I(X^*(t)) \quad \text{and} \quad \left.\frac{\partial V(X,t)}{\partial X}\right|_{X=X^*(t)} = \left.\frac{\mathrm{d}\,I(X)}{\mathrm{d}\,X}\right|_{X=X^*(t)},$$

respectively. Both $V(X,t)$ and $X^*(t)$ can be computed effectively using the fully implicit finite difference scheme together with the Projected Successive-Over-Relaxation method (Kwok, 2008).

There is one exception of the real option value function that is not formulated as an optimal stopping model. The least-cost separating equilibrium strategy of the high-type firm is dictated by investing at $\min(X_h^*(t),\overline{X}_l^s(t))$. The corresponding real option value function $V_h^s(X,t)$ is governed by

$$\frac{\partial V_h^s}{\partial t}+\mu X\frac{\partial V_h^s}{\partial X}+\frac{\sigma^2}{2}X^2\frac{\partial^2 V_h^s}{\partial X^2}-rV_h^s=0, \quad 0<X<\min(X_h^*(t),\overline{X}_l^s(t)),\, 0<t<T, \tag{5.2}$$

with terminal payoff:

$$V_h^s(X,T)=(\lambda_h X-F-I)^+$$

and boundary conditions:

$$V_h^s(0,t)=0, \quad V_h^s(\min(X_h^*(t),\overline{X}_l^s(t)),t)=(\lambda_h\min(X_h^*(t),\overline{X}_l^s(t))-F-I)^+, \quad 0<t<T.$$

Once the time dependent barrier $\min(X_h^*(t), \overline{X}_l^s(t))$ has been found from earlier calculations, the value function $V_h^s(X,t)$ can be solved numerically using the fully implicit finite difference scheme, with specific treatment of the numerical boundary condition that deals with the time dependent barrier feature.

5.1 Separating equilibrium

In Section 3, we have analyzed the time dependent properties of the binding thresholds $\overline{X}_h^s(t)$ and $\overline{X}_l^s(t)$, and the investment threshold X_{inv} of the high-type firm under separating equilibrium in our finite time real option signalling game model of equity financing. The characterization of the least-cost separating equilibrium has been summarized in Proposition 3. We performed numerical studies to examine and verify these theoretical results.

Binding thresholds and first-best investment threshold

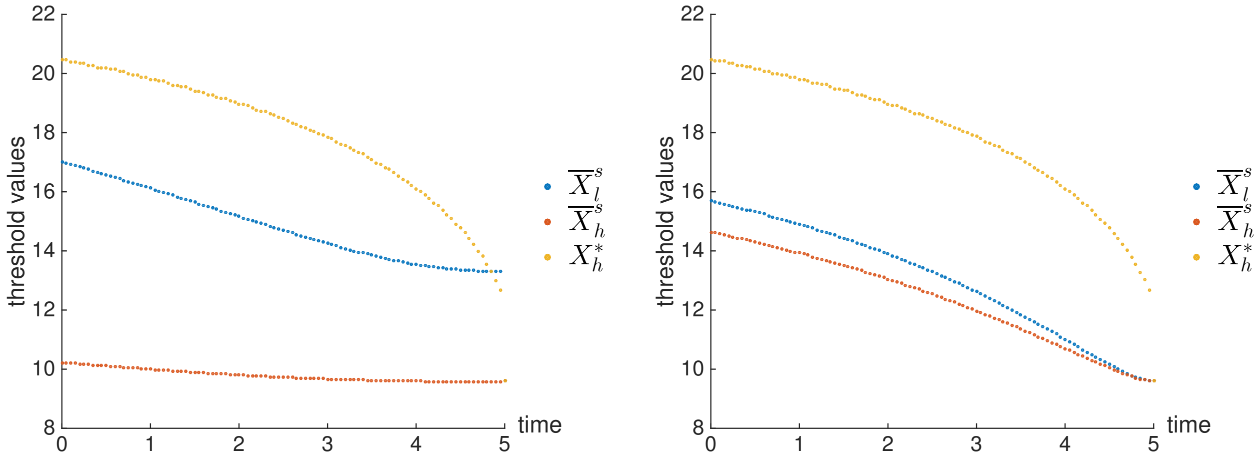
First, we would like to examine the relative positions among the binding thresholds $\overline{X}_l^s(t)$ and $\overline{X}_h^s(t)$ and the first-best investment threshold $X_h^*(t)$ as time evolves. In Figure 3(a, b), we show the numerical plots of $\overline{X}_l^s(t)$, $\overline{X}_h^s(t)$ and $X_h^*(t)$ against time t at two levels of λ_l, namely, (a) $\lambda_l = 0.6$, (b) $\lambda_l = 1.1$. The other parameter values are chosen as follows: $r = 5\%$, $\mu = 1\%$, $\sigma = 25\%$, $\lambda_h = 1.25$, $F = 200$, $I = 100$, $T = 5$ and $p = 0.5$. As revealed in Figure 3(a) with $\lambda_l = 0.6$ and $\lambda_h = 1.25$, where λ_l is well below λ_h, we observe that $\overline{X}_h^s(t)$ remains well below $\overline{X}_l^s(t)$ at all times. Since $[\overline{X}_h^s(t), \overline{X}_l^s(t)]$ is a finite interval at all times, the high-type firm can separate from the low-type firm. In particular, we observe that at time sufficiently close to expiry, $\overline{X}_l^s(t)$ may stay above $X_h^*(t)$. Let $\hat{t}$ be the time at which $\overline{X}_l^s(t) = X_h^*(t)$. At $t < \hat{t}$, the high-type firm invests aggressively at $\overline{X}_l^s(t)$ before $X_h^*(t)$; while at $t \geq \hat{t}$, the high-type firm can choose to invest optimally at its first-best threshold $X_h^*(t)$ at zero separating cost. The width of the interval $[\overline{X}_h^s(t), \overline{X}_l^s(t)]$ decreases with increasing λ_l and time t. When $\lambda_l = 0.6$ and $\lambda_h = 1.25$, we have $\frac{F(r-\mu)}{\lambda_h} > \frac{(F+I)(r-\mu)}{\lambda_l}$. This gives $\overline{X}_l^s(T^-) = \frac{40}{3} > \overline{X}_h^s(T^-) = 9.6$. However, with $\lambda_l = 1.1$ and $\lambda_h = 1.25$, we have $\frac{F(r-\mu)}{\lambda_l} < \frac{(F+I)(r-\mu)}{\lambda_h}$. This gives

$$\lim_{t \to T^-} \overline{X}_l^s(t) = \max\left(\frac{(F+I)(r-\mu)}{\lambda_h}, \frac{F(r-\mu)}{\lambda_l}\right) = 9.6,$$

$$\lim_{t \to T^-} \overline{X}_h^s(t) = \frac{(F+I)(r-\mu)}{\lambda_h} = 9.6,$$

so that the two binding thresholds coincide at $t \to T^-$ [see Figure 3(b)]. Relative positions of $\overline{X}_l^s(t)$ and $X_h^*(t)$ also depend on λ_l. When λ_l is closer to λ_h [say, $\lambda_l = 1.1$ and $\lambda_h = 1.25$ as in Figure 3(b)], $X_h^*(t)$ always stays above $\overline{X}_l^s(t)$. In this case, under the least-cost separating equilibrium, the high-type firm always chooses to invest aggressively at $\overline{X}_l^s(t)$ at all times and never invests at its first-best strategy $X_h^*(t)$.

