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CURRENCY-TRANSLATED FOREIGN EQUITY OPTIONS WITH PATH DEPENDENT FEATURES AND THEIR MULTI-ASSET EXTENSIONS

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Currency-translated foreign equity options (quanto options) are designed for investors who would like to manage different types of risk in international equity investments. The terminal payoffs of quanto options depend on the price of a foreign currency denominated asset (or stock index) and the exchange rate in different combinations of choices. This paper presents a systematic framework to derive pricing formulas for different European-style quanto options with path-dependent payoff functions. The path dependent features can be the barrier feature associated with the underlying asset price movement, the averaging feature of the exchange rate over the life of the option, etc. In many cases, the pricing formulas for quanto options can be inferred from their vanilla counterparts by applying the quanto-prewashing technique of making modifications on the risk neutralized drift rates and volatility rates. The extension of the pricing formulations to multi-asset extremum options with the quanto feature is also considered. The pricing behaviors of the joint quanto options and the Asian quanto options are examined.

Keywords: Quanto options, quanto-prewashing, path dependent features, multi-asset options.

JEL classification code: G130

1. Introduction

With the growth in globalization of investments in recent years, the currency-translated foreign equity options (quanto options) have gained wider popularity. Quanto options are contingent claims where the payoff is determined by a financial price or index in one currency but the actual payout is done in another currency. The payoffs of these quanto options can be structured in a variety of combinations of linking foreign asset price and exchange rate, thus generating a rich set of choices of investment and hedging opportunities. Besides the choice of either fixed or floating exchange rate, their payoff structures can be made more exotic by introducing the barrier or Asian feature on either the underlying asset price or the exchange rate or both. These wider classes of payoff structures allow investors to hedge a specific risk or bet on a particular speculation in their international equity investment. The exposition on the uses and hedging properties of vanilla type quanto options can be found in [1, 6–8].

In this paper, we derive the pricing formulas and examine the pricing behaviors of European-style quanto options with exotic path-dependent payoff structures in the Black–Scholes world. The pricing formulations are also extended to multi-asset extremum options. Although we follow similar "quanto-prewashing" technique of making modifications on the risk neutralized drift rates and volatilities [8], this is a non-trivial extension of a number of earlier works [1, 6–8], where only vanilla type payoff functions were considered in those papers. For example, the pricing formulas for the joint quanto options (with and without barrier) and Asian quanto options (single-asset and multi-asset) are obtained in our work. The pricing behaviors of these new classes of quanto options are also examined.

This paper is organized as follows. In the next section, various versions of the partial differential equation formulation of the quanto option models are derived. The required modifications on the risk neutralized drift rates and volatility rates in the quanto-prewashing process are summarized in a succinct fashion. The pricing formulas for several standard quanto options with vanilla payoffs are obtained as illustrations of the effectiveness of the formulations. The pricing formulas and pricing behaviors of quanto options with barrier feature and Asian feature are presented in Secs. 3 and 4, respectively. The barrier feature and the Asian feature can be on the asset price process or the exchange rate process. The extension of the pricing formulations to multi-asset extremum options with the quanto feature is given in Sec. 5. Summary of results and conclusive remarks are given in the last section.

2. Partial Differential Equation Formulations

We would like to derive the various versions of the partial differential equation formulation of quanto option models. Apparently, there are four independent variables, namely, the domestic currency price of one unit of foreign currency F, the asset price in foreign currency S, the asset price in domestic currency S^*, and time t. Note that S and S^* are related by

$$S^* = FS\,, \tag{1}$$

and so the quanto option prices can be functions of either the set of independent variables: S^*, F and t, or the other set: S, F and t, or even the third set: S, S^* and t. The usual lognormal distributions for the stochastic state variables are assumed, where

$$\frac{dS}{S} = \mu_S dt + \sigma_S dZ_S \tag{2a}$$

$$\frac{dF}{F} = \mu_F dt + \sigma_F dZ_F \tag{2b}$$

$$\frac{dS^*}{S^*} = \mu_{S^*} dt + \sigma_{S^*} dZ_{S^*}\,. \tag{2c}$$

Here, μ_S, μ_F and μ_{S^*} are the constant drift rates, σ_S, σ_F and σ_{S^*} are the constant volatilities, and dZ_S, dZ_F and dZ_{S^*} are the Wiener processes of the respective stochastic variables. Also, we write the correlation coefficient between dZ_S and dZ_F as ρ_{SF}, and similar meaning for ρ_{S^*F} and ρ_{SS^*}. Since S^*, F and S are related by Eq. (1), and from Ito's lemma, we obtain

$$\mu_{S^*} = \mu_S + \mu_F + \rho_{SF}\sigma_S\sigma_F \tag{3a}$$

$$\sigma_{S^*}^2 = \sigma_S^2 + \sigma_F^2 + 2\rho_{SF}\sigma_S\sigma_F\,. \tag{3b}$$

Further, the correlation coefficients are related by

$$\rho_{S^*F} = \frac{\sigma_F + \rho_{SF}\sigma_S}{\sigma_{S^*}} \tag{4a}$$

$$\rho_{SS^*} = \frac{\sigma_S + \rho_{SF}\sigma_F}{\sigma_{S^*}}\,. \tag{4b}$$

2.1. *Domestic currency world*

The usual assumptions of the Black–Scholes environment are adopted. Let $V_d(S^*, F, t)$ denote the price of a quanto option in domestic currency using S^*, F and t as the independent variables. By using the standard argument of forming a riskless hedging portfolio containing appropriate units of the underlying asset and foreign currency and selling short one unit of the quanto option, the governing equation for $V_d = V_d(S^*, F, t)$ is found to be

$$\begin{aligned}\frac{\partial V_d}{\partial t} + \frac{\sigma_{S^*}^2}{2}S^{*2}\frac{\partial^2 V_d}{\partial S^{*2}} + \rho_{S^*F}\sigma_{S^*}\sigma_F S^* F\frac{\partial^2 V_d}{\partial S^*\partial F} + \frac{\sigma_F^2}{2}F^2\frac{\partial^2 V_d}{\partial F^2} + (r_d - q)S^*\frac{\partial V_d}{\partial S^*}\\ + (r_d - r_f)F\frac{\partial V_d}{\partial F} - r_d V_d = 0, \qquad S^* > 0, F > 0, t > 0\,,\end{aligned} \tag{5}$$

where q is the dividend yield of the asset and $r_f(r_d)$ is the foreign (domestic) riskless interest rate.

Let $\delta_{S^*}^d$ and δ_F^d denote the risk neutralized drift rates for S^* and F in the domestic currency world, respectively. It can be observed easily from the drift terms in above governing equation that

$$\delta_{S^*}^d = r_d - q \qquad \text{and} \qquad \delta_F^d = r_d - r_f\,. \tag{6a}$$

The risk neutralized drift rate for S in the domestic currency world, δ_S^d, is then given by [see Eq. (3a)]

$$\delta_S^d = \delta_{S^*}^d - \delta_F^d - \rho_{SF}\sigma_S\sigma_F = r_f - q - \rho_{SF}\sigma_S\sigma_F\,. \tag{6b}$$

The derivation of δ_S^d using financial argument can be found in [8].

There are quanto options whose terminal payoff function depends only on one stochastic state variable S, and so the corresponding option price depends on S and t only. By performing the calculus of changing the independent variables in Eq. (5), the governing equation for $V_d = V_d(S,t)$ reduces to the usual one-dimensional Black–Scholes equation, namely,

$$\frac{\partial V_d}{\partial t} + \frac{\sigma_S^2}{2}S^2\frac{\partial^2 V_d}{\partial S^2} + \delta_S^d S\frac{\partial V_d}{\partial S} - r_d V_d = 0\,, \qquad S>0, t>0\,, \tag{7}$$

where the risk neutralized drift rate is simply δ_S^d, which is given in Eq. (6b).

