Optimal initiation of Guaranteed Lifelong Withdrawal Benefit with dynamic withdrawals

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Abstract

We consider pricing Guaranteed Lifelong Withdrawal Benefit (GLWB) that consists of the early phase of accumulation of benefit base and the later income phase of annuities. The most recent form of the GLWB provides flexibility in allowing additional purchases in the accumulation phase, dynamic withdrawals in the income phase, dynamic initiation into the income phase and complete surrender right throughout the life of the contract. The policyholder chooses the initiation of the income phase optimally based on a combination of factors, like the age-dependent scheduled withdrawal rates, penalty charge rate, bonus and ratchet provisions. Using the bang-bang control analysis, we show that the strategy space of the optimal policies is limited to four choices: maximum allowable purchase, zero withdrawal, withdrawal at the contractual amount or complete surrender. We construct the Fourier transform algorithm for effective pricing of GLWB products with policy fund value under the general two-dimensional Markov process of the fund value and its variance. Our pricing model includes complex path dependent features arising from the ratchet and bonus provisions, dynamic control of withdrawals and additional purchases, together with optimality in the time of initiation of the income phase. We also analyze various pricing properties of the GLWB based on the effective and accurate Fourier transform algorithms. In particular, we examine the impact of various contractual specifications of the GLWB on the optimal decision of initiation of the income phase and optimal withdrawal strategies.

JEL Classification: G22, C50

Keywords: variable annuities, lifelong withdrawal guarantees, optimal initiation, bang-bang analysis, Fourier transform algorithm

1 Introduction

Variable annuities are long-term unit-linked insurance products that offer various types of guarantees. In 2005, variable annuities with Guaranteed Lifelong Withdrawal Benefit (GLWB) were introduced with the unique features that combine the longevity protection of an income benefit and periodic withdrawal benefits. By 2016, the GLWB rider is structured in about half of new variable annuities sales in the US markets. These guarantees are funded by the rider charges (proportional fees), which are paid annually by the policyholder from the policy account. S-ince the embedded guarantees may be too costly to the issuers and they are difficult to be

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hedged, many insurers of variable annuities faced record levels of breakage of their risk hedging strategies. This problem became more acute in the period after the financial tsunami in 2008. The types of risks faced by insurers in the actual implementation of hedging strategies include policyholder behavior, basis risk and execution risk from poor liquidity of hedging instruments.

In a typical contractual design of GLWB, the policyholder first accumulates assets during the accumulation phase and later receives annuities payments during the income phase. The policyholder pays an upfront premium and the amount is then invested in her own choice of portfolio of mutual funds. Policyholder can be allowed to place additional purchases of funds during the accumulation phase beyond the initial upfront premium. In some GLWB contracts, policyholders may be allowed to have a limited number of withdrawals (say, once in every rider year) during the accumulation phase without initiating the income phase. The policy account is the ongoing value of the mutual funds held by the policyholder. Besides changes due to investment returns and withdrawal amounts, the policy account is also subject to periodic deduction due to payment of the rider charges and increment in value due to additional purchases of funds during the accumulation phase. The GLWB's benefit base is not the same as the policy account value, except at initiation of the GLWB contract where the benefit base is set equal to the policy account value. The ratchet (step-up) provision increases the benefit base periodically if the policyholder's policy account has increased to a value higher than the benefit base on an event date. During the accumulation phase, the benefit base is increased under the bonus (roll-up) provision if the policyholder does not withdraw in a given year. The bonus provision on the benefit base become ineffective once the accumulation phase ends. Rider charges and contractual withdrawal amounts are typically calculated based on a fixed proportion of the benefit base. There are GLMB contracts that may charge the rider charges based on the maximum among the benefit base and policy account value. During the income phase, the policyholder is entitled to receive the guaranteed lifelong annual income calculated based on the product of the benefit base and the scheduled withdrawal rate. The scheduled withdrawal rate is dependent on the age of the policyholder in the year of entry into the income phase. The scheduled withdrawal rate is locked in and never changes in the remaining life of the contract. The initiation of the income phase is chosen optimally by the policyholder based on various considerations, like the age-dependent scheduled withdrawal rates, bonus rates and others. It is common to set an upper bound on the period length of the accumulation phase, say, 20 years. Beyond the allowable time period, the income phase will be activated automatically. Throughout the whole life of the policy, the policyholder is allowed to withdraw more than the contractual amount (up to the policy account value) by paying a penalty charge based on the excess withdrawals. Also, the excess withdrawals would reduce the benefit base. Indeed, the complete withdrawal of the policy account means surrender (early termination) of the contract. Empirical studies have revealed various forms of policyholders' behaviors that induce surrender. The surrender of the contract incurs various costs to the issuer, like upfront costs of finding new customers, adverse selection problem of keeping only poor-insurability policies and liquidity issues of coping with cash payments.

In the literature, there have been numerous papers that discuss pricing and hedging of various types of GLWB products. Chi and Lin (2012) study flexible premium variable annuities that allow additional contributions. Bernard *et al.* (2017) analyze the risk of mispricing by insurers that offer guarantees on flexible premiums in variable annuities. Fung *et al.* (2014) examine how financial and demographic parameters would affect the fair guaranteed fee charged for a GLWB. Their studies are limited to the plain GLWB that only allows static withdrawals and the withdrawals start immediately without any deferment period. Their results show that financial risk is dominant for GLWB while the effect of systematic mortality risk can be significant. Steinorth and Mitchell (2015) adopt an expected utility framework to examine how a risk-averse decision maker would choose her optimal withdrawal policies in a GLWB. They show that the ratchet provision may make the policyholder behavior more predictable. Feng and Jing (2017) manage to derive analytic solution for the risk neutral valuation and delta hedging of the plain vanilla GLWB by neglecting structural features like dynamic withdrawals and existence of the accumulation phase. They claim that their analytical solution can provide benchmark against which finite difference and simulation algorithms can be tested. They also demonstrate new techniques for fitting a sum of exponentials to the probability density function of mortality.

On the design of effective numerical algorithms for pricing and hedging GLWB products, Forsyth and Vetzal (2014) construct finite difference schemes for solving a coupled system of one-dimensional partial differential equations to compute the hedging cost for a GLWB when the underlying fund value process follows a Markov regime switching process. They also consider the different costs of hedging under various withdrawal policies, like the optimal withdrawal policies that maximize policyholder's expected value of discounted cash flows and the sub-optimal withdrawal policies that are dependent on moneyness of the surrender option. Though the policyholder may be allowed to withdraw any portion of the account according to the contractual withdrawal guarantees, Azimzadeh and Forsyth (2015) show by using the bang-bang control theory that a holder can maximize the issuer's costs by only choosing either zero withdrawal, withdrawal at the contractual rate or complete surrender for GLWB products. The success of their bang-bang analysis rests on the choices of contractual features such that the solution to the optimal control model can be formulated as maximizing a convex objective function, together with satisfaction of the technical condition that the underlying fund value process preserves convexity and monotonicity. Interestingly, these conditions are satisfied for the GLWB products but not the related Guaranteed Minimum Withdrawal Benefits (GMWB) products. Huang and Kwok (2016) develop an effective regression-based Monte Carlo simulation algorithm for solving the stochastic control models associated with pricing of GLWB products. With the simplification of the strategy space of optimal withdrawal policies to only three choices, the solution of the stochastic control GLWB model by the regression-based Monte Carlo simulation algorithm becomes feasible. They perform sensitivity analysis of the GLWB price function with respect to different parameter values in the stochastic control models of GLWB. Their numerical calculations show that high bonus rate and short cycle of ratchet event add more value to the GLWB price, and reveal the downward trend in the adoption of the zero withdrawal as the optimal strategy when the policyholder ages. On the other hand, the adoption of the contractual guaranteed withdrawal exhibits an upward trend over the calendar time. They also show how a high penalty rate suppresses the propensity of adopting the strategy of complete surrender. Also, there may be significant difference in the GLWB prices under different assumptions of the policyholder's withdrawal behavior.

Our paper addresses the relatively less explored issue of optimal initiation of the income phase in GLWB products. For simpler insurance contracts (not GLWB products) that allow policyholders to switch from a financial investment to a life annuity, Hainaut and Deelstra (2014) consider the optimal timing for annuitization under jump diffusion process of the underlying fund and stochastic mortality. Their analysis is based on maximizing the market value of discounted cash flows. For GLWB products, Huang *et al.* (2014) analyze the optimal time that the policyholder should end the accumulation phase and initiate the income phase. Their GLWB model assumes that payments of fees, bonuses, steps-up and withdrawals occur continuously, while ignoring other contractual features like dynamic withdrawals and surrender right of the policyholders. Also, constant interest rate and volatility, and deterministic mortality are assumed for the sake of simplicity in their model formulation. They show that the guarantee rider in a GLWB is more valuable when the probability of ruin (zero policy fund value) is higher. For GLWBs that are in-the-money (value of the benefit base is higher than that of the account value), their numerical studies reveal that it is optimal for the policyholders to initiate the income phase in their late 50s and certainly in early 60s, except when the age-dependent guaranteed scheduled withdrawal rate is about to increase to a higher age band. Under such scenario, the optimal policy of initiation is to wait until the new guaranteed withdrawal rate is hit and then initiate immediately into the income phase.

The contribution of this paper is four-fold. Firstly, we present the full formulation of the stochastic control models for pricing the GLWB products with both the accumulation and income phases, the ratchet and bonus provisions, additional purchases, dynamic controls of withdrawals, surrender right and initiation into the income phase. Secondly, we perform the bang-bang analysis of the set of control policies and show that the strategy space of the optimal withdrawal policies and additional purchases is limited to a finite discrete values from the set of continuous values. Thirdly, we construct efficient and accurate Fourier transform algorithms for solving the stochastic control models associated with pricing of GLWB products when the underlying fund value process follows the Heston stochastic volatility model. The numerical evaluation of the value function can be performed over successive event dates (typically one year) in single step, without the necessity of performing time-marching evaluations over multiple time steps between successive event dates as in typical finite difference calculations. Lastly, we examine how the optimal initiation regions and optimal choices of withdrawal policies are affected by various structural features in the GLWB, like the age-dependent scheduled withdrawal rate, bonus rate, additional purchase and penalty charge rate.

This paper is organized as follows. In the next section, we present a detailed product description of GLWB, in particular, the different bonus and ratchet features in the accumulation phase and income phase. We then discuss the model formulation of GLWB under the general framework of two-dimensional Markov process for the underlying policy fund value and its variance process. Special attention is paid to consider the jump conditions on the policy fund value and benefit base across the event dates of additional purchases, initiation of the income phase, withdrawals, death payment event, bonus and ratchet provisions. In Section 3, we present the details of the bang-bang analysis of the strategy space of optimal policies. For the optimal polices, we show that there are only four possible choices: maximum allowable purchase, zero withdrawal, withdrawal at the contractual amount and complete surrender. In Section 4, we discuss the construction of the Fourier transform algorithms for pricing GLWB products under the Heston stochastic volatility model. In Section 5, we present the numerical studies that analyze how the GLWB price, optimal withdrawal policies and optimal initiation regions, and their sensitivities with respect to various contractual features and model parameters of the GLWB. The last section contains summary of results and conclusive remarks.

2 Formulation

We start with the product description of the GLWB in a variable annuity contract. At initiation of the contract, the policyholder pays an upfront single premium into her policy account, which is then invested in mutual funds of her own choice. The initial policy account value is set to be the initial premium paid by the policyholder. The rider charges paid by the policyholder throughout the policy life for the provision of the guarantees are calculated based on a fixed proportion of the benefit base, or alternatively, maximum value of the benefit base and account value. These rider charges are taken from the policy account periodically through the cancellation of fund units. The benefit base is set to be the upfront premium initially, which can be adjusted upward via the ratchet provision (step-up) or bonus feature (roll-up).

The ratchet mechanism increases the benefit base to the level of the policy account at the time right after the ratchet event date if the policy account value after the withdrawal and payment of rider charges exceeds the benefit base. If the policyholder chooses not to withdraw any amount on a withdrawal date in the accumulation phase, then the benefit base is increased proportionally by the bonus rate. The policyholder is allowed to have additional purchases of fund units in the accumulation phase. In the income phase, the contractual withdrawal amount is a fixed proportion of the benefit base. The policyholder is also allowed to withdraw more than the contractual withdrawal amount and the excess withdrawal amount is subject to a proportional penalty charge. If the policyholder withdraw the whole policy account, then this signifies complete surrender. Indeed, complete surrender is allowed in both the accumulation phase and income phase. Another event that causes the termination of the contract is the death of the policyholder. The value that remains in the policy account will be passed to a beneficiary. We assume that all events of additional purchase, initiation of the income phase, death payment event, withdrawals, surrender, bonus and ratchet provisions are limited to a predetermined set of event dates.

The initiation of the GLWB contract starts in the accumulation phase. After then, the policyholder has the right to activate the income phase or stay in the accumulation phase at each of the later event dates. Once leaving the accumulation phase, the GLWB contract stays in the income phase for the remaining life of the contract. In the accumulation phase, the policyholder is allowed to make additional purchase or withdraw any nonnegative amount up to full depletion of the fund (surrender), otherwise there is no contractual withdrawal. The benefit base enjoys the bonus feature if no withdrawal or additional purchase is made. There is a maximum length of the period of the accumulation phase, where the GLWB contract is mandated to move into the income phase beyond a specified date. In the income phase, additional purchase is not allowed while the policyholder is guaranteed to receive the contractual withdrawal amount even when the policy account value is fully depleted.

We present the following list of notations used in our later discussion.

Notations

T:	maximum remaining longevity of the policyholder
$\mathcal{T}$:	set of the annual event dates, where $\mathcal{T} = \{1, 2, \ldots, T-1\}$
$\mathcal{T}_e$:	set of the ratchet event dates, $\mathcal{T}_e \subseteq \mathcal{T}$
T_a:	the last event date on which the policyholder can remain in the accumulation phase, beyond which the income phase will be activated automatically
Γ:	characterized by the vector $(\gamma_1, \gamma_2, \ldots, \gamma_{T-1})$, where γ_i is the annual withdrawal amount or additional purchase (considered as negative withdrawal) on the withdrawal date i.
W_t:	time-t policy fund value process
A_t:	time-t benefit base process
B:	cap multiplier of the benefit base that fixes the upper bound of additional purchase
f_i^A:	cash flow received by the policyholder in the accumulation phase in year i.
f_i^I:	cash flow received by the policyholder in the income phase in year i.

η_b: percentage of the benefit base charged on the policy fund value as the annual rider fee

τ_I: $\mathcal{F}_t$-stopping time at which the policyholder activates the income phase

τ_S: $\mathcal{F}_t$-stopping time at which the policyholder chooses to surrender the contract

$G(\tau_I)$: percentage of the benefit base for calculating the annual contractual withdrawal amounts with dependence on the initiation time of the income phase τ_I

k_i: proportional penalty charge applied on the excess of withdrawal amount over the contractual withdrawal at year i

b_i: bonus rate at year i

${}_{x_0}p_i$: probability that an x_0-year old policyholder survives in the next i years (written as p_i for notational convenience for fixed x_0)

q_{x_0+i}: probability that a policyholder at age x_0+i dies within the next year (written as q_i for notational convenience for fixed x_0)

2.1 Bonus and ratchet features

The updating procedures of the policy fund value and benefit base observe different bonus and ratchet provisions in the two different phases, the details of which are presented below.

