Saddlepoint Approximation Methods for Pricing VIX Derivatives and Options on Realized Variance

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Agenda

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   - Steepest descent approach
   - Lugannani-Rice formula for approximating tail probability: $P[X \geq K]$
   - Exponential tilting approach for approximating tail expectation: $E[(X - K)^+]$
2. Pricing VIX derivatives
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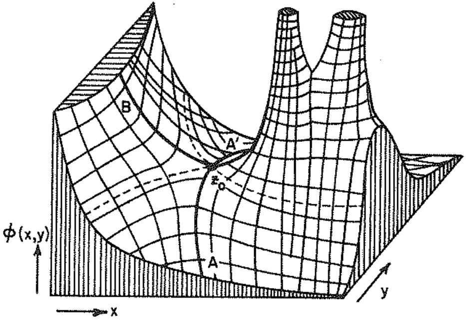
   - Approximate analytic pricing under affine stochastic volatility models with jumps
3. Pricing options on discrete realized variance
   - Analytic approximation based on small time asymptotic approximation of the Laplace transform of discrete realized variance

Reference

Saddlepoint Approximation Methods in Financial Engineering, Y.K. Kwok and W.D. Zheng, Springer (2018).

Saddlepoint point

Inside the domain of analyticity of $h(z) = \phi(x, y) + i\psi(x, y)$, $z = x + iy$, any stationary point of $\phi(x, y) = \text{Re } h(z)$, where $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial y} = 0$, is a saddlepoint.



Topography of the surface $\phi = \text{Re}h(z)$ near the saddle point z_0, for a typical function $h(z)$. The heavy solid curves follow the centers of the ridges and valleys from the saddle point, and the dashed curves follow level contours, $\phi = \phi(x_0, y_0) =$ constant. The curve AA' is the path of steepest descent.

Complex moment generating functions and cumulant generating functions

The complex moment generating function of a random variable X is defined by

$$M(z) = \int_{-\infty}^{\infty} e^{zu} f(u)\, du, \quad \text{where } z = x + iy,$$

where $f(u)$ is the density function of X. This may be considered as the generalized bilateral Laplace transform of $f(u)$.

(i) When $y = 0$, we recover the real moment generating function $M(x)$.

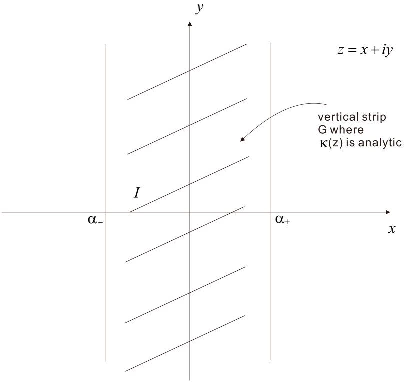
(ii) When $x = 0$, $M(iy) = \int_{-\infty}^{\infty} e^{iyu} f(u)\, du$ is the characteristic function.

We take $M(z)$ to be analytic in some open vertical strip G containing the imaginary axis. In this way, the generalized characteristic function is also analytic.

The cumulant generating function $\kappa(z)$ is $\log M(z)$.

Domain of analyticity of $\kappa(z)$

$\kappa(z)$ is assumed to be analytic in some open vertical strip $G = \{z : \alpha_- < \text{Re}(z) < \alpha_+\}$ in the complex plane that contains the imaginary axis, where $\alpha_- < 0$ and $\alpha_+ > 0$; and both α_- and α_+ can be infinite.



The Bromwich path is a vertical infinite line segment that lies completely inside the domain of analyticity G.

$\kappa_X(z)$ and tail probabilities and expectation

Let X be any random variable with cumulant generating function $\kappa_X(z)$. For any $K \in \mathbb{R}$, the density, tail probability, tail expectation and expected shortfall of X are given by the following Laplace integrals:

Density

$$p_X(K) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\kappa_X(z) - zK} \, \mathrm{d}z, \qquad \gamma \in (\alpha_-, \alpha_+);$$

Tail probability

$$P[X > K] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\kappa_X(z) - zK}}{z} \, \mathrm{d}z, \qquad \gamma \in (0, \alpha_+);$$

Tail expectation (call option pricing)

$$E\left[(X - K)^+\right] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\kappa_X(z) - zK}}{z^2} \, \mathrm{d}z, \qquad \gamma \in (0, \alpha_+);$$

Expected shortfall (risk measure in credit portfolios)

$$E\left[X\mathbf{1}_{\{X > K\}}\right] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \kappa_X'(z) \frac{e^{\kappa_X(z) - zK}}{z} \, \mathrm{d}z, \qquad \gamma \in (0, \alpha_+).$$

Relation between tail expectation and expected shortfall

$$\begin{aligned} E[(X-K)^+] &= E[X\mathbb{1}_{\{X>K\}}] - KP[X>K] \\ &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{\kappa_X(z)-zK}}{z}[\kappa_X'(z)-K]\,dz \\ &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{\kappa_X(z)-zK}}{z}(\mu-K)\,dz + \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\kappa_X(z)-zK}\frac{\kappa_X'(z)-\mu}{z}\,dz. \end{aligned}$$

In the last three Bromwich integrals, the choice of $\gamma > 0$ or $\gamma < 0$ would require adjustment due to the contribution from the isolated singularity $z = 0$.

For example, the modified tail expectation formula is

$$E\left[(X-K)^+\right] = \kappa_X''(0) - K + \int_{\hat{\gamma}-i\infty}^{\hat{\gamma}+i\infty} \frac{e^{\kappa_X(z)-zK}}{z^2}\,\mathrm{d}z,\ \hat{\gamma}\in(\gamma_-,0).$$

The saddlepoint approximation approach is used to derive approximation formulas for the above Bromwich integrals.

We derive an asymptotic expansion of the complex integral

$$I(\lambda) = \int_C h(z)e^{\lambda f(z)}\, \mathrm{d}z,$$

where C is a contour in the complex plane, $h(z)$ and $f(z)$ are analytic functions in a domain $\mathcal{D}$ containing C. Recall that the zeros of $f'(z)$ are the saddlepoints of $f(z)$. Here, λ is taken to be real positive and large in value.

As λ is positive and large, we expect that the value of $I(\lambda)$ is dominated by the saddlepoint point with the largest value in Re f. Let that be the simple saddlepoint z_0.