Suppose the independent stochastic asset variable is changed from S^* to S, the governing equation for $V_d = V_d(S,F,t)$ can be inferred directly from Eq. (5) and this gives

$$\begin{aligned}\frac{\partial V_d}{\partial t} &+ \frac{\sigma_S^2}{2}S^2\frac{\partial^2 V_d}{\partial S^2} + \rho_{SF}\sigma_S\sigma_F SF\frac{\partial^2 V_d}{\partial S\partial F} + \frac{\sigma_F^2}{2}F^2\frac{\partial^2 V_d}{\partial F^2}\\ &+ \delta_S^d S\frac{\partial V_d}{\partial S} + \delta_F^d F\frac{\partial V_d}{\partial F} - r_d V_d = 0\,, \qquad S>0, F>0, t>0\,.\end{aligned} \tag{8}$$

In one of the barrier quanto option models analyzed later [see Eq. (22)], it is most convenient to choose S, S^* and t as the independent variables. The corresponding governing equation for $V_d = V_d(S,S^*,t)$ is given by

$$\begin{aligned}\frac{\partial V_d}{\partial t} &+ \frac{\sigma_S^2}{2}S^2\frac{\partial^2 V_d}{\partial S^2} + \rho_{SS^*}\sigma_S\sigma_{S^*} SS^*\frac{\partial^2 V_d}{\partial S\partial S^*} + \frac{\sigma_{S^*}^2}{2}S^{*2}\frac{\partial^2 V_d}{\partial {S^*}^2}\\ &+ \delta_S^d S\frac{\partial V_d}{\partial S} + \delta_{S^*}^d S^*\frac{\partial V_d}{\partial S^*} - r_d V_d = 0\,, \qquad S>0, S^*>0, t>0\,.\end{aligned} \tag{9}$$

2.2. *Foreign currency world*

Let $V_f = V_f(S^*,F,t)$ denote the option price in foreign currency, where

$$V_f(S^*,F,t) = V_d(S^*,F,t)/F\,. \tag{10}$$

By performing the calculus of transformation of variables in Eq. (5), the governing differential equation for $V_f = V_f(S^*,F,t)$ can be found to be

$$\begin{aligned}\frac{\partial V_f}{\partial t} &+ \frac{\sigma_{S^*}^2}{2}{S^*}^2\frac{\partial^2 V_f}{\partial {S^*}^2} + \rho_{S^*F}\sigma_{S^*}\sigma_F S^*F\frac{\partial^2 V_f}{\partial S^*\partial F} + \frac{\sigma_F^2}{2}F^2\frac{\partial^2 V_f}{\partial F^2}\\ &+ (r_d - q + \rho_{S^*F}\sigma_{S^*}\sigma_F)S^*\frac{\partial V_f}{\partial S^*} + (r_d - r_f + \sigma_F^2)F\frac{\partial V_f}{\partial F} - r_f V_f = 0\,,\\ &S^*>0, F>0, t>0\,.\end{aligned} \tag{11}$$

By observing the drift terms in the above governing equation, one again deduces that the risk neutralized drift rates for S^* and F in the foreign currency world are given by (see Appendix)

$$\delta_{S^*}^f = r_d - q + \rho_{S^*F}\sigma_{S^*}\sigma_F \qquad \text{and} \qquad \delta_F^f = r_d - r_f + \sigma_F^2\,, \tag{12a}$$

respectively. The risk neutralized drift rate of S in the foreign currency world is known to be

$$\delta_S^f = r_f - q\,. \tag{12b}$$

The alternative approach of deriving these risk neutralized drift rates in the foreign currency world is presented in the Appendix.

By observing the general pattern shown in Eqs. (5), (8), (9) and (11), one can deduce immediately the governing equation for the option price in either currency world and any set of independent state variables by simply placing the "quanto-prewashing" drift rates and volatilities as coefficients in the corresponding drift terms and volatility terms. This "quanto-prewashing" technique is reminiscent of the two numeraire theorems stated in [8].

Once the risk neutralized drift rates for S^*, S and F in both the domestic and foreign currency worlds are available [see Eqs. (6a), (6b), (12a) and (12b)], the pricing formulas for quanto options with vanilla payoff structures can be inferred directly from their non-quanto counterparts. As for illustration, we consider the following three types of quanto options analyzed in Reiner's paper [6]:

(i) Fixed exchange rate foreign equity call

$$\text{terminal payoff}: V_d^{(1)}(S_T, T) = F_0 \max(S_T - X_f, 0)\,,$$

where F_0 is some predetermined fixed exchange rate and X_f is the strike price in foreign currency.

Since the payoff depends on S and it is denominated in the domestic currency world, the risk neutralized drift rate adopted should be δ_S^d. Hence, the corresponding price formula is given by

$$V_d^{(1)}(S,t) = F_0 e^{-r_d\tau}[Se^{\delta_S^d\tau}N(d_1) - X_f N(d_2)]\,, \qquad \tau = T - t\,, \tag{13a}$$

where

$$d_1 = \frac{\ln\frac{S}{X_f} + \left(\delta_S^d + \frac{\sigma_S^2}{2}\right)\tau}{\sigma_S\sqrt{\tau}}\,, \qquad d_2 = d_1 - \sigma_S\sqrt{\tau}\,. \tag{13b}$$

The price formula does not depend on the exchange rate F since the exchange rate has been chosen to be at the fixed value F_0. Note that the writer of the option has an exposure of the foreign currency of amount $\max(S_T - X_f, 0)$ at expiry. The dependence of the option price on the exchange rate volatility σ_F and the correlation coefficient ρ_{SF} (through δ_S^d) reflects this exposure.

(ii) Call on foreign equity denominated in domestic currency

$$\text{terminal payoff}: V_d^{(2)}(S_T, F_T, T) = \max(F_T S_T - X_d, 0) = \max(S_T^* - X_d, 0)\,,$$

where X_d is the strike price in domestic currency.

The corresponding price formula is easily seen to be

$$V_d^{(2)}(S^*,t) = S^* e^{-q\tau}N(\hat{d}_1) - X_d e^{-r_d\tau}N(\hat{d}_2)\,, \qquad \tau = T - t\,, \tag{14a}$$

where

$$\hat{d}_1 = \frac{\ln \frac{S^*}{X_d} + \left(\delta^d_{S^*} + \frac{\sigma^2_{S^*}}{2}\right)\tau}{\sigma_{S^*}\sqrt{\tau}}, \qquad \hat{d}_2 = \hat{d}_1 - \sigma_{S^*}\sqrt{\tau}. \tag{14b}$$

(iii) Floating exchange rate foreign equity call

$$\text{terminal payoff}: V_d^{(3)}(S_T, F_T, T)/F_T = V_f^{(3)}(S_T, T) = \max(S_T - X_f, 0).$$

This call option behaves like the usual vanilla call option in the foreign currency world, and so the corresponding price formula is given by

$$V_d^{(3)}(S, F, t)/F = V_f^{(3)}(S, t)$$
$$= Se^{-q\tau}N(\bar{d}_1) - X_f e^{-r_f\tau}N(\bar{d}_2), \qquad \tau = T - t, \tag{15a}$$

where

$$\bar{d}_1 = \frac{\ln \frac{S}{X_f} + \left(\delta^f_S + \frac{\sigma^2_S}{2}\right)\tau}{\sigma_S\sqrt{\tau}}, \qquad \bar{d}_2 = \bar{d}_1 - \sigma_S\sqrt{\tau}. \tag{15b}$$

A genuine two-dimensional quanto option model is the "joint" quanto option where the exchange rate F is guaranteed to have at least the floor value F_0 [7].

(iv) Joint quanto option

$$\text{terminal payoff}: V_d^{(4)}(S_T, F_T, T) = \max(F_T, F_0)\max(S_T - X_f, 0).$$

Let $G_d(S, F, \tau; S_T, F_T)$ be the Green function of the governing Eq. (8), where (see [4, p. 105])

$$G_d(S, F, \tau; S_T, F_T) = \frac{e^{-r_d\tau}}{2\pi\tau}\frac{1}{\sqrt{1-\rho^2_{SF}}\sigma_S\sigma_F S_T F_T}$$
$$\times \exp\left(-\frac{x_S^2 - 2\rho_{SF}x_S x_F + x_F^2}{2(1-\rho^2_{SF})}\right), \tag{16}$$

where

$$x_S = \frac{1}{\sigma_S\sqrt{\tau}}\left[\ln\frac{S_T}{S} - \left(\delta^d_S - \frac{\sigma^2_S}{2}\right)\tau\right] \tag{17a}$$

$$x_F = \frac{1}{\sigma_F\sqrt{\tau}}\left[\ln\frac{F_T}{F} - \left(\delta^d_F - \frac{\sigma^2_F}{2}\right)\tau\right]. \tag{17b}$$

The price of the joint quanto option is then given by

$$V_d^{(4)}(S, F, t) = F_0\int_0^{F_0}\int_{X_f}^{\infty}(S_T - X_f)G_d(S, F, \tau; S_T, F_T)\, dS_T dF_T$$
$$+ \int_{F_0}^{\infty}\int_{X_f}^{\infty}F_T(S_T - X_f)G_d(S, F, \tau; S_T, F_T)\, dS_T dF_T$$
$$= F_0 e^{-r_d\tau}[Se^{\delta^d_S\tau}N_2(d_1, -f_1; -\rho_{SF}) - X_f N_2(d_2, -f_2; -\rho_{SF})]$$
$$+ Fe^{-r_f\tau}[Se^{\delta^f_S\tau}N_2(\bar{d}_1, \bar{f}_1; \rho_{SF}) - X_f N_2(\bar{d}_2, \bar{f}_2; \rho_{SF})], \tag{18}$$

where

$$\bar{f}_2 = \frac{1}{\sigma_F\sqrt{\tau}}\left[\ln\frac{F}{F_0} + \left(\delta_F^d - \frac{\sigma_F^2}{2}\right)\tau\right], \qquad f_1 = f_2 + \rho_{SF}\sigma_S\sqrt{\tau}, \tag{19a}$$

$$\bar{f}_2 = \frac{1}{\sigma_F\sqrt{\tau}}\left[\ln\frac{F}{F_0} + \left(\delta_F^f - \frac{\sigma_F^2}{2}\right)\tau\right], \qquad \bar{f}_1 = \bar{f}_2 + \rho_{SF}\sigma_S\sqrt{\tau}. \tag{19b}$$

Note that the first (second) term in the price formula (18) involves risk neutralized drift rates in the domestic (foreign) currency world.