Jump of benefit base and policy fund value across a withdrawal date in the income phase

In the income phase, when the withdrawal amount γ_i at year i chosen by the policyholder does not exceed the contractual withdrawal amount $G(\tau_I)A_i$, then the benefit base would not be reduced and the withdrawal is not subject to penalty charge. When γ_i exceeds $G(\tau_I)A_i$, the benefit base decreases proportionally according to the amount of excess withdrawal. More specifically, the ratio of decrease is given by $\frac{\gamma_i - G(\tau_I)A_i}{W_i - \eta_b A_i - G(\tau_I)A_i}$ so that the updated benefit base is given by $\frac{W_i - \eta_b A_i - \gamma_i}{W_i - \eta_b A_i - G(\tau_I)A_i} A_i$. On the other hand, the updated benefit base may benefit from the ratchet provision when the policy fund value after withdrawal and payment of rider charge exceeds the updated benefit base. The jump conditions of the benefit base and policy fund value across the withdrawal date at year i are presented below:

$$W_{i^+} = \Big((W_i - \eta_b A_i)^+ - \gamma_i \Big)^+ \quad 0 \le \gamma_i \le \max(W_i - \eta_b A_i, G(\tau_I)A_i); \tag{2.1a}$$

$$A_{i^+} = \begin{cases} \max\left(A_i, ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) & \text{if } 0 \le \gamma_i \le G(\tau_I)A_i \\ \max\left(\frac{W_i - \eta_b A_i - \gamma_i}{W_i - \eta_b A_i - G(\tau_I)A_i} A_i, ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) & \text{if } G(\tau_I)A_i < \gamma_i \le W_i - \eta_b A_i \end{cases}. \tag{2.1b}$$

Note that W_i has zero value as the floor and the rider charge $\eta_b A_i$ is deducted from the policy fund value W_i before the policyholder makes the withdrawal γ_i. Since the excess withdrawal beyond the contractual withdrawal amount $G(\tau_I)A_i$ is charged at the proportional penalty rate k_i, the actual cash amount received by the policyholder in the income phase as resulted from the withdrawal amount γ_i is given by

$$f_i^I(\gamma_i; A_i, G(\tau_I)) = \begin{cases} \gamma_i & \text{if } 0 \le \gamma_i \le G(\tau_I)A_i \\ G(\tau_I)A_i + (1-k_i)[\gamma_i - G(\tau_I)A_i] & \text{if } G(\tau_I)A_i < \gamma_i \le W_i - \eta_b A_i \end{cases}. \tag{2.2}$$

Jump of benefit base and policy fund value across a withdrawal date in the accumulation phase

In the accumulation phase at year i, the benefit base A_i rolls up by predetermined bonus rate b_i if there is no withdrawal. The policyholder is allowed to have an additional purchase of the fund units, which would increase both the benefit base and policy account value. Otherwise, any positive withdrawal taken by the policyholder reduces both the policy fund value and benefit base, and the withdrawal amount is subject to penalty charge. Also, γ_i may assume negative value up to $-BA_i$, which indicates that an additional purchase can be up to the cap multiplier B times the benefit base A_i. The jump conditions on the benefit base and policy fund value across year i are summarized as follows:

$$W_{i^+} = \left((W_i - \eta_b A_i)^+ - \gamma_i\right)^+ \qquad -BA_i \le \gamma_i \le (W_i - \eta_b A_i)^+; \tag{2.3a}$$

$$A_{i^+} = \begin{cases} \max\left(A_i(1+b_i) - \gamma_i, ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) & \text{if } -BA_i \le \gamma_i \le 0 \\ \max\left(\frac{W_i - \eta_b A_i - \gamma_i}{W_i - \eta_b A_i} A_i, ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) & \text{if } 0 < \gamma_i \le (W_i - \eta_b A_i)^+ \end{cases}. \tag{2.3b}$$

The cash flow $f_i^A(\gamma_i; A_i)$ received by the policyholder as resulted from the withdrawal amount γ_i is given by

$$f_i^A(\gamma_i; A_i) = \begin{cases} \gamma_i & \text{if } -BA_i \le \gamma_i \le 0 \\ (1-k_i)\gamma_i & \text{if } 0 < \gamma_i \le (W_i - \eta_b A_i)^+ \end{cases}. \tag{2.4}$$

The vector functions $(W_i^+, A_i^+) = \boldsymbol{h}_i^A(W_i, A_i, \gamma_i)$ in the accumulation phase and $(W_i^+, A_i^+) = \boldsymbol{h}_i^I\left(W_i, A_i, \gamma_i; G(\tau_I)\right)$ in the income phase are introduced to characterize the jump conditions of the policy fund value and benefit base associated with the withdrawal amount γ_i in the accumulation phase ($i < \tau_I$) and income phase ($i \ge \tau_I$), respectively.

(i) In the accumulation phase where $i < \tau_I$, we have

$$\begin{aligned}(W_{i^+}, A_{i^+}) &= \boldsymbol{h}_i^A(W_i, A_i, \gamma_i) \\ &= \begin{cases} \begin{pmatrix} ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \\ \max\left(A_i(1+b_i) - \gamma_i, ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) \end{pmatrix}^T & \text{if } -BA_i \le \gamma_i \le 0 \\ \begin{pmatrix} ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \\ \max\left(\frac{W_i - \eta_b A_i - \gamma_i}{W_i - \eta_b A_i} A_i, ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) \end{pmatrix}^T & \text{if } 0 < \gamma_i \le (W_i - \eta_b A_i)^+. \end{cases}\end{aligned} \tag{2.5a}$$

(ii) In the income phase where $i \ge \tau_I$, we have

$$\begin{aligned}(W_{i^+}, A_{i^+}) &= \boldsymbol{h}_i^I\left(W_i, A_i, \gamma_i; G(\tau_I)\right) \\ &= \begin{cases} \begin{pmatrix} ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \\ \max\left(A_i, ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) \end{pmatrix}^T & \text{if } 0 \le \gamma_i \le G(\tau_I)A_i \\ \begin{pmatrix} ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \\ \max\left(\frac{W_i - \eta_b A_i - \gamma_i}{W_i - \eta_b A_i - G(\tau_I)A_i} A_i, ((W_i - \eta_b A_i)^+ - \gamma_i)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) \end{pmatrix}^T & \text{if } G(\tau_I)A_i < \gamma_i \le (W_i - \eta_b A_i)^+. \end{cases}\end{aligned} \tag{2.5b}$$

2.2 Pricing formulation

The full amount of the policy fund value W_i is given to the beneficiary at year i as the death payment when the policyholder dies within year $i-1$ and year i. The set of the control variables do not include τ_S since τ_S is implicitly dictated by the optimal choices of Γ and τ_I. Let $\mathcal{E}$ be the admissible strategy set for the pair of control variables (Γ, τ_I). Taking mortality risk into account, the value function $V(W, A, v, 0)$ at initiation is determined by assuming the policyholder to choose the control variables (Γ, τ_I) so as to maximize the expectation of the discounted cash flows. Correspondingly, hedging of the GLWB contract by the issuer would be most costly. Under this assumption on the policyholder's policy of activation and withdrawals, we can use the standard hedging argument to derive the value function. As a result, by virtue of the risk neutral valuation principle, the value function is computed under a risk neutral measure $\mathbb{Q}$. Further details on the justification of the use of risk neutral valuation under the assumption that the policyholder chooses the optimal strategy to maximize the monetary value of the contract can be found in Forsyth and Vetzal (2014).

The value function of the GLWB product is formally given by

$$\begin{aligned} V(W, A, v, 0) = \sup_{(\Gamma,\tau_I)\in\mathcal{E}} E_{\mathbb{Q}}\Bigg[& \sum_{i=1}^{\tau_S\wedge(T-1)} e^{-ri} p_{i-1} q_{i-1} W_i + \sum_{i=1}^{(\tau_I-1)\wedge\tau_S} e^{-ri} p_i f_i^A(\gamma_i; A_i) \\ & + \sum_{i=\tau_I}^{\tau_S\wedge(T-1)} e^{-ri} p_i f_i^I\big(\gamma_i; A_i, G(\tau_I)\big) + \mathbf{1}_{\{\tau_S > T-1\}} e^{-rT} p_{T-1} W_T \Bigg]. \end{aligned} \tag{2.6}$$

The first summation term represents the death payment weighted by the probability of mortality from the first withdrawal date to the complete surrender time τ_S or $T-1$, whichever comes earlier. The second summation term gives the sum of discounted withdrawal cash flows from the initiation date of the contract to the last withdrawal date in the accumulation phase or the complete surrender time τ_S, whichever comes earlier. The third summation term gives the sum of discounted withdrawal cash flows from the activation time of the income phase to the complete surrender time τ_S or $T-1$, whichever comes earlier. The last single term is the discounted cash flow received by the policyholder at the maximum remaining life T provided that complete surrender has never been adopted throughout the whole life of the policyholder. In our subsequent exposition, we drop the subscript $\mathbb{Q}$ in the expectation operator $E_{\mathbb{Q}}$ for brevity.

In our pricing model, the joint process of policy fund value and its stochastic variance $\{(W_t, v_t)\}_{0\le t\le T}$ is assumed to be a two-dimensional càdlàg Markov process defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{Q})$ between any two consecutive withdrawal dates, $i < t < i+1$, $i = 0, 1, \ldots, T-1$. We choose the joint process $\{(W_t, v_t)\}_{0\le t\le T}$ to be a càdlàg Markov process so as to satisfy the technical condition on the underlying fund value process in the convexity analysis of the value function of the GLWB product in the bang-bang analysis. The general two-dimensional càdlàg Markov process nests most popular stochastic models, such as the geometric Brownian motion, Heston's model, Merton's jump diffusion model and the double-exponential jump diffusion model.

The mortality risk is assumed to be diversifiable across a large number of policyholders. The optimal complete surrender time is dictated by the optimal choice of the withdrawal amount γ_i, where

$$\tau_S = \inf\{i \in \mathcal{T} | \gamma_i = W_i - \eta_b A_i > 0\}.$$

Implicitly, the complete surrender amount $W_i - \eta_b A_i$ should be larger than the contractual withdrawal amount $G(\tau_I)A_i$ in the income phase for activating the optimal complete surrender.

Dynamic programming procedure

We write GLWB$^{(A)}$ and GLWB$^{(I)}$ to represent the GLWB rider in the accumulation phase and income phase, respectively. The time-t value function of GLWB$^{(I)}$, denoted by $V^{(I)}(W, A, v, t; G_0)$, is seen to have dependence on the guaranteed withdrawal rate $G(\tau_I)$. Since the contractual withdrawal rate depends on the activation time of the income phase τ_I, it is necessary to calculate a set of $V^{(I)}(W, A, v, t; G_0)$ with G_0 being set to be $G(i)$, $i = 1, 2, \cdots, T_a + 1$. For example, suppose the policyholder purchases the GLWB at the age of 50 years old, the GLWB contract sets the contractual withdrawal rate to be 5% if the activation time is between age 65 and age 70; 5.5% if the activation time is between age 70 and age 75; 6% if the activation time is between age 75 and age 80; 6.5% if the activation time is beyond age 80. It is necessary to calculate $V^{(I)}(W, A, v, t; G_0)$ with G_0 being equal to the four separate cases: 5%, 5.5%, 6% and 6.5%, respectively. For notational convenience, we let $\{G_{n_1}, \cdots, G_{n_K}\}$ denote all possible outcomes for G_0.

Using the dynamic programming principle of backward induction, we compute $V^{(I)}(W, A, v, i; G_0)$ as follows:

$$\begin{aligned}
V^{(I)}(W, A, v, T; G_0) &= p_{T-1}W_T, \\
V^{(I)}(W, A, v, i; G_0) &= p_{i-1}q_{i-1}W_i + \sup_{\gamma_i \in [0, \max(W_i - \eta_b A_i, G_0 A_i)]} \{p_i f_i^I(\gamma_i; A_i, G_0) \\
&\quad + e^{-r}E[V^{(I)}(W, A, v, i+1; G_0) | (W_{i^+}, A_{i^+}) = \boldsymbol{h}_i^I(W_i, A_i, \gamma_i; G_0), v_{i^+} = v_i]\},
\end{aligned} \tag{2.7}$$

where $i = 1, 2, \cdots, T-1$ and $G_0 = G_{n_k},\ k = 1, \cdots, K$. Since the GLWB rider is in the accumulation phase at the initiation of the contract, the calculation of the value function of GLWB$^{(I)}$ at time 0 is not required for pricing the GLWB rider. In Section 3, we show the bang-bang analysis for GLWB$^{(I)}$; and in Section 4, we present an efficient Fourier transform algorithm to calculate the values of $V^{(I)}(W, A, v, t; G_{n_k})$ for $k = 1, \cdots, K$.

Similarly, we let $V^{(A)}(W, A, v, t)$ be the time-t value function of GLWB$^{(A)}$. Since T_a is the last withdrawal date on which the GLWB contract may stay in the accumulation phase, we start from the withdrawal date T_a and calculate $V^{(A)}(W, A, v, T_a)$. Let $V_C^{(A)}(T_a)$ be the continuation value at year T_a conditional on the policyholder choosing to remain in the accumulation phase on T_a. Also, we let $V_C^{(I)}(T_a)$ be the continuation value at year T_a conditional on the policyholder's choice of activating the income phase. We then have

$$V^{(A)}(W, A, v, T_a) = p_{T_a-1}q_{T_a-1}W_{T_a} + \max\{V_C^{(A)}(T_A), V_C^{(I)}(T_a)\}, \tag{2.8}$$

where

$$\begin{aligned}
V_C^{(A)}(T_a) &= \sup_{\gamma_{T_a} \in [-BA_{T_a}, (W_{T_a} - \eta_b A_{T_a})^+]} \{p_{T_a} f_{T_a}^A(\gamma_{T_a}; A_{T_a}) \\
&\quad + e^{-r}E[V^{(I)}(W, A, v, T_a + 1; G(T_a + 1)) | (W_{T_a^+}, A_{T_a^+}) = \boldsymbol{h}_{T_a}^A(W_{T_a}, A_{T_a}, \gamma_{T_a}), v_{T_a^+} = v]\}, \\
V_C^{(I)}(T_a) &= \sup_{\gamma_{T_a} \in [0, \max((W_{T_a} - \eta_b A_{T_a})^+, G(T_a)A_{T_a})]} \{p_{T_a} f_{T_a}^I(\gamma_{T_a}; A_{T_a}, G(T_a)) \\
&\quad + e^{-r}E[V^{(I)}(W, A, v, T_a + 1; G(T_a)) | (W_{T_a^+}, A_{T_a^+}) = \boldsymbol{h}_{T_a}^I(W_{T_a}, A_{T_a}, \gamma_{T_a}; G(T_a)), v_{T_a^+} = v]\}.
\end{aligned}$$

For $V_C^{(A)}(T_a)$, since the income phase is mandated to be activated on $T_a + 1$, we then set the guaranteed withdrawal rate to be $G(T_a+1)$ in the conditional expectation of $V^{(I)}\big(W,A,v,T_a+1;G(T_a+1)\big)$. On the other hand, for $V_C^{(I)}(T_a)$, since the income phase is activated on T_a, the guaranteed withdrawal rate is set to be $G(T_a)$. As a remark, if $V_C^{(A)}(T_a) \geq V_C^{(I)}(T_a)$, then it is optimal for the policyholder to choose to remain in the accumulation phase on the event date T_a. Otherwise, it is optimal to initiate the income phase at year T_a and the control variable τ_I is set to be T_a accordingly. Besides the optimal timing of activation of the income phase, the policyholder can also determine the optimal withdrawal strategy γ_{T_a} for the value function $V^{(A)}(W,A,v,T_a)$ for fixed W and A at year T_a using the Fourier transform method, details of which are discussed in Section 5.

For an earlier event date, $1 \leq i \leq T_a - 1$, we have

$$V^{(A)}(W,A,v,i) = \ p_{i-1}q_{i-1}W_i + \max\{V_C^{(A)}(i), V_C^{(I)}(i)\}, \tag{2.9}$$

where

$$\begin{aligned}
V_C^{(A)}(i) = & \sup_{\gamma_i \in [-BA_i,(W_i-\eta_b A_i)^+]} \{p_i f_i^A(\gamma_i;A_i) \\
& + e^{-r} E[V^{(A)}(W,A,v,i+1) | (W_{i^+},A_{i^+}) = \boldsymbol{h}_i^A(W_i,A_i,\gamma_i), v_{i^+} = v]\}, \\
V_C^{(I)}(i) = & \sup_{\gamma_i \in [0,\max((W_i-\eta_b A_i)^+,G(i)A_i)]} \{p_i f_i^I(\gamma_i;A_i,G(i)) \\
& + e^{-r} E[V^{(I)}\big(W,A,v,i+1;G(i)\big) | (W_{i^+},A_{i^+}) = \boldsymbol{h}_i^I(W_i,A_i,\gamma_i;G(i)), v_{i^+} = v_i]\}.
\end{aligned}$$

Here, $V_C^{(A)}(i)$ corresponds to the case that the policyholder chooses not to activate the income phase at year i. Since the policyholder is entitled to choose to stay in the accumulation phase or activate the income phase in the next year $i+1$, we evaluate the conditional expectation of $V^{(A)}(W,A,v,i+1)$. On the other hand, note that $V_C^{(I)}(i)$ corresponds to the case that the policyholder chooses to activate the income phase in year i, so we consider computing the conditional expectation of $V^{(I)}\big(W,A,v,i+1;G(i)\big)$. Lastly, since the GLWB contract on the initiation date, where $i=0$, is in the accumulation phase and there is no withdrawal event at the beginning of the contract, we have

$$V(W,A,v,0) = e^{-r}E[V^{(A)}(W,A,v,1)]. \tag{2.10}$$

3 Bang-bang analysis

The design of the numerical algorithm would be much simplified if the choices of the optimal withdrawal amount γ_i are limited to a finite number of discrete values. Azimzadeh and Forsyth (2015) provide a rigorous proof on the existence of optimal bang-bang controls for various variable annuities with guaranteed withdrawals, where the optimal withdrawal policies are either zero withdrawal, withdrawal at the contractual amount or complete surrender. Here, we would like to perform a rigorous bang-bang controls of GLWB under more general form of the underlying asset price process and additional structural features (like optional initiation and additional purchases). The technical analysis relies on the convexity and monotonicity properties of the value function. As part of the technical procedure, it is necessary to require the two-dimensional Markov process $\{(W_t,v_t)\}_t$ to observe the following mathematical properties:

Property 1 *(Convexity preservation) For any convex terminal payoff function $\Phi(W_T)$, the* *corresponding European price function as defined by*

$$\phi(w,v,t) = e^{-r(T-t)} E\left[\Phi(W_T)\right| W_t = w, v_t = v], \ t \leq T,$$

is also convex with respect to w.