In summary, these numerical plots of the binding thresholds $\overline{X}_l^s(t)$ and $\overline{X}_h^s(t)$, and the first-best investment threshold $X_h^*(t)$ at varying values of λ_l reveal interesting time dependent features of the separating equilibrium. The plots verify that $\overline{X}_h^s(t) < \overline{X}_l^s(t)$ at all time $t < T$, while $\overline{X}_h^s(t) \to \overline{X}_l^s(t)$ at the limit $t \to T^-$ when $\lambda_l \geq \frac{F}{F+I}\lambda_h$. When λ_l is much lower than λ_h so that the mimicking cost of the low-type firm is high, it may occur that $X_h^*(t) < \overline{X}_l^s(t)$ when the remaining time of investment opportunity is sufficiently short. In this case, the high-type firm may invest at its first-best investment threshold at zero separating cost.



$3(a)$ $\lambda_l = 0.6$ $\qquad$ $3(b)$ $\lambda_l = 1.1$

Figure 3: The binding thresholds $\overline{X}_l^s(t)$ and $\overline{X}_h^s(t)$ and the first-best investment threshold $X_h^*(t)$ are plotted against time t at two values of λ_l.

Information cost

Positivity of the cost of separating implies that the high-type firm value function V_h^c under complete information is higher than V_h^s under the least-cost separating equilibrium. On the other hand, under low value of λ_l, zero cost of separating may occur when the remaining time of investment opportunity is sufficiently short, corresponding to $V_h^c = V_h^s$ as $t \to T^-$. We plot $V_h^c(X,t)$ and $V_h^s(X,t)$ with respect to time t under $\lambda_l = 0.6$ and $\lambda_l = 1.1$ at $X = 8$ [see Figure $4(a,b)$]. The other parameters are chosen to be the same as those in Figure 3.

We define the information cost of the high-type firm to be the difference of the real option value functions under complete information and separating equilibrium, where

$$\text{cost}(X,t) = V_h^c(X,t) - V_h^s(X,t) > 0. \tag{5.3}$$

At time $t \to T^-$, the two option value functions V_h^s and V_h^c always converge to the same terminal payoff since

$$\lim_{t\to T^-} V_h^c(X,t) = \lim_{t\to T^-} V_h^s(X,t) = [\lambda_h \Pi(X) - F - I]^+$$

for all revenue flow rate level X. In Figure $4(a,b)$, we observe that the two option values coincide at zero value since $[\lambda_h \Pi(X) - F - I]^+|_{X=8} = 0$ on the expiry date T.

In Figure $4(c,d)$, we plot the information cost against t under $\lambda_l = 0.6$ and $\lambda_l = 1.1$ at $X = 8$. The time evolution of the information cost is driven by two factors: (i) The gap between the two binding thresholds $\overline{X}_h^s(t)$ and $\overline{X}_l^s(t)$ becomes narrower as time moves closer to expiry. The high-type firm may become more aggressive to invest and attempts to separate from the low-type firm. This would increase the information cost as time evolves. (ii) The first-best investment threshold $X_h^*(t)$ of the high-type gets closer to the binding threshold $\overline{X}_l^s(t)$ with increasing time. As time evolves, the high-type firm may become less aggressive to adopt the separating strategy due to narrowing of the gap between $X_h^*(t)$ and $\overline{X}_l^s(t)$.

(i) When λ_l is sufficiently small [say $\lambda_l = 0.6$ in Figure $4(c)$], the gap between $\overline{X}_h^s(t)$ and $\overline{X}_l^s(t)$ is larger than that between $X_h^*(t)$ and $\overline{X}_l^s(t)$. In this case, the second factor dominates and the information cost of the high-type firm is decreasing as time evolves. The information cost becomes zero when t goes beyond $\hat{t}$, where $\overline{X}_l^s(\hat{t}) = X_h^*(\hat{t})$.

(ii) When λ_l is sufficiently large [say $\lambda_l = 1.1$ in Figure 4(d)], the interval $[\overline{X}_h^s(t), \overline{X}_l^s(t)]$ is narrow and the first-best threshold $X_h^*(t)$ is quite far from the binding threshold $\overline{X}_l^s(t)$. In this case, the first factor dominates at an earlier time and the information cost is increasing when the time is far from expiry. At time $t \to T^-$, the information cost is pulled to zero since the two option value functions V_l^c and V_h^s are both equal to the terminal payoff $[\lambda_h \Pi(X) - F - I]$. Subsequently, the second factor dominates when t is sufficiently close to expiry.

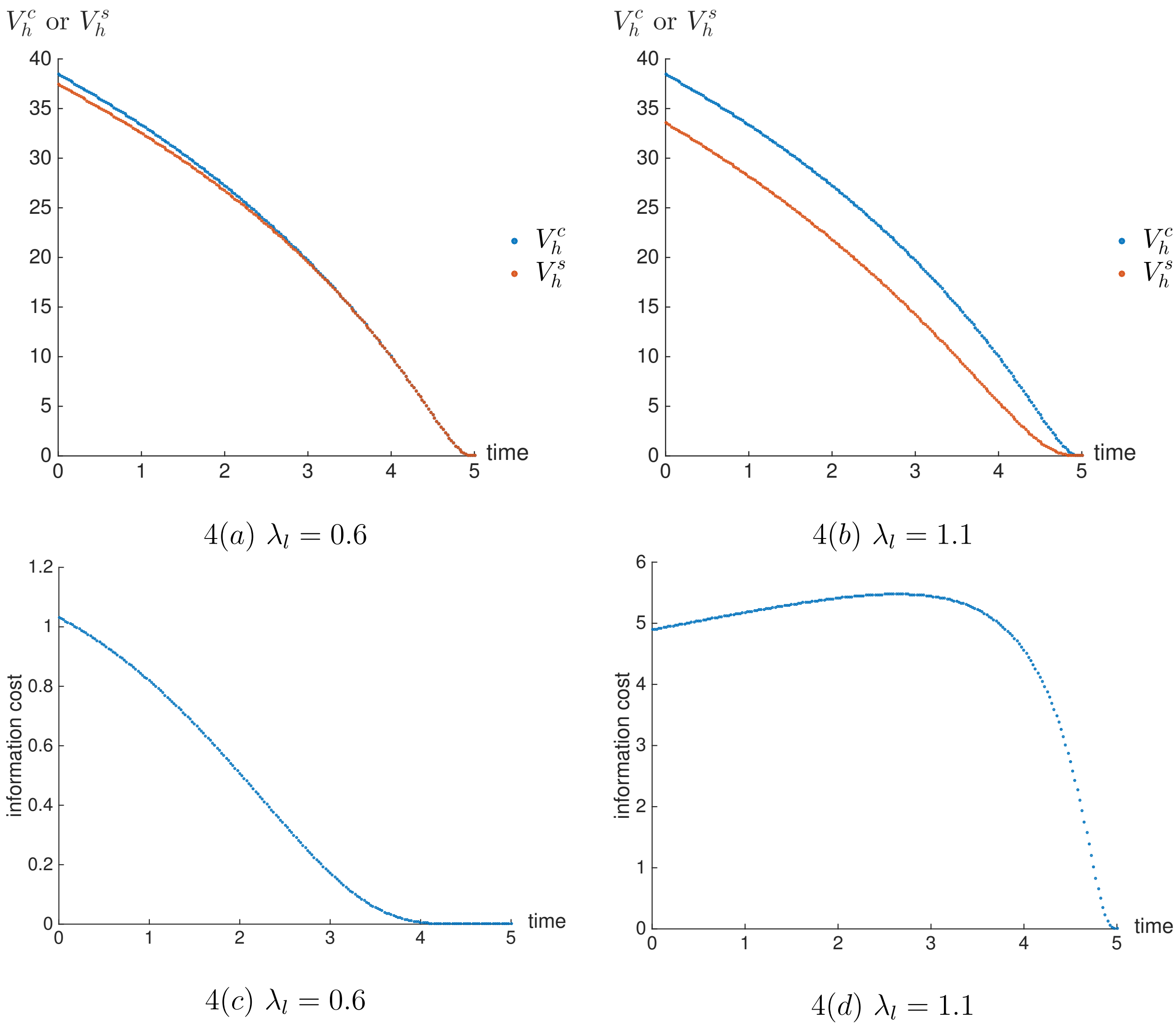


Figure 4: The option values V_h^s and V_h^c and information cost are plotted against time t at two values of λ_l.