2.3. *Pricing behaviors of the vanilla quanto options*

Since the exchange rate in the terminal payoff of the joint quanto option is chosen to be $\max(F_0, F_T)$, and so we expect that the price of the joint quanto option is more expensive than those of the quanto options with either fixed exchange rate F_0 or floating exchange rate F_T, that is,

$$V_d^{(4)}(S,F,t) > \max(V_d^{(1)}(S,t), V_d^{(3)}(S,F,t)). \tag{20}$$

It would be interesting to explore the dependence of the prices of the quanto options on the correlation coefficient ρ_{SF}. For the floating exchange rate option, the option price function $V_d^{(3)}$ is independent of ρ_{SF} and so we have $\frac{\partial V_d^{(3)}}{\partial \rho_{SF}} = 0$. On the other hand, since the effective dividend yield of the foreign asset in the domestic currency world becomes $q + \rho_{SF}\sigma_S\sigma_F$ [see δ_S^d in Eq. (6b)], and so the price function of the fixed exchange rate call, $V_d^{(1)}(S,t)$, is a decreasing function of ρ_{SF}, that is, $\frac{\partial V_d^{(1)}}{\partial \rho_{SF}} < 0$.

Since the exchange rate in the joint quanto option is chosen to be a hybrid of fixed and floating exchange rates, and so the rates of change in option prices with respect to ρ_{SF} should satisfy

$$\frac{\partial V_d^{(1)}}{\partial \rho_{SF}} < \frac{\partial V_d^{(4)}}{\partial \rho_{SF}} < \frac{\partial V_d^{(3)}}{\partial \rho_{SF}} = 0. \tag{21}$$

The properties of the price functions of the quanto options as summarized in Eqs. (20) and (21) are succinctly illustrated in Fig. 1. The parameter values used in the calculations are: $r_d = 9\%$, $r_f = 7\%$, $q = 8\%$, $\sigma_S = \sigma_F = 20\%$, $S = 1.2$, $F = 1.5$, $F_0 = 1.5$, $X_f = 1.0$, $\tau = 0.5$. Since F_0 is chosen to be equal to F at the current time, it is likely that $F_T > F_0$ as ρ_{SF} moves closer to 1 and so $V_d^{(4)}$ tends to $V_d^{(3)}$ as $\rho_{SF} \to 1$.

The price of the call on foreign equity denominated in domestic currency, $V_d^{(2)}(S,F,t)$, is an increasing function of ρ_{SF} since the effective volatility of the asset price in the domestic currency world, σ_{S^*}, increases with increasing ρ_{SF} [see Eq. (3b)].

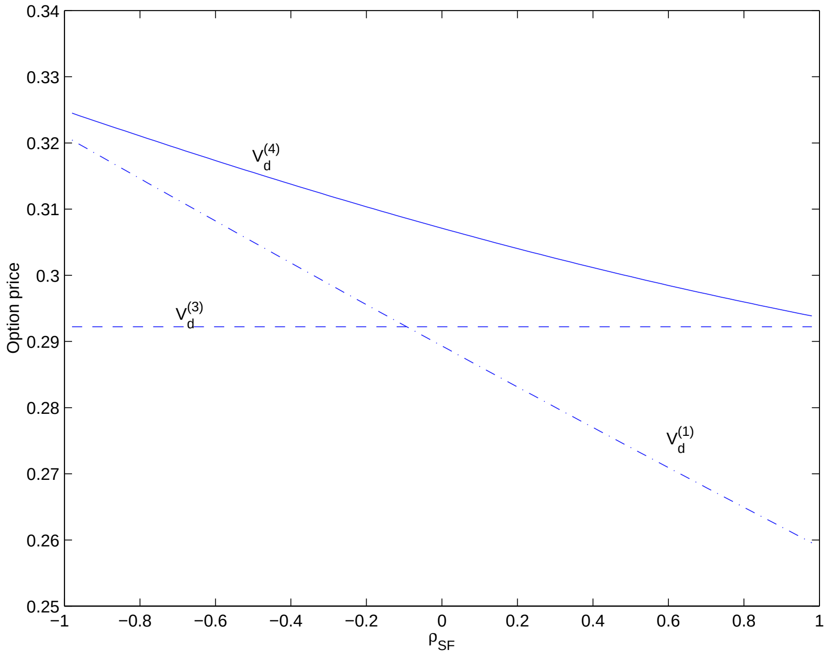

Fig. 1. The dependence of the price functions of the quanto options on the correlation coefficient ρ_{SF}.

3. Barrier Quanto Options

A barrier option is a path-dependent option that is either nullified, activated or exercised if the price of the underlying asset breaches a barrier during the life of the option. For barrier quanto options, the barrier feature can be either on the exchange rate process or on the asset value process or both. Suppose the barrier feature is on the exchange rate F while the payoff function does not depend explicitly on F, then F can be considered as an external barrier variable. The pricing behaviors of single-asset options [2] and multi-asset options [5] with an external barrier have been fully analyzed in the literature.

Suppose the asset price process S has a down-and-out barrier: $b(\tau) = b_0 e^{-\alpha\tau}$, where b_0 and α are constant, so that the quanto option is nullified when S falls below $b(\tau)$. Since the terminal payoffs in $V_d^{(1)}(S,t)$ and $V_f^{(3)}(S,t)$ are independent of F_T, the quanto option models remain to be one-dimensional, and so the price formulas can be inferred directly from those of the corresponding non-quanto barrier options. However, some efforts are required to derive the corresponding pricing formulas for $V_d^{(2)}(S,F,t)$ and $V_d^{(4)}(S,F,t)$.

The terminal payoff function of $V_d^{(2)}(S,F,t)$ depends on the state variable S^* while the barrier depends on S only. Hence, it is preferable to use the set of independent variables: S, S^* and t and the corresponding governing equation for

$V_d^{(2)}(S, S^*, t)$ is given by Eq. (9). Now, S can be considered as the external barrier variable and S^* as the payoff variable. This barrier quanto option model resembles the option model with an external barrier analyzed in [2] and [5]. By mimicking directly the price formula in [5], the price formula $V_d^{(2)}(S, S^*, t)$ (with the down-and-out barrier $b(\tau)$) is found to be

$$V_d^{(2)}(S, S^*, t) = S^* e^{-q\tau}[N_2(e_1, \hat{d}_1; \rho_{SS^*}) - \exp(-2(\gamma_d + \rho_{SS^*}\sigma_{S^*})\hat{x}_S) N_2(e_3, \hat{d}_3; \rho_{SS^*})] - X_d e^{-r_d\tau}[N_2(e_2, \hat{d}_2; \rho_{SS^*}) - \exp(-2\gamma_d \hat{x}_S) N_2(e_4, \hat{d}_4; \rho_{SS^*})], \quad (22)$$

where $\hat{d}_1$ and $\hat{d}_2$ are defined in Eq. (14b), and

$$\hat{x}_S = \frac{1}{\sigma_S} \ln \frac{S}{b(\tau)}, \qquad \gamma_d = \frac{1}{\sigma_S}\left(\delta_S^d - \alpha - \frac{\sigma_S^2}{2}\right), \quad (23a)$$

$$\hat{d}_3 = \hat{d}_1 - 2\rho_{SS^*}\frac{\hat{x}_S}{\sqrt{\tau}}, \qquad \hat{d}_4 = \hat{d}_2 - 2\rho_{SS^*}\frac{\hat{x}_S}{\sqrt{\tau}}, \quad (23b)$$

$$e_1 = \frac{\ln \frac{S}{b(\tau)} + \left(r_d - q - \frac{\sigma_S^2}{2}\right)\tau}{\sigma_S\sqrt{\tau}}, \qquad e_2 = e_1 - \rho_{SS^*}\sigma_{S^*}\sqrt{\tau}, \quad (23c)$$

$$e_3 = e_1 - \frac{2\hat{x}_S}{\sqrt{\tau}}, \qquad e_4 = e_2 - \frac{2\hat{x}_S}{\sqrt{\tau}}. \quad (23d)$$

