Mathematics behind volatility index and trading on volatility

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Agenda

1. Volatility index (VIX) on S&P 500 index (Chicago Board Options Exchange)

   – square root of the risk neutral expectation of the integrated variance of the S&P 500 index over the next 30 calendar days, reported on an annualized basis

   – Volatility index as fear gauze (market confidence)

   – extracted from prices of traded S&P 500 index options

2. Pricing of VIX derivatives

   – directly invest in volatility as an asset class

   – VIX futures and options (most actively traded contracts at CBOE)

3. Derivative products on discrete realized variance of stock price

   – Discrete realized variance: $\sum_{i=1}^{N} \left(\ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2$ as the underlying payoff variable

   – Gamma swaps, timer options

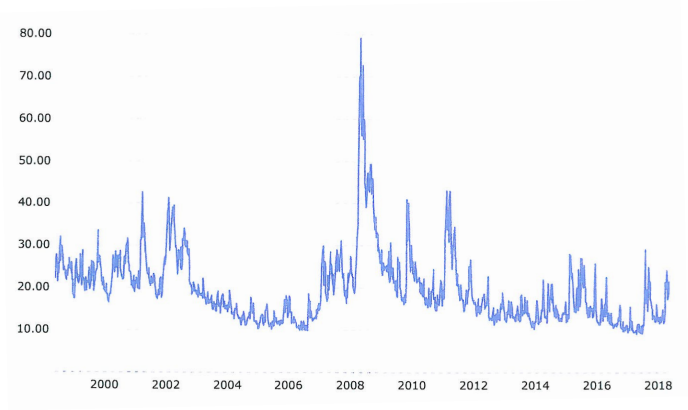
Characteristics of volatility (hidden stochastic process)

Market volatility (standard derivation of log return) is a crucial determinant of investment decisions. Some investors may trade on volatility, rather than directional bet on prices.

Chicago Board Options Exchange (CBOE) volatility index (ticker symbol VIX) on S&P 500 index has become the standard measure of volatility risk in the US stock market.

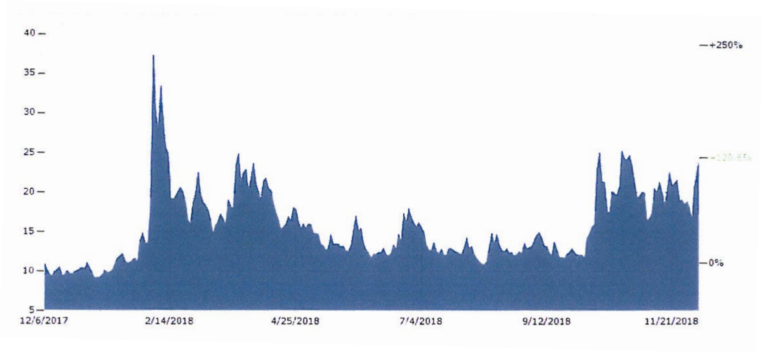
- Volatility is a hidden (latent) stochastic process that is not directly observable, unlike the index value. Volatility is likely to grow when uncertainty and risk increase.
- Volatilities appear to revert to the mean (non-linear drift).
  - After a large volatility spike, the volatility can potentially decrease rapidly.
  - After a low volatility period, it may start to increase slowly.
- Volatility is often negatively correlated with stock or index level. VIX tends to stay high after large downward moves of index value.

Plot of VIX for the past two decades



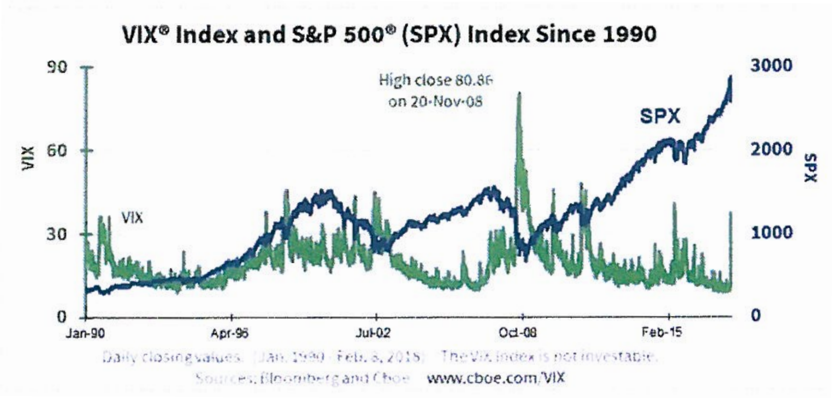
Volatility is bursty in nature: tendency of high volatility to come in bursts. VIX spiked in 2008: period of financial tsunami.

Plot of VIX in 2018



Volatility is negatively correlated to the S&P index return. Fear of inflation that might prompt the Federal Reserve to raise interest rate in February. Deepening of the trade war between China and US pushes up the VIX since September. Reversion of the long-term and short-term yield curves shocked the market on December 4, 2018.