We deform C to $\widetilde{C}$ that passes through the saddlepoint z_0. For z on $\widetilde{C}$ that is near z_0, we approximate $f(z)$ and $h(z)$ by

$$f(z) \approx f(z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 \quad \text{and} \quad h(z) \approx h(z_0).$$

The choice of the local quadratic approximation of $f(z)$ at z_0 implicitly implies the use of the Gaussian base distribution as an approximation.

On the deformed path $\widetilde{C}$ that passes z_0, we write

$$z - z_0 = re^{i\theta} \quad \text{and} \quad f''(z_0) = |f''(z_0)|e^{i\psi},$$

So that $f''(z_0)(z - z_0)^2 = \left|f''(z_0)\right| e^{i(2\theta+\psi)}$. We choose θ along the steepest descent path where $e^{i(2\theta+\psi)} = -1$ or $\theta = -\frac{\psi}{2} \pm \frac{\pi}{2}$. For large positive λ and on the steepest descent path, the dominant contribution to the integral can be computed by a local computation in the neighborhood of z_0 since the modulus of $e^{\lambda f(z)}$ is negligible elsewhere.

We extend the limits of integration with respect to r to be infinity. As $\lambda \to \infty$, we have

$$I(\lambda) \approx h(z_0)e^{\lambda f(z_0)} \int_{-\infty}^{\infty} e^{\frac{\lambda}{2}|f''(z_0)|r^2 e^{i(2\theta+\psi)}} e^{i\theta} \, \mathrm{d}r.$$

By following the steepest descent path and evaluating the Gaussian integral, we obtain the asymptotic expansion

$$I(\lambda) \approx h(z_0)e^{\lambda f(z_0)}e^{i\theta} \int_{-\infty}^{\infty} e^{-\frac{\lambda}{2}|f''(z_0)|r^2} \, \mathrm{d}r = h(z_0)e^{\lambda f(z_0)}e^{i\theta}\sqrt{\frac{2\pi}{\lambda|f''(z_0)|}}.$$

The Lugannani-Rice saddlepoint approximation formula for $P[X > x]$ is given by

$$P(X > x) \approx \begin{cases} 1 - \Phi(\hat{w}) + \phi(\hat{w})\left[\dfrac{1}{\hat{z}\sqrt{\kappa''(\hat{z})}} - \dfrac{1}{\hat{w}}\right], & x \neq E[X] = \kappa'(0) \\ \dfrac{1}{2} - \dfrac{1}{6\sqrt{2\pi}}\dfrac{\kappa'''(0)}{\kappa''(0)^{3/2}}, & x = E[X] = \kappa'(0) \end{cases}$$

where $\hat{z}$ solves the saddlepoint equation:

$$\kappa'(z) = x,$$

and $\hat{w} = \mathrm{sgn}(\hat{z})\sqrt{2[\hat{z}x - \kappa(\hat{z})]}$. Here, $\Phi(\cdot)$ and $\phi(\cdot)$ are the normal distribution and normal density, respectively.

Note that when $x = E[X] = \kappa'(0)$, we have $\hat{z} = \hat{w} = 0$. The alternative formula under this scenario is derived by taking the asymptotic limits: $\hat{z} \to 0$ and $\hat{w} \to 0$.

Remark The formula is derived using the Gaussian base distribution approximation of X around the saddlepoint. It can be generalized to other base distributions that may fit the underlying base distribution better.

Proof. Provided that the saddlepoint $\hat{z}$ exists in $(0, \alpha_+)$, we choose the Bromwich path to pass through $\hat{z}$ and observe

$$P[X > x] = \frac{1}{2\pi i} \int_{\hat{z}-i\infty}^{\hat{z}+i\infty} \frac{e^{\kappa(z)-zx}}{z} \, \mathrm{d}z.$$

Under the choice of the Gaussian base distribution (quadratic polynomial approximation of mgf), we set

$$[\kappa(z) - zx] - [\kappa(\hat{z}) - \hat{z}x] = \frac{(w - \hat{w})^2}{2}$$

and

$$\frac{\hat{w}^2}{2} = -[\kappa(\hat{z}) - \hat{z}x].$$

so the two functions match their values at $w = 0$ with $z = 0$ and $w = \hat{w}$ with $z = \hat{z}$.

Also, we force $\hat{w}$ and $\hat{z}$ to have the same sign, so $\hat{w} = \mathrm{sgn}(\hat{z})\sqrt{2[\hat{z}x - \kappa(\hat{z})]}$.

We adopt the change of variable from z to w where $\kappa(z) - zx = \frac{w^2}{2} - w\hat{w}$. This gives

$$P[X > x] = \frac{1}{2\pi i} \int_{\hat{w}-i\infty}^{\hat{w}+i\infty} e^{\frac{w^2}{2} - w\hat{w}} \frac{1}{z(w)} \frac{\mathrm{d}z(w)}{\mathrm{d}w} \, \mathrm{d}w.$$

Since $z = 0$ when $w = 0$, the term $\frac{1}{z(w)} \frac{\mathrm{d}z(w)}{\mathrm{d}w}$ has a singularity at $w = 0$. Near $z = \hat{z}$ and $w = \hat{w}$, it can be shown that z and w have linear relationship.

We perform the following decomposition to isolate the singularity at $w = 0$, where

$$P[X > x] = \frac{1}{2\pi i}\int_{\hat{w}-i\infty}^{\hat{w}+i\infty} \frac{e^{\frac{w^2}{2}-w\hat{w}}}{w}\,\mathrm{d}w + \frac{1}{2\pi i}\int_{\hat{w}-i\infty}^{\hat{w}+i\infty} e^{\frac{w^2}{2}-w\hat{w}}\left[\frac{1}{z(w)}\frac{\mathrm{d}z(w)}{\mathrm{d}w} - \frac{1}{w}\right]\mathrm{d}w.$$

The value of the first integral is known to be $1 - \Phi(\hat{w})$, where $\Phi(\cdot)$ is the standard normal distribution function.