Next, we compute the price formula $V_d^{(4)}(S, F, t)$ for the joint quanto option with the down-and-out barrier $b(\tau)$. The joint density of S_T and F_T, given S and F at time t, and S has a down-and-out barrier $b(\tau)$, is found to be [5]

$$\psi_{SF}(S_T, F_T; S, F | S_u > b_0 e^{-\alpha(T-u)}) = \frac{e^{-r_d\tau}}{2\pi\tau}\frac{1}{\sqrt{1-\rho_{SF}^2}\,\sigma_S\sigma_F S_T F_T}\left[\exp\left(-\frac{x_S^2 - 2\rho_{SF}x_S x_F + x_F^2}{2(1-\rho_{SF}^2)}\right) - \exp(-2\gamma_d\hat{x}_S)\exp\left(-\frac{\tilde{x}_S^2 - 2\rho_{SF}\tilde{x}_S\tilde{x}_F + \tilde{x}_F^2}{2(1-\rho_{SF}^2)}\right)\right], \quad (24)$$

where x_S and x_F are defined in Eqs. (17a), (17b), and

$$\tilde{x}_S = x_S + \frac{2\hat{x}_S}{\sqrt{\tau}}, \qquad \tilde{x}_F = x_F + \frac{2\rho_{SF}\hat{x}_S}{\sqrt{\tau}}. \quad (25)$$

Correspondingly, the price of the joint quanto option, where S has the down-and-out barrier $b(\tau)$, is given by

$$V_d^{(4)}(S, F, t) = F_0 \int_0^{F_0}\int_{X_f}^{\infty}(S_T - X_f)\psi_{SF}\, dS_T dF_T + \int_{F_0}^{\infty}\int_{X_f}^{\infty} F_T(S_T - X_f)\psi_{SF}\, dS_T dF_T$$

$$
\begin{aligned}
&= F_0 e^{-r_d\tau}\{e^{\delta_S^d \tau} S[N_2(d_1, -f_1; -\rho_{SF}) \\
&\quad - \exp(-2(\gamma_d + \sigma_S)\hat{x}_S) N_2(g_1, -h_1; -\rho_{SF})] \\
&\quad - X_f[N_2(d_2, -f_2; -\rho_{SF}) - \exp(-2\gamma_d \hat{x}_S) N_2(g_2, -h_2; -\rho_{SF})]\} \\
&\quad + F\{S e^{-q\tau}[N_2(\bar{d}_1, \bar{f}_1; \rho_{SF}) - \exp(-2(\gamma_f + \sigma_S)\hat{x}_S) N_2(\bar{g}_1, \bar{h}_1; \rho_{SF})] \\
&\quad - X_f e^{-r_f\tau}[N_2(\bar{d}_2, \bar{f}_2; \rho_{SF}) - \exp(-2\gamma_f \hat{x}_S) N_2(\bar{g}_2, \bar{h}_2; \rho_{SF})]\}\,, \qquad (26)
\end{aligned}
$$

where $\bar{d}_1$ and $\bar{d}_2$ are defined in Eq. (15b), $f_1, f_2, \bar{f}_1$ and $\bar{f}_2$ are defined in Eqs. (19a) and (19b), $\hat{x}_S$ and γ_d are defined in Eq. (23a), and

$$g_1 = d_1 + \frac{2\hat{x}_S}{\sqrt{\tau}}\,, \qquad h_1 = f_1 + \frac{2\rho_{SF}\hat{x}_S}{\sqrt{\tau}}\,, \tag{27a}$$

$$g_2 = g_1 - \sigma_S\sqrt{\tau}\,, \qquad h_2 = h_1 - \rho_{SF}\sigma_S\sqrt{\tau}\,, \tag{27b}$$

$$\bar{g}_1 = g_1 + \rho_{SF}\sigma_F\sqrt{\tau}\,, \qquad \bar{h}_1 = h_1 + \sigma_F\sqrt{\tau}\,, \tag{27c}$$

$$\bar{g}_2 = \bar{g}_1 - \sigma_S\sqrt{\tau}\,, \qquad \bar{h}_2 = \bar{h}_1 - \rho_{SF}\sigma_S\sqrt{\tau}\,, \tag{27d}$$

$$\gamma_f = \frac{1}{\sigma_S}\left(\delta_S^f - \alpha - \frac{\sigma_S^2}{2}\right)\,. \tag{27e}$$

4. Asian Quanto Options

The Asian (or averaging) feature in an option model refers to the dependence of the payoff function on some form of averaging of the underlying state variable over a part or the whole life of the option. The averaging procedure can be taken either discretely or continuously. The common forms of averaging in option contracts can be either geometric or arithmetic average of the underlying state variables. Note that lognormal distributions are assumed for both the asset price process and foreign exchange rate process in the Black–Scholes world. Since the product of lognormal densities remains to be lognormal, for the goal of achieving analytical tractability, we assume continuous geometric averaging in the analysis of the following Asian quanto options.

Let the averaging period be $[0, T]$, where T is the expiration date of the option, and let t denote the current time, $0 < t \le T$. The continuous geometric average of the asset price process S over $[0, t]$ is defined to be

$$G_S^t = \exp\left(\frac{1}{t}\int_0^t \ln\, S(\tau)d\tau\right)\,, \qquad 0 < t \le T\,, \tag{28a}$$

and similar definition for the continuous geometric average of the foreign exchange rate F, where

$$G_F^t = \exp\left(\frac{1}{t}\int_0^t \ln\, F(\tau)d\tau\right)\,, \qquad 0 < t \le T\,. \tag{28b}$$

The pricing formulas of the following Asian quanto options are to be derived:

(v) Fixed exchange rate foreign equity call with averaged strike

$$\text{terminal payoff}: V_d^{(5)}(S_T, G_S^T, T) = F_0 \max(S_T - G_S^T, 0)$$

(vi) Averaged exchange rate foreign equity call

$$\text{terminal payoff}: V_d^{(6)}(S_T, G_F^T, T) = G_F^T \max(S_T - X_f, 0)$$

(vii) Averaged exchange rate foreign equity call with averaged strike

$$\text{terminal payoff}: V_d^{(7)}(S_T, G_S^T, G_F^T, T) = G_F^T \max(S_T - G_S^T).$$

Apparently, G_S appears as an independent variable in $V_d^{(5)}$. It is observed that dG_S is deterministic where

$$dG_S = \left(\frac{G_S}{t} \ln \frac{S}{G_S}\right) dt. \tag{29}$$

In the evaluation of the differential $dV_d(S, G_S, t)$, an extra term $(\frac{G_S}{t} \ln \frac{S}{G_S}) \frac{\partial V_d}{\partial G_S} dt$ also appears. Hence, the governing equation for $V_d = V_d(S, G_S, t)$ can be inferred easily from Eq. (7) to be (see [9])

$$\frac{\partial V_d}{\partial t} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V_d}{\partial S^2} + \delta_S^d S \frac{\partial V_d}{\partial S} + \left(\frac{G_S}{t} \ln \frac{S}{G_S}\right) \frac{\partial V_d}{\partial G_S} - r_d V_d = 0,$$
$$S > 0, \qquad G_S > 0, \qquad t > 0. \tag{30}$$

The following set of similarity variables are chosen:

$$x = t \ln \frac{G_S}{S} \qquad \text{and} \qquad U(x, t) = \frac{V_d(S, G_S, t)}{S}, \tag{31}$$

where the asset price S is used as the numeraire. In terms of the new similarity variables, Eq. (30) is reduced to one-dimensional and takes the form

$$\frac{\partial U}{\partial t} + \frac{\sigma_S^2}{2} t^2 \frac{\partial^2 U}{\partial x^2} - \left(\delta_S^d + \frac{\sigma_S^2}{2}\right) t \frac{\partial U}{\partial x} + (\delta_S^d - r_d) U = 0, \qquad -\infty < x < \infty, t > 0. \tag{32}$$

The terminal payoff function of the fixed exchange rate foreign equity call with averaged strike becomes

$$U(x, T) = F_0 \max(1 - e^{x/T}, 0). \tag{33}$$

The Green function of Eq. (32) is found to be (see [9])

$$G(x, t; \xi, T) = e^{(\delta_S^d - r_d)\tau} n \left(\frac{\xi - x + \left(\delta_S^d + \frac{\sigma_S^2}{2}\right) \int_t^T u\, du}{\sigma_S \sqrt{\int_t^T u^2\, du}} \right), \tag{34}$$

where $n(x)$ is the standard normal density function. The solution to Eq. (32) augmented with the auxiliary condition in Eq. (33) can be formally represented by

$$U(x,t) = F_0\, e^{(\delta_S^d - r_d)\tau} \int_{-\infty}^{\infty} \max(1 - e^{\xi/T}, 0) G(x,t;\xi,T)\, d\xi\,. \tag{35}$$

By direct integration of the above integral, the value of the fixed exchange rate foreign equity call with averaged strike is found to be

$$V_d^{(5)}(S, G_S, t) = F_0 S e^{(\delta_S^d - r_d)\tau} \left[N(k_1) - \left(\frac{G_S}{S}\right)^{t/T} e^{-Q(t;T)\tau} N(k_2) \right], \tag{36}$$

where

$$k_1 = \frac{-t \ln \frac{G_S}{S} + \left(\delta_S^d + \frac{\sigma_S^2}{2}\right) \frac{T^2 - t^2}{2}}{\sigma_S \sqrt{\frac{T^3 - t^3}{3}}}, \tag{37a}$$

$$k_2 = k_1 - \sigma_S \sqrt{\frac{T^3 - t^3}{3T^2}}, \tag{37b}$$

$$Q(t;T) = \left(\delta_S^d + \frac{\sigma_S^2}{2}\right) \frac{T+t}{2T} - \sigma_S^2 \frac{T^2 + Tt + t^2}{6T^2}\,. \tag{37c}$$