Property 2 *(Scaling) For any positive K, the two stochastic processes $\{(W_t, v_t)\}_t$ and $\{(\frac{W_t}{K}, v_t)\}_t$ have the same distribution law given that their initial values agree almost surely.*

Ekström and Tysk (2007) analyze the property of convexity preservation for option prices in models with jump and show that many popular jump diffusion models, such as Merton's jump diffusion model and Kou's model, satisfy Property 1. Hobson (2010) gives a sufficient condition for convexity preservation in stochastic volatility models and the corresponding condition required for the Heston model also satisfies Property 1. Moreover, it is obvious that all the above mentioned models satisfy Property 2. The class of two-dimensional Markov processes that observes both Properties 1 and 2 indeed includes most popular models of asset price processes.

By virtue of Property 2, together with the invariant forms of the bonus rate b, guaranteed withdrawal G, cap multiplier for additional purchase B and penalty charge k, the value functions $V^{(I)}$ and $V^{(A)}$ satisfy the following scaling properties for any positive scalar K:

$$V^{(I)}(KW, KA, v, t; G_0) = KV^{(I)}(W, A, v, t; G_0) \tag{3.1a}$$

$$V^{(A)}(KW, KA, v, t) = KV^{(A)}(W, A, v, t). \tag{3.1b}$$

By virtue of the above scaling properties, we can achieve reduction in dimensionality of the pricing model by one when we calculate the conditional expectations in the dynamic programming procedure. The scaling properties are also crucial in establishing the bang-bang control analysis.

Our main results on the bang-bang control strategies for GLWB$^{(I)}$ and GLWB$^{(A)}$ are summarized in Theorem 3.

Theorem 3 *Assume that $\{(W_t, v_t)\}_t$ satisfies both Properties 1 and 2, GLWB$^{(I)}$ and GLWB$^{(A)}$ observe the following strategy space of optimal withdrawal, respectively.*

1 *On any withdrawal date i, the optimal withdrawal strategy γ_i for GLWB$^{(I)}$ in the income phase with a positive guaranteed rate G_0 is limited to (i) $\gamma_i = 0$; (ii) $\gamma_i = G_0 A_i$; or (iii) $\gamma_i = W_i - \eta_b A_i$.*

2 *On any withdrawal date i, the optimal strategy on this withdrawal date for GLWB$^{(A)}$ in the accumulation phase is either*

   (2a) *to initiate the income phase on this withdrawal date if $V_C^{(I)}(i) > V_C^{(A)}(i)$.*

   (2b) *or to remain in the accumulation phase on this withdrawal date if $V_C^{(I)}(i) \leq V_C^{(A)}(i)$ and the optimal withdrawal strategy γ_i is limited to (i) $\gamma_i = -BA_i$; (ii) $\gamma_i = 0$; or (iii) $\gamma_i = W_i - \eta_b A_i$.*

In summary, when the policy is already in the income phase, the withdrawal policies are limited to zero withdrawal, withdrawal at the contractual rate or complete surrender. When the policy is in the accumulation phase, the policyholder may choose to enter into the income phase or stay in the accumulation phase. The subsequent optimal policies while staying in the accumulation phase are limited to maximum allowable purchase, zero withdrawal or complete surrender. The proof of Theorem 3 is presented Appendix A. Interestingly, one may deduce the following set of dominated strategies as stated in Corollary 4.

Corollary 4 *When positive bonus rate and penalty charge rate are applied on the withdrawal date i and the income phase has not been activated before date i, the strategy of staying in the accumulation phase on this withdrawal date and choosing zero withdrawal $\gamma_i = 0$ dominates the strategy of activating the income phase on this withdrawal date and choosing zero withdrawal $\gamma_i = 0$. Furthermore, when $W_i - \eta A_i < G(\tau_I)A_i$ in the income phase, the guaranteed withdrawal strategy $\gamma_i = G(\tau_I)A_i$ dominates the complete surrender $\gamma_i = W_i - \eta_b A_i$.*

Corollary 4 identifies the set of dominated strategies and it is a direct consequence of Theorem 3. These results are useful in the design of efficient Fourier transform algorithm since we eliminate the dominated strategies when one searches for the optimal strategies in the numerical calculation procedures. Consequently, the strategy space of the optimal withdrawal policies is limited to the following choices: (i) maximum allowable purchase, zero withdrawal or complete surrender in the accumulation phase; (ii) zero withdrawal, withdrawal at the contractual amount or complete surrender in the income phase.

4 Fourier transform algorithms

In this section, we construct the efficient Fourier transform algorithms for pricing the GLWB product with policy fund value under the Heston model and complex path dependent features arising from the ratchet and bonus events, dynamic control of withdrawals and additional purchases, together with optimality in the time of initiation of the income phase. By the recursive backward induction procedure, the computation starts with the discounted expectation of the value function in the income phase. We then proceed backward in time to compute the value function in the accumulation phase. Thanks to the bang-bang analysis, it suffices to consider the choice set of γ_i at year i to be $\{-BA_i, 0, (W_i - \eta_b A_i)^+\}$ and $\{0, (W_i - \eta_b A_i)^+, G(i)A_i\}$ in the accumulation phase and income phase, respectively. Assuming annualized event dates and adopting the dynamic programming procedure, the value function $V^{(A)}(W, A, v, i)$ on the withdrawal date i in the accumulation phase can be expressed as

$$V^{(A)}(W, A, v, i) = \; p_{i-1}q_{i-1}W_i + \max\{V_C^{(A)}(i), V_C^{(I)}(i)\}, \tag{4.1}$$

where

$$\begin{aligned}
V_C^{(A)}(i) = & \sup_{\gamma_i \in \{-BA_i, 0, (W_i - \eta_b A_i)^+\}} \{p_i f_i^A(\gamma_i; A_i) \\
& + e^{-r} E[V^{(A)}(W, A, v, i+1) | (W_{i^+}, A_{i^+}) = \boldsymbol{h}_i^A(W_i, A_i, \gamma_i), v_{i^+} = v]\}, \\
V_C^{(I)}(i) = & \sup_{\gamma_i \in \{0, (W_i - \eta_b A_i)^+, G(i)A_i\}} \{p_i f_i^I(\gamma_i; A_i, G(i)) \\
& + e^{-r} E[V^{(I)}\big(W, A, v, i+1; G(i)\big) | (W_{i^+}, A_{i^+}) = \boldsymbol{h}_i^I\big(W_i, A_i, \gamma_i; G(i)\big), v_{i^+} = v_i]\}.
\end{aligned}$$

We observe continuity of the value function across the two phases at $T_a + 1$, where

$$V^{(A)}(W, A, v, T_a + 1) = V^{(I)}\big(W, A, v, T_a + 1; G(T_a + 1)\big).$$

Since the benefit base A_t remains unchanged between consecutive event dates, we achieve dimensionality reduction by defining the normalized policy fund value $\widetilde{W}_t$ and normalized value functions $\widetilde{V}^{(I)}\big(\widetilde{W},v,t;G(i)\big)$ and $\widetilde{V}^{(A)}(\widetilde{W},v,t)$ as follows:

$$\begin{aligned}
\widetilde{W}_t &= W_t/A_t, \\
\widetilde{V}^{(I)}\big(\widetilde{W},v,t;G(i)\big) &= V^{(I)}\big(W,A,v,t;G(i)\big)/A_t, \\
\widetilde{V}^{(A)}(\widetilde{W},v,t) &= V^{(A)}(W,A,v,t)/A_t.
\end{aligned}$$

By considering the possible choices of withdrawal strategies and the jump conditions on $\widetilde{W}_i$ and A_i across the event date i under the respective withdrawal strategy, the normalized value function in the accumulation phase $\widetilde{V}^{(A)}(\widetilde{W},v,i)$ can be written as

$$\begin{aligned}
&\widetilde{V}^{(A)}(\widetilde{W},v,i) \\
= \;& p_{i-1}q_{i-1}\widetilde{W}_i + \max\Bigg\{ p_i\left[G(i) + (1-\kappa_i)(\widetilde{W}_i - \eta_b - G(i))\right], \\
& p_i G(i) + e^{-r}\max\Big(1, [\widetilde{W}_i - \eta_b - G(i)]\mathbf{1}_{\{i\in\mathcal{T}_e\}}\Big) \\
& \qquad E\left[\widetilde{V}^{(I)}(\widetilde{W},v,i+1;G(i))\Big|\widetilde{W}_{i+} = \frac{(\widetilde{W}_i - \eta_b - G(i))^+}{\max\Big(1, [\widetilde{W}_i - \eta_b - G(i)]\mathbf{1}_{\{i\in\mathcal{T}_e\}}\Big)}, v_i\right], \\
& - p_i B + e^{-r}\max\Big((1+b_i)+B, [(\widetilde{W}_i-\eta_b)^+ + B]\mathbf{1}_{\{i\in\mathcal{T}_e\}}\Big) \\
& \qquad E\left[\widetilde{V}^{(A)}(\widetilde{W},v,i+1)\Big|\widetilde{W}_{i+} = \frac{(\widetilde{W}_i-\eta_b)^+ + B}{\max\Big((1+b_i)+B, [(\widetilde{W}_i-\eta_b)^+ + B]\mathbf{1}_{\{i\in\mathcal{T}_e\}}\Big)}, v_i\right], \\
& e^{-r}\max\Big(1+b_i, (\widetilde{W}_i-\eta_b)\mathbf{1}_{\{i\in\mathcal{T}_e\}}\Big)\, E\left[\widetilde{V}^{(A)}(\widetilde{W},v,i+1)\Big|\widetilde{W}_{i+} = \frac{(\widetilde{W}_i-\eta_b)^+}{\max\Big(1+b_i, (\widetilde{W}_i-\eta_b)\mathbf{1}_{\{i\in\mathcal{T}_e\}}\Big)}, v_i\right]\Bigg\}. \qquad (4.2)
\end{aligned}$$

For notational convenience, we define the following functions

$$\begin{aligned}
\phi_i^{(1)}(x) &= \frac{(x-\eta_b-G(i))^+}{\max\big(1, [x-\eta_b-G(i)]\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big)}, \\
\phi_i^{(2)}(x) &= \frac{(x-\eta_b)^+ + B}{\max\big((1+b_i)+B, [(x-\eta_b)^+ + B]\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big)}, \\
\phi_i^{(3)}(x) &= \frac{(x-\eta_b)^+}{\max\big(1+b_i, (x-\eta_b)\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big)}, \\
\psi_i^{(1)}(x) &= \max\big(1, [x-\eta_b-G(i)]\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big), \\
\psi_i^{(2)}(x) &= \max\big((1+b_i)+B, [(x-\eta_b)^+ + B]\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big), \\
\psi_i^{(3)}(x) &= \max\big(1+b_i, (x-\eta_b)\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big).
\end{aligned}$$

Here, $\{\phi_i^{(j)}(\widetilde{W}_i)\}_{j=1,2,3}$ relate $\widetilde{W}_{i+}$ and $\widetilde{W}_i$ across the withdrawal date i under the three respective withdrawal strategies while $\{\psi_i^{(j)}(\widetilde{W}_i)\}_{j=1,2,3}$ give the multiplier for the benefit base arising from the corresponding jump condition. In terms of these functions, the normalized value function in the accumulation phase $\widetilde{V}^{(A)}(\widetilde{W}, v, i)$ can be rewritten into a more concise representation:

$$\begin{aligned}
&\widetilde{V}^{(A)}(\widetilde{W}, v, i) \\
&= p_{i-1}q_{i-1}\widetilde{W}_i + \max\Big\{ p_i \left[G(i) + (1-\kappa_i)(\widetilde{W}_i - \eta_b - G(i))\right], \\
&\quad p_i G(i) + e^{-r}\psi_i^{(1)}(\widetilde{W}_i) E\left[\widetilde{V}^{(I)}\big(\widetilde{W}, v, i+1; G(i)\big)\Big| \widetilde{W}_{i^+} = \phi_i^{(1)}(\widetilde{W}_i), v_i\right], \\
&\quad - p_i B + e^{-r}\psi_i^{(2)}(\widetilde{W}_i) E\left[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\Big| \widetilde{W}_{i^+} = \phi_i^{(2)}(\widetilde{W}_i), v_i\right], \\
&\quad e^{-r}\psi_i^{(3)}(\widetilde{W}_i) E\left[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\Big| \widetilde{W}_{i} = \phi_i^{(3)}(\widetilde{W}_i), v_i\right] \Big\}, \quad i = 1, 2, \ldots, T_a. \qquad (4.3)
\end{aligned}$$

Recall that $\{G_{n_1}, \cdots, G_{n_K}\}$ denote all possible outcomes for the contractual withdrawal rate G_0. We introduce another set of functions that serve to capture the corresponding jump conditions on $\widetilde{W}_i$ and A_i across the withdrawal date i in the income phase as follows:

$$\begin{aligned}
\phi_i^{(4)}(x) &= \frac{(x-\eta_b)^+}{\max\big(1, (x-\eta_b)\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big)}, \\
\phi_i^{(5)}(x; G_{n_k}) &= \frac{(x-\eta_b - G_{n_k})^+}{\max\big(1, (x-\eta_b - G_{n_k})\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big)}, \\
\psi_i^{(4)}(x) &= \max\big(1, (x-\eta_b)\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big), \\
\psi_i^{(5)}(x; G_{n_k}) &= \max\big(1, (x-\eta_b - G_{n_k})\mathbf{1}_{\{i\in\mathcal{T}_e\}}\big).
\end{aligned}$$

In terms of these functions, the normalized value function in the income phase $\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k})$ can be expressed as follows

$$\begin{aligned}
&\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k}) \\
&= p_{i-1}q_{i-1}\widetilde{W}_i + \max\Big\{ p_i \left[G_{n_k} + (1-\kappa_i)(\widetilde{W}_i - \eta_b - G_{n_k})\right], \\
&\quad e^{-r}\psi_i^{(4)}(\widetilde{W}_i) E\left[\widetilde{V}^{(I)}(\widetilde{W}, v, i+1; G_{n_k})\Big| \widetilde{W}_{i^+} = \phi_i^{(4)}(\widetilde{W}_i), v_i\right], \\
&\quad p_i G_{n_k} + e^{-r}\psi_i^{(5)}(\widetilde{W}_i; G_{n_k}) E\left[\widetilde{V}^{(I)}(\widetilde{W}, v, i+1; G_{n_k})\Big| \widetilde{W}_{i^+} = \phi_i^{(5)}(\widetilde{W}_i), v_i\right] \Big\}, \quad i = 1, 2, \ldots, T-1. \qquad (4.4)
\end{aligned}$$

As a result, we can also evaluate $\widetilde{V}^{(A)}(\widetilde{W}, v, i)$ in an alternative way as follows:

$$\begin{aligned}
&\widetilde{V}^{(A)}(\widetilde{W}, v, i) \\
&= p_{i-1}q_{i-1}\widetilde{W}_i + \max\Big\{ - p_{i-1}q_{i-1}\widetilde{W}_i + \widetilde{V}^{(I)}\big(\widetilde{W}, v, i; G(i)\big) \\
&\quad - p_i B + e^{-r}\psi_i^{(2)}(\widetilde{W}_i) E\left[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\Big| \widetilde{W}_{i^+} = \phi_i^{(2)}(\widetilde{W}_i), v_i\right], \\
&\quad e^{-r}\psi_i^{(3)}(\widetilde{W}_i) E\left[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\Big| \widetilde{W}_{i} = \phi_i^{(3)}(\widetilde{W}_i), v_i\right] \Big\}, \quad i = 1, 2, \ldots, T_a. \qquad (4.5)
\end{aligned}$$

We formulate the backward induction calculations combined with the dynamic programming procedure for the normalized value functions in the income and accumulation phases as follows:

1. The backward induction procedure is initiated by observing the following terminal condition corresponding to the respective scheduled withdrawal rate G_{n_k}:

$$\widetilde{V}^{(I)}(\widetilde{W}, v, T; G_{n_k}) = p_{T-1}\widetilde{W}_T,$$

   where $k = 1, \cdots, K$.

2. Time-stepping calculations between the consecutive event dates
   First, we evaluate $\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k})$ recursively by eq. (4.4) for $T_a + 1 \leq i \leq T - 1$ and $1 \leq k \leq K$. Next starting i from T_a to 1, we calculate $\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k})$ and $\widetilde{V}^{(A)}(\widetilde{W}, v, i)$ according to eqs. (4.4) and (4.3), respectively.