For the second integral, the integrand is analytic in the vertical strip in the complex plane. We consider the local approximation at $z = \hat{z}$ (or equivalent $w = \hat{w}$), where

$$\frac{1}{z(w)}\frac{\mathrm{d}z(w)}{\mathrm{d}w} - \frac{1}{w} \approx \frac{1}{\hat{z}}\frac{\mathrm{d}z(\hat{w})}{\mathrm{d}w} - \frac{1}{\hat{w}} = \frac{1}{\hat{z}}\frac{1}{\sqrt{\kappa''(\hat{z})}} - \frac{1}{\hat{w}}.$$

We differentiate the transformation relation twice and substitute $w = \hat{w}$, which gives

$$\left.\frac{\mathrm{d}z}{\mathrm{d}w}\right|_{w=\hat{w}} = \frac{1}{\sqrt{\kappa''(\hat{z})}}.$$

Esscher exponential tilting technique

We apply the exponential tilting technique to find the relation between tail expectation and tail probability. The distribution $F(x;\theta)$ of the θ-tilted distribution of X is related to $F(x)$ by

$$\mathrm{d}F(x;\theta) = e^{\theta x - \kappa(\theta)}\,\mathrm{d}F(x).$$

The cgf of the Q-tilted distribution $\kappa_\theta(z)$ is related to $\kappa(z)$ via

$$\kappa_\theta(z) = \kappa(z+\theta) - \kappa(\theta).$$

When $\theta = 0$, we observe $\kappa(0) = 0$ and recover $F(x;0) = F(x)$. Also, we have

$$1 - F(K;\theta) = \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} \frac{e^{\kappa(z+\theta)-\kappa(\theta)-zK}}{z}\,\mathrm{d}z, 0 < \xi < \gamma_+ - \theta.$$

Tail expectation and derivative of the θ-tilted distribution function

Differentiating both sides of the above equation with respect to θ and setting $\theta = 0$, we obtain

$$-\left.\frac{\partial F(K;\theta)}{\partial \theta}\right|_{\theta=0} = \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} \frac{e^{\kappa(z)-zK}}{z}[\kappa'(z) - \kappa'(0)]\, \mathrm{d}z.$$

Combining these results, the tail expectation is given by

$$\begin{aligned} E[(X-K)^+] &= [\kappa'(0) - K][1 - F(K)] - \frac{\partial F(K;\theta)}{\partial \theta}\Big|_{\theta=0} \\ &\approx (\mu - K)P[X > K] - \frac{\partial \tilde{F}(K;\theta)}{\partial \theta}\Big|_{\theta=0}, \end{aligned}$$

where we approximate $F(K;\theta)$ by $\tilde{F}(K;\theta)$, taken to be the saddlepoint approximation to the distribution function of the exponentially Q-tilted distribution.

Extension to any arbitrary base distribution

We extend the saddlepoint approximation formula for tail expectation to an arbitrary base distribution, whose cgf is denoted by $\kappa_0(z)$. We apply the modified Legendre-Fenchel transformation based on $\kappa_0(z)$, where the transformation of variables from z to w is defined by

$$\kappa_0(w) - w\kappa_0'(\hat{w}) = \kappa(z) - zK,$$

where $\hat{w}$ is first determined via the solution of the following equation

$$\kappa_0(w) - w\kappa_0'(w) = \kappa(\hat{z}) - \hat{z}K.$$

Here, $\hat{z}$ is the saddlepoint that solves $\kappa'(z) = K$.

It is necessary to compute $\frac{\partial \tilde{F}(K;\theta)}{\partial \theta}$, where $\tilde{F}(K;\theta)$ is the saddlepoint approximation to the distribution function of the exponentially θ-tilted distribution using the base distribution with cgf $\kappa_0(z)$.

Let $f_0(x)$ and $F_0(x)$ denote the respective density function and distribution function of the non-Gaussian base distribution with cgf $\kappa_0(z)$. For $E[X] \neq K$, the saddlepoint approximation to tail expectation $E[(X-K)^+]$ based on $\kappa_0(z)$ is found to be

$$
\begin{aligned}
& E[(X-K)^+] \\
\approx & [\kappa'(0)-K][1-\tilde{F}(K)] \\
& + f_0\left(\kappa_0'(\hat{w})\right)\left\{[K-\kappa'(0)]\left[\frac{1}{\hat{w}}-\frac{1}{\hat{w}^3\kappa_0''(\hat{w})}-\frac{\kappa_0'''(\hat{w})}{2\hat{w}\kappa_0''(\hat{w})^{\frac{3}{2}}\hat{\mu}}\right]+\frac{\sqrt{\kappa_0''(\hat{w})}}{\hat{z}\hat{\mu}}\right\} \\
& + f_0'\left(\kappa_0'(\hat{w})\right)[K-\kappa'(0)]\left[\frac{1}{\hat{w}^2}-\frac{\sqrt{\kappa_0''(\hat{w})}}{\hat{w}\hat{\mu}}\right], \qquad E[X]\neq K,
\end{aligned}
$$

where $\hat{w}$ is the solution to eq.(2.5), $\hat{\mu}=\hat{z}\sqrt{\kappa''(\hat{z})}$, $\hat{z}$ is the saddlepoint that satisfies $\kappa'(z)=K$ and

$$
\tilde{F}(K)=F_0(\kappa_0'(\hat{w}))+f_0(\kappa_0'(\hat{w}))\left\{\frac{1}{\hat{w}}-\frac{1}{\hat{z}}\left[\frac{\kappa_0''(\hat{w})}{\kappa''(\hat{z})}\right]^{\frac{1}{2}}\right\}.
$$

Here $\tilde{F}(K)$ is the saddlepoint approximation formula of tail probability.

When $E[X] = K$, the above saddlepoint approximation formula becomes degenerate since $\hat{z} = \hat{w} = 0$. By considering the asymptotic limits under $\hat{w} \to 0$ and $\hat{z} \to 0$, the corresponding saddlepoint approximation formula becomes

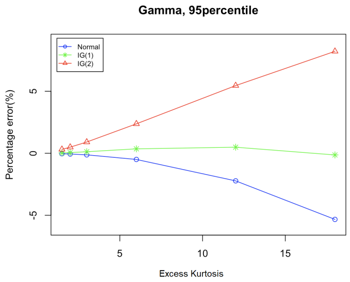
$$E[(X-K)^+] \approx f_0(\kappa_0'(0)) \left\{ \frac{\sqrt{\kappa_0''(0)}}{24\sqrt{\kappa''(0)}} \left[\frac{\kappa'''(0)^2}{\kappa''(0)^2} - \frac{\kappa''''(0)}{\kappa''(0)} \right] + \frac{\sqrt{\kappa''(0)}}{8\sqrt{\kappa_0''(0)}} \left[\frac{\kappa_0'''(0)^2}{\kappa_0''(0)^2} - \frac{\kappa_0''''(0)}{\kappa_0''(0)} \right] + \frac{1}{12} \frac{\kappa_0'''(0)}{\kappa_0''(0)} \frac{\kappa'''(0)}{\kappa''(0)} + \sqrt{\kappa_0''(0)\kappa''(0)} \right\} + \frac{f_0'(\kappa_0'(0))\sqrt{\kappa''(0)}\kappa_0''(0)}{6} \left[\frac{\kappa'''(0)}{\kappa''(0)^{\frac{3}{2}}} - \frac{\kappa_0'''(0)}{\kappa_0''(0)^{\frac{3}{2}}} \right], \qquad E[X] = K.$$

Relative errors for calculating tail expectation

Underlying random variable is Gamma (α, β), where density is

$$f_G(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad \Gamma(\alpha) = \int_0^\infty e^t t^{\alpha-1}\,\mathrm{d}t.$$

Comparison of relative errors for calculating tail expectation against different levels of kurtosis. "Normal" is the relative error using the standard Gaussian base, "IG (1)" and "IG(2)" are the relative errors computed by the Zheng-Kwok formula and Huang-Oosterlee formula using the inverse Gaussian base distribution, respectively.