In the foreign currency world, the terminal payoff of the averaged exchange rate foreign equity call can be expressed as

$$V_f^{(6)}(S_T, F, G_F^T, T) = \frac{G_F^T}{F} \max(S_T - X_f, 0)\,. \tag{38}$$

Now, we choose the similarity variables:

$$\tilde{U}(S,t) = \frac{V_f(S, F, G_F, t)}{(G_F/F)^{t/T}}, \tag{39}$$

the governing equation for $\tilde{U}(S,t)$ can be found to be

$$\frac{\partial \tilde{U}}{\partial t} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 \tilde{U}}{\partial S^2} + \left(r_f - q - \frac{t}{T}\rho_{SF}\sigma_S\sigma_F\right) S \frac{\partial \tilde{U}}{\partial S}$$
$$- \left[r_f - \left(r_f - r_d - \frac{\sigma_F^2}{2}\right)\frac{t}{T} - \frac{\sigma_F^2}{2}\frac{t^2}{T^2}\right]\tilde{U} = 0\,, \qquad 0 < S < \infty, t > 0\,. \tag{40}$$

The corresponding terminal condition can be expressed as

$$\tilde{U}(S_T, T) = \max(S_T - X_f, 0)\,. \tag{41}$$

It is seen that Eq. (40) resembles the governing equation for the Black–Scholes call model with time dependent coefficients. We define

$$\alpha(t;T) = \frac{1}{T-t} \int_t^T \left(r_f - q - \frac{\xi}{T}\rho_{SF}\sigma_S\sigma_F\right) d\xi$$
$$= r_f - q - \frac{T+t}{2T}\rho_{SF}\sigma_S\sigma_F \tag{42a}$$

$$\beta(t;T) = \frac{1}{T-t}\int_t^T \left[r_f - \left(r_f - r_d - \frac{\sigma_F^2}{2}\right)\frac{\xi}{T} - \frac{\sigma_F^2}{2}\frac{\xi^2}{T^2}\right] d\xi$$

$$= r_f - \left(r_f - r_d - \frac{\sigma_F^2}{2}\right)\frac{T+t}{2T} - \frac{\sigma_F^2}{6}\frac{T^2+Tt+t^2}{T^2}\,. \tag{42b}$$

The solution for $\tilde{U}(S,t)$ is then given by

$$\tilde{U}(S,t) = e^{-\beta(t;T)\tau}[e^{\alpha(t;T)\tau}SN(a_1) - X_f N(a_2)]\,, \tag{43}$$

where

$$a_1 = \frac{\ln\frac{S}{X_f} + \left[\alpha(t;T) + \frac{\sigma_S^2}{2}\right]\tau}{\sigma_S\sqrt{\tau}}\,, \qquad a_2 = a_1 - \sigma_S\sqrt{\tau}\,. \tag{44}$$

Lastly, the value of the averaged exchange rate foreign equity call is found to be

$$V_d^{(6)}(S,F,G_F,t) = F(G_F/F)^{t/T}e^{-\beta(t;T)\tau}\left[e^{\alpha(t;T)\tau}SN(a_1) - X_f N(a_2)\right]\,. \tag{45}$$

We use the combination of the techniques used in the derivation of $V_d^{(5)}(S,G_S,t)$ and $V_d^{(6)}(S,F,G_F,t)$ to obtain $V_d^{(7)}(S,F,G_S,G_F,t)$. Suppose we choose the following similarity variables as the independent variables in the option model:

$$\hat{U}(x,t) = \frac{V_f(S,F,G_F,t)}{S(G_F/F)^{t/T}} \qquad \text{and} \qquad x = t\,\ln\frac{G_S}{S}\,, \tag{46}$$

the governing equation for $\hat{U}(x,t)$ is found to be

$$\frac{\partial\hat{U}}{\partial t} + \frac{\sigma_S^2}{2}t^2\frac{\partial^2\hat{U}}{\partial x^2} - \left(r_f - q - \frac{t}{T}\rho_{SF}\sigma_S\sigma_F + \frac{\sigma_S^2}{2}\right)t\frac{\partial\hat{U}}{\partial x}$$

$$+ \left[-q + \left(r_f - r_d - \frac{\sigma_F^2}{2} - \rho_{SF}\sigma_S\sigma_F\right)\frac{t}{T} + \frac{\sigma_F^2}{2}\frac{t^2}{T^2}\right]\hat{U} = 0\,,$$

$$-\infty < x < \infty\,, \qquad t > 0\,. \tag{47}$$

The corresponding terminal payoff function for $V_d^{(7)}(S_T, G_S^T, G_F^T, T)$ is transformed to become

$$\hat{U}(x,T) = \max(1 - e^{x/T}, 0)\,. \tag{48}$$

Accordingly, we define

$$\hat{\alpha}(t;T) = \frac{1}{T-t}\int_t^T \left(r_f - q + \frac{\sigma_S^2}{2} - \frac{\xi}{T}\rho_{SF}\sigma_S\sigma_F\right)\xi d\xi$$

$$= \left(r_f - q + \frac{\sigma_S^2}{2}\right)\frac{T+t}{2} - \rho_{SF}\sigma_S\sigma_F\frac{T^2+Tt+t^2}{3T}\,, \tag{49a}$$

$$\hat{\beta}(t;T) = \frac{1}{T-t}\int_t^T \left[q - \left(r_f - r_d - \rho_{SF}\sigma_S\sigma_F - \frac{\sigma_F^2}{2}\right)\frac{\xi}{T} - \frac{\sigma_F^2}{2}\frac{\xi^2}{T^2}\right] d\xi$$

$$= q - \left(r_f - r_d - \rho_{SF}\sigma_S\sigma_F - \frac{\sigma_F^2}{2}\right)\frac{T+t}{2T} - \frac{\sigma_F^2}{6}\frac{T^2+Tt+t^2}{T^2}, \qquad (49b)$$

$$\hat{Q}(t;T) = \frac{\hat{\alpha}(t;T)}{T} - \sigma_S^2 \frac{T^2+Tt+t^2}{6T^2}. \qquad (49c)$$

The price formula for the averaged exchange rate foreign equity call with averaged strike is finally found to be

$$V_d^{(7)}(S,F,G_S,G_F,t) = SF\left(\frac{G_F}{F}\right)^{t/T} e^{-\hat{\beta}(t;T)\tau} \times \left[N(\hat{k}_1) - \left(\frac{G_S}{S}\right)^{t/T} e^{-\hat{Q}(t;T)\tau} N(\hat{k}_2)\right], \qquad (50)$$

where

$$\hat{k}_1 = \frac{-t\ln\frac{G_S}{S} + \hat{\alpha}(t;T)\tau}{\sigma_S\sqrt{\frac{T^3-t^3}{3}}} \quad \text{and} \quad \hat{k}_2 = \hat{k}_1 - \sigma_S\sqrt{\frac{T^3-t^3}{3T^2}}. \qquad (51)$$

4.1. *Pricing behaviors of the Asian quanto options*

It has been discussed earlier that the fixed exchange rate foreign equity call has no dependence on F but the option price function $V_d^{(1)}(S,t)$ [see Eq. (13a)] depends on σ_F and ρ_{SF} through δ_S^d. When the exchange rate is changed from fixed value F_0 to the averaged value G_F^T, we would expect a reducing level of dependence of the option price on ρ_{SF}. This is reflected in the adjusted risk neutralized drift rate in the averaged exchange rate foreign equity call option. From Eq. (40), it is observed that the adjusted risk neutralized drift rate δ_S^{f,G_F} is given by

$$\delta_S^{f,G_F} = r_f - q - \frac{t}{T}\rho_{SF}\sigma_S\sigma_F. \qquad (52)$$

Due to the presence of the factor $\frac{t}{T}$ in front of ρ_{SF}, the option price $V_d^{(6)}(S,F,G_F,t)$ [see Eq. (45)] is less sensitive to the change of ρ_{SF} in comparison to $V_d^{(1)}(S,t)$.

The more interesting aspect of the pricing behaviors of the Asian quanto option models is the dependence of the option price function on time t. Compared to options with vanilla payoff structures, the price functions of the Asian quanto option models have more sophisticated functional dependence on the time variable since the averaging processes evolve in time in a complicated manner.