3. The fair value of the normalized value function at initiation is obtained by setting

$$\widetilde{V}^{(A)}(\widetilde{W}_0, v_0, 0) = e^{-r}E\left[\widetilde{V}^{(A)}(\widetilde{W}, v, 1)\right].$$

Remark
The above backward induction works for the general class of two-dimensional càdlàg Markov processes. Though our proposed Fourier transform algorithm is constructed under the Heston model, the formulation can be applicable for the 3/2 stochastic volatility model, Merton's jump diffusion model and the double-exponential jump diffusion model with some slight modifications.

In the numerical valuation of the conditional expectation of the normalized value functions, two technical challenges remain. Firstly, the Fourier transforms of the normalized value functions may be not well defined since the normalized value functions do not tend to zero at the two ends of the domain of definition. Secondly, how does one perform effective conditional expectation calculation of the normalized value functions in the variance domain? Next we show how to circumvent these difficulties.

Normalized value functions at low policy fund value
Recall that in the above backward induction in calculating $E\big[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\big|\widetilde{W}_{i^+}, v_i\big]$, we have to consider the two separate cases: (1) $\widetilde{W}_{i^+} > 0$ (2) $\widetilde{W}_{i^+} = 0$. For the first case, we present the Fourier transform algorithm to calculate the two-dimensional expectation in our later discussion. For the second case, since

$$E\big[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\big|\widetilde{W}_{i^+} = 0, v_i\big] = E\big[\widetilde{V}^{(A)}(0, v, i+1)\big|v_i\big],$$

so one has to calculate the solution for the normalized value functions at zero policy fund value $\widetilde{V}^{(A)}(0, v, i+1)$.

In fact, the normalized value functions do not decay to zero when W_i approaches to 0 and ∞. As a result, any choice of the damping factor cannot guarantee the existence of the Fourier transform of the damped normalized value functions. Recall that η_b is the percentage of the benefit base charged on the policy fund value as the annual rider fee. Fortunately, one can always find the solutions for the normalized value functions when $\widetilde{W}_i \leq \eta_b$, which plays an important role in constructing the new functions based on the normalized value functions such that the Fourier transforms of these damped new functions are well defined. Therefore, let us first show how to derive the solutions for the normalized value functions when $\widetilde{W}_i \leq \eta_b$.

For any $i \in \{1, 2, \cdots, T\}$, we restrict our attention to the special case that $\widetilde{W}_i \leq \eta_b$. Actually, $\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k})$ can be shown to have the closed form representation for $1 \leq k \leq K$ in this special case. In the income phase, since $\widetilde{W}_{i^+} = 0$, we have $\widetilde{W}_t = 0$ for any $t > i$ due to no additional purchase. As a result, complete surrender is excluded on all subsequent withdrawal dates. In fact, it is always optimal to withdraw starting from the withdrawal date i. This leads to

$$\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k}) = p_{i-1}q_{i-1}\widetilde{W}_i + \sum_{j=i}^{T-1} p_j G_{n_k} e^{-r(j-i)}, \quad \text{if } \widetilde{W}_i \leq \eta_b. \tag{4.6}$$

Here, the second term is the sum of the discounted expected withdrawal amounts. Alternatively, the above closed form solution can also be obtained directly from eq. (4.4). We take $q_{T-1} = 1$ so that

$$\widetilde{V}^{(I)}\big(\widetilde{W}, v, T; G_{n_k}\big) = p_{T-1}\widetilde{W}_T.$$

For notational convenience, we write

$$g^{(I)}(i; G_{n_k}) = \sum_{j=i}^{T-1} p_j G_{n_k} e^{-r(j-i)},$$

so that

$$\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k}) = p_{i-1}q_{i-1}\widetilde{W}_i + g^{(I)}(i; G_{n_k}), \quad \text{if } \widetilde{W}_i \leq \eta_b. \tag{4.7}$$

In addition,

$$E\big[\widetilde{V}^{(I)}\big(\widetilde{W}, v, i+1; G_{n_k}\big)\big|\widetilde{W}_{i^+} = 0, v_i\big] = g^{(I)}(i+1; G_{n_k}). \tag{4.8}$$

According to eq. (4.7), the closed form solution for $\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k})$ shows no dependence on the current variance in the above special case.

Unfortunately, $\widetilde{V}^{(A)}(\widetilde{W}, v, i)$ does not retain the above nice analytic tractability except at $T_a + 1$. However, we can define

$$\widetilde{V}^{(A)}(\widetilde{W}, v, i) = p_{i-1}q_{i-1}\widetilde{W}_i + g^{(A)}(v, i), \quad \text{if } \widetilde{W}_i \leq \eta_b. \tag{4.9}$$

Provided that $\widetilde{W}_i \leq \eta_b$ for $i = 1, \cdots, T_a$, based on eqs. (4.5) (4.7) and (4.9), we can derive

$$\begin{aligned} & g^{(A)}(v, i) \\ = \; & \max\Big\{ g^{(I)}\big(i; G(i)\big), \\ & -p_i B + e^{-r}[(1+b_i) + B] E\left[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\middle|\widetilde{W}_{i^+} = \frac{B}{(1+b_i)+B}, v_i\right], \\ & e^{-r}(1+b_i) E\left[g^{(A)}(v_{i+1}, i+1)\middle|v_i\right] \Big\}, \quad \text{if } \widetilde{W}_i \leq \eta_b. \end{aligned} \tag{4.10}$$

Here, $g^{(A)}(v, i)$ does not admit an analytical representation except when $B = 0$. Fortunately, starting with the terminal condition $g^{(A)}(v, T_a + 1) = g^{(I)}\big(T_a + 1; G(T_a + 1)\big)$, $g^{(A)}(v, i)$ can be calculated using the Fourier transform method to be shown later.

It is desirable to construct a new set of modified normalized value functions with the property that they are equal to zero once $\widetilde{W}_i \leq \eta_b$. This would guarantee that the generalized Fourier transforms of these two modified functions with respect to $\log \widetilde{W}_t$ are well defined by adopting some proper damping factors. We define the two modified normalized value functions by

$$\begin{aligned} U^{(A)}(\widetilde{W}, v, i) &= \widetilde{V}^{(A)}(\widetilde{W}, v, i) - \big(p_{i-1}q_{i-1}\widetilde{W}_i + g^{(A)}(v, i)\big), \\ U^{(I)}(\widetilde{W}, v, i; G_{n_k}) &= \widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k}) - \big(p_{i-1}q_{i-1}\widetilde{W}_i + g^{(I)}(i; G_{n_k})\big). \end{aligned} \tag{4.11}$$

4.1 Expectation calculations of the normalized value functions

Now we are ready to calculate the conditional expectations of the normalized value functions, which is a key step in the backward induction. In order to compute the conditional expectation $E\big[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\big|\widetilde{W}_{i^+}, v_i\big]$, provided that $\widetilde{W}_{i^+} > 0$, one has to evaluate the two-dimensional expectation integral $E\big[U^{(A)}(\widetilde{W}, v, i+1)\big|\widetilde{W}_{i^+}, v_i\big]$. The latter can be calculated relatively easily since the generalized Fourier transforms of the modified normalized value functions are guaranteed to exist. We would like to apply the Fourier transform method in the log normalized policy fund value dimension and a quadrature rule in the variance dimension since the transition density of the variance v_t has an analytic form. However, the Feller condition is difficult to satisfy in practice, and the density of variance grows extremely fast in the left tail when the Feller condition fails. To resolve this difficulty, Fang and Oosterlee (2011) propose to transform the density function from the variance domain to the log-variance domain. Interested readers may refer to Fang and Oosterlee (2011), Zeng and Kwok (2014) for more details.

More specifically, based on the dynamics for the policy fund value process W_t, an application of eq. (4.11) gives

$$E\big[\widetilde{V}^{(A)}(\widetilde{W}, v, i+1)\big|\widetilde{W}_{i^+}, v_i\big] = E\big[U^{(A)}(\widetilde{W}, v, i+1)\big|\widetilde{W}_{i^+}, v_i\big] + p_i q_i e^r \widetilde{W}_{i^+} + E[g^{(A)}(v_{i+1}, i+1)|v_i]. \tag{4.12}$$

We define the log-variance $\gamma_t = \ln v_t$. By the tower property and conditional on the log-variance process at time $i+1$, we obtain

$$E\big[U^{(A)}(\widetilde{W}, v, i+1)\big|\widetilde{W}_{i^+}, v_i\big] = E\left[E\big[U^{(A)}(\widetilde{W}, e^{\gamma}, i+1)\big|\widetilde{W}_{i^+}, \gamma_{i+1}, \gamma_i\big]\middle|\widetilde{W}_{i^+}, \gamma_i\right].$$

Now we apply an appropriate J-point quadrature integration rule (say, the Gauss-Legendre quadrature rule) to evaluate the outer expectation integral, which involves integration over the density function $p_\gamma(\gamma_{i+1}|\gamma_i)$. By performing discretization along the dimension of γ_{i+1} at the discrete nodes $\zeta_j,\ j = 1, 2, \cdots, J$, we have

$$E\big[U^{(A)}(\widetilde{W}, v, i)\big|\widetilde{W}_{i^+}, v_i\big] \approx \sum_{j=1}^{J} w_j p_\gamma(\zeta_j|\gamma_i) E\left[U^{(A)}(\widetilde{W}, e^{\zeta_j}, i+1)|\widetilde{W}_{i^+}, \gamma_{i+1} = \zeta_j, \gamma_i\right], \tag{4.13}$$

where w_j is the weight at the quadrature node $\zeta_j,\ j = 1, 2, \cdots, J$.

To perform the inner expectation calculation, we adopt the Fourier transform method that is widely used in option pricing (Lord *et al.*, 2008; Kwok *et al.*, 2012). Let $X_t = \log \widetilde{W}_t$, the generalized Fourier transform of $U^{(A)}(\widetilde{W}, e^{\zeta_j}, i+1)$ with respect to X_{i+1} is defined by

$$\begin{aligned} \widehat{U}^{(A)}(\beta, e^{\zeta_j}, i+1) &= \int_{-\infty}^{\infty} e^{(\alpha+i\beta)X_{i+1}} U^{(A)}(\widetilde{W}, e^{\zeta_j}, i+1)\, \mathrm{d}X_{i+1} \\ &= \int_{\log \eta_b}^{\infty} e^{(\alpha+i\beta)X_{i+1}} U^{(A)}(e^{X_{i+1}}, e^{\zeta_j}, i+1)\, \mathrm{d}X_{i+1}. \end{aligned} \tag{4.14}$$

Here, the parameter α is a damping factor, which should be properly chosen to insure the existence of the generalized Fourier transform of $U^{(A)}(\widetilde{W}, e^{\zeta_j}, i+1)$. With reference to the conditional moment generating function $\Psi(\omega, \gamma_t, \gamma_s) = E[e^{\omega(X_t - X_s)}|\gamma_t, X_s, \gamma_s]$, the renowned

Parseval relation leads to the following inverse Fourier transform representation:

$$
\begin{aligned}
& E\left[U^{(A)}(\widetilde{W}, e^{\zeta_j}, i+1)\middle|\widetilde{W}_{i^+}, \gamma_{i+1} = \zeta_j, \gamma_i\right] \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\alpha+i\beta)\log \widetilde{W}_{i^+}} \widehat{U}^{(A)}(\beta, e^{\zeta_j}, i+1)\Psi\left(-\alpha - i\beta, \zeta_j, \gamma_i\right) \mathrm{d}\beta.
\end{aligned} \tag{4.15}
$$

Combining eqs. (4.12), (4.13) and (4.15), and interchanging the order of integration and summation, we obtain

$$
\begin{aligned}
& E\big[\widetilde{V}^{(A)}\big(\widetilde{W}, v, i+1\big)\big|\widetilde{W}_{i^+}, v_i\big] \\
= & \; p_i q_i e^r \widetilde{W}_{i^+} + \sum_{j=1}^{J} g^{(A)}(e^{\zeta_j}, i+1) p_\gamma(\zeta_j|\gamma_i) w_j \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\alpha+i\beta)\log \widetilde{W}_{i^+}} \sum_{j=1}^{J} \widehat{U}^{(A)}(\beta, e^{\zeta_j}, i+1)\widetilde{\Psi}\left(-\alpha - i\beta, \zeta_j, \gamma_i\right) w_j \,\mathrm{d}\beta, \quad i = 0, 1, \cdots, T_a.
\end{aligned} \tag{4.16}
$$

Here, $\widetilde{\Psi}\left(\omega, \gamma_t, \gamma_s\right) = \Psi\left(\omega, \gamma_t, \gamma_s\right) p_\gamma(\gamma_t|\gamma_s)$ and $\widetilde{\Psi}(\omega, \gamma_t, \gamma_s)$ admits a closed form representation (Fang and Oosterlee, 2011; Zeng and Kwok, 2014). For notational convenience, we have suppressed the dependency of $p_\gamma(\gamma_t|\gamma_s)$, $\Psi\left(\omega, \gamma_t, \gamma_s\right)$ and $\widetilde{\Psi}\left(\omega, \gamma_t, \gamma_s\right)$ on $t - s$. Similarly, for $i = 1, \cdots, T-1$, we have

$$
\begin{aligned}
& E\big[\widetilde{V}^{(I)}\big(\widetilde{W}, v, i+1; G_{n_k}\big)\big|\widetilde{W}_{i^+}, v_i\big] \\
= & \; p_i q_i e^r \widetilde{W}_{i^+} + g^{(I)}(i+1; G_{n_k}) \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\alpha+i\beta)\log \widetilde{W}_{i^+}} \sum_{j=1}^{J} \widehat{U}^{(I)}(\beta, e^{\zeta_j}, i+1; G_{n_k})\widetilde{\Psi}\left(-\alpha - i\beta, \zeta_j, \gamma_i\right) w_j \,\mathrm{d}\beta.
\end{aligned} \tag{4.17}
$$

The expectation corresponding to $i = T-1$ admits a simple closed form representation

$$
E\big[\widetilde{V}^{(I)}(\widetilde{W}, v, T; G_{n_k})\big|\widetilde{W}_{T-1^+}, v_{T-1}\big] = p_{T-1} e^r \widetilde{W}_{T-1^+}. \tag{4.18}
$$

Alternatively, this conditional expectation can be derived easily based on eq. (4.17) by observing that $U^{(I)}(\widetilde{W}, v, T; G_{n_k})$ is a zero function.

Implementation procedures

We would like to present the details of the implementation procedure of the Fourier transform algorithm for computing the value functions. Substituting eq. (4.17) into eq. (4.4) and taking advantage of the relation (4.11) between $U^{(I)}(\widetilde{W}, v, i; G_{n_k})$ and $\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k})$, we manage to construct the Fourier transform algorithm that calculates $U^{(I)}(\widetilde{W}, v, i; G_{n_k})$. Note that $U^{(A)}(\widetilde{W}, v, i)$ can be evaluated in a similar manner. The detailed steps in constructing these recursive equations and the terminal condition are provided in Appendix B. The initial value function $V^{(A)}(W_0, A_0, v_0, 0)$ can be obtained at the last time step. The implementation procedures of the Fourier transform algorithm for calculating the value functions by applying the Fourier transform method to the log policy fund value process and the quadrature rule to the log-variance dimension are summarized as below.

Let the infinite Fourier domain for β be truncated to the finite truncation domain $[-Mh, Mh]$, and consider $X_t = l_0 + l\Delta$, $l = 1, 2, \cdots, L$. Here, M and L are referred as the truncation levels, and l_0 represents the lower truncation boundary. Later, we drop l_0 and write X_t as $l\Delta$ for brevity, where $l = 1, 2, \cdots, L$. By definitions of $U^{(I)}(\widetilde{W}, v, i; G_{n_k})$ and $U^{(A)}(\widetilde{W}, v, i)$, it is only necessary to evaluate these two functions when $\widetilde{W} > \eta_b$.

Step 1: Preparation at $T-1$
Calculate the generalized Fourier transform using the following formula

$$\widehat{U}^{(I)}(mh, e^{\zeta_j}, T-1; G_{n_k}) = \sum_{l=1}^{L} e^{(\alpha+imh)l\Delta} p_{T-1}(e^{l\Delta} - \eta_b - G_{n_k})^+ \,\Delta, \tag{4.19}$$

where $m = -M, \cdots, M$, $j = 1, \cdots, J$ and $k = 1, \cdots, K$. Here, α is the damping factor and $\alpha < -1$ is required to guarantee the existence of the generalized Fourier transform.