It is remarkable that the relative errors fall within 0.2% even with excessive kurtosis.

VIX (volatility index) is calculated as 100 times the square root of the expected 30-day variance of the rate of return of the forward price of the S&P 500 index, where

$$\text{VIX} = 100\sqrt{\text{forward price of realized cumulative variance}}.$$

Suppose the forward price F_t of the S&P index under a risk measure Q follows

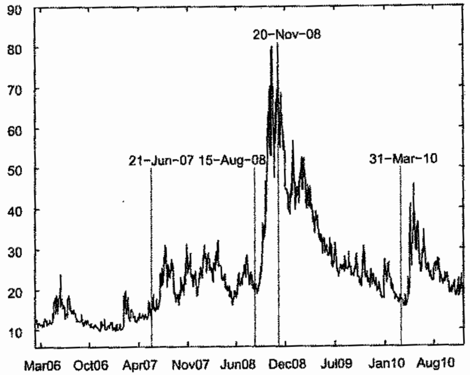
$$\frac{dF_t}{F_t} = \sigma_t \; dW_t \text{ so that } d\ln F_t = -\frac{\sigma_t^2}{2} \; dt + \sigma_t \; dW_t.$$

Subtracting the two equations, we obtain the cumulative variance over $[0, T]$ under continuous time model as follows:

$$\frac{dF_t}{F_t} - d\ln F_t = \frac{\sigma_t^2}{2} \; dt, \text{ so } \int_0^T \sigma_t^2 \; dt = 2\left[\int_0^T \frac{dF_t}{F_t} - \ln\frac{F_T}{F_0}\right].$$

We compute $E_Q\left[\int_0^T \sigma_t^2 \; dt\right]$, visualized as the forward price of the realized cumulative variance of S&P index over $[0, T]$.

Historical evolution of the VIX index (Year 2006-2010)



- The lowest value was 9.89 on Jan. 24, 2007, called "calm before the storm".
- On October 24, 2008, the VIX reached an intraday value of 89.53.
- In January 2010, we witnessed the Greek debt crisis.

As specified by the Chicago Board of Options Exchange, the formal definition of $\text{VIX}(t, t+\hat{\tau})$ is given by

$$\text{VIX}^2(t, t+\hat{\tau}) = \frac{2}{\hat{\tau}} \sum_i \frac{\Delta X_i}{X_i^2} e^{r\hat{\tau}} Q_i(X_i) - \frac{1}{\hat{\tau}} \left[\frac{F_t(t+\hat{\tau})}{X_0} - 1 \right]^2,$$

where $\hat{\tau} = 30/365$, $Q_i(X_i)$ is the price of the out-of-the-money SPX option with strike X_i, and X_0 is the highest strike below the index forward price $F_t(t+\hat{\tau})$. Here, all index options and forward are maturing at $t+\hat{\tau}$.

We consider the Taylor expansion of $\ln \dfrac{S_{t+\hat{\tau}}}{F_t}$ and the integral representation of the remainder term

$$\ln \frac{S_{t+\hat{\tau}}}{F_t} = \frac{S_{t+\hat{\tau}} - F_t}{F_t} - \int_0^{F_t} \frac{1}{K^2}(K - S_{t+\hat{\tau}})^+ \mathrm{d}K - \int_{F_t}^{\infty} \frac{1}{K^2}(S_{t+\hat{\tau}} - K)^+ \mathrm{d}K.$$

By taking the continuous limit of the above discretized sum of the out-of-the-money SPX options, we obtain

$$\text{VIX}_t^2 = -\frac{2}{\hat{\tau}} E_t^Q \left[\ln \frac{S_{t+\hat{\tau}}}{S_t e^{r\hat{\tau}}} \right].$$

Heston affine stochastic volatility models with simultaneous jumps (SVSJ)

Under a risk neutral measure Q, the joint dynamics of stock price S_t and its instantaneous variance V_t under the affine SVSJ model assumes the form

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= (r - \lambda m)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t^S + (e^{J^S} - 1)\,\mathrm{d}N_t, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\,\mathrm{d}t + \varepsilon\sqrt{V_t}\,\mathrm{d}W_t^V + J^V \mathrm{d}N_t, \end{aligned}$$

where W_t^S and W_t^V are a pair of correlated standard Brownian motions with $\mathrm{d}W_t^S \mathrm{d}W_t^V = \rho\,\mathrm{d}t$, and N_t is a Poisson process with constant intensity λ that is independent of the two Brownian motions.

- J^S and J^V denote the random jump sizes of the log price and variance, respectively.
- These random jump sizes are assumed to be independent of W_t^S, W_t^V and N_t.

The above conditional expectation can be shown to be (Zhu and Lian, 2012)

$$\mathrm{VIX}_t^2 = aV_t + b,$$

where

$$a = \frac{1 - e^{-\kappa\hat{\tau}}}{\kappa\hat{\tau}}, \; b = 2\lambda\left[\bar{\mu} - (\mu_S + \rho_J\mu_V)\right] + \left(\theta + \frac{\mu_V\lambda}{\kappa}\right)(1 - a)\,.$$

The time-t price of the T-maturity VIX futures is given by

$$F(V_t, t) = E_t^Q[\mathrm{VIX}_T] = E_t^Q[\sqrt{aV_T + b}].$$

Provided that V_t process is affine, we can derive a system of ordinary differential equations to determine $a(t)$ and $b(t)$.