The fundamental similarity state variables in the Asian quanto option models are

$$R_S = \frac{G_S}{S} \quad \text{and} \quad R_F = \frac{G_F}{F}. \qquad (53)$$

We plot the price functions of the Asian quanto options against time t with fixed values of R_S and/or R_F [see Figs. 2, 3, 4(a)–(d)]. The parameter values used in the calculations are: $r_d = 9\%$, $r_f = 7\%$, $q = 8\%$, $\sigma_S = \sigma_F = 20\%$, $S = 1.0$, $F = 1.5, T = 1$, $F_0 = 1.5$, $X_f = 1.0$ and $\rho_{SF} = 0.5$.

The option price function $V_d^{(5)}(S, G_S, t)$ shows strong dependence on R_S due to the presence of the factor $R_S^{t/T}$ in the price formula in Eq. (36). For fixed values of R_S, the plots in Fig. 2 show that $V_d^{(5)}(S, G_S, t)$ is an increasing (decreasing) function of t when $R_S < 1$ ($R_S \geq 1$). To justify the pricing behaviors by financial argument, we consider two fixed exchange rate averaged strike call options which have the same values of S and G_S but different values of time to expiry. Three separate cases are considered:

(i) $R_S < 1$, that is, $G_S < S$
The options are in-the-money but the gap between S and G_S becomes narrower as time evolves since G_S will increase in value steadily given that $S > G_S$ at the current time. It is likely that the option with the longer time to expiry will expire less in-the-money than the shorter-lived counterpart, and thus leads to $V_d^{(5)}(S, G_S, t_1) < V_d^{(5)}(S, G_S, t_2)$, $t_1 < t_2$.

(ii) $R_S > 1$, that is, $G_S > S$
The options are now out-of-the-money. The shorter-lived option has the lower chance to expire in-the-money and so $V_d^{(5)}(S, G_S, t_1) > V_d^{(5)}(S, G_S, t_2)$, $t_1 < t_2$.

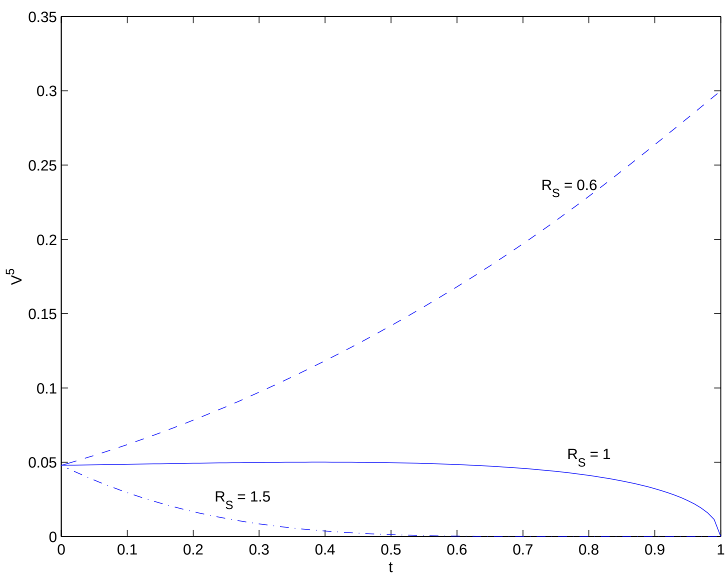

Fig. 2. Plots of $V_d^{(5)}(S, G_S, t)$ against t for $R_S = 0.6$, $R_S = 1$ and $R_S = 1.5$.

(iii) $R_S = 1$, that is, $G_S = S$

The options are now at-the-money. The option price is not quite sensitive to the remaining time to expiry except when the time is sufficiently almost to expiry. The option price drops almost to the zero value as the time is approaching expiration.

The price formula $V_d^{(6)}(S, F, G_F, t)$ can be decomposed into the product of two parts. The part $e^{-\beta(t;T)\tau}[e^{\alpha(t;T)\tau} SN(a_1) - X_f N(a_2)]$, which is independent of F and G_F, resembles the usual Black–Scholes formula, except that the drift rate and discount rate are time dependent functions. The drift rate function $\alpha(t;T)$ and the discount rate function $\beta(t;T)$ [see Eqs. (42a) and (42b)] are decreasing functions of time. The time dependence nature of the other part, $F(G_F/F)^{t/T}$, is more apparent. For a fixed value of $R_F = G_F/F$, the value of $(G_F/F)^{t/T}$ changes from 1 to R_F as t increases from 0 to T.

The time dependence properties of at-the-money averaged exchange rate foreign equity call options are revealed in Fig. 3. The option price function $V_d^{(6)}(S, F, G_F, t)$ corresponding to lower fixed value of R_F decreases at a faster rate in time. With sufficiently high fixed value of R_F, the price function can be an increasing function of time t for options with longer time to expiry.

The plots shown in Figs. 4(a)–4(d) illustrate that the time dependence properties of $V_d^{(7)}(S, F, G_S, G_F, t)$ have close resemblance to those of $V_d^{(5)}(S, G_S, t)$ and $V_d^{(6)}(S, F, G_F, t)$. This is not surprising since $V_d^{(7)}(S, F, G_S, G_F, t)$ is a hybrid of $V_d^{(5)}(S, G_S, t)$ and $V_d^{(6)}(S, F, G_F, t)$. When R_F assumes the fixed value 1, the plots

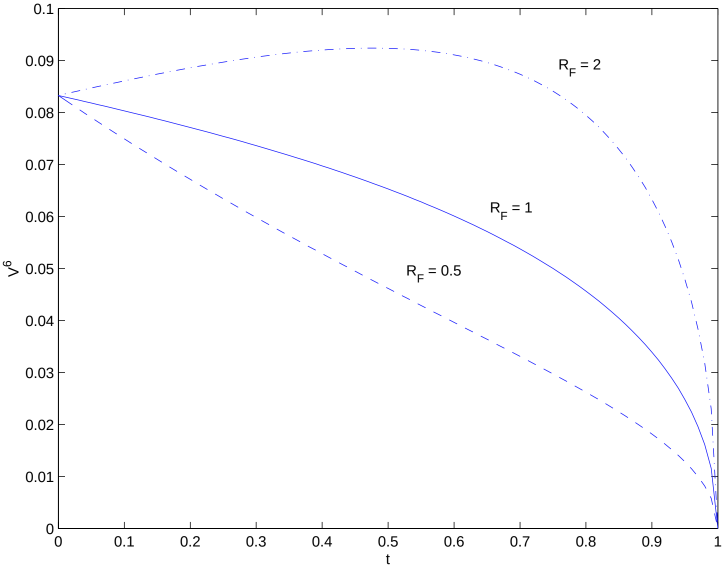

Fig. 3. Plots of $V_d^{(6)}(S, F, G_F, t)$ against t for $R_F = 0.5$, $R_F = 1$ and $R_F = 2$.

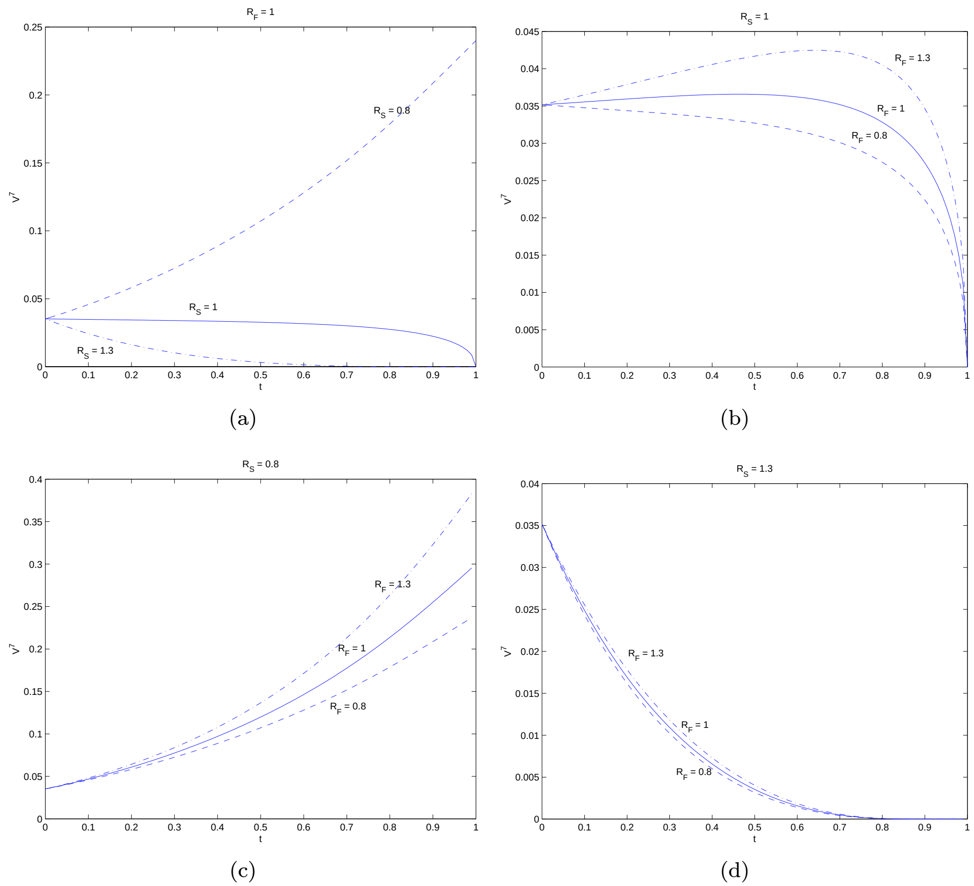

Fig. 4. (a) Plots of $V_d^{(7)}(S, F, G_S, G_F, t)$ against t for $R_S = 0.8$, $R_S = 1$ and $R_S = 1.3$, while R_F assumes the fixed value 1. (b) Plots of $V_d^{(7)}(S, F, G_S, G_F, t)$ against t for $R_F = 0.8$, $R_F = 1$ and $R_F = 1.3$, while R_S assumes the fixed value 1. (c) Plots of $V_d^{(7)}(S, F, G_S, G_F, t)$ against t for $R_F = 0.8$, $R_F = 1$ and $R_F = 1.3$, while R_S assumes the fixed value 0.8. (d) Plots of $V_d^{(7)}(S, F, G_S, G_F, t)$ against t for $R_F = 0.8$, $R_F = 1$ and $R_F = 1.3$, while R_S assumes the fixed value 1.3.