Abnormal return

Under separating equilibrium, the information released at the time when the high-type firm invests at $X_h^s(t) = \min(\overline{X}_l^s(t), X_h^*(t))$ triggers a positive (negative) jump on the firm value of the high-type (low-type) firm. Following Morellec and Schürhoff (2011), we define the abnormal return of the high-type firm and low-type firm to be

$$
\begin{aligned}
R_h^a(X_h^s(t),t) &= \frac{V_h^s(X_h^s(t),t) - V_m^s(X_h^s(t),t)}{V_m^s(X_h^s(t),t)} \\
&= \frac{\lambda_h \Pi(X_h^s(t)) - F - I}{p[\lambda_h \Pi(X_h^s(t)) - F - I] + (1-p)V_l^c(X_h^s(t),t)} - 1 > 0, \\
R_l^a(X_h^s(t),t) &= \frac{V_l^c(X_h^s(t),t) - V_m^s(X_h^s(t),t)}{V_m^s(X_h^s(t),t)} \\
&= \frac{V_l^c(X_h^s(t),t)}{p[\lambda_h \Pi(X_h^s(t)) - F - I] + (1-p)V_l^c(X_h^s(t),t)} - 1 < 0, \qquad (5.4)
\end{aligned}
$$

respectively. The ratio of the abnormal return of the high-type firm to that of the low-type firm is seen to be

$$\frac{R_h^a}{R_l^a} = -\frac{1-p}{p},$$

which is always negative for $0 < p < 1$.

Figure 5 shows the plots of the abnormal returns of both types of the firm with respect to time t at $\lambda_l = 0.6$. The other parameters are chosen to be the same as those in Figure 3. We observe that the abnormal return of the high-type (low-type) firm is monotonously increasing (decreasing) with respect to time t. To explain the monotonicity property of the abnormal return of the high-type firm, we rewrite R_h^a in an alternative analytic form:

$$R_h^a(X_h^s(t),t) = \frac{1}{p + (1-p)\frac{V_l^c(X_h^s(t),t)}{\lambda_h \Pi(X_h^s(t)) - F - I}} - 1.$$

The monotonic increasing property of R_h^a stems from the monotonic decreasing property of $\frac{V_l^c(X_h^s(t),t)}{\lambda_h \Pi(X_h^s(t)) - F - I}$. Note that the denominator is a linear function of $X_h^s(t)$ while the numerator is the option value $V_l^c(X_h^s(t),t)$. Both quantities have dependence on $X_h^s(t)$, which is decreasing in value with increasing time. However, the option value $V_l^c(X_h^s(t),t)$ has a higher rate of decrease with time since there is an additional loss of time value of the option. Therefore, we expect that $\frac{V_l^c(X_h^s(t))}{\lambda_h \Pi(X_h^s(t)) - F - I}$ decreases in value with time, in particular when the remaining time of investment opportunity becomes shorter.

Next, we consider the asymptotic behavior of the abnormal return at time near expiry under the two cases: (i) $\frac{F(r-\mu)}{\lambda_l} > \hat{X}_h^0$, (ii) $\frac{F(r-\mu)}{\lambda_l} \le \hat{X}_h^0$. At time $t \to T^-$, recall that

$$\lim_{t \to T^-} X_h^s(t) = \min\left(\max\left(\frac{F(r-\mu)}{\lambda_l}, \hat{X}_h^0\right), \frac{r(F+I)}{\lambda_h}\right).$$

(i) With $\lambda_l = 0.6$, $\frac{F(r-\mu)}{\lambda_l} > \hat{X}_h^0$ is observed, so we have

$$X_h^s(T^-) = \min\left(\frac{F(r-\mu)}{\lambda_l}, \frac{r(F+I)}{\lambda_h}\right) \in (\hat{X}_h^0, \hat{X}_l^0).$$

This implies

$$V_h^s(X_h^s(T^-),T^-) = \lambda_h \Pi(X_h^s(T^-)) - F - I > 0$$

and

$$V_l^c(X_h^s(T^-),T^-) = [\lambda_l \Pi(X_h^s(T^-)) - F - I]^+ = 0.$$

The abnormal returns of both types of firm near expiry are found to be

$$R_h^a(X_h^s(T^-),T^-) = \frac{1}{p} - 1 \quad \text{and} \quad R_l^a(X_h^s(T^-),T^-) = 0 - 1 = -1.$$

These asymptotic results are verified in the plots in Figure 5, where R_h^a tends to 1 as $t \to T^-$ when $p = 0.5$.

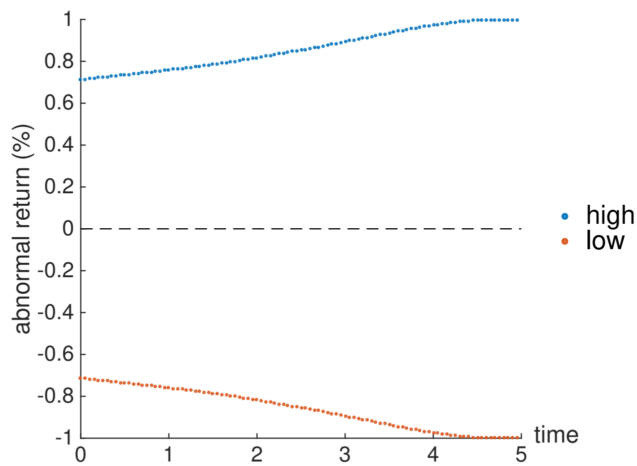
(ii) With $\lambda_l = 1.1$, $\frac{F(r-\mu)}{\lambda_l} \leq \hat{X}_h^0$ is observed, so we have $X_h^s(T^-) = \hat{X}_h^0$. This implies

$$V_h^s(X_h^s(T^-), T^-) = \lambda_h \Pi(X_h^s(T^-)) - F - I = 0$$

and

$$V_l^c(X_h^s(T^-), T^-) = [\lambda_l \Pi(X_h^s(T^-)) - F - I]^+ = 0.$$

Since the investment threshold at time near expiry for the high-type firm is the zero-NPV threshold $\hat{X}_h^0$ under separating equilibrium, the high-type firm would not choose to invest as $t \to T^-$. The outside investors cannot distinguish the firm type when investment would not occur. It becomes irrelevant to quantify abnormal return under this scenario.