Step 2: Backward induction from $T-2$ to T_a+1

$$\begin{aligned}
& U^{(I)}\left(e^{l\Delta}, e^{\zeta_p}, i; G_{n_k}\right) \\
= & -g^{(I)}(i; G_{n_k}) + \max\Bigg\{ p_i G_{n_k} + p_i(1-\kappa_i)(e^{l\Delta} - \eta_b - G_{n_k}), \\
& e^{-r}\psi_i^{(4)}(e^{l\Delta}) \Bigg[p_i q_i e^r \phi_i^{(4)}(e^{l\Delta}) + g^{(I)}(i+1; G_{n_k}) + \frac{1}{2\pi} \sum_{m=-M}^{M} e^{-(\alpha+imh)\log\phi_i^{(4)}(e^{l\Delta})} \\
& \quad * \sum_{j=1}^{J} \widehat{U}^{(I)}(mh, e^{\zeta_j}, i+1; G_{n_k}) \widetilde{\Psi}\left(-\alpha - imh, \zeta_j, \zeta_p\right) w_j h \Bigg], \\
& p_i G_{n_k} + e^{-r}\psi_i^{(5)}(e^{l\Delta}; G_{n_k}) \Bigg[p_i q_i e^r \phi_i^{(5)}(e^{l\Delta}; G_{n_k}) + g^{(I)}(i+1; G_{n_k}) + \mathbb{1}_{\left\{\phi_i^{(5)}(\widetilde{W}_i; G_{n_k})>0\right\}} \\
& \quad * \frac{1}{2\pi} \sum_{m=-M}^{M} e^{-(\alpha+imh)\log\phi_i^{(5)}(e^{l\Delta}; G_{n_k})} \sum_{j=1}^{J} \widehat{U}^{(I)}(mh, e^{\zeta_j}, i+1; G_{n_k}) \widetilde{\Psi}\left(-\alpha - imh, \zeta_j, \zeta_p\right) w_j h \Bigg] \Bigg\},
\end{aligned} \tag{4.20a}$$

where $l = 1, \cdots, L$, $p = 1, \cdots, J$ and $k = 1, \cdots, K$. The corresponding generalized Fourier transform is calculated according to the following formula

$$\widehat{U}^{(I)}(mh, e^{\zeta_p}, i; G_{n_k}) = \sum_{l=1}^{L} e^{(\alpha+imh)l\Delta} U^{(I)}(e^{l\Delta}, e^{\zeta_p}, i; G_{n_k})\Delta. \tag{4.20b}$$

Repeat Step 2 for $i = T-2, \cdots, T_a+1$.

Remark
One may employ the usual fast Fourier transform method to facilitate the computation of various summation terms in eq. (4.20a), which take the form $\frac{1}{2\pi}\sum_{m=-M}^{M} e^{-(\alpha+imh)l\Delta} f(m)h$ for some $f(m)$. In fact, numerical evaluation of the summation terms with respect to m in eq. (4.20a) is achieved by combining the fast Fourier transform technique and spline interpolation of the log normalized policy fund value (which exhibits jump across an event date).

Step 3: Preparation at T_a+1

The mandated activation to the income phase at T_a+1 leads to

$$\begin{aligned}\widehat{U}^{(A)}(mh, e^{\zeta_p}, T_a+1) &= \sum_{l=1}^{L} e^{(\alpha+imh)l\Delta} U^{(I)}\big(e^{l\Delta}, e^{\zeta_p}, T_a+1; G(T_a+1)\big)\Delta, \\ g^{(A)}(e^{\zeta_p}, T_a+1) &= g^{(I)}\big(T_a+1; G(T_a+1)\big).\end{aligned} \tag{4.21}$$

Step 4: Backward induction from T_a to 1

(1) Calculate the function $g^{(A)}(e^{\zeta_p}, i)$ based on the following formula:

$$\begin{aligned}&g^{(A)}(e^{\zeta_p}, i) \\ =\ & \max\Bigg\{ g^{(I)}\big(i; G(i)\big), \\ &- p_i B + e^{-r}\big((1+b_i)+B\big)\Bigg[\frac{Bp_iq_ie^r}{(1+b_i)+B} + \sum_{j=1}^{J} g^{(A)}(e^{\zeta_j}, i+1)p_\gamma(\zeta_j|\zeta_p)w_j \\ &+ \mathbb{1}_{\{B>0\}}\frac{h}{2\pi}\sum_{m=-M}^{M} e^{-(\alpha+imh)\log\frac{B}{(1+b_i)+B}} \sum_{j=1}^{J} \widehat{U}^{(A)}(mh, e^{\zeta_j}, i+1)\widetilde{\Psi}\left(-\alpha-imh, \zeta_j, \zeta_p\right)w_j\Bigg], \\ &e^{-r}(1+b_i)\sum_{j=1}^{J} g^{(A)}(e^{\zeta_j}, i+1)p_\gamma(\zeta_j|\zeta_p)w_j\Bigg\},\end{aligned} \tag{4.22}$$

where $p = 1, 2, \cdots, J$.

(2) Calculate $U^{(I)}\left(e^{l\Delta}, e^{\zeta_p}, i; G_{n_k}\right)$ and $\widehat{U}^{(I)}(mh, e^{\zeta_p}, i; G_{n_k})$ as shown in Step 2.

(3) Calculate $U^{(A)}\left(e^{l\Delta}, e^{\zeta_p}, i\right)$ and $\widehat{U}^{(A)}(mh, e^{\zeta_p}, i)$ based on the following formulas:

$$\begin{aligned}&U^{(A)}\left(e^{l\Delta}, e^{\zeta_p}, i\right) \\ =\ & - g^{(A)}(e^{\zeta_p}, i) + \max\Bigg\{U^{(I)}\left(e^{l\Delta}, e^{\zeta_p}, i; G(i)\right) + g^{(I)}(i; G(i)), \\ &- p_iB + e^{-r}\psi_i^{(2)}(e^{l\Delta})\Bigg[p_iq_ie^r\phi_i^{(2)}(e^{l\Delta}) + \sum_{j=1}^{J} g^{(A)}(e^{\zeta_j}, i+1)p_\gamma(\zeta_j|\zeta_p)w_j \\ &+ \frac{1}{2\pi}\sum_{m=-M}^{M} e^{-(\alpha+imh)\log\phi_i^{(2)}(e^{l\Delta})} \sum_{j=1}^{J} \widehat{U}^{(A)}(mh, e^{\zeta_j}, i+1)\widetilde{\Psi}\left(-\alpha-imh, \zeta_j, \zeta_p\right)w_jh\Bigg], \\ &e^{-r}\psi_i^{(3)}(e^{l\Delta})\Bigg[p_iq_ie^r\phi_i^{(3)}(e^{l\Delta}) + \sum_{j=1}^{J} g^{(A)}(e^{\zeta_j}, i+1)p_\gamma(\zeta_j|\zeta_p)w_j \\ &+ \frac{1}{2\pi}\sum_{m=-M}^{M} e^{-(\alpha+imh)\log\phi_i^{(3)}(e^{l\Delta})} \sum_{j=1}^{J} \widehat{U}^{(A)}(mh, e^{\zeta_j}, i+1)\widetilde{\Psi}\left(-\alpha-imh, \zeta_j, \zeta_p\right)w_jh\Bigg]\Bigg\},\end{aligned} \tag{4.23a}$$

and

$$\widehat{U}^{(A)}(mh, e^{\zeta_p}, i) = \sum_{l=1}^{L} e^{(\alpha+imh)l\Delta} U^{(A)}(e^{l\Delta}, e^{\zeta_p}, i)\Delta, \tag{4.23b}$$

where $l = 1, 2, \cdots, L$, $p = 1, 2, \cdots, J$ and $m = -M, \cdots, M$.

Repeat Step 3 for $i = T_a, T_a - 1, \cdots, 1$.

Step 4: Inversion of the Fourier transform at the final step to recover the value function

$$\begin{aligned} & V^{(A)}(W_0, A_0, e^{\zeta_p}, 0) \\ = \; & e^{-r}\Big[p_0 q_0 e^r W_0 + \sum_{j=1}^{J} g^{(A)}(e^{\zeta_j}, 1) p_\gamma(\zeta_j|\zeta_p) w_j A_0 + \frac{A_0}{2\pi} \sum_{m=-M}^{M} e^{-(\alpha+imh)\log W_0/A_0} \\ & \quad * \sum_{j=1}^{J} \widehat{U}^{(A)}(mh, e^{\zeta_j}, 1) \widetilde{\Psi}\left(-\alpha - imh, \zeta_j, \zeta_p\right) w_j h\Big], \end{aligned} \tag{4.24}$$

where $p = 1, 2, \cdots, J$. The initial value function $V^{(A)}(W_0, A_0, v_0, 0)$ can be obtained using spline interpolation.

Remarks

1. Since the benefit base A_t is substituted into the pricing formulation only in the final step of the algorithm through the initial benefit base A_0, the Fourier transform algorithm can be used to find the value function at varying values of A_0 simultaneously with minimal additional computational cost.

2. Huang and Kwok (2016) present the regression-based Monte Carlo simulation algorithms for pricing and hedging of the GLWB in variable annuities without considering the optimal initiation, age-dependent scheduled withdrawal rates and additional purchases. With the level of complexities under the general framework considered in this paper, the regression-based Monte Carlo method would not be effective for pricing the GLWB products. At least for one consideration, the effective implementation of the regression-based Monte-Carlo simulation method requires the knowledge of normalized value function at zero policy fund value. Unfortunately, when additional purchases are allowed, the normalized value function at zero policy fund value does not admit an analytic representation. On the contrary, our newly derived Fourier transform algorithm can be employed for their GLWB formulation by making some slight modification. The corresponding modified version is much easier compared to our Fourier transform algorithm in this paper since we do not need to calculate the two normalized value functions at each time step in the modified version.

3. There may be potential loss of monotonicity in a finite term Fourier expansion of the value function, thus raising the query on the justification of the bang-bang result under such scenario. Indeed, a similar issue has been considered in Huang and Kwok (2016) on the regression-based Monte Carlo simulation algorithm, where the expansion of the value function in terms of finite number of basis functions may lose monotonicity as well. The key consideration is to establish uniform convergence of the approximation in finite number of basis functions to the value function as the number of basis functions tends to infinity [see Assumption H1 in proving Proposition 3 in Huang and Kwok (2016)]. In our

r	κ	ϵ	θ	v_0	ρ
0.04	1.15	0.39	0.0348	0.0348	-0.64

Table 1: Parameter values of the Heston model.

Parameter	Notation	Value
Penalty for excess withdrawal	$k(t)$	$0 \leq t \leq 1 : 6\%$, $1 < t \leq 2 : 5\%$, $2 \leq t \leq 3 : 4\%$, $3 < t \leq 4 : 3\%$, $4 \leq t \leq 5 : 2\%$, $5 < t \leq 25 : 1\%$, $25 < t \leq T : 0\%$
Expiry time	T	57 (years)
Initial investment	W_0	100
Initial benefit base	A_0	100
Insurance fee (for benefit base)	η_b	1%
Mortality		DAV 2004R (65 year old male) (Pasdika and Wolff, 2005)
Mortality payments		At year end
Bonus (no withdrawal)	b_i	0.06 annual
Ratchet cycle		yearly
Withdrawal strategy		Optimal
Withdrawal dates		yearly

Table 2: Contract parameter values of the GLWB product.

Fourier transform algorithm, based on known results on uniform convergence of finite term Fourier series, we observe that the numerical conditional expectation calculated by the Fourier algorithm converges uniformly to the true solution as the number of Fourier terms and the computational domain go to infinity. It is then seen that the proof of Proposition 3 in Huang and Kwok (2016) can be extended to show the desired convergence results in the Fourier transform algorithm.

5 Numerical results

In this section, we first demonstrate the high level of accuracy and efficiency of the Fourier transform algorithm for pricing GLWB under the Heston model through some carefully designed test cases. The numerical performance of the Fourier transform algorithm is compared with that of a modified regression-based Monte Carlo algorithm based on an extended version of Huang and Kwok (2016). Secondly, we show the performance of our Fourier transform algorithm under a general framework and perform the sensitivity analysis of the GLWB price with respect to varying model parameters and contractual features. We also compute the fair rider fees under various parameter values of the Heston model and different contractual specifications. Furthermore, we explore the characterization of the optimal withdrawal strategy regions for GLWB in the $\widetilde{W}$-v plane. Finally, we investigate the optimal initiation regions with respect to the calendar time and the optimal initiation regions with respect to the initial age from the perspective of diverse policyholders. In particular, we examine the impact of model parameters and contractual features on these two kinds of optimal initiation regions.

Refinement	Base		No bonus		No surrender		No ratchet	
	Price	CPU	Price	CPU	Price	CPU	Price	CPU
$L = 2M = 2^6, J = 2^5$	100.18997	0.3	100.12132	0.4	98.50320	0.3	99.40426	0.3
$L = 2M = 2^7, J = 2^6$	100.19916	1.4	100.12770	1.5	98.50740	1.5	99.40978	1.5
$L = 2M = 2^8, J = 2^7$	100.20326	6.7	100.12695	6.8	98.50659	6.7	99.40972	6.7
$L = 2M = 2^9, J = 2^8$	100.20365	58.7	100.12703	58.8	98.50695	59.1	99.40971	59.2
Monte Carlo	100.22 (0.020)		100.13 (0.021)		98.52 (0.023)		——	

Table 3: Numerical results for GLWB obtained from the Fourier transform algorithm for various levels of refinement under $B = 0,\ T_a = \infty$ and $G(t) = 0.05$ for any t. Here, M, L and J denote the truncation parameter for the Fourier transform of the normalized policy fund value, the normalized policy fund value and discretization parameter for the log-variance, respectively. The CPU times (seconds) required in the computations are listed. Numerical results obtained from the regression-based Monte Carlo algorithm are provided in the last row and the standard deviation are listed in brackets. The regression-based Monte Carlo method fails to give reasonable numerical results when there is no ratchet feature in the contract.

5.1 Numerical accuracy and efficiency

Firstly, we compare numerical accuracy and computational efficiency of the Fourier transform algorithm with the regression-based Monte Carlo algorithm in pricing GLWB with the optimal initiation feature. In our sample calculations, we consider the simplified scenario where $B = 0,\ T_a = \infty$ and $G(t) = 0.05$ for any t; that is, additional purchases is not allowed and there is no restriction for the policyholder to activate the income phase by a given age.

We assume that the policy fund value dynamics is governed by the Heston model as follows:

$$\begin{aligned} \frac{\mathrm{d}W_t}{W_t} &= r\,\mathrm{d}t + \sqrt{v_t}(\rho\,\mathrm{d}B_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}B_t^2), \\ \mathrm{d}v_t &= \kappa(\theta - v_t)\,\mathrm{d}t + \epsilon\,\mathrm{d}B_t^1, \end{aligned} \tag{5.1}$$

where B_t^1 and B_t^2 are two independent Brownian motions under the risk neutral measure $\mathbb{Q}$. Tables 1 and 2 list the parameter values in the Heston model and the relevant contractual features in the GLWB product, which are regarded as the "Base" case in our sample calculations. We adopt the values of the Heston model parameters obtained by Bakshi *et al.* (1997). Their calibration was performed based on minimizing the sum of squared pricing errors between the market prices of S&P 500 options and the model-determined prices.

In Table 3, we list the prices of the GLWB product with the optimal initiation feature under four different scenarios: "Base", "No bonus", "No surrender" and "No ratchet" using the Fourier transform algorithm and the regression-based Monte Carlo algorithm. For the first three scenarios, good agreement of numerical results obtained from the two numerical methods is observed. This confirms high accuracy of the Fourier transform algorithm. When the ratchet feature is not included, the regression-based Monte Carlo algorithm fails to provide reasonably stable numerical results even with a large number of simulation paths. It took about 3000 seconds to generate numerical results using the regression-based Monte Carlo algorithm when the number of the simulation paths was taken to be 10^6. The CPU times (seconds) required using the Fourier transform algorithm for various levels of refinement of the truncation parameters L, M and J are also listed. The numerical GLWB prices obtained by the Fourier transform algorithm exhibit penny accuracy (5 significant figures accuracy) at relatively low values of L, M and J (say, $L = 2M = 2^8, J = 2^7$). By comparing the CPU times, we observe that the fast Fourier transform method is more computationally efficient than the regression-based Monte Carlo method. The CPU times are consistent with the order of complexity of the Fourier

Withdrawal rate	$G(t)$
Withdraw 1	$0 \le t \le T$: 5%
Withdraw 2	$0 \le t \le 5$: 5%, $6 \le t \le 10$: 5.5%, $11 \le t \le T$: 6%
Withdraw 3	$0 \le t \le 15$: 5%+0.1%t, $16 \le t \le T$: 6.5%

Table 4: Values of the contractual withdrawal rate $G(t)$ of the GLWB product.