Let $X_t = \ln S_t$. The joint moment generating function (mgf) of X_t and V_t is defined to be

$$E[\exp(\phi X_T + bV_T + \gamma)],$$

where ϕ, b and γ are constant parameters.
Let $U(X_t, V_t, t)$ denote the non-discounted time-t value of a contingent claim with the terminal payoff function: $U_T(X_T, V_T)$, where T is the maturity date. Let $\tau = T - t$, $U(X, V, \tau)$ is governed by the following partial integro-differential equation (PIDE):

$$\begin{aligned}\frac{\partial U}{\partial \tau} &= \left(r - m\lambda - \frac{V}{2}\right)\frac{\partial U}{\partial X} + \kappa(\theta - V)\frac{\partial U}{\partial V} \\ &+ \frac{V}{2}\frac{\partial^2 U}{\partial X^2} + \frac{\varepsilon^2 V}{2}\frac{\partial^2 U}{\partial V^2} + \rho\varepsilon V\frac{\partial^2 U}{\partial X \partial V} \\ &+ \lambda E\left[U(X + J^S, V + J^V, \tau) - U(X, V, \tau)\right].\end{aligned}$$

Thanks to the affine structure, $U(X, V, \tau)$ admits an analytic solution of the form

$$U(X, V, \tau) = \exp(\phi X + B(\tau, \mathbf{q})V + \Gamma(\tau, \mathbf{q}) + \Lambda(\tau, \mathbf{q})),$$

where $\mathbf{q} = (\phi\ b\ \gamma)^T$.

The parameter functions $B(\tau,\mathbf{q})$, $\Gamma(\tau,\mathbf{q})$ and $\Lambda(\tau,\mathbf{q})$ satisfy the following Riccati system of ordinary differential equations:

$$\frac{\partial B}{\partial \tau} = -\frac{1}{2}(\phi - \phi^2) - (\kappa - \rho\varepsilon\phi)B + \frac{\varepsilon^2}{2}B^2$$
$$\frac{\partial \Gamma}{\partial \tau} = r\phi + \kappa\theta B$$
$$\frac{\partial \Lambda}{\partial \tau} = \lambda\big(E[\exp(\phi J^S + BJ^V) - 1] - m\phi\big)$$

with the initial conditions: $B(0) = b$, $\Gamma(0) = \gamma$ and $\Lambda(0) = 0$.

Suppose we assume that $J^V \sim \exp(1/\eta)$ and J^S follows

$$J^S | J^V \sim \text{Normal}(\nu + \rho_J J^V, \delta^2),$$

which is the Gaussian distribution with mean $\nu + \rho_J J^V$ and variance δ^2, we obtain

$$m = E[e^{J^S} - 1] = \frac{e^{\nu + \delta^2/2}}{1 - \eta\rho_J} - 1,$$

provided that $\eta\rho_J < 1$. Under the above assumptions on J^S and J^V, the parameter functions can be found analytically.

Let $f(z;\tau,V_t)$ denote the marginal mgf of V_T, where $\tau = T - t$ is the time to maturity. Thanks to the affine structure of the governing dynamics equation of V_t, we are able to obtain

$$f(z;\tau,V_t) = E_t^Q\left[e^{zV_T}\right] = e^{B(z;\tau)V_t+\Gamma(z;\tau)+\Lambda(z;\tau)}, \quad \text{Re } z < \alpha_+,$$

where

$$B(z;\tau) = \frac{2\kappa z}{\sigma_V^2(1-e^{\kappa\tau})z + 2\kappa e^{\kappa\tau}}$$

$$\Gamma(z;\tau) = -\frac{2\kappa\theta}{\sigma_V^2}\log\left(1 + \frac{\sigma_V^2 z}{2\kappa}\left(e^{-\kappa\tau}-1\right)\right)$$

$$\Lambda(z;\tau) = \frac{2\lambda\mu_V}{2\kappa\mu_V - \sigma_V^2}\log\left(1 + \frac{z(\sigma_V^2 - 2\kappa\mu_V)}{2\kappa(1-\mu_V z)}(e^{-\kappa\tau}-1)\right),$$

and α_+ is determined by requiring the arguments of the above logarithm terms to be greater than zero.

The corresponding mgf of X is seen to be

$$E\left[e^{zX}\right] = e^{bz}E\left[e^{azV_T}\right] = e^{bz}f(az;\tau,V_t), \quad \text{Re } z < \alpha_+;$$

and the cgf of X is given by

$$\begin{aligned}\kappa(z) &= \log E\left[e^{zX}\right] = bz + \log f(az;\tau,V_t)\\ &= bz + B(az;\tau)V_t + \Gamma(az;\tau) + \Lambda(az;\tau), \quad \text{Re } z < \alpha_+.\end{aligned}$$

To find the saddlepoint approximation of $E[\sqrt{X}]$, we start with the Bromwich integral representation of $E[\sqrt{X}]$. It can be shown that

$$\begin{aligned}E[\sqrt{X}] &= \int_0^\infty \sqrt{x}p(x)\ \mathrm{d}x = \int_0^\infty \sqrt{x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\kappa(z)-zx}\ \mathrm{d}z\mathrm{d}x\\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\kappa(z)}\int_0^\infty \sqrt{x}e^{-zx}\ \mathrm{d}z\mathrm{d}x\\ &= \frac{1}{4\sqrt{\pi}i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\kappa(z)}}{z^{3/2}}\ \mathrm{d}z\\ &= \frac{1}{4\sqrt{\pi}i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\kappa(z)-\frac{3}{2}\log z}\ \mathrm{d}z, \quad \gamma\in(0,\alpha_+).\end{aligned}$$

Saddlepoint equation

The fractional power $\dfrac{1}{z^{3/2}}$ is absorbed into the exponent as $e^{-\frac{3}{2}\log z}$. We find the positive saddlepoint $\hat{z}$ within the domain $(0, \alpha_+)$ that solves the modified saddlepoint equation:

$$\kappa'(z) - \frac{3}{2z} = 0,$$

where

$$\kappa'(z) = b + a\frac{\mathrm{d}B}{\mathrm{d}z}(az;\tau)V_t + a\frac{\mathrm{d}\Gamma}{\mathrm{d}z}(az;\tau) + a\frac{\mathrm{d}\Lambda}{\mathrm{d}z}(az;\tau).$$