of $V_d^{(7)}(S, F, G_S, G_F, t)$ against t for varying fixed values of R_S resemble those of $V_d^{(5)}(S, G_S, t)$ (see Figs. 4(a) and 2). On the other hand, when R_S assumes the fixed value 1, the plots of $V_d^{(7)}(S, F, G_S, G_F, t)$ against t for varying fixed values of R_F resembles those of $V_d^{(6)}(S, F, G_F, t)$. The other plots shown in Figs. 4(c) and 4(d) reveal similar features as those in Fig. 2.

5. Multi-asset Extremum Options with the Quanto Feature

In this section, the above pricing formulations are extended to multi-asset extremum options with the quanto feature. The discussion is restricted to European call options on the maximum of several risky assets. The generalization to other types of multi-asset quanto options is quite straightforward.

Let $S_i, i = 1, \ldots, m$, denote the price of the ith underlying asset of the maximum call option. Here, m is the total number of underlying assets. We consider the following three types of quanto feature associated with the maximum call option:

(viii) Fixed exchange rate foreign equity maximum call

$$\begin{aligned}\text{terminal payoff}: &V_d^{(8)}(S_{1T}, \ldots, S_{mT}, T)\\ &= F_0 \max(\max(S_{1T}, \ldots, S_{mT}) - X_f, 0)\end{aligned}$$

(ix) Joint quanto maximum call

$$\begin{aligned}\text{terminal payoff}: &V_d^{(9)}(S_{1T}, \ldots, S_{mT}, F_T, T)\\ &= \max(F_0, F_T) \max(\max(S_{1T}, \ldots, S_{mT}) - X_f, 0)\end{aligned}$$

(x) Averaged exchange rate foreign equity maximum call

$$\begin{aligned}\text{terminal payoff}: &V_d^{(10)}(S_{1T}, \ldots S_{mT}, G_F^T, T)\\ &= G_F^T \max(\max(S_{1T}, \ldots, S_{mT}) - X_f, 0)\end{aligned}$$

In the risk neutral world, the underlying asset price processes and the exchange rate process are assumed to follow the lognormal distributions:

$$\frac{dS_i}{S_i} = (r_f - q_i)dt + \sigma_i dZ_i\,, \qquad i = 1, \ldots, m\,, \tag{54a}$$

$$\rho_{ij} dt = dZ_i dZ_j\,, \qquad i \neq j\,, \qquad i, j = 1, \ldots, m\,, \tag{54b}$$

$$\frac{dF}{F} = (r_d - r_f)dt + \sigma_F dZ_F\,, \tag{54c}$$

where q_i and σ_i are the dividend yield and volatility of the ith asset, respectively. By extending the partial differential equation formulation in Sec. 2 [see Eq. (7)] to multi-state option models, the governing equation for $V_d = V_d(S_1, \ldots, S_m, t)$ is found to be

$$\begin{aligned}&\frac{\partial V_d}{\partial t} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V_d}{\partial S_i \partial S_j} + \sum_{i=1}^{m} \delta_{S_i}^d S_i \frac{\partial V_d}{\partial S_i} - r_d V_d = 0\,,\\ &\qquad 0 < S_i < \infty, i = 1, \ldots, m\,, \qquad t > 0\,,\end{aligned} \tag{55}$$

where the risk neutralized drift rates are given by

$$\delta_{S_i}^d = r_f - q_i - \rho_{S_i F}\sigma_i\sigma_F\,, \qquad i = 1, \ldots, m\,. \tag{56}$$

The governing equation and the payoff function of the fixed exchange rate foreign equity maximum call resemble those of its non-quanto counterpart, except that the drift rates appearing in the governing equation become $\delta_{S_i}^d$ [see Eq. (56)] instead of

r_d. Hence, the pricing formula for $V^{(8)}(S_1, \ldots, S_m, t)$ can be inferred directly from that of the non-quanto counterpart (see [3]). We then have

$$V_d^{(8)}(S_1, \ldots, S_m, t) = F_0 \sum_{i=1}^{m} [S_i e^{(\delta_{S_i}^d - r_d)\tau} N_m(d_1^i, e^{i1}, \ldots, e^{i,i-1}, e^{i,i+1}, \ldots, e^{im};$$
$$\rho_{i12}, \ldots, \rho_{i1m}, \rho_{i23}, \ldots, \rho_{i2m}, \ldots, \rho_{i,m-1,m})]$$
$$- F_0 X_f e^{-r_d\tau}[1 - N_m(-d_2^1, \ldots, -d_2^m; \rho_{12}, \ldots, \rho_{1m},$$
$$\rho_{23}, \ldots, \rho_{2m}, \ldots, \rho_{m-1,m})], \tag{57}$$

where

$$d_1^i = \frac{\ln \frac{S_i}{X_f} + \left(\delta_{S_i}^d + \frac{\sigma_i^2}{2}\right)\tau}{\sigma_i\sqrt{\tau}}, \qquad d_2^i = d_1^i - \sigma_i\sqrt{\tau}, \qquad i = 1, \ldots, m, \tag{58a}$$

$$e^{ij} = \frac{\ln \frac{S_i}{S_j} + \left(\delta_{S_i}^d - \delta_{S_j}^d + \frac{\sigma_{ij}^2}{2}\right)\tau}{\sigma_{ij}\sqrt{\tau}}, \qquad i, j = 1, \ldots, m, \qquad i \neq j, \tag{58b}$$

$$\sigma_{ij}^2 = \sigma_i^2 + 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2, \qquad i, j = 1, \ldots, m, \qquad i \neq j, \tag{58c}$$

$$\rho_{ijk} = \frac{\rho_{jk}\sigma_j\sigma_k - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \sigma_i^2}{\sigma_{ij}\sigma_{ik}}, \qquad i, j, k = 1, \ldots, m. \tag{58d}$$

Since the terminal payoff of the joint quanto maximum call involves F as well, the governing equation should include F as one of the independent variables. The generalization of the governing equation as posed in Eq. (8) to multi-asset option models is deduced to be:

$$\frac{\partial V_d}{\partial t} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V_d}{\partial S_i \partial S_j} + \sum_{i=1}^{m} \rho_{S_iF}\sigma_i\sigma_F S_i F \frac{\partial^2 V_d}{\partial S_i \partial F} + \frac{\sigma_F^2}{2}F^2\frac{\partial^2 V_d}{\partial F^2}$$
$$+ \sum_{i=1}^{m} \delta_{S_i}^d S_i \frac{\partial V_d}{\partial S_i} + \delta_F^d F \frac{\partial V_d}{\partial F} - r_d V_d = 0,$$
$$0 < S_i < \infty, i = 1, \ldots, m, 0 < F < \infty, t > 0, \tag{59}$$

where $V_d = V_d(S_1, \ldots, S_m, F, t)$.