B	Withdrawal rate	$J = 2^5$		$J = 2^6$		$J = 2^7$		$J = 2^8$	
		Price	CPU	Price	CPU	Price	CPU	Price	CPU
0.3	1	100.23609	0.9	100.23685	1.9	100.23643	6.0	100.23640	21.1
	2	103.60179	1.8	103.60036	3.9	103.59954	11.1	103.59953	37.4
	3	105.62601	6.6	105.62304	15.9	105.62338	45.1	105.62324	158.2
0.5	1	100.26566	0.9	100.26725	2.0	100.26639	5.7	100.26649	20.7
	2	104.81688	1.7	104.81303	3.9	104.81017	11.0	104.81065	37.1
	3	108.51604	6.5	108.50796	16.1	108.50925	44.9	108.50835	157.8

		$L = 2M = 2^6$		$L = 2M = 2^7$		$L = 2M = 2^8$		$L = 2M = 2^9$	
		Price	CPU	Price	CPU	Price	CPU	Price	CPU
0.3	1	100.22046	1.9	100.23176	3.0	100.23643	5.5	100.23693	9.5
	2	103.53241	3.6	103.55795	6.0	103.59954	11.2	103.59916	18.9
	3	105.53063	14.3	105.56298	24.0	105.62338	44.8	105.62300	80.8
0.5	1	100.24946	1.8	100.26080	2.9	100.26639	5.4	100.26695	9.5
	2	104.70765	3.6	104.74723	5.9	104.81017	11.1	104.80963	19.0
	3	108.33800	14.3	108.39753	23.9	108.50925	45.1	108.50832	81.6

Table 5: Numerical results for the GLWB prices obtained from the Fourier transform algorithm with respect to varying contractual withdrawal rates and upper bounds on additional purchases. The CPU times (seconds) required in the Fourier transform calculations are also listed. The truncation level parameters are set to be $L = 2M = 2^8$ and $J = 2^7$, except as noted.

transform algorithm when M and J go beyond 2^7. At low values of M and J, the order of the complexity is less apparent since a significant portion of CPU time is used in the initiation step in the Fourier transform algorithm.

5.2 Pricing behaviors of the GLWB

Next, we present the performance of the Fourier transform algorithm for pricing GLWB under the generalized cases of inclusion of all contractual features, where the regression-based Monte Carlo method may fail to give reasonably stable numerical results. In our calculations, we set $T_a = 20$ and consider different choices on the contractual withdrawal rate $G(t)$ and cap multiplier of the benefit base B for additional purchases (see Tables 4 and 5).

In Table 5, we present the numerical results of the GLWB prices obtained from the Fourier transform algorithm with varying values of J and varying values of M, respectively. The CPU times required for the Fourier transform calculations are also reported. As expected, increasing the additional purchase parameter B or the contractual withdrawal rate would lead to a higher GLWB price. In addition, though the Cox-Ingersoll-Ross model parameters fail to satisfy the

Case	$\theta = 0.0225$	$\theta = 0.0286$	$\theta = 0.0348$
	Fair rider fee (bps)	Fair rider fee (bps)	Fair rider fee (bps)
Base	68.6	86.1	103.8
No bonus	67.5	85.0	102.5
No surrender	51.3	69.2	83.5
No ratchet	59.7	74.6	89.4

Table 6: The effect of contractual provisions on the fair rider fee under varying values of θ, the mean reversion level of the variance in the Heston model. Here, $T_a = 20$, $B = 0.3$ and the contractual withdrawal rate is chosen to be "Withdraw 2".

Case	$\epsilon = 0.25$	$\epsilon = 0.32$	$\epsilon = 0.39$
	Fair rider fee (bps)	Fair rider fee (bps)	Fair rider fee (bps)
Base	109.5	106.9	103.8
No bonus	108.1	105.5	102.5
No surrender	88.4	86.0	83.5
No ratchet	92.1	90.7	89.4

Table 7: The effect of contractual provisions on the fair rider fee under varying values of ϵ, the volatility of the variance in the Heston model. Here, $T_a = 20$, $B = 0.3$ and the contractual withdrawal rate is chosen to be "Withdraw 2".

Feller condition, our fast Fourier transform algorithm remains to converge rapidly for the log-variance dimension even under such scenario. Finally, though the time dependent feature of the contractual withdrawal rate adds one additional dimension to our pricing problem, the CPU times do not increase substantially. This is because the computational time of our Fourier transform is mainly attributed to the calculations of the kernel function $\widetilde{\Psi}\left(-\alpha - imh, \zeta_j, \zeta_p\right)$, which involves the valuation of the modified Bessel function. Interested readers may refer to Zeng and Kwok (2014) for more details.

Next, we conduct sensitivity analysis of the GLWB price function with respect to the contractual features and model parameters. Without loss of generality, we set $T_a = 20$, $B = 0.3$ and the contractual withdrawal rate to be "Withdraw 2".

Fair rider fees

The fair rider fee is determined by setting the GLWB price at initiation to be equal to the initial account value. We solve for the rider fee η_b from the following equation:

$$V^{(A)}(W_0, A_0, v_0, 0) = W_0.$$

In Tables 6 and 7, we examine the effect of contractual provisions on the fair rider fee under varying values of θ and ϵ. Here, θ represents the mean reversion level of the variance and ϵ is the volatility of variance in the Heston model. The fair rider fee is more sensitive to θ but less sensitive to ϵ. Since the GLWB prices increase with θ and decrease with ϵ (expected variance of the policy fund value process is known to be a decreasing function of ϵ), so the fair rider fees increase with θ and decrease with ϵ.

Cycle of ratchet events and penalty charge

The cycle of ratchet events refers to the number of years lapsed between successive ratchet event dates. In Figure 1, we plot the GLWB price against the cycle of ratchet events under two penalty charge schemes: Penalty 1 and Penalty 2. Here, "Penalty 1" refers to the penalty charge setting taken from Table 2, while "Penalty 2" is obtained by adding 3% to the penalty charge fee $k(t)$ when $t \leq 25$. The GLWB price is seen to decrease with longer cycle of ratchet events and higher penalty charge. The plots also reveal that the ratchet provision and penalty charge scheme may have strong impact on the GLWB price.

Correlation coefficient and volatility of variance

Figure 2 examines the impact of the correlation coefficient ρ and volatility of variance ϵ on the price of GLWB. The GLWB price decreases with an increasing value of ϵ. The GLWB price is seen to be an increasing function of the correlation coefficient ρ.

Optimal withdrawal strategy regions in the $\widetilde{W}$-v plane

Now we would like to explore the characterization of the optimal withdrawal strategy regions for GLWB using the Fourier transform algorithm. In particular, we examine the impact of contractual withdrawal rate and additional purchases on the optimal withdrawal strategy regions. At a fixed time, the separation of the optimal withdrawal strategy regions is characterized by the normalized policy fund value $\widetilde{W}$ and variance v. Figure 3 illustrates the separation of optimal withdrawal strategy regions in the $\widetilde{W}$-v plane on the first withdrawal date under four different scenarios. As revealed from Figure 3, when the contractual withdrawal rate is an increasing function of time, GLWB has a smaller withdrawal region. This is because the policyholder would choose not to withdraw prematurely and prefer to wait until a higher contractual withdrawal rate at a later time. At the same time, the region of optimal additional purchase increases under higher contractual withdrawal rate. In addition, a larger value of B would enlarge the region of additional purchase. As a final remark, the region of optimal additional purchases increases as the variance v increases while the optimal withdrawal region decreases with higher variance.

5.3 Optimal initiation policies

Optimal initiation boundary against the calendar time t

By fixing the variance, for each t, we consider the optimal value of $\widetilde{W}$ at which the value function in the accumulation phase equals that in the income phase. Similar to the optimal exercise boundary for an American option, we plot the optimal initiation boundary as a function of time t. We examine the effect of the variance, bonus rate and contractual withdrawal rate on the optimal initiation boundary in the $\widetilde{W}$-t plane. We let the colored region denote the optimal initiation region in which the income phase should be initiated. In Figures 4a-4c, we assume a constant contractual withdrawal rate and plot the optimal initiation regions in the $\widetilde{W}$-t plane for different values of the variance and bonus rate. The optimal initiation region decreases as the variance or the bonus rate increases. Note that it is optimal to stay in the accumulation phase only in the areas with high normalized policy fund value $\widetilde{W}$ and earlier withdrawal dates. Especially when the bonus rate is low, as observed from Figure 4c, it is always optimal to initiate the income phase immediately. This is because the incentive for the policyholder to choose zero withdrawal or additional purchase is low when the bonus is very small.

We investigate the effect of the contractual withdrawal rate on the optimal initiation region. We assume the same parameter values as in Figure 4a, except that the contractual withdrawal rate increases on some specified dates (triggering dates). In our test, the contractual withdrawal rate rises from 5% to 5.5% on the triggering date $t = 6$ and it increases to 6% at $t = 11$. Figure

Parameter	Notation	Value
Initial age	x_0	$50 \le x_0 \le 120$
Penalty for excess withdrawal	${}_{x_0}k_t$	$0 \le x_0+t \le 66 : 6\%,\ \ 66 < x_0+t \le 67 : 5\%,$ $67 \le x_0+t \le 68 : 4\%,\ \ 68 < x_0+t \le 69 : 3\%,$ $69 \le x_0+t \le 70 : 2\%,\ \ 70 < x_0+t \le 90 : 1\%,$ $90 < x_0+t \le 122 : 0\%$
Expiry time	T_{x_0}	$122 - x_0$ (years)
Survival probability	${}_{x_0}p_t$	$e^{-e^{\frac{x_0+t-87.25}{9.5}}+e^{\frac{x_0-87.25}{9.5}}}$

Table 8: Contract parameter values of the GLWB product with diverse policyholders.

4d reveals that when the time goes beyond the latest of the triggering dates, the plot agrees with that of Figure 4a. Otherwise, the increase in the contractual withdrawal rate provides a strong incentive for the policyholder to delay initiation of the income phase. Especially when the calendar time is approaching a triggering date of changes of the contractual withdrawal rate, the optimal initiation boundary is zero; so initiation becomes non-optimal at any level of normalized policy fund value.

Optimal initiation regions in the $\widetilde{W}$-x_0 plane

We would like to study the optimal initiation of a GLWB from the perspective of diverse policyholders. We let ${}_tp_{x_0}$, ${}_tk_{x_0}$ and $G_{x_0}(t)$ denote the survival probability, the penalty charge rate and contractual withdrawal rate at time t for a policyholder with an initial age x_0, respectively. The model and contract parameter values are taken from Tables 1 and 2, except those listed in Table 8.

Similar to Huang *et al.* (2014), we can determine the optimal initiation region with respect to the initial age x_0. Compared to Huang *et al.* (2014), here we consider a discrete set of event dates and allow for stochastic volatility, dynamic withdrawal, additional purchase and mandated initiation time. Also, we construct the Fourier transform algorithm to determine the optimal initiation region rather than using the finite difference method. On the first withdrawal date ($t = 1$), for a fixed variance, we plot the optimal initiation region with respect to the initial age x_0. The effects of the investment, the penalty charge rate and the contractual withdrawal rate on the optimal initiation are revealed in Figures 5a-5d. It is optimal for young policyholders to accumulate regardless of the level of $\widetilde{W}$ when more additional purchase is allowed. The additional purchase parameter B has a pronounced impact on young policyholders. Secondly, setting the penalty charge rate for excess withdrawal to be 100% is equivalent to ruling out the complete surrender feature. The optimal initiation region becomes larger when compared with the scenario where the complete surrender is allowed and the penalty charge rate is a decreasing function of age. In fact, the lowering of the penalty rate motivates the policyholder to delay the initiation until a smaller penalty charge rate kicks in. Finally, the effect of the contractual withdrawal rate on this optimal initiation region is similar to that on the optimal initiation region in the $\widetilde{W}$-t plane. We consider the contractual withdrawal rate as an increasing function of age. In our test calculations, we assume that the contractual withdrawal rate rises from 5% to 5.5% at age 71 (triggering age) and jumps from 5.5% to 6% at age 76. Figure 5d shows that an increase in the contractual withdrawal rate motivates the policyholders who are younger than the last triggering age to delay initiation. This effect becomes more profound for policyholders at an age immediately before any triggering age since the corresponding optimal initiation boundary is zero.

6 Conclusion

We present the comprehensive pricing model for the GLWB product with the accumulation phase and income phase, additional purchases, age-dependent scheduled withdrawal rate, bonus and ratchet provisions under the Heston stochastic volatility process for the policy fund value. The pricing model includes the optimal stopping rule of initiation and dynamic withdrawal as the stochastic control process. Through a rigorous bang-bang analysis, we show that the strategy space of the optimal policies is limited to four choices, thus simplifying the construction of the Fourier transform algorithm for pricing GLWB products with complex path dependent features. The success of the bang-bang analysis relies on the convexity and monotonicity properties of the price function. The results are applicable to many common processes for the policy fund value. In the design of the Fourier transform algorithm, the dimension of the pricing model is reduced by one since the benefit base remains constant between two consecutive event dates. We then apply the Fourier transform in the log normalized policy fund value dimension and a quadrature rule in the log-variance dimension. The Fourier transform algorithm is seen to be efficient, accurate and reliable even under the level of complexities of path dependence in our comprehensive pricing model of the GLWB while the regression-based Monte Carlo simulation algorithm may fail to give a reliable numerical solution. The CPU time required for numerical evaluation of the price function to achieve 5 significant figures accuracy using the Fourier transform algorithm is typically within a few seconds.

We perform sensitivity analysis of the GLWB price function with respect to various contractual features and model parameters. We consider the impact of the fair rider fees under various parameter values of the Heston stochastic volatility model. We also examine the impact of the contractual withdrawal rate and upper bound of additional purchases on the optimal initiation policies. The optimal initiation policies are seen to depend sensibly on the age-dependent contractual withdrawal rate. Policyholders would wait for a more favorable contractual withdrawal rate for optimal entry into the income phase.

Appendix A - Proof of Theorem 3

We perform the bang-bang analysis for GLWB$^{(I)}$ in the income phase, then extend the analysis in a similar manner to GLWB$^{(A)}$ in the accumulation phase. The proofs requires the following two technical results in convex analysis, stated as Property A.1 and Property A.2 below:

Property A.1 *Let $\mathcal{A}$ be a convex set, and let $\mathcal{B}$ and $\mathcal{C}$ be vector spaces over $\mathbb{R}$. If $g : \mathcal{A} \to \mathcal{B}$ is convex, $h : \mathcal{B} \to \mathcal{C}$ is convex and monotonic increasing, then $h \circ g$ is convex on set $\mathcal{A}$.*

Property A.2 *Suppose we have a function $f \colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ that satisfies (i) $f(\cdot, y)$ is convex for any fixed $y \in \mathbb{R}^+$; (ii) for any positive constant K, $f(Kx, Ky) = Kf(x, y)$, then $f(\cdot, \cdot)$ is convex.*

The proof of Property A.1 can be found in Boyd and Vandenberghe (2004). The proof of Property A.2 is presented at the end of Appendix A.

1. Proof of the bang-bang controls for GLWB$^{(I)}$
We would like to prove Part 1 of Theorem 3, which is restated as below: On any withdrawal date $1 \leq i \leq T-1$, the optimal strategy γ_i for GLWB$^{(I)}$ with any positive guaranteed rate G_0 is limited to (i) $\gamma_i = 0$; (ii) $\gamma_i = G_0 A_i$ (iii) $\gamma_i = W_i - \eta_b A_i$. Moreover $V^{(I)}(\cdot, \cdot, i; G_0)$ is convex.

The proof requires several intermediate results as stated in Lemmas A.3 and A.4. The detailed proofs of these lemmas are presented at the end of Appendix A.

Lemma A.3 *If $V^{(I)}(\cdot,\cdot,v,i+1;G_0)$ is convex, the optimal strategy γ_i on the withdrawal date i for GLWB$^{(I)}$ is limited to (i) $\gamma_i = 0$; (ii) $\gamma_i = G_0 A_i$; (iii) $\gamma_i = W_i - \eta_b A_i$.*

Lemma A.4 *If $V^{(I)}(\cdot,\cdot,v,i+1;G_0)$ is convex, then $V^{(I)}(\cdot,\cdot,v,i;G_0)$ is also convex.*

To complete the proof of Part 1 of Theorem 3, we argue as follows:

1. Suppose $V^{(I)}(\cdot,\cdot,v,i+1;G_0)$ is convex, the optimal strategy on the withdrawal date i for GLWB$^{(I)}$ is limited to a finite number of choices (see the details in Lemma A.3) and $V^{(I)}(\cdot,\cdot,v,i;G_0)$ is also convex (see the details in Lemma A.4).

2. Since the terminal payoff $V^{(I)}(\cdot,\cdot,v,T;G_0)$ is convex, Lemmas A.3 and A.4 can be applied inductively to establish the bang-bang control for GLWB$^{(I)}$ on any withdrawal date.