The corresponding first order derivatives of B, Γ and Λ are found to be

$$\frac{dB}{dz}(z;\tau) = \frac{4\kappa^2 e^{\kappa\tau}}{[\sigma_V^2(1-e^{\kappa\tau})z + 2\kappa e^{\kappa\tau}]^2}$$

$$\frac{d\Gamma}{dz}(z;\tau) = \frac{2\kappa\theta(e^{\kappa\tau}-1)}{\sigma_V^2(1-e^{\kappa\tau})z + 2\kappa e^{\kappa\tau}}$$

$$\frac{d\Lambda}{dz}(z;\tau) = \frac{2\lambda\mu_V(e^{\kappa\tau}-1)}{\left\{\left[\sigma_V^2(1-e^{\kappa\tau}) - 2\kappa\mu_V\right]z + 2\kappa e^{\kappa\tau}\right\}(1-\mu_V z)}.$$

Taylor expansion of the exponent in the Bromwich integral for VIX futures

Suppose that the positive saddlepoint $\hat{z}$ lies within the domain $(0, \alpha_+)$, we perform the deformation of the Bromwich contour through $\hat{z}$ to give

$$E[\sqrt{X}] = \frac{1}{4\sqrt{\pi}i} \int_{\hat{z}-i\infty}^{\hat{z}+i\infty} e^{\kappa(z) - \frac{3}{2}\log z} \,\mathrm{d}z.$$

The conventional saddlepoint approximation methods work with the approximation of $\kappa(z) - zK$. With the fractional power $z^{3/2}$ in the denominator, we introduce $-3/2 \log z$ into the exponent. The Taylor expansion of the exponent in the above integrand at $z = \hat{z}$ is seen to be

$$\kappa(z) - \frac{3}{2}\log z = \kappa(\hat{z}) - \frac{3}{2}\log \hat{z} + \left[\kappa''(\hat{z}) + \frac{3}{2\hat{z}^2}\right] \frac{(z-\hat{z})^2}{2} + \cdots .$$

Substituting the above Taylor expansion into the Bromwich integral and performing evaluation of the resulting Gaussian integral yields the first order saddlepoint approximation formula:

$$E[\sqrt{X}] \approx \frac{1}{4\sqrt{\pi}i}\int_{\hat{z}-i\infty}^{\hat{z}+i\infty} e^{\kappa(\hat{z})-\frac{3}{2}\log\hat{z}+\left[\kappa''(\hat{z})+\frac{3}{2\hat{z}^2}\right]\frac{(z-\hat{z})^2}{2}}\,\mathrm{d}z$$

$$= \frac{1}{4\sqrt{\pi}}\, e^{\kappa(\hat{z})-\frac{3}{2}\log\hat{z}}\int_{-\infty}^{\infty} e^{-\left[\kappa''(\hat{z})+\frac{3}{2\hat{z}^2}\right]\frac{y^2}{2}}\,\mathrm{d}y = \frac{\sqrt{2}}{4}\frac{e^{\kappa(\hat{z})}/\hat{z}^{3/2}}{\sqrt{\kappa''(\hat{z})+\frac{3}{2\hat{z}^2}}}.$$

By expanding $\kappa(z)-\dfrac{3}{2}\log z$ up to fourth order power in $(z-\hat{z})$, the corresponding second order saddlepoint approximation formula is given by

$$E[\sqrt{X}] \approx \frac{\sqrt{2}}{4}\frac{e^{\kappa(\hat{z})}/\hat{z}^{3/2}}{\sqrt{\kappa''(\hat{z})+\frac{3}{2\hat{z}^2}}}(1+R),$$

where

$$R = \frac{1}{8}\frac{\kappa''''(\hat{z})+\frac{9}{\hat{z}^4}}{\left[\kappa''(\hat{z})+\frac{3}{2\hat{z}^2}\right]^2} - \frac{5}{24}\frac{\left[\kappa'''(\hat{z})-\frac{3}{\hat{z}^3}\right]^2}{\left[\kappa''(\hat{z})+\frac{3}{2\hat{z}^2}\right]^3}.$$

With symbolic computer languages, it is still manageable to perform evaluation of higher order derivatives at $\hat{z}$.

Maturity τ (year)	0.2	0.4	0.6	0.8	1
SPA1	15.2739	16.0059	16.7024	17.3687	18.0087
(PE%)	(4.7429)	(3.9977)	(3.3728)	(2.8444)	(2.3926)
SPA2	14.5473	15.3521	16.1141	16.8390	17.5316
(PE%)	(-0.2399)	(-0.2504)	(-0.2682)	(-0.2918)	(-0.3203)
NI	14.5823	15.3906	16.1574	16.8883	17.5879

Fair values of VIX futures with varying maturities under the SVJJ model.

The percentage errors (PE%) using the second order saddlepoint approximation formula is well within 0.4%.

VIX options

The VIX call option price is

$$\begin{aligned} C(V_t, t) &= e^{-r(T-t)} E_t^Q\left[(VIX_T - K)^+\right] \\ &= e^{-r(T-t)} E_t^Q\left[(\sqrt{aV_T + b} - K)^+\right] \\ &= e^{-r(T-t)} E_t^Q\left[(\sqrt{X} - K)^+\right]. \end{aligned}$$

Let X be a nonnegative random variable, whose cgf as denoted by $\kappa(z)$ is analytic in some open vertical strip $\{z \in \mathbb{C} : \alpha_- < \text{Re } z < \alpha_+\}$, where $\alpha_- < 0$ and $\alpha_+ > 0$. The Bromwich integral representation of $E[(\sqrt{X} - K)^+]$ is given by

$$\begin{aligned} E\left[(\sqrt{X} - K)^+\right] &= \int_{K^2}^{\infty} (\sqrt{x} - K) p(x) \, \mathrm{d}x \\ &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\kappa(z)} \left(\int_{K^2}^{\infty} \sqrt{x} e^{-zx} \, \mathrm{d}x - \int_{K^2}^{\infty} K e^{-zx} \, \mathrm{d}x \right) \mathrm{d}z \\ &= \frac{1}{4\sqrt{\pi} i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\kappa(z)} \left[1 - \text{erf}(\sqrt{z} K)\right]}{z^{3/2}} \, \mathrm{d}z, \quad \gamma \in (0, \alpha_+), \end{aligned}$$

where $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2} \, \mathrm{d}t$.