VIX index and SPX



S&P 500 index and VIX are negatively correlated. Growth of SPX over the last 10 years since 2008 financial tsunami with low level of VIX.

VIX expresses volatility in percentage points. It is calculated as 100 times the square root of the expected 30-day variance of the rate of return of the forward price of the S&P 500 index.

$$\text{VIX} = 100\sqrt{\text{forward price of realized cumulative variance}}$$

Suppose the forward price F_t of the S&P index under a risk neutral measure Q follows

$$\frac{dF_t}{F_t} = \sigma_t \, dW_t \text{ so that } d\ln F_t = -\frac{\sigma_t^2}{2}\, dt + \sigma_t \, dW_t.$$

Here, the instantaneous volatility function σ_t is stochastic.

When one computes the differential of a function of F_t, an additional drift term $-\frac{\sigma_t^2}{2}dt$ in $d\ln F_t$ arises from the Ito lemma, where

$$\frac{1}{2}\sigma_t^2 F_t^2 (dW_t)^2 \frac{\partial^2}{\partial F_t^2} \ln F_t = -\frac{\sigma_t^2}{2}\, dt.$$

Subtracting the two dynamic equations, we obtain the cumulative variance over $[0, T]$ under continuous time model as follows:

$$\frac{dF_t}{F_t} - d\ln F_t = \frac{\sigma_t^2}{2}\, dt, \text{ so } \int_0^T \sigma_t^2\, dt = 2\left[\int_0^T \frac{dF_t}{F_t} - \ln\frac{F_T}{F_0}\right].$$

Our goal is to find the mathematical formula for $E_Q\left[\int_0^T \sigma_t^2\, dt\right]$, visualized as the forward price of the realized cumulative variance over $[0, T]$.

Since the expectation of an Ito process with zero drift rate is zero, so

$$E_Q\left[\int_0^T \frac{dF_t}{F_t}\right] = E_Q\left[\int_0^T \sigma_t\, \mathrm{d}W_t\right] = 0.$$

We obtain

$$E_Q\left[\int_0^T \sigma_t^2\, \mathrm{d}t\right] = -2E_Q\left[\ln\frac{F_T}{F_0}\right].$$

The log contract with the terminal payoff $\ln\frac{F_T}{F_0}$ appears. However, log contract is not traded. How to relate the log contract with the commonly traded call and put options?

Taylor expansion formula

For any twice-differentiable function $f\colon \mathbb{R} \to \mathbb{R}$, and any non-negative S_*, we have

$$f(S_T) = f(S_*) + f'(S_*)(S_T - S*) + \int_0^{S_*} f''(K)(K - S_T)^+ \, dK$$
$$+ \int_{S_*}^{\infty} f''(K)(S_T - K)^+ \, dK.$$

The sum of the two integrals is the integral representation of the remainder term in the Taylor expansion of $f(S_T)$ up to the first power term.

In this integral representation, we manage to exhibit terminal payoffs of put and call: $(K - S_T)^+$ and $(S_T - K)^+$.

Rewriting F_T and F_0 for S_T and S^*, respectively, we have

$$f(F_T) - f(F_0) = f'(F_0)(F_T - F_0) + \int_0^{F_0} f''(K)(K - F_T)^+ dK + \int_{F_0}^{\infty} f''(K)(F_T - K)^+ dK.$$

Here, F_0 is the time-0 forward price of the S&P index, an observable known quantity. Taking $f(F_T) = \ln F_T$, we have

$$\ln \frac{F_T}{F_0} = \frac{F_T - F_0}{F_0} - \int_0^{F_0} \frac{(K - F_T)^+}{K^2}\, dK - \int_{F_0}^{\infty} \frac{(F_T - K)^+}{K^2}\, dK.$$

Under Q, since $\mathrm{d}F_t = F_t \sigma_t \, \mathrm{d}W_t$, F_t is a martingale. We obtain $E_Q[F_T] = F_0$ and this gives $E_Q\left[\frac{F_T}{F_0} - 1\right] = 0$. We then have

$$E_Q\left[\int_0^T \sigma_t^2 \, dt\right] = 2E_Q\left[\int_0^{F_0} \frac{(K - F_T)^+}{K^2}\, dK + \int_{F_0}^{\infty} \frac{(F_T - K)^+}{K^2}\, dK\right].$$

Continuum of out-of-the-money calls and puts

By the risk neutral expectation formula, where discounted security prices are Q-martingales, we have

$$e^{-rT} E_Q[(K - F_T)^+] = \text{put}_K,$$

where put_K is the time-0 price of a put on the S&P index with strike K. At T, forward price and the underlying index coincide in value. Similarly, we have

$$e^{-rT} E_Q[(F_T - K)^+] = \text{call}_K.$$

We then obtain

$$E_Q\left[\int_0^T \sigma_t^2 \, \mathrm{d}t\right] = 2e^{rT}\left[\int_0^{F_0} \frac{\text{put}_K}{K^2}\, \mathrm{d}K + \int_{F_0}^{\infty} \frac{\text{call}_K}{K^2}\, \mathrm{d}K\right].$$

The two terms represent continuum of puts whose strikes are below F_0 and calls whose strikes are above F_0, respectively.