2. Proof of the bang-bang controls for GLWB$^{(A)}$
We define $C^{(I)}_{i,v,G_0}(\cdot,\cdot)$: $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $C^{(A)}_{i,v}(\cdot,\cdot)$: $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ as follows:

$$C^{(I)}_{i,v,G_0}(x,y) = E[V^I(W,A,v,i+1;G_0)|W_{i^+}=x, A_{i^+}=y, v_{i^+}=v], \quad i=1,2,\ldots,T-1,$$

and

$$C^{(A)}_{i,v}(x,y) = E[V^A(W,A,v,i+1)|W_{i^+}=x, A_{i^+}=y, v_{i^+}=v], \quad i=1,2,\ldots,T_a.$$

Suppose the income phase has not been initiated before T_a, then $V^{(A)}(W,A,v,T_a)$ in eq. (2.8) can be written as

$$V^{(A)}(W,A,v,T_a) = \ p_{T_a-1}q_{T_a-1}W_{T_a} + \max\{V_C^{(A)}(T_a), V_C^{(I)}(T_a)\}, \tag{A.1}$$

where

$$\begin{aligned}
V_C^{(A)}(T_a) &= \sup_{\gamma_{T_a}\in[-BA_{T_a},(W_{T_a}-\eta_b A_{T_a})^+]} \{p_{T_a} f^A_{T_a}(\gamma_{T_a};A_{T_a}) + e^{-r} C^{(I)}_{T_a,v,G(T_a+1)} \circ \boldsymbol{h}^A_{T_a}(W_{T_a},A_{T_a},\gamma_{T_a})\},\\
V_C^{(I)}(T_a) &= \sup_{\gamma_{T_a}\in[0,\max((W_{T_a}-\eta_b A_{T_a})^+,G(T_a)A_{T_a})]} \{p_{T_a} f^I_{T_a}\big(\gamma_{T_a};A_{T_a},G(T_a)\big)\\
&\qquad + e^{-r} C^{(I)}_{T_a,v,G(T_a)} \circ \boldsymbol{h}^I_{T_a}\big(W_{T_a},A_{T_a},\gamma_{T_a};G(T_a)\big)\}.
\end{aligned}$$

To show the calculation of $V_C^{(A)}(T_a)$, we observe that $f^A_{T_a}(\cdot;A_{T_a})$ and $\boldsymbol{h}^A_{T_a}(W_{T_a},A_{T_a},\cdot)$ are both convex on $[-BA_{T_a},0]$ and $[0,(W_{T_a}-\eta_b A_{T_a})^+]$ by eqs. (2.4) and (2.5a). Since $V^{(I)}\big(\cdot,\cdot,v,T_a+1;G(T_a+1)\big)$ is convex by the proof in part 1 of Theorem 3, we can similarly show that $C^{(I)}_{T_a,v,G(T_a+1)} \circ \boldsymbol{h}^A_{T_a}(W_{T_a},A_{T_a},\cdot)$ is convex on $[-BA_{T_a},0]$ and $[0,(W_{T_a}-\eta_b A_{T_a})^+]$. By Corollary 32.3.2 in Rockafellar (1997), the optimal strategy γ_{T_a} for $V_C^{(A)}(T_a)$ is limited to: (i) $\gamma_{T_a} = -BA_{T_a}$; (ii) $\gamma_{T_a}=0$; (iii) $\gamma_{T_a} = (W_{T_a}-\eta_b A_{T_a})^+$. Also, one can show that $V_C^{(A)}(T_a)$ is convex with respect to (W_{T_a},A_{T_a}) using a similar argument of proving Lemma A.4 (see below).

Similarly for $V_C^{(I)}(T_a)$, the optimal strategy γ_{T_a} for $V_C^{(I)}(T_a)$ is limited to: (i) $\gamma_{T_a}=0$; (ii) $\gamma_{T_a} = G(T_a)A_{T_a}$; (iii) $\gamma_{T_a} = W_{T_a}-\eta_b A_{T_a}$. Moreover, $V_C^{(I)}(T_a)$ is convex with respect to (W_{T_a},A_{T_a}).

We define the two mappings $F^A_{T_a}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $F^I_{T_a}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ as follows:

$$F^A_{T_a}(x,y) = V_C^{(A)}(T_a) \text{ with } W_{T_a}=x \text{ and } A_{T_a}=y$$

and

$$F^I_{T_a}(x,y) = V^{(I)}_C(T_a) \text{ with } W_{T_a} = x \text{ and } A_{T_a} = y$$

Also, we define the mapping $\boldsymbol{F}_{T_a} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ by

$$\boldsymbol{F}_{T_a}(x,y) := \left(\begin{array}{c} F^A_{T_a}(x,y) \\ F^I_{T_a}(x,y) \end{array} \right),$$

where $F^A_{T_a}$ and $F^I_{T_a}$ are seen to be convex. Then eq. (A.1) can be rewritten as

$$V^{(A)}(W,A,v,T_a) = p_{T_a-1}q_{T_a-1}W_{T_a} + \max \circ \boldsymbol{F}_{T_a}(W_{T_a}, A_{T_a}),$$

where the operator "max" is defined by $\max(x,y) = x\mathbf{1}_{\{x\geq y\}} + y\mathbf{1}_{\{x<y\}}$. By applying Property A.1, $V^{(A)}(\cdot,\cdot,v,T_a)$ is convex due to convexity of the two operators: max and $\boldsymbol{F}_{T_a}$.

The above arguments can be applied inductively to obtain Parts (2a) and (2b) in Theorem 3.

Proof of Property A.2

To show Property A.2, it suffices to show

$$f(\widehat{x},\widehat{y}) \leq \theta f(x_1,y_1) + (1-\theta) f(x_2,y_2), \quad \forall \theta \in (0,1), \tag{A.2}$$

where

$$\left(\begin{array}{c} \widehat{x} \\ \widehat{y} \end{array} \right) = \theta \left(\begin{array}{c} x_1 \\ y_1 \end{array} \right) + (1-\theta) \left(\begin{array}{c} x_2 \\ y_2 \end{array} \right).$$

By virtue of the homogeneous property in Property A.2, eq. (A.2) is equivalent to

$$f\left(\frac{\widehat{x}}{\widehat{y}}, 1\right) \leq \widehat{\theta} f\left(\frac{x_1}{y_1}, 1\right) + (1-\widehat{\theta}) f\left(\frac{x_2}{y_2}, 1\right), \tag{A.3}$$

with $\widehat{\theta} = \dfrac{\theta y_1}{\widehat{y}} \in (0,1)$. Indeed, by observing $\frac{\widehat{x}}{\widehat{y}} = \widehat{\theta}\frac{x_1}{y_1} + (1-\widehat{\theta})\frac{x_2}{y_2}$ and convexity of $f(\cdot,1)$, we can establish eq. (A.2).

Proof of Lemma A.3

On any withdrawal date, $1 \leq i \leq T-1$, we deduce from eq. (2.7) that

$$\begin{aligned} V^{(I)}(W,A,v,i;G_0) = \; & p_{i-1}q_{i-1}W_i + \sup_{\gamma_i \in [0, \max(W_i - \eta_b A_i, G_0 A_i)]} \{ p_i f^I_i(\gamma_i; A_i, G_0) \\ & + e^{-r} C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}^I_i(W_i, A_i, \gamma_i; G_0) \} \end{aligned} \tag{A.4}$$

If $V^{(I)}(\cdot,\cdot,v,i+1;G_0)$ is convex, then $C^{(I)}_{i,v,G_0}(\cdot,\cdot)$ is convex due to Property 1, eq. (3.1a) and Property A.2. Notice that $\boldsymbol{h}^I_i(W_i, A_i, \cdot; G_0)$ is convex on $[0, G_0 A_i]$ and $[G_0 A_i, W_i - \eta_b A_i]$ (the later interval becomes empty when $W_i - \eta_b A_i < G_0 A_i$). Together with monotonicity of $C^{(I)}_{i,v,G_0}(x,y)$ on both x and y, $C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}^I_i(W_i, A_i, \cdot; G_0)$ is convex on $[0, G_0 A_i]$ and $[G_0 A_i, W_i - \eta_b A_i]$ by Property A.1. As a result, we can easily show that $p_i f^I_i(\cdot; A_i, G_0) + e^{-r} C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}^I_i(W_i, A_i, \cdot; G_0)$ is convex on $[0, G_0 A_i]$ and $[G_0 A_i, W_i - \eta_b A_i]$.

Since the supremum of a convex function on a closed bounded convex set must occur at one of the extreme points of the set (Corollary 32.3.2 in Rockafellar, 1997), the optimal strategy γ_i shown in (A.4) is limited to (i) $\gamma_i = 0$, (ii) $\gamma_i = G_0 A_i$ and (iii) $\gamma_i = W_i - \eta_b A_i$. Hence, Claim A.3 is proved.

Proof of Lemma A.4

It suffices to show that $V^{(I)}(\cdot, A, v, i; G_0)$ is convex due to eq. (3.1a) and Property A.2. We define

$$\widehat{W} = \theta W_1 + (1-\theta)W_2, \quad \forall \theta \in (0,1),$$

and let $\widehat{\gamma}^*$ be the optimal strategy for $V^{(I)}(\widehat{W}, A, v, i; G_0)$. Also, we let γ_1 and γ_2 be the candidate strategies (not necessary to be optimal) for $V^{(I)}(W_1, A, v, i; G_0)$ and $V^{(I)}(W_2, A, v, i; G_0)$, respectively. We observe that $\widehat{\gamma}^*$ is limited to the following three choices according to Claim A.3: (i) $\widehat{\gamma}^* = 0$; (ii) $\widehat{\gamma}^* = G_0 A_i$; (iii) $\widehat{\gamma}^* = \widehat{W} - \eta_b A_i$.

For the case of $\widehat{\gamma}^* = \widehat{W} - \eta_b A_i$, we set $\gamma_1 = \max(W_1 - \eta_b A_i, G_0 A_i)$ and $\gamma_2 = \max(W_2 - \eta_b A_i, G_0 A_i)$. We have

$$\begin{aligned}
V^{(I)}(\widehat{W}, A, v, i; G_0) &= p_{i-1}q_{i-1}\widehat{W} + p_i f_i^I(\widehat{\gamma}^*; A_i, G_0) \\
&= p_{i-1}q_{i-1}\widehat{W} + p_i\big[(\widehat{W} - \eta_b A_i - G_0 A_i)(1-k_i) + G_0 A_i\big] \\
&= \theta\big[p_{i-1}q_{i-1}W_1 + p_i(W_1 - \eta_b A_i - G_0 A_i)(1-k_i) + p_i G_0 A_i\big] \\
&\quad +(1-\theta)\big[p_{i-1}q_{i-1}W_2 + p_i(W_2 - \eta_b A_i - G_0 A_i)(1-k_i) + p_i G_0 A_i\big] \\
&\le \theta[p_{i-1}q_{i-1}W_1 + p_i f_i^I(\gamma_1; A_i, G_0)] \\
&\quad +(1-\theta)[p_{i-1}q_{i-1}W_2 + p_i f_i^I(\gamma_2; A_i, G_0)] \\
&\le \theta V^{(I)}(W_1, A, v, i; G_0) + (1-\theta)V^{(I)}(W_2, A, v, i; G_0).
\end{aligned}$$

The above inequalities hold since γ_1 and γ_2 are the admissible strategies for $V^{(I)}(W_1, A, v, i; G_0)$ and $V^{(I)}(W_2, A, v, i; G_0)$, respectively.

For the case of $\widehat{\gamma}^* = G_0 A_i$, we set $\gamma_1 = \gamma_2 = G_0 A_i$. Since $\boldsymbol{h}_i^I(\cdot, A_i, G_0 A_i; G_0)$ is convex by virtue of eq. (2.5b), so $C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}_i^I(\cdot, A_i, G_0 A_i; G_0)$ is also convex. Since $\widehat{\gamma}^*$ is the adopted optimal strategy, we have

$$\begin{aligned}
& V^{(I)}(\widehat{W}, A, v, i; G_0) \\
= \; & p_{i-1}q_{i-1}\widehat{W} + p_i f_i^I(G_0 A_i; A_i, G_0) \\
& +e^{-r}E[V^{(I)}(W, A, v, i+1; G_0) | (W_{i^+}, A_{i^+}) = \boldsymbol{h}_i^I(\widehat{W}, A_i, G_0 A_i; G_0), v_{i^+} = v] \\
= \; & p_{i-1}q_{i-1}\widehat{W} + p_i f_i^I(G_0 A_i; A_i, G_0) + e^{-r} C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}_i^I(\widehat{W}, A_i, G_0 A_i; G_0) \\
\le \; & p_{i-1}q_{i-1}\widehat{W} + p_i f_i^I(G_0 A_i; A_i, G_0) + e^{-r}[\theta C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}_i^I(W_1, A_i, G_0 A_i; G_0) \\
& +(1-\theta) C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}_i^I(W_2, A_i, G_0 A_i; G_0)] \\
= \; & \theta[p_{i-1}q_{i-1}W_1 + p_i f_i^I(G_0 A_i; A_i, G_0) + e^{-r} C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}_i^I(W_1, A_i, G_0 A_i; G_0)] \\
& +(1-\theta)[p_{i-1}q_{i-1}W_2 + p_i f_i^I(G_0 A_i; A_i, G_0) + e^{-r} C^{(I)}_{i,v,G_0} \circ \boldsymbol{h}_i^I(W_2, A_i, G_0 A_i; G_0)] \\
\le \; & \theta V^{(I)}(W_1, A, v, i; G_0) + (1-\theta)V^{(I)}(W_2, A, v, i; G_0).
\end{aligned}$$

For the last case where $\widehat{\gamma}^* = 0$, we can establish in a similar manner that

$$V^{(I)}(\widehat{W}, A, v, i; G_0) \le \theta V^{(I)}(W_1, A, v, i; G_0) + (1-\theta)V^{(I)}(W_2, A, v, i; G_0).$$

Hence, Lemma A.4 is proved. $\square$.

Appendix B - Backward induction for calculating the modified normalized value functions and terminal condition

We show how to derive a backward induction for calculating $U^{(I)}(\widetilde{W}, v, i; G_{n_k})$ and $U^{(A)}(\widetilde{W}, v, i)$ and derive the terminal condition for $\widehat{U}^{(I)}(\widetilde{W}, v, T-1; G_{n_k})$. Recall that when $\widetilde{W}_i \leq \eta_b$, $U^{(I)}(\widetilde{W}, v, i, G_{n_k})$ equals to zero. Therefore we can restrict our attention to the condition that $\widetilde{W}_i > \eta_b$. Based on eqs. (4.4), (4.17), and (4.8), we can express the normalized value functions in terms of the Fourier transform integrals. When we compute $E[\widetilde{V}^{(I)}(\widetilde{W}, v, i+1; G_{n_k}) | \widetilde{W}_{i^+} = \phi_i^{(5)}(\widetilde{W}_i), v_i]$ in eq. (4.4), it is necessary to distinguish the two cases by comparing $\phi_i^{(5)}(\widetilde{W}_i; G_{n_k})$ with zero. An unified formula for $V^{(I)}(\widetilde{W}, v, i; G_{n_k})$ can be expressed as follows

$$
\begin{aligned}
&\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k}) \\
&= p_{i-1}q_{i-1}\widetilde{W}_i + \max\Big\{ p_i \left[G_{n_k} + (1-\kappa_i)(\widetilde{W}_i - \eta_b - G_{n_k}) \right], \\
&\quad e^{-r}\psi_i^{(4)}(\widetilde{W}_i) \Big[p_i q_i e^{r-\eta_p} \phi_i^{(4)}(\widetilde{W}_i) + g^{(I)}(i+1; G_{n_k}) \\
&\qquad + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\alpha+i\beta)\log \phi_i^{(4)}(\widetilde{W}_i)} \sum_{j=1}^{J} \widehat{U}^{(I)}(\beta, e^{\zeta_j}, i+1) \widetilde{\Psi}\left(-\alpha - i\beta, \zeta_j, \gamma_i\right) w_j \, \mathrm{d}\beta \Big], \\
&\quad p_i G + e^{-r}\psi_i^{(5)}(\widetilde{W}_i; G_{n_k}) \Big[p_i q_i e^{r-\eta_p} \phi_i^{(5)}(\widetilde{W}_i; G_{n_k}) + g^{(I)}(i+1; G_{n_k}) + \mathbb{1}_{\left\{\phi_i^{(5)}(\widetilde{W}_i; G_{n_k}) > 0\right\}} \\
&\qquad * \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\alpha+i\beta)\log \phi_i^{(5)}(\widetilde{W}_i; G_{n_k})} \sum_{j=1}^{J} \widehat{U}^{I}(\beta, e^{\zeta_j}, i+1) \widetilde{\Psi}\left(-\alpha - i\beta, \zeta_j, \gamma_i\right) w_j \, \mathrm{d}\beta \Big] \Big\}.
\end{aligned} \tag{B.1}
$$

The last term is included conditional on $\phi_i^{(5)}(\widetilde{W}_i; G_{n_k}) > 0$. Using the relation (4.11) between $U^{(I)}(\widetilde{W}, v, i; G_{n_k})$ and $\widetilde{V}^{(I)}(\widetilde{W}, v, i; G_{n_k})$, and performing the computation on a set of nodes, we can obtain a recursive equation (4.20a) for $U^{(I)}(\widetilde{W}, v, i; G_{n_k})$. Likewise, $\widetilde{V}^{(A)}(\widetilde{W}, v, i)$ can be derived easily by combining eqs. (4.5) and (4.16) when $\widetilde{W}_i > \eta_b$. Therefore, an application of eq. (4.11) gives the result for $\widetilde{U}^{(A)}(\widetilde{W}, v, i)$ in eq. (4.23a).