Saddlepoint formula for VIX options

To derive the saddlepoint approximation formula, we start by expressing the integrand of the Bromwich integral as an exponential function as follows:

$$E\left[(\sqrt{X}-K)^+\right] = \frac{1}{4\sqrt{\pi}i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\kappa(z)+g(z)-\frac{3}{2}\log z}\,\mathrm{d}z,$$

where $g(z) = \log\left(1-\mathrm{erf}(\sqrt{z}K)\right)$. The form of the Bromwich integral of the VIX call option is similar to that of the VIX futures, except with an additional term $g(z)$. Accordingly, the modified saddlepoint equation is

$$\kappa'(z) + g'(z) - \frac{3}{2z} = 0,$$

where

$$g'(z) = -\frac{Ke^{-zK^2}}{\sqrt{\pi}\sqrt{z}\left[1-\mathrm{erf}(\sqrt{z}K)\right]}.$$

Suppose that there exists a saddlepoint $\hat{z}$ that lies within the domain $(0, \alpha_+)$, by deforming the contour to the vertical line that passes through $\hat{z}$, we obtain an alternative integral representation as follows:

$$E\left[(\sqrt{X} - K)^+\right] = \frac{1}{4\sqrt{\pi}i} \int_{\hat{z}-i\infty}^{\hat{z}+i\infty} e^{\kappa(z)+g(z)-\frac{3}{2}\log z}\,\mathrm{d}z, \quad \hat{z} \in (0, \alpha_+).$$

The time-t price of the VIX call option is

$$C(V_t, t) \approx e^{-r(T-t)} \frac{\sqrt{2}}{4} \frac{e^{\kappa(\hat{z})+g(\hat{z})} \Big/ \hat{z}^{3/2}}{\sqrt{\kappa''(\hat{z}) + g''(\hat{z}) + \frac{3}{2\hat{z}^2}}}.$$

One may apply similar procedure as that of the VIX futures to derive the second order saddlepoint approximation formula.

Numerical tests for pricing VIX options

Maturity τ (year)			0.4		
SPA1	2.8726	2.1803	1.5219	0.9360	0.4789
(PE%)	(-4.2284)	(-6.1146)	(-7.9000)	(-8.8180)	(-8.8189)
SPA2	2.9649	2.2841	1.6184	1.0051	0.5130
(PE%)	(-1.1522)	(-1.6461)	(-2.0591)	(-2.0921)	(-2.3188)
NI	2.9994	2.3223	1.6524	1.0266	0.5252
Maturity τ (year)			0.6		
SPA1	3.5588	2.8699	2.2030	1.5785	1.0279
(PE%)	(-4.6550)	(-6.1970)	(-7.7834)	(-9.0078)	(-9.5155)
SPA2	3.6857	3.0081	2.3384	1.6937	1.1089
(PE%)	(-1.2574)	(-1.6798)	(-2.1179)	(-2.3637)	(-2.3841)
NI	3.7326	3.0595	2.3890	1.7347	1.1360

Compared to VIX futures, the numerical accuracy of the saddlepoint approximation formulas for pricing VIX options is less promising.

The terminal payoff of a put option on the discrete realized variance of asset price is given by $\max(K - V_d(0, T; N), 0)$, where K is the strike price and

$$V_d(0, T; N) = \frac{A}{N} \sum_{i=1}^{N} \left(\ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2.$$

The annualized factor A is taken to be 252 for daily monitoring and $\{t_0, t_1, \ldots, t_N\}$ is the set of monitoring instants.

The moment generating function of $V_d(0, T; N)$ cannot be found readily under stochastic volatility models, unlike that of the continuous realized variance.

Though the analytic expression of the cgf is not available, we deduce a useful analytic approximation using the small time asymptotic approximation of the Laplace transform of the discrete realized variance as a control.

Laplace integrals

Let $\kappa(\theta)$ and $\kappa_0(\theta)$ denote the CGF of the random discretely sampled realized variance I and $I-K$, respectively, where K represents the fixed strike. The two CGFs are related by $\kappa_0(\theta)=\kappa(\theta)-K\theta$. We write $X=I-K$ and $F_0(x)$ as the distribution function of X.

Recall that the Fourier transform of the payoff of a call option on discrete realized variance $E[(I-K)\mathbf{1}_{\{I>K\}}]=E[X\mathbf{1}_{\{X>0\}}]$ is $\frac{e^{\kappa_0(t)}}{t^2}$. Consider the following tail expectations expressed in terms of the following Laplace integrals:

$$
\begin{aligned}
\Xi_1 &= E[X\mathbf{1}_{\{X>0\}}]=\frac{1}{2\pi i}\int_{\tau_1-i\infty}^{\tau_1+i\infty}\frac{e^{\kappa_0(t)}}{t^2}\,dt, \quad \tau_1\in(0,\alpha_+) \text{ where } \alpha_+>0;\\
\Xi_2 &= -E[X\mathbf{1}_{\{X<0\}}]=\frac{1}{2\pi i}\int_{\tau_2-i\infty}^{\tau_2+i\infty}\frac{e^{\kappa_0(t)}}{t^2}\,dt, \quad \tau_2\in(\alpha_-,0) \text{ where } \alpha_-<0.
\end{aligned}
$$

The contour is taken to be along a vertical line parallel to the imaginary axis. We write the integrand as $e^{\kappa_0(t)-2\ln t}$.

The first order saddlepoint approximation to Ξ_j, $j = 1, 2$, is given by

$$\Xi_j \approx \hat{\hat{\Xi}}_j = \frac{e^{\kappa_0(\hat{t}_j)}/\hat{t}_j^2}{\sqrt{2\pi \left[\frac{2}{\hat{t}_j^2} + \kappa_0^{(2)}(\hat{t}_j)\right]}}, \ j = 1, 2,$$

where $\hat{t}_1 > 0$ ($\hat{t}_2 < 0$) is the positive (negative) root in (α_-, α_+) of the saddlepoint equation:

$$\kappa_0'(t) - 2/t = 0.$$

Note that $\Xi_1 - \Xi_2 = \mu_X$, which is consistent with the put-call parity in option pricing theory.

Suppose both roots $\hat{t}_1$ and $\hat{t}_2$ exist, we can use either the saddlepoint approximation $\hat{\hat{\Xi}}_1$ ($\tilde{\tilde{\Xi}}_1$) or $\mu_X + \hat{\hat{\Xi}}_2$ ($\mu_X + \tilde{\tilde{\Xi}}_2$) to approximate the value of the call option.