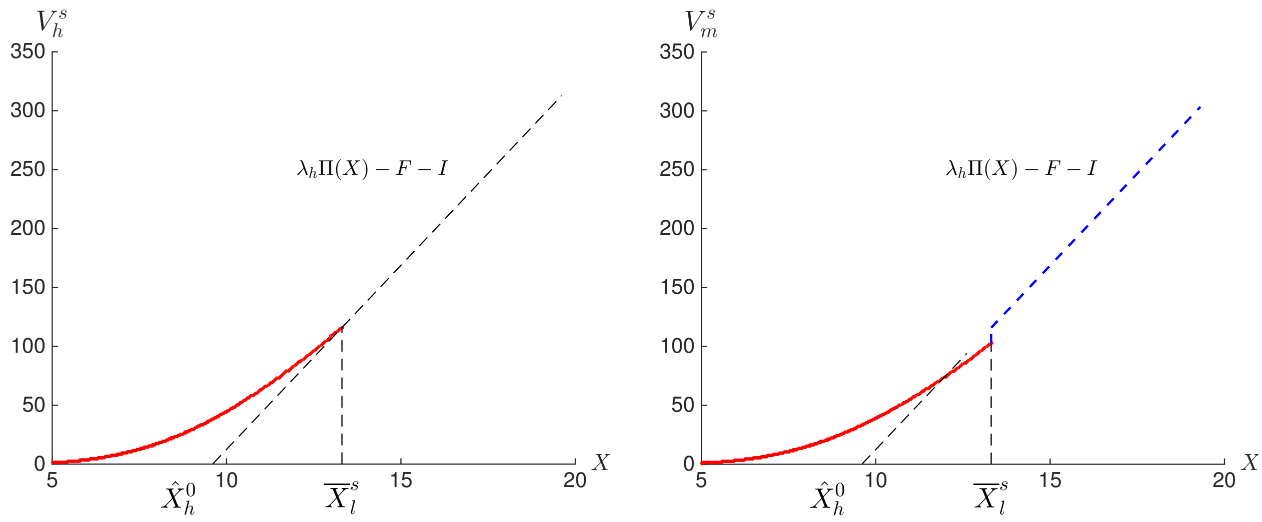
$\lambda_l = 0.6$

Figure 5: The abnormal returns of both types of the firm are plotted against time t.

Real option value against X

We also examine the behavior of the real option values V_h^s and V_m^s under separating equilibrium against revenue flow rate X at a fixed time. Figure 6(a) plots the real option value V_h^s of the high-type firm under separating equilibrium against revenue flow rate X with $\lambda_l = 1.1$ at $t = 2.5$. The other parameters are chosen to be the same as those in Figure 3. We observe that V_h^s does not exhibit the smooth-pasting property at its investment threshold $\overline{X}_l^s$ since it is a binding threshold rather than an optimal stopping threshold.

Figure 6(b) plots the market value V_m^s against X under separating equilibrium of the high-type firm. At the investment threshold $\overline{X}_l^s$, the exact firm type is revealed to outside investors, resulting in an upward jump of real option value of the high-type firm. The updated firm value is equal to the exercise payoff: $\lambda_h \Pi(X) - F - I$. The upward jump is consistent with the positive jump of the abnormal return of the high-type firm discussed earlier.



6(a) V_h^s against X 6(b) V_m^s against X

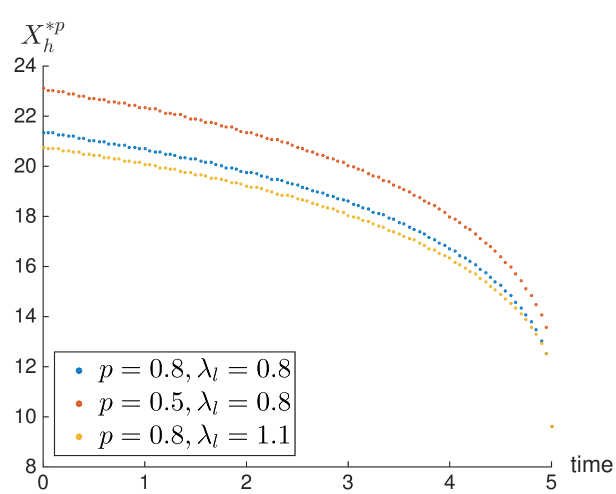
Figure 6: Plots of V_h^s and V_m^s against revenue flow rate X at a fixed time. Since investment has occurred at $X \geq \overline{X}_l^s$, both option values become equal to the exercise payoff: $\lambda_h \Pi(X) - F - I$.

5.2 Pooling equilibrium

The mathematical characterization of pooling equilibrium has been summarized in Proposition 4. We performed numerical studies to verify the theoretical results on the investment thresholds and value functions under pooling equilibrium, and examine the time evolution of the information cost.

Optimal threshold and value functions

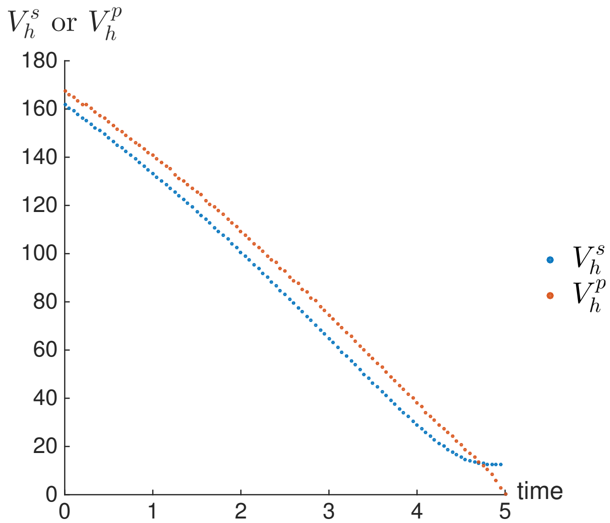
The firm may choose to invest at the optimal threshold $X_h^{*p}(t)$ of the high-type firm under pooling equilibrium regardless of its type.



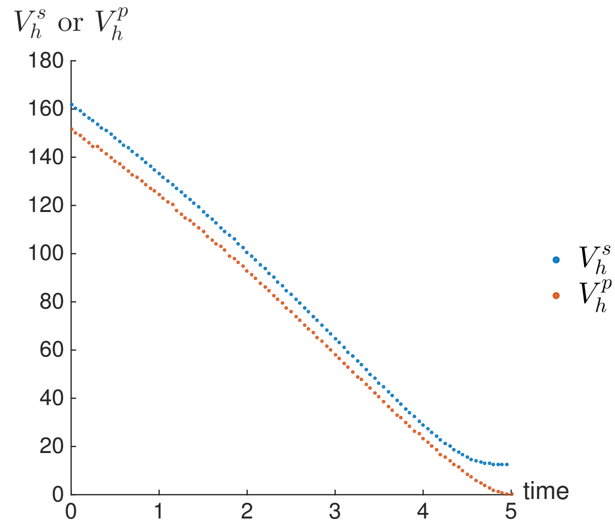
$p = 0.8,\ \lambda_l = 0.8$

Figure 7: The optimal threshold $X_h^{*p}(t)$ under pooling equilibrium is plotted against time t at various values of λ_l and p.

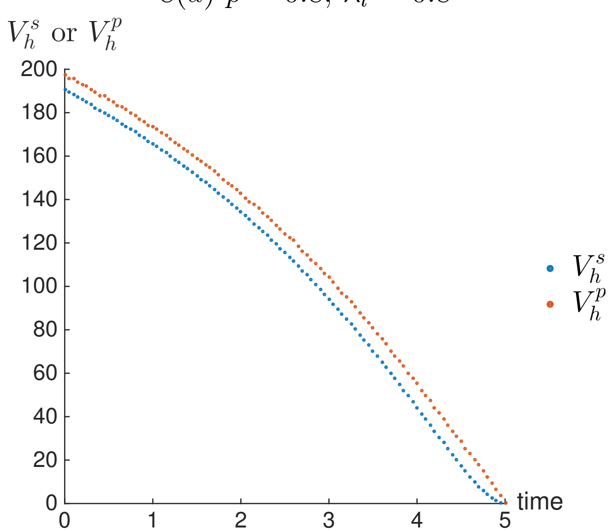
The optimal threshold X_h^{*p} under pooling equilibrium with different sets of parameter values: (i) $p = 0.8$, $\lambda_l = 0.8$, (ii) $p = 0.5$, $\lambda_l = 0.8$ and (iii) $p = 0.8$, $\lambda_l = 0.8$, are plotted against time t [see Figure 7]. The other parameters are chosen to be the same as those in Figure 3. Figure 7 shows that the firm is accelerating its investment gradually under pooling equilibrium as time becomes closer to maturity.