At time $T-1$, based on eqs. (4.4) and (4.18), one can easily obtain the closed form representation for the normalized value function as follows

$$
\begin{aligned}
&\widetilde{V}^{(I)}(\widetilde{W}, v, T-1; G_{n_k}) \\
&= p_{T-2}q_{T-2}\widetilde{W}_{T-1} + \max\Big\{ p_{T-1}G_{n_k} + p_{T-1}(1-\kappa_{T-1})(\widetilde{W}_{T-1} - \eta_b - G_{n_k}), \\
&\quad p_{T-1}(\widetilde{W}_{T-1} - \eta_b)^+, \quad p_{T-1}G_{n_k} + p_{T-1}(\widetilde{W}_{T-1} - \eta_b - G_{n_k})^+ \Big\} \\
&= p_{T-1}G_{n_k} + p_{T-1}(\widetilde{W}_{T-1} - \eta_b - G_{n_k})^+.
\end{aligned}
$$

We then have

$$
\begin{aligned}
U^{(I)}(\widetilde{W}, v, T-1; G_{n_k}) &= \widetilde{V}^{(I)}(\widetilde{W}, v, T-1; G_{n_k}) - \left(p_{T-2}q_{T-2}\widetilde{W}_{T-1} + g^{(I)}(T-1; G_{n_k})\right) \\
&= p_{T-1}(\widetilde{W}_{T-1} - \eta_b - G_{n_k})^+,
\end{aligned}
$$

leading to eq. (4.19) for $\widehat{U}^{(I)}(mh, e^{\zeta_j}, T-1; G_{n_k})$.

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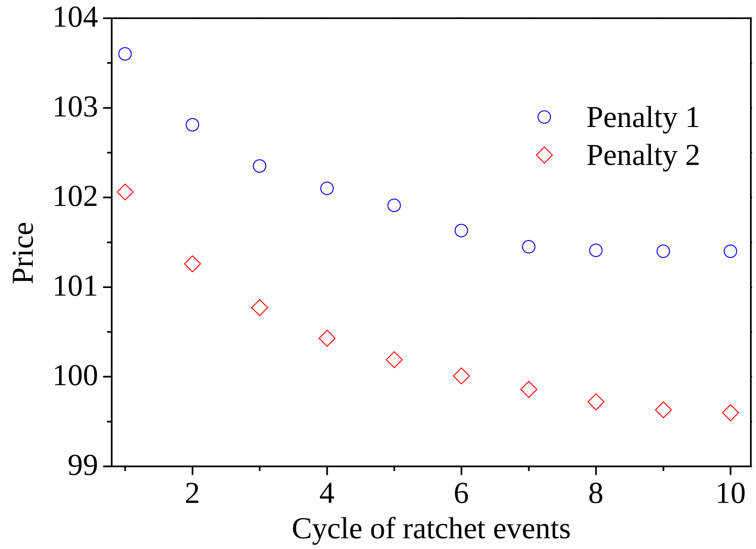

Figure 1: Plot of the price of GLWB against the cycle of ratchet events under two penalty schemes: Penalty 1 and Penalty 2.

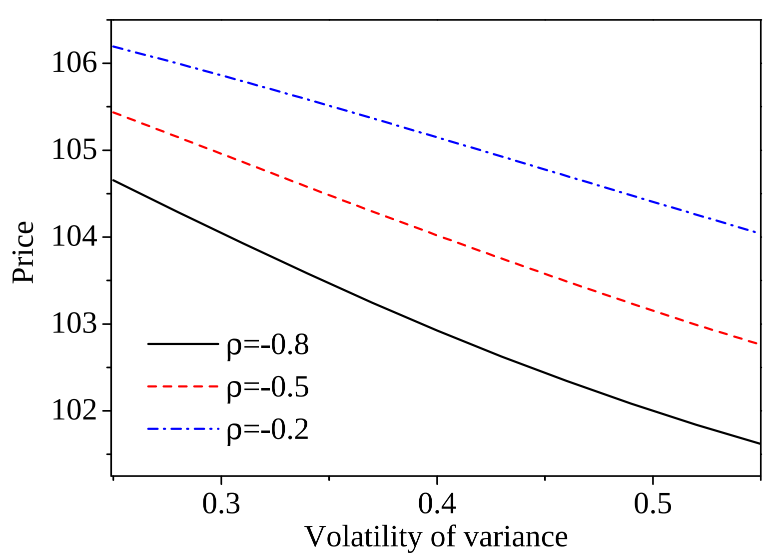

Figure 2: Plot of the price of GLWB against the volatility of variance ϵ under varying values of correlation coefficient ρ.

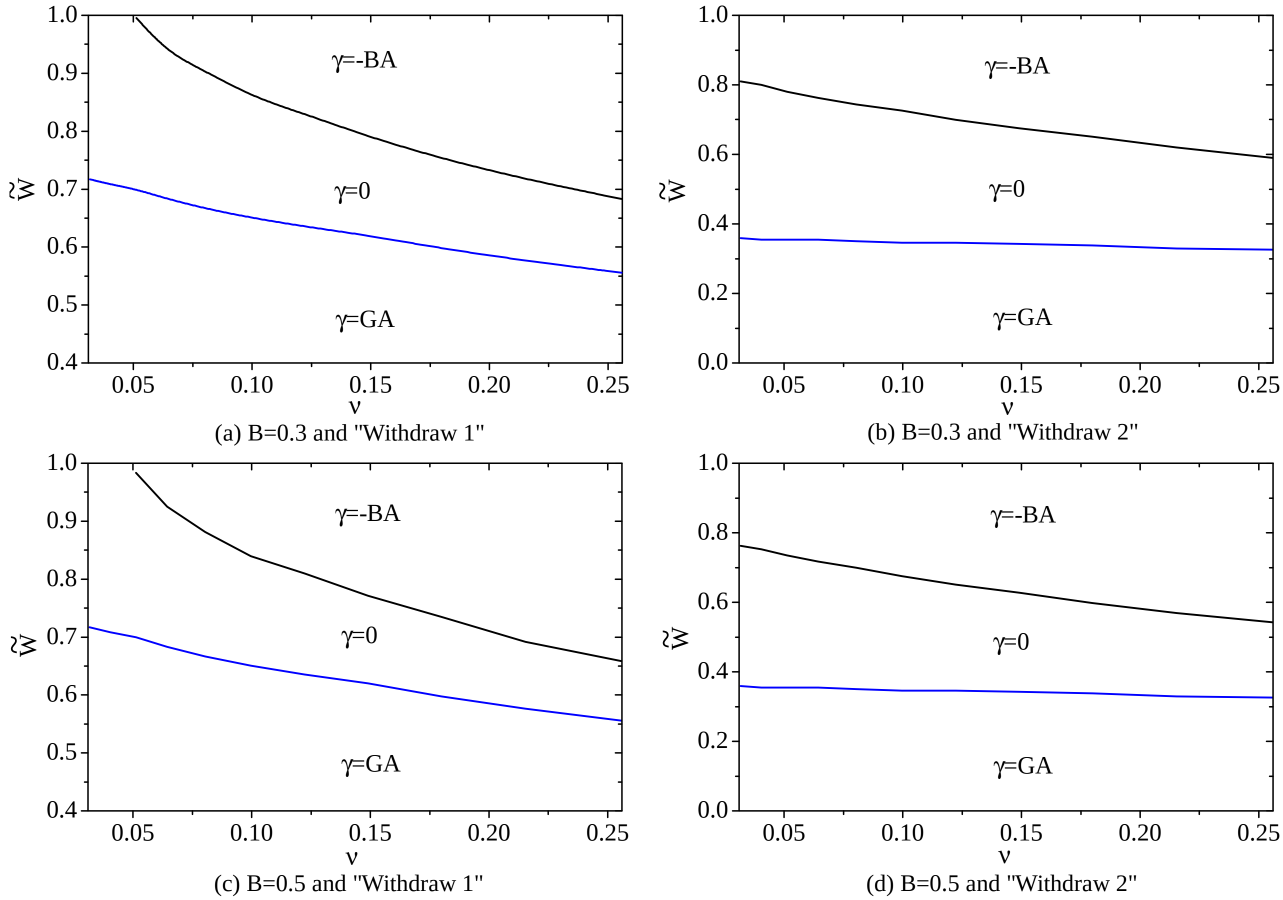

Figure 3: Plots of the optimal withdrawal boundaries for GLWB at time $t = 1$ under different values for the contractual withdrawal rate and upper bound of additional purchase. Here, "Withdraw 1" and "Withdraw 2" are depicted in Table 4, $T_a = 20$, and other model and contract parameter values are shown in Tables 1 and 2.

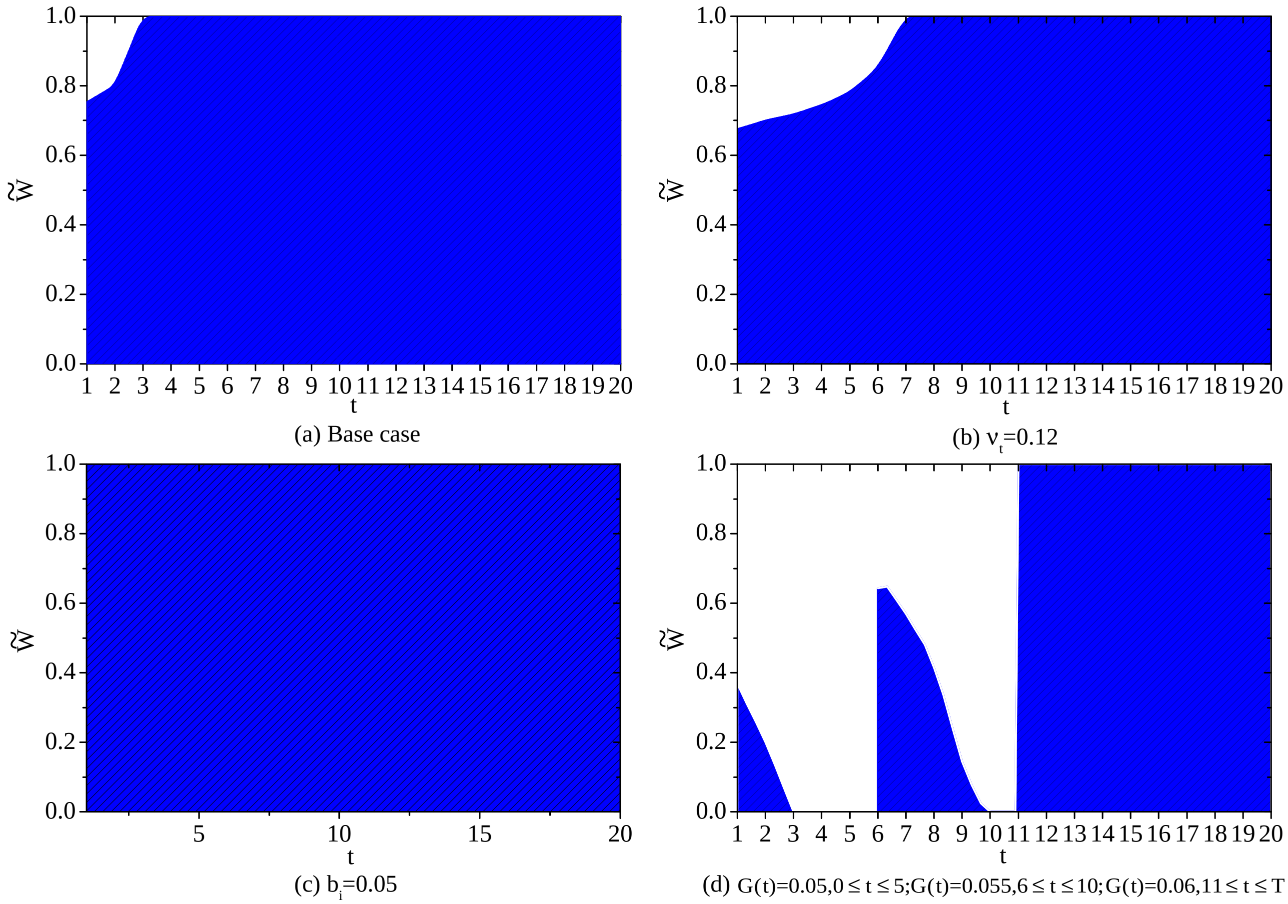

Figure 4: Plots of the optimal initiation regions in the $\widetilde{W}$-t plane under varying values of the contractual withdrawal rate $G(t)$, bonus rate b_i and variance v_t. In the "Base case" shown in Figure 4a, we choose $b_i = 0.06$, $v_t = 0.04$, $G(t) = 0.05$. We modify one parameter from the "Base case" as labelled in each of Figures 4b-4d. Here, $B = 0.3$, $T_a = 20$, and the other parameter values are shown in Tables 1 and 2.

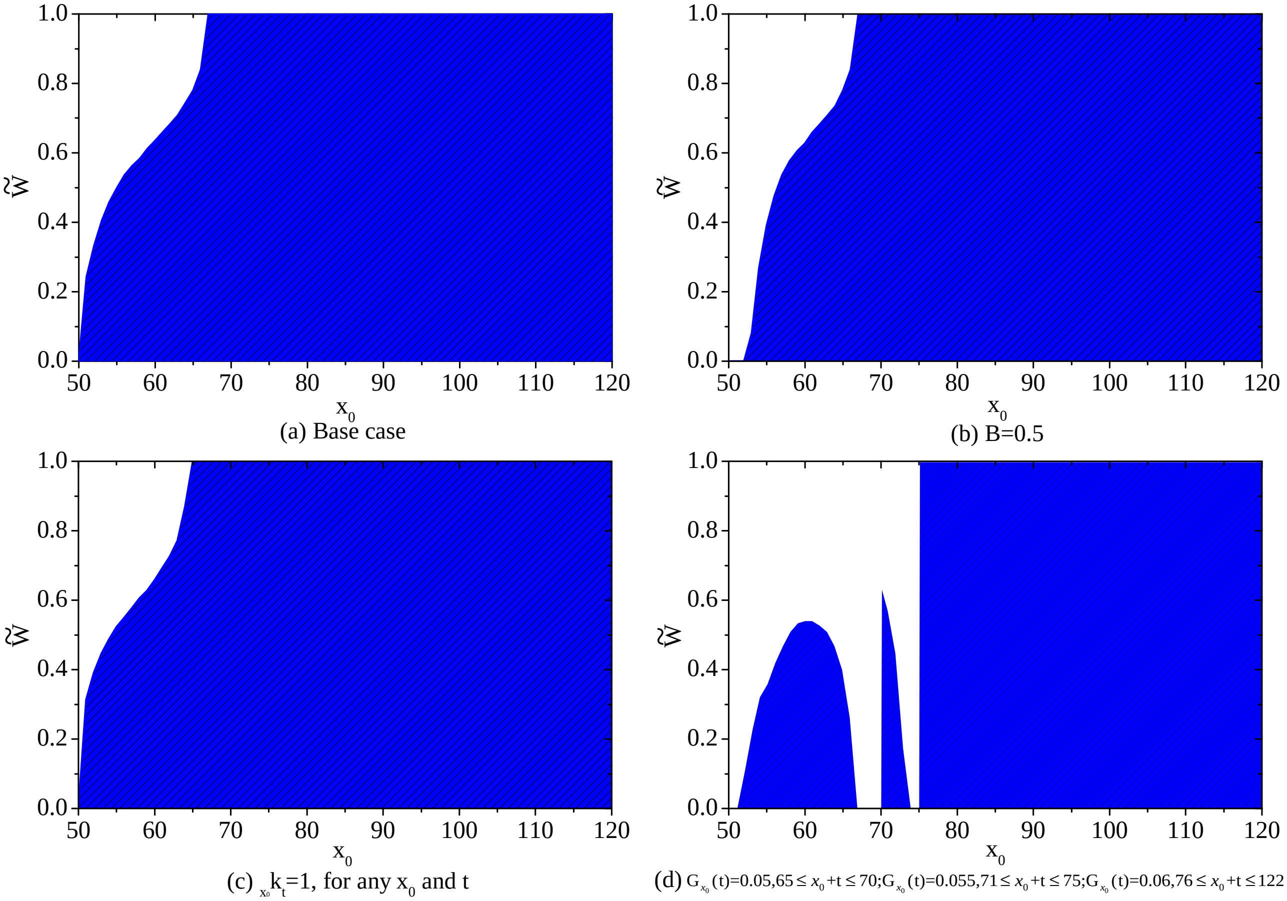

Figure 5: Plots of the optimal initiation region in the $\widetilde{W}$-x_0 plane on the first withdrawal date under varying values of the contractual withdrawal rate $G_{x_0}(t)$, additional purchase parameter B and penalty charge rate $_{x_0}k_t$. In the "Base case" shown in Figure 5(a), we choose $B = 0.3$, $G_{x_0}(t) = 0.05$ and the penalty scheme is taken from Table 8. We modify one parameter from the "Base case" as labelled in each of Figures 5b-5d. Here, $T_a = 20$, $v_t = 0.04$ and the other parameter values are shown in Tables 1, 2 and 8.