By performing the Taylor expansion of $\kappa_0(t) - 2\ln t$ up to the fourth order, we manage to derive the second order saddlepoint approximation formulas. The second order saddlepoint approximation to Ξ_j is given by

$$\tilde{\hat{\Xi}}_j = \hat{\Xi}_j(1 + R_j), \quad j = 1, 2,$$

where the adjustment term R_j is given by

$$R_j = \frac{1}{8}\frac{\kappa_0^{(4)}(\hat{t}_j) + 12\hat{t}_j^{-4}}{[\kappa_0^{(2)}(\hat{t}_j) + 2\hat{t}_j^{-2}]^2} - \frac{5}{24}\frac{[\kappa_0^{(3)}(\hat{t}_j) - 4\hat{t}_j^{-3}]^2}{[\kappa_0^{(2)}(\hat{t}_j) + 2\hat{t}_j^{-2}]^3}, \; j = 1, 2.$$

We consider the small time asymptotic approximation to the MGFs of the quadratic variation process $I(0,T;\infty) = \frac{1}{T}[\ln S_T, \ln S_T]$ and discretely sampled realized variance $I(0,T;N)$. The asymptotic limit of $V_d(0,T;N)$ is gamma distributed with shape parameter $N/2$ and scale parameter $2V_0/N$.

For any $u \leq 0$, their mgfs are found to be

$$\lim_{T\to 0^+} M_{I(0,T;\infty)}(u) = e^{uV_0},$$

$$\lim_{T\to 0^+} M_{I(0,T;N)}(u) = \left(1 - \frac{2V_0 u}{N}\right)^{-N/2}.$$

The difference $M_{I(0,T;N)}(u) - M_{I(0,T;\infty)}(u)$ is seen to be almost invariant with respect to T. We use the above difference as a control and propose the following approximate MGF formula:

$$\hat{M}_{I(0,T;N)}(u) = M_{I(0,T;\infty)}(u) + \left(1 - \frac{2V_0 u}{N}\right)^{-N/2} - e^{uV_0}, \quad u \in \mathbb{C}_-.$$

Under an affine stochastic volatility model, $M_{I(0,T;\infty)}(u)$ can be derived analytically by solving a Riccati system of equations.

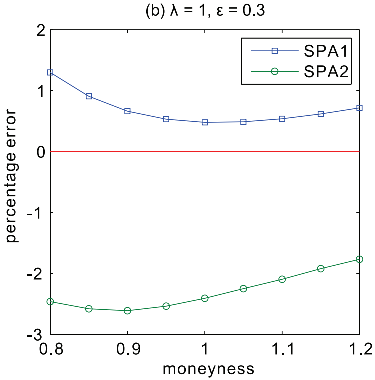
$$\begin{aligned}
\hat{\kappa}_{I(0,T;N)}(u) &= \ln \hat{M}_{I(0,T;N)}(u), \\
\hat{\kappa}'_{I(0,T;N)}(u) &= \frac{M'_{I(0,T;\infty)}(u) + f_1(u)}{M_{I(0,T;N)}(u)}, \\
\hat{\kappa}''_{I(0,T;N)}(u) &= \frac{M''_{I(0,T;\infty)}(u) + f_2(u)}{\hat{M}_{I(0,T;N)}(u)} - \frac{[M'_{I(0,T;\infty)}(u) + f_1(u)]^2}{[\hat{M}_{I(0,T;N)}(u)]^2},
\end{aligned}$$

where the sequence of functions $f_n(u)$ is defined by

$$f_n(u) = V_0^k \frac{\frac{N}{2}\left(\frac{N}{2}+1\right)\cdots\left(\frac{N}{2}+n\right)}{\left(\frac{N}{2}\right)^n}\left(1 - \frac{2V_0 u}{N}\right)^{-N/2-n}, \quad n = 1, 2, \cdots .$$

maturity (days)	5	10	15	20	40	60
strike (OTM)	0.9037	0.9222	0.9399	0.9568	1.0174	1.0683
SPA1	0.2885	0.2556	0.2530	0.2577	0.2840	0.3071
SPA2	0.2851	0.2545	0.2500	0.2500	0.2741	0.2986
MC	0.2794	0.2463	0.2404	0.2441	0.2732	0.2992
SE	0.0008	0.0007	0.0006	0.0006	0.0006	0.0007
strike (ATM)	1.1296	1.1527	1.1748	1.1960	1.2717	1.3354
SPA1	0.4579	0.4334	0.4367	0.4459	0.4865	0.5188
SPA2	0.4500	0.4255	0.4262	0.4336	0.4729	0.5079
MC	0.4490	0.4286	0.4309	0.4406	0.4828	0.5154
SE	0.0010	0.0009	0.0008	0.0008	0.0008	0.0008
strike (ITM)	1.3555	1.3833	1.4098	1.4352	1.5261	1.6024
SPA1	0.6483	0.6352	0.6455	0.6597	0.7129	0.7517
SPA2	0.6367	0.6240	0.6322	0.6450	0.6978	0.7385
MC	0.6402	0.6330	0.6429	0.6574	0.7094	0.7465
SE	0.0012	0.0010	0.0009	0.0008	0.0008	0.0009

Prices of put options on the daily sampled realized variance with varying strike prices and maturities under the SVSJ model.



Plots of the percentage errors against moneyness for the one-month (20 days) put options on daily sampled realized variance with different model parameters of the SVSJ model.

Conclusion

- Effective pricing of VIX futures and VIX options under stochastic volatility models. Potential enhancement of numerical accuracy if non-Gaussian base distribution is employed.

- We use the small time asymptotic approximation of the Laplace transform of discrete realized variance as control to obtain the saddlepoint approximation method for pricing options on discrete realized variance.

- In general, the second order saddlepoint approximation formulas provide reasonably good approximation (with a small percentage error) to the values of derivatives even when the option is deep out-of-the-money and under the choices of extreme parameter values (high values of jump intensity λ and volatility of variance ε).

- The saddlepoint approximation formulas compute well with the Fourier transform algorithm since the Fourier transform algortihm may give negative option prices when the options are deep out-of-the-money.

- Our saddlepoint formulas rely on the availability of cgf and their higher order derivatives, together with the efficient solution of the saddlepoint equation. All these analytic calculations can be achieved at acceptable efficiency with the use of symbolic programming languages. On the other hand, the implementation of the Fourier transform algorithms in option pricing may face with the challenge of finding the appropriate damping factors.

- Ait-Sahalia and Yu (2006) use the Taylor expansion in small time (time interval of the transition density of the time-homogeneous process) to obtain approximations of the cgf and saddlepoints.

- Glasserman and Kim (2009) implement the saddlepoint approximation under the general affine models by developing a procedure of numerically solving the Riccati systems of ODEs and approximating the saddlepoints using the series inversion technique.