$8(a)$ $p = 0.8$, $\lambda_l = 0.8$



$8(b)$ $p = 0.5$, $\lambda_l = 0.8$



$8(c)$ $p = 0.8$, $\lambda_l = 1.1$

Figure 8: The option values V_h^s and V_h^p are plotted against time t at various values of λ_l and p.

We examine the existence of pooling equilibrium at different times and parameter values of λ_l and p. The low-type firm always prefers pooling equilibrium to separating equilibrium since it achieves a higher option value through mimicking the high-type firm, provided that the conditions required in Proposition 4 for pooling equilibrium are met. We check the incentive compatibility constraint of the high-type firm by plotting option values $V_h^s(X,t)$ and $V_h^p(X,t)$ against time t, where X is evaluated at $\min(\overline{X}_l^s(t), X_h^*(t))$. In our numerical studies, three pairs of parameter values of p and λ_l are chosen: (i) $p = 0.8$, $\lambda_l = 0.8$, (ii) $p = 0.5$, $\lambda_l = 0.8$ and (iii) $p = 0.8$, $\lambda_l = 1.1$. The other parameters are chosen to be the same as those in Figure 3.

(i) With $p = 0.8$ and $\lambda_l = 0.8$, we observe that V_h^p stays above V_h^s at an earlier time and falls below V_h^s when the remaining time of investment opportunity is sufficiently short [see Figure 8(a)]. This implies that the high-type firm may choose to pool at earlier time, then chooses to separate when time is approaching closer to expiry.

(ii) With $p = 0.5$ and $\lambda_l = 0.8$, we observe that V_h^p stays below V_h^s at all time [see Figure 8(b)]. It implies that the high-type firm always prefers separating to pooling. We conclude that pooling equilibrium does not exist at any time when probability p is sufficiently low.

(iii) With $p = 0.8$ and $\lambda_l = 1.1$, we observe that V_h^p stays above V_h^s at all time [see Figure 8(c)]. This implies that the high-type firm always prefers pooling to separating. We conclude that pooling equilibrium may exist at all times when λ_l is sufficiently close to λ_h.

Information cost

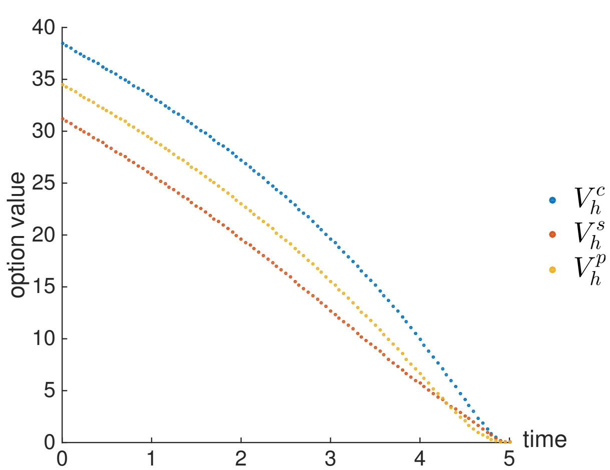
We plot $V_h^c(X,t)$, $V_h^s(X,t)$ and $V_h^p(X,t)$ with respect to time t under $p = 0.8$ and $\lambda_l = 0.8$ at $X = 8$ [see Figure 9(a)]. The other parameters are chosen to be the same as those in Figure 3. Since we consider the scenario where the firm chooses between separating and pooling equilibrium, the information cost of the high-type firm is defined to be

$$\text{cost}(X,t) = V_h^c(X,t) - \max(V_h^s(X,t), V_h^p(X,t)). \tag{5.5}$$

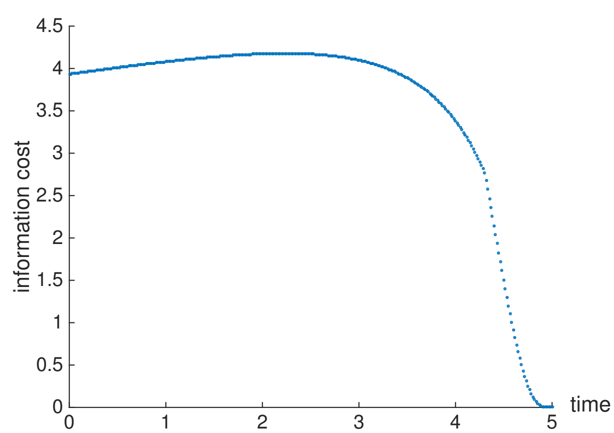
At time $t \to T^-$, the option value function V_h^p converges to the terminal payoff

$$\lim_{t \to T^-} V_h^p(X,t) = H_h^p(X) = \frac{\lambda_h \Pi(X) - F}{\lambda_p \Pi(X) - F}[\lambda_p \Pi(X) - F - I]^+$$

for all revenue flow rate X. In Figure 9(a), we observe that the three option values, namely, V_h^c, V_h^s and V_h^p, coincide at zero value since $[\lambda_h \Pi(X) - F - I]^+|_{X=8} = H_h^p(X)|_{X=8} = 0$ on the expiry date T. We plot the information cost against t under $p = 0.8$ and $\lambda_l = 0.8$ evaluated at $X = 8$ [see Figure 9(b)]. We observe similar time dependent behavior of the information cost as that under separating equilibrium. In particular, the high-type firm may choose to separate from the low-type firm at time close to expiry since the option value V_h^s becomes higher than V_h^p at some time between $t = 4$ and $t = 5$. This is revealed by the appearance of a cusp point at the same time in the plot of the information cost against t [see Figure 9(b)].



9(a) $p = 0.8$, $\lambda_l = 0.8$



9(b) $p = 0.8$, $\lambda_l = 0.8$

Figure 9: The option values V_h^c, V_h^p and V_h^s and information cost are plotted against time t.

6 Conclusion

We extend the dynamic real option signaling game models of financing decisions of a firm on a risky project from perpetuity to a finite time horizon. Asymmetric information is assumed, where the firm quality is known to the firm management but not outside investors. Under separating equilibrium, the high-type firm may credibly signal its type to outside investors through investment timing. However, this would impose an information cost on the high-type firm. We analyze the incentive compatibility constraints and perform mathematical characterization of the existence and uniqueness of the time dependent binding thresholds and optimal investment thresholds of both types of firm. The nature of the least-cost separating equilibrium and the associated belief system is examined. The low-type firm may mimic the investment strategy of the high-type firm. We also analyze the nature of pooling equilibrium of equity financing, where firm of either type invests at the same threshold and issues the same number of equity shares to finance the risky project.

In addition, our contributions include detailed studies of the time dependent behaviors of the separating and pooling equilibriums, and the time evolution of the binding thresholds and investment thresholds of both types of the firm. In particular, we show that when the mimicking cost of the low-type firm is high, it may occur that the high-type firm can invest at its first-best optimal threshold at zero information cost when the remaining time of the investment opportunity is sufficiently short. Our theoretical studies produce various theoretical results for the information cost, abnormal return and investment thresholds under separating equilibrium at the time right before the expiry of the investment opportunity. Interestingly, when the investment opportunity almost evaporates soon, the information cost is pulled to zero. Also, the abnormal return is dependent only on the probability of assessment of firm type but independent of the revenue flow rate level. In summary, our finite time signaling game model encompasses the usual perpetual model and the classical Myers-Majluf model of short-lived investment opportunity, without imposing the usual unrealistic assumptions in real options investment models that the investment opportunity either lasts forever or evaporates soon.