To derive $V_d^{(9)}$, we use the combination of the approaches used to derive $V_d^{(4)}$ [see Eq. (18)] and $V_d^{(8)}$ [see Eq. (57)]. The price of the joint quanto maximum call is found to be

$$V_d^{(9)}(S_1, \ldots, S_m, F, t)$$
$$= F_0 \sum_{i=1}^{m} [S_i e^{(\delta_{S_i}^d - r_d)\tau} N_{m+1}(d_1^i, -f^i, e^{i1}, \ldots, e^{i,i-1}, e^{i,i+1}, \ldots, e^{im};$$
$$- \rho_{S_1F}, \ldots, -\rho_{S_mF}, \rho_{i12}, \ldots, \rho_{i1m}, \rho_{i23}, \ldots, \rho_{i2m}, \ldots, \rho_{i,m-1,m})]$$

$$
\begin{aligned}
&- F_0 X_f e^{-r_d\tau}[1 - N_{m+1}(-d_1^1, \ldots, -d_1^m, f_2; -\rho_{S_1F}, \ldots, -\rho_{S_mF}\,,\\
&\rho_{12}, \ldots, \rho_{1m}, \rho_{23}, \ldots, \rho_{2m}, \ldots, \rho_{m-1,m})]\\
&+ F\sum_{i=1}^{m}[S_i e^{-q_i\tau} N_{m+1}(\bar{d}_1^i, \bar{f}^i, k^{i1}, \ldots, k^{i,i-1}, k^{i,i+1}, \ldots, k^{im}\,;\\
&\rho_{S_1F}, \ldots, \rho_{SmF}, \rho_{i12}, \ldots, \rho_{i1m}, \rho_{i23}, \ldots, \rho_{i2m}, \ldots, \rho_{i,m-1,m})]\\
&- F X_f e^{-r_f\tau}[1 - N_{m+1}(-\bar{d}_2^1, \ldots, -\bar{d}_2^m, -\bar{f}_2; \rho_{S_1F}, \ldots, \rho_{S_mF}\,,\\
&\rho_{12}, \ldots, \rho_{1m}, \rho_{23}, \ldots, \rho_{2m}, \ldots, \rho_{m-1,m})]\,, \qquad (60)
\end{aligned}
$$

where f_2 and $\bar{f}_2$ are defined in Eq. (19a) and Eq. (19b), respectively, and

$$
f^i = f_2 + \rho_{S_iF}\sigma_i\sqrt{\tau}\,, \qquad \bar{f}^i = \bar{f}_2 + \rho_{S_iF}\sigma_i\sqrt{\tau}\,, \qquad i = 1, \ldots, m\,, \qquad (61a)
$$

$$
k^{ij} = \frac{\ln\frac{S_i}{S_j} + \left(\delta_{S_i}^f - \delta_{S_j}^f + \frac{\sigma_{ij}^2}{2}\right)\tau}{\sigma_{ij}\sqrt{\tau}} \qquad i, j = 1, \ldots, m\,, \qquad i \neq j\,, \qquad (61b)
$$

$$
\bar{d}_1^i = \frac{\ln\frac{S_i}{X_f} + \left(\delta_{S_i}^f + \frac{\sigma_i^2}{2}\right)\tau}{\sigma_i\sqrt{\tau}}\,, \qquad \bar{d}_2^i = \bar{d}_1^i - \sigma_i\sqrt{\tau}\,, \qquad i = 1, \ldots, m\,. \quad (61c)
$$

To derive the price formula for the averaged exchange rate foreign equity maximum call, we first consider the generalization of the governing equation for $\tilde{U}$ [see Eq. (40)] to multi-asset option models. Here, $\bar{U} = \bar{U}(S_1, \ldots, S_m, t)$ is defined by

$$
\bar{U}(S_1, \ldots, S_m, t) = \frac{V_f(S_1, \ldots, S_m, F, G_F, t)}{(G_F/F)^{t/T}}\,, \qquad (62)
$$

and its governing equation takes the form

$$
\begin{aligned}
&\frac{\partial \bar{U}}{\partial t} + \sum_{i=1}^{m}\frac{\sigma_i^2}{2}S_i^2\frac{\partial^2\bar{U}}{\partial S_i^2} + \sum_{i=1}^{m}\left(r_f - q_i - \frac{t}{T}\rho_{S_iF}\sigma_i\sigma_F\right)S_i\frac{\partial\bar{U}}{\partial S_i}\\
&\quad - \left[r_f - \left(r_f - r_d - \frac{\sigma_F^2}{2}\right)\frac{t}{T} - \frac{\sigma_F^2}{2}\frac{t^2}{T^2}\right]\bar{U} = 0\,,\\
&\qquad 0 < S_i < \infty, i = 1, \ldots, m, t > 0\,. \qquad (63)
\end{aligned}
$$

The corresponding terminal condition for $V_d^{(10)}(S_1, \ldots, S_m, G_F, F, t)$ is transformed to become

$$
\bar{U}(S_{1T}, \ldots, S_{mT}, T) = \max(\max(S_{1T}, \ldots, S_{mT}) - X_f, 0)\,. \qquad (64)
$$

Similar to $\alpha(t;T)$ defined in Eq. (42a), we define

$$
\alpha_i(t;T) = \frac{1}{T-t}\int_t^T\left(r_f - q_i - \frac{\xi}{T}\rho_{S_iF}\sigma_i\sigma_F\right)d\xi = r_f - q_i - \frac{T+t}{2T}\rho_{S_iF}\sigma_i\sigma_F\,,
$$

$$
i = 1, \ldots, m\,. \qquad (65)
$$

Following similar lines of approach for deriving $V_d^{(6)}$ and $V_d^{(8)}$ [see Eq. (45) and Eq. (57), respectively], the value of the averaged exchange rate foreign equity maximum call is deduced to be

$$\begin{aligned} &V_d^{(10)}(S_i,\ldots,S_m,F,G_F,t) \\ &= F(G_F/F)^{t/T}e^{-\beta(t;T)\tau}\Bigg\{\sum_{i=1}^{m}\Bigg[e^{\alpha_i(t;T)\tau}S_iN_m(a_1^i,l^{i1},\ldots,l^{i,i-1},l^{i,i+1},\ldots,l^{im}; \\ &\quad \rho_{i12},\ldots,\rho_{i1m},\rho_{i23},\ldots,\rho_{i2m},\ldots,\rho_{i,m-1,m})\Bigg] \\ &\quad - X_f\left[1-N_m\left(-a_2^1,\ldots,-a_2^m;\rho_{12},\ldots,\rho_{1m},\rho_{23},\ldots,\rho_{2m},\ldots,\rho_{m-1,m}\right)\right]\Bigg\}, \end{aligned} \tag{66}$$

where $\beta(t;T)$ is defined in Eq. (42b) and

$$l^{ij} = \frac{\ln\frac{S_i}{S_j} + \left[\alpha_i(t;T) - \alpha_j(t;T) + \frac{\sigma_{ij}^2}{2}\right]\tau}{\sigma_{ij}\sqrt{\tau}}, \qquad i,j = 1,\ldots,m, \qquad i \neq j. \tag{67a}$$

$$a_1^i = \frac{\ln\frac{S_i}{X_f} + \left[\alpha_i(t;T) + \frac{\sigma_i^2}{2}\right]\tau}{\sigma_i\sqrt{\tau}}, \qquad a_2^i = a_1^i - \sigma_i\sqrt{\tau}, \qquad i = 1,\ldots,m. \tag{67b}$$

6. Conclusion

A general framework for pricing the class of quanto options with path dependent features has been presented in this paper. Except for the three price formulas given in Eqs. (13a), (14a) and (15a), all the other price formulas derived in this paper are believed to be new additions to the dictionary of price formulas of derivative products. In the analytic formulation of the quanto option models, one always has to adjust the risk neutralized drift rates in the governing equation, according to whether the valuation of the payoff function is considered in the domestic or foreign currency world. In those quanto option models where the payoff function can be formulated such that the exchange rate F does not appear, the corresponding pricing formula of the quanto option can be inferred from that of its non-quanto counterpart, except that the risk neutralized drift rate and volatility value have to be modified accordingly. This desirable property of inference of pricing formulas persists even the quanto option contains path dependent feature, although the analytic complexity of deriving the corresponding adjusted risk neutralized drift rates and volatilities may increase quite substantially. In the joint quanto option models where the payoff function depends on both the exchange rate and the underlying asset prices, the dimensionality of the quanto option will be increased by one, with F as an additional independent state variable. As compared to their non-quanto counterparts, the analytic tractability of quanto option models is not much affected by the presence of the quanto feature.

Appendix A

The risk neutralized drift rates can be obtained in an alternative approach by using Eq. (3a). We define $F' = \frac{1}{F}$, and it is obvious that $\delta^f_{F'} = r_f - r_d, \rho_{FF'} = -1, \sigma_{F'} = \sigma_F$, $\rho_{S^*F'} = -\rho_{S^*F}$, $\delta^f_S = r_f - q$ and the risk neutralized drift rate of FF' is zero. Using Eq. (3a), we deduce that

$$0 = \delta^f_F + \delta^f_{F'} - \sigma^2_F \qquad \text{so that} \qquad \delta^f_F = r_d - r_f + \sigma^2_F\,.$$

Further, by observing $S = S^*F'$ and applying Eq. (3a), we obtain

$$\delta^f_S = \delta^f_{S^*} + \delta^f_{F'} + \rho_{S^*F'}\sigma_{S^*}\sigma_{F'}\,, \tag{A.1}$$

and upon rearranging, we have

$$\delta^f_{S^*} = (r_f - q) - (r_f - r_d) - \rho_{S^*F'}\sigma_{S^*}\sigma_{F'} = r_d - q + \rho_{S^*F}\sigma_{S^*}\sigma_F\,. \tag{A.2}$$

The use of financial argument to show Eq. (A.2) can be found in [8].

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