References

[1] Bouvard, M. (2014) Real option financing under asymmetric information, *Review of Financial Studies*, 27(1): 180-210.

[2] Clausen, S., Flor, C.R. (2015) The impact of assets-in-place on corporate financing and investment decision, *Journal of Banking and Finance*, 61: 64-80.

[3] Connelly, B.L., Certo, S.T., Ireland, R.D., Reutgel, C.R. (2011) Signaling Theory: A review and assessment, *Journal of Management*, 37(1): 39-67.

[4] Dixit, A., Pindyck, R. (1994) *Investment under Uncertainty*, Princeton University Press, Princeton, New Jersey.

[5] Grenadier, S.R., Wang, N. (2005) Investment timing, agency and information, *Journal of Financial Economics*, 75: 493-533.

[6] Grenadier, S.R. (2011) Real option signaling games with applications to corporate finance, *Review of Financial Studies*, 24(12):3993-4036.

[7] Gryglewicz, S., Huisman, K.J.M., Kort, P.M. (2008) Finite project life and uncertainty effects on investment, *Journal of Economic Dynamics and Control*, 32: 2191-2213.

[8] Kwok, Y.K. (2008) *Mathematical Models of Financial Derivatives*, second edition, Springer, Verlag.

[9] Leung, C.M., Kwok, Y.K. (2018) Real options signaling game models for dynamic acquisition under information asymmetry, to appear in *Decisions in Economics and Finance*.

[10] Morellec, E., Schürhoff, N. (2011) Corporate investment and financing under asymmetric information, *Journal of Financial Economics*, 99: 262-288.

[11] Myers, S.C., Majluf, N.S. (1984) Corporate financing and investment decisions when firms have information that investors do not have, *Journal of Financial Economics*, 13: 187-221.

[12] Nishihara, M. (2016) Selling out or going public? A real options signaling approach, *Working paper of Osaka University*.

[13] Nishihara, M., Shibata, T. (2017) Default and liquidation timing under asymmetric information, *European Journal of Operational Research*, 263: 321-336.

[14] Strebulaev, I.A., Zhu, H.X., Zryumov, P. (2014) Dynamic information asymmetry, financing, and investment decisions, *Working paper of Stanford University*.

[15] Watanabe, T. (2016) Real options and signaling in strategic investment games, *Working paper of Tokyo Metropolitan University*.

[16] Xu, R., Li, S. (2010) Belief updating, debt pricing and financial decisions under asymmetric information, *Research in International Business and Finance*, 24(2): 123-137.

Appendix A - Proof of Lemma 1

According to ineq. (3.2), the satisfaction of the incentive compatibility constraint (ICC) of the low-type firm is related to positivity of the following function:

$$G_l^s(X,t) = H_l(X) - V_l^c(X,t), \quad \hat{\hat{X}}_l^0 \le X \le X_l^*(t).$$

The binding threshold $\overline{X}_l^s(t)$ for satisfying the ICC of the low-type firm is given by the solution to $G_l^s(X,t) = 0$.

First, we show that solution to $G_l^s(X,t) = 0$ exists within $(\hat{\hat{X}}_l^0, X_l^*(t))$. We consider the values of $G_l^s(X,t)$ at the end points, $\hat{\hat{X}}_l^0$ and $X_l^*(t)$. Since real option value should be positive, so $V_l^c(\hat{\hat{X}}_l^0, t) > 0$; and by virtue of eqs. (2.5) and (3.1), $H_l(\hat{\hat{X}}_l^0) = 0$. We have $G_l^s(\hat{\hat{X}}_l^0, t) < 0$. On the other hand, by the value matching condition of $V_l^c(X,t)$ at its optimal threshold $X_l^*(t)$, we obtain

$$\begin{aligned} V_l^c(X_l^*(t),t) &= \lambda_l \Pi(X_l^*(t)) - F - I \\ &= [\lambda_l \Pi(X_l^*(t)) - F]\left[1 - \frac{I}{\lambda_l \Pi(X_l^*(t)) - F}\right] \\ &< [\lambda_l \Pi(X_l^*(t)) - F]\left[1 - \frac{I}{\lambda_h \Pi(X_l^*(t)) - F}\right] = H_l(X_l^*(t)). \end{aligned}$$

We then have $G_l^s(X_l^*(t),t) > 0$. Since $G_l^s(X,t)$ is continuous in X, by the mean value theorem, we guarantee the existence of the root of $G_l^s(X_l^*(t),t)$ within $(\hat{\hat{X}}_l^0, X_l^*(t))$.

Next, we use the geometric properties of $G_l^s(X,t)$ to show that the root of $G_l^s(X,t)$ is unique. The first and second order derivatives of $H_l(X)$ are found to be

$$\begin{aligned} \frac{\partial H_l}{\partial X} &= \frac{\lambda_l}{r-\mu} - (\lambda_h - \lambda_l)\frac{IF(r-\mu)}{[\lambda_h X - F(r-\mu)]^2}, \\ \frac{\partial^2 H_l}{\partial X^2} &= 2\lambda_h(\lambda_h - \lambda_l)\frac{IF(r-\mu)}{[\lambda_h X - F(r-\mu)]^3} > 0, \quad \hat{\hat{X}}_l^0 \le X \le X_l^*(t). \end{aligned}$$

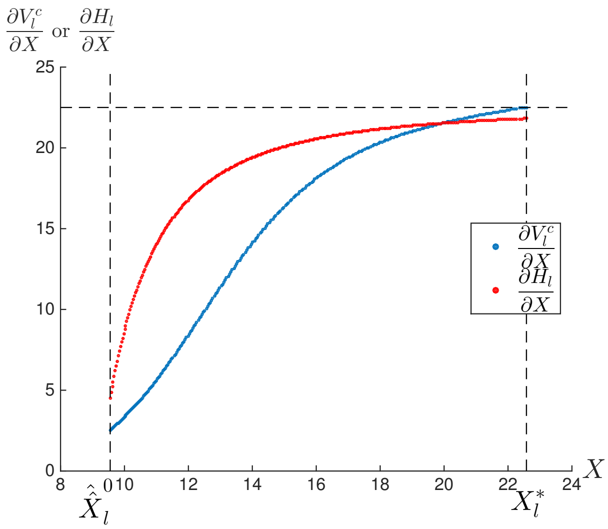
Note that $\frac{\partial H_l}{\partial X}\Big|_{X=X_l^*(t)} < \frac{\lambda_l}{r-\mu}$ and $\frac{\partial H_l}{\partial X} \to \frac{\lambda_l}{r-\mu}$ as $X \to \infty$. On the other hand, by virtue of the smooth pasting condition of $V_l^c(X,t)$ at the optimal threshold $X_l^*(t)$, we have $\frac{\partial V_l^c}{\partial X}\Big|_{X=X_l^*(t)} = \frac{\lambda_l}{r-\mu}$. The plots of $\frac{\partial H_l}{\partial X}$ and $\frac{\partial V_l^c}{\partial X}$ against X, $X \in [\hat{\hat{X}}_l^0, X_l^*(t)]$ are shown in Figures A1(a,b) under the two different cases (i) $\frac{\partial H_l}{\partial X}\Big|_{X=\hat{\hat{X}}_l^0} \ge \frac{\partial V_l^c}{\partial X}\Big|_{X=\hat{\hat{X}}_l^0}$, (ii) $\frac{\partial H_l}{\partial X}\Big|_{X=\hat{\hat{X}}_l^0} < \frac{\partial V_l^c}{\partial X}\Big|_{X=\hat{\hat{X}}_l^0}$, respectively. By considering the property of $\frac{\partial G_l^s}{\partial X} = \frac{\partial H_l}{\partial X} - \frac{\partial V_l^c}{\partial X}$ over $[\hat{\hat{X}}_l^0, X_l^*(t)]$, we can deduce the following behavior of $G_l^s(X,t)$:

(i) When $\frac{\partial G_l^s}{\partial X}\Big|_{X=\hat{\hat{X}}_l^0} \ge 0$, $G_l^s(X,t)$ achieves its positive maximum value at one maximum point within $[\hat{\hat{X}}_l^0, X_l^*(t)]$. The plot of $G_l^s(X,t)$ against X intersects the X-axis at one point [see Figure A2(a)].

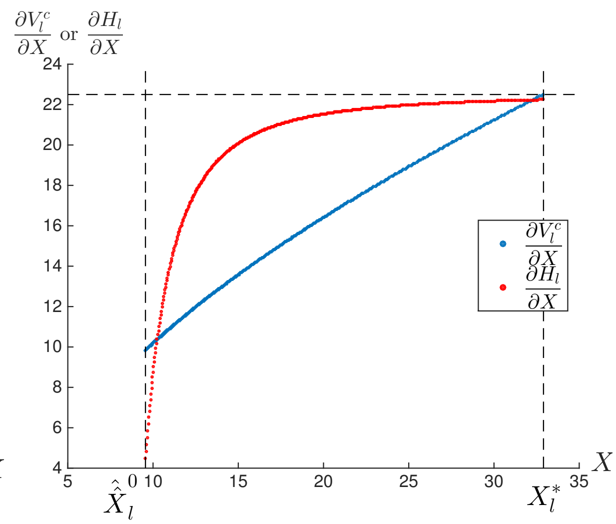
(ii) When $\frac{\partial G_l^s}{\partial X}\Big|_{X=\hat{\hat{X}}_l^0} < 0$, as X increases, $\frac{\partial G_l^s}{\partial X}$ remains negative for a while, then becomes positive and finally stays negative as X approaches $X_l^*(t)$. Consequently, $G_l^s(X,t)$ decreases from a negative value at $\hat{\hat{X}}_l^0$, reaches the unique minimum point below the

X-axis, then increases and intersects the X-axis once, reaches the unique maximum point above the X-axis, and finally decreases to reach a positive value at $X = X_l^*(t)$ [see Figure A2(b)].

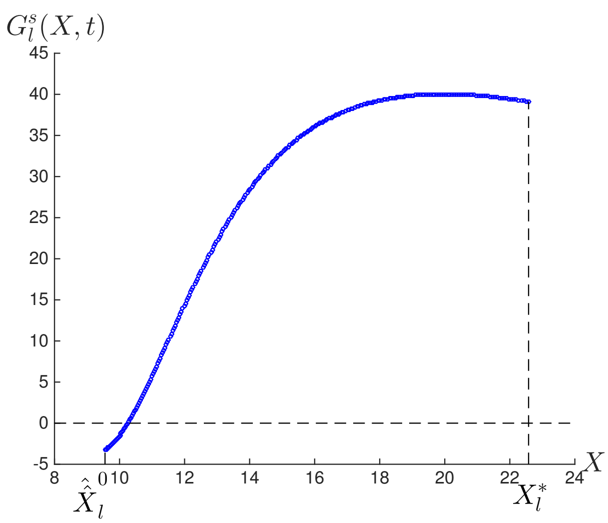
In conclusion, when $X \leq \overline{X}_l^s(t)$, the ICC of the low-type firm is violated since $H_l(X) \leq V_l^c(X,t)$. The low-type firm has no incentive to mimic the investment decision of the high-type firm.



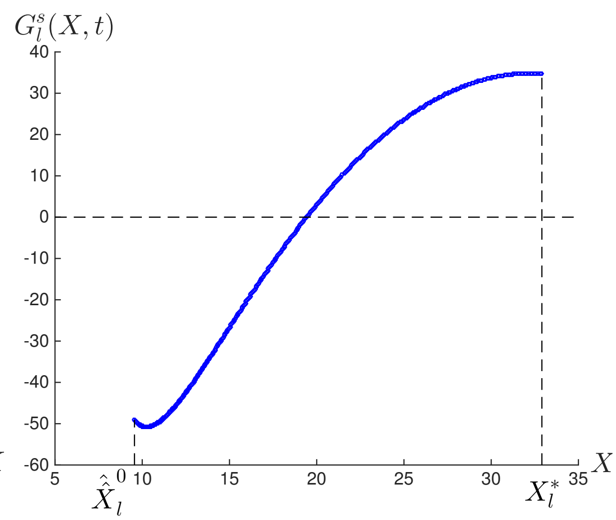
A1(a)



A1(b)



A2(a)



A2(b)

To generate the plots in Figures A1(a) and A2(a), we choose the following parameter values: risk-free rate $r = 5\%$, volatility and growth rate of the revenue flow rate $\sigma = 25\%$ and $\mu = 1\%$, respectively, present value of total operating expenses $F = 10/r$, fixed cost $I = 100$, probability of being high-type $p = 0.5$, maturity $T = 20$, $\lambda_h = 1.25$ and $\lambda_l = 0.9$, current time $t = 19$ (close to maturity). To generate the plots in Figures A1(b) and A2(b), the parameter values are the same as those for Figures A1(a) and A2(a) except that the current time is changed to $t = 0$.

Appendix B - Proof of Lemma 2

For the proof of existence of root of $G_h^s(X,t)$ within $(\hat{X}_h^0, X_h^{*m}(t))$, it suffices to show

$$G_h^s(\hat{X}_h^0, t) = -V_h^m(\hat{X}_h^0, t) < 0 \qquad (B1(a))$$

and

$$G_h^s(X_h^{*m}(t),t) = \lambda_h \Pi(X_h^{*m}(t),t) - F - I - [\lambda_h \Pi(X_h^{*m}(t),t) - F]\left[1 - \frac{I}{\lambda_l \Pi(X_h^{*m}(t),t) - F}\right]$$
$$> (\frac{\lambda_h}{\lambda_l} - 1)I > 0. \qquad (B1(b))$$

To show the validity of ineq. B1(b), we consider $G_h^s(X,t)$ evaluated at $X = X_h^{*m}(t)$ and write $X_h^{*m}(t)$ as $\hat{X}$ for notional convenience, then

$$\begin{aligned} G_h^s(\hat{X}) &= \lambda_h \Pi(\hat{X}) - F - I - \frac{\lambda_h \Pi(\hat{X}) - F}{\lambda_l \Pi(\hat{X}) - F}[\lambda_l \Pi(\hat{X}) - F - I] \\ &= \left(\frac{\lambda_h \hat{X}}{r-\mu} - F\right)\left(\frac{1}{\frac{\lambda_l \hat{X}}{r-\mu} - F} - 1\right) I. \end{aligned}$$

It is observed that $G_h^s(\hat{X})$ is a monotonically decreasing function of $\hat{X}$ and has the asymptotic limit $(\frac{\lambda_h}{\lambda_l} - 1)I$ as $\hat{X} \to \infty$. By virtue of ineqs. B1(a,b), and since $G_h^s(X,t)$ is continuous in X, the mean value theorem dictates existence of root of $G_h^s(X,t)$.

Again, we use the geometric properties of $G_h^s(X,t)$ to establish uniqueness of root of $G_h^s(X,t)$ within $(\hat{X}_h^0, X_h^{*m}(t))$. We consider

$$\frac{\partial G_h^s(X,t)}{\partial X} = \frac{\lambda_h}{r-\mu} - \frac{\partial V_h^m(X,t)}{\partial X}, \quad \hat{X}_h^0 \le X \le X_h^{*m}(t).$$

The real option value $V_h^m(X,t)$ of the high-type firm at its optimal threshold $X = X_h^{*m}(t)$ under pooling equals its intrinsic value $H_h(X)$, where

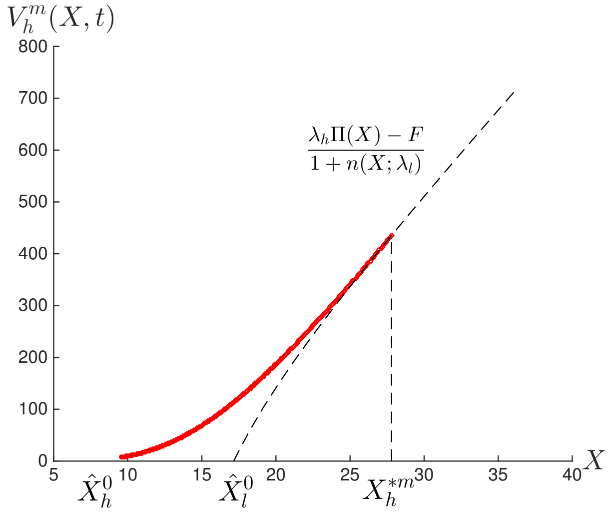
$$H_h(X) = \frac{\lambda_h \Pi(X) - F}{\lambda_l \Pi(X) - F}[\lambda_l \Pi(X) - F - I].$$

By the smooth pasting condition at the optimal threshold $X_h^{*m}(t)$, we obtain

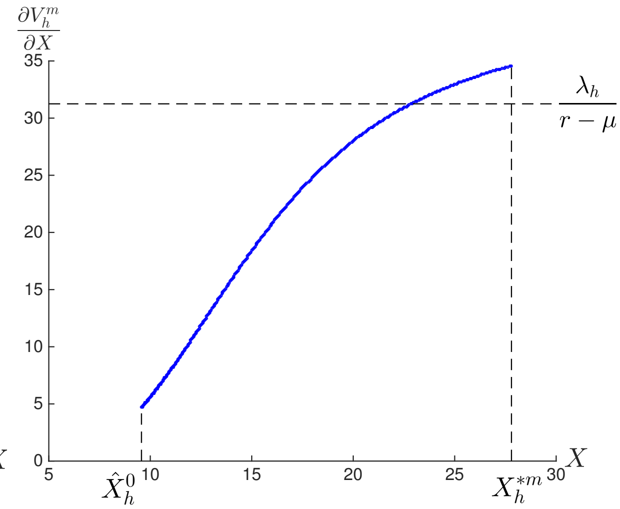
$$\begin{aligned} \left.\frac{\partial V_h^m(X,t)}{\partial X}\right|_{X=X_h^{*m}(t)} &= \left.\frac{\partial H_h(X)}{\partial X}\right|_{X=X_h^{*m}(t)} \\ &= \frac{\lambda_h}{r-\mu} + (\lambda_h - \lambda_l)\frac{IF(r-\mu)}{[\lambda_l X_h^{*m}(t) - F(r-\mu)]^2} > \frac{\lambda_h}{r-\mu}. \end{aligned}$$

We can deduce that $\frac{\partial V_h^m(X,t)}{\partial X}$ starts at value below $\frac{\lambda_h}{r-\mu}$ at $X = \hat{X}_h^0$, increases as X increases and goes above $\frac{\lambda_h}{r-\mu}$ as X increases up to $X_h^{*m}(t)$ [see Figure B1(b)]. Correspondingly, $G_h^s(X,t)$ starts at negative value at $X = \hat{X}_h^0$, increases monotonically as X increases, reaches unique maximum value above zero, then decreases monotonically but remains staying above $(\frac{\lambda_h}{\lambda_l} - 1)I$ as X increases up to $X_h^{*m}(t)$. As a result, the plot of $G_h^s(X,t)$ against X intersects the X-axis only once within $[\hat{X}_h^0, X_h^{*m}(t)]$ [see Figure B2].

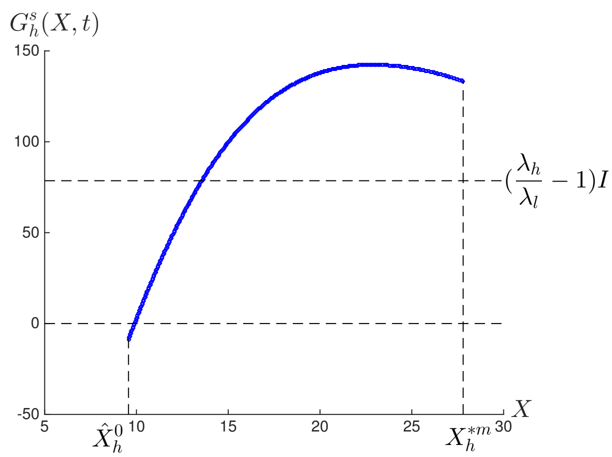
To generate the plots in Figures B1(a,b) and B2, the parameter values are the same as those for the figures in Appendix A, except that the maturity date T is changed to $T = 2.5$, the current time is $t = 0$ and $\lambda_l = 0.7$.



B1(a)



B1(b)



B2

Appendix C - Perfect Bayesian Equilibrium

A sufficient condition for a feasible strategy to constitute a Perfect Bayesian Equilibrium is that the high-type firm has no incentive to defect to other strategies given a set of out-of-equilibrium belief. In the current context, we would like to establish that the high-type firm has no incentive to defect from the separating strategy $X^s(t) = \overline{X}_l^s(t)$ to choose some other strategy $\hat{X}^s(t) > \overline{X}_l^s(t)$ when $\overline{X}_l^s(t) < X_h^*(t)$ under the pessimistic belief: $\Lambda = \lambda_l$. Let $\hat{V}_h(X,t;\hat{X})$ denote the value function of the high-type firm that follows the alternative strategy $\hat{X}$ while the firm is perceived as low-type. Given the belief $\Lambda = \lambda_l$, it suffices to show that the high-type firm's value of investment at $\overline{X}_l^s(t)$ is higher than its value function at $\overline{X}_l^s(t)$ using an alternative strategy $\hat{X}$; that is,

$$\lambda_h\Pi(\overline{X}_l^s(t)) - F - I > \hat{V}_h(\overline{X}_l^s(t),t;\hat{X}). \tag{C1}$$

Comparing $\hat{V}_h(X,t;\hat{X})$ with $V_h^m(X,t)$ [see eq. (3.4)], where both value functions share the same payoff upon investment, we argue that $\hat{V}_h(X,t;\hat{X}) < V_h^m(X,t)$ for any alternative strategy $\hat{X}$ since the optimal stopping rule is applied in valuation of the real option value function $V_h^m(X,t)$. If we define

$$\hat{G}_h(X,t) = \lambda_h \Pi(X) - F - I - \hat{V}_h(X,t;\hat{X}),$$

we deduce that

$$\hat{G}_h(X,t) > G_h^s(X,t) = \lambda_h \Pi(X) - F - I - V_h^m(X,t).$$

Since $G_h^s(\overline{X}_l^s(t),t) > 0$, we have $\hat{G}_h(\overline{X}_l^s(t),t) > 0$; so ineq. (C1) is established.