They represent out-of-the-money options with respect to the current forward price F_0. The CBOE's choice is sensible since out-of-the-money options tend to be more liquid.

CBOE's VIX formula

We multiply the above expected realized cumulative variance by the product of the annualization conversion factor $\frac{365}{30}$ and percentage point factor 100 to obtain

$$\begin{aligned} & VIX_t^2 \\ = \quad & 100^2 \left\{ \frac{2}{30/365} \sum_i \frac{\Delta K_i}{K_i^2} e^{r(30/365)} Q(K_i) - \frac{1}{30/365} \left(\frac{F_0}{K_0} - 1 \right)^2 \right\}. \end{aligned}$$

Here, K_0 is the first strike below the forward index level F_0, $Q(K_i)$ is the time-t out-of-the-money option price with strike K_i.

The last term $\frac{1}{30/365} \left(\frac{F_0}{K_0} - 1 \right)^2$ is the adjustment required since K_0 is adopted instead of F_0.

VIX futures and options

Investors directly invest in volatility as an asset class by mean of VIX derivatives. CBOE began trading in futures on VIX on March 26, 2004 and European options on February 24, 2006. Both are cash settled. The contract multiplier for each VIX futures contract is $1000, while that of VIX option is $100.

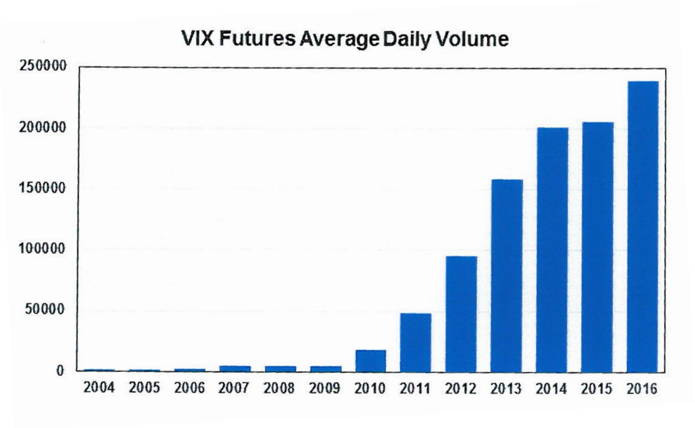
VIX derivatives can be used to hedge the risks of investments in the S&P 500 index and/or achieve exposure to S&P 500 volatility without having to delta hedge their S&P 500 option positions with the stock index.

Year-to-date through the end of July, 2017, average daily volume in VIX futures was 283,342 contracts, 20 percent ahead of the same period a year ago.

In VIX options at CBOE, a reported 2,562,477 contracts traded on August 10, 2017 , surpassing the previous single-day record of 2,382,752 contracts on February 3, 2014.

Year-to-date through the end of July, 2017, average daily volume in VIX options was 687,181 contracts, 11 percent ahead of the same period a year ago.

Trading volumes of VIX futures



On June 24, 2016, in reaction to the Brexit referendum, over $721,000$ VIX futures contracts changed hands and on November 9, 2016, the volume was $644,892$ in reaction to the surprise outcome of the US election.

Every asset class deserves its own volatility index, including VIX itself. The VVIX is an indicator of the expected volatility of the 30-day forward price of the VIX. This volatility drives nearby VIX option prices.

Approximate fair values of VIX futures prices and their standard deviations can be inferred from the VVIX term structure.

Direct modeling of VIX dynamics (Bates, 2012)

$$\begin{aligned} \mathrm{d}V_t &= \kappa_V[\theta(t) - V(t)]\mathrm{d}t + \sigma\sqrt{V(t)}\,\mathrm{d}W_V^Q(t) \\ \mathrm{d}\theta_t &= \kappa_\theta[\overline{\theta} - \theta(t)]\mathrm{d}t + \overline{\sigma}\sqrt{\theta(t)}\,\mathrm{d}W_\theta^Q(t) \end{aligned}$$

The class of stochastic volatility models that use a constant-elasticity-of-variance (CEV) process for the instantaneous variance exhibit nice analytical tractability when the CEV parameter γ takes a few specific values $(0, 1/2, 1, 3/2)$.

$$dV_t = \kappa V_t^{\alpha}(\theta - V_t)dt + \epsilon V_t^{\gamma} dZ_t$$

- Heston's model corresponds to $\gamma = 1/2$. Though exhibiting the best analytical tractability, it leads to downward sloping volatility of variance smiles, contradicting with empirical findings from market data.

- The choice of $\gamma = 3/2$ fits best with the estimation of the CEV power in the instantaneous variance process using S&P 500 daily returns data. It maintains a certain level of analytical tractability. It can capture the volatility skew evolution better than the Heston model.

Since VIX is driven by S&P 500 index, it is more common to use the dynamics of S&P 500 index and its volatility as fundamental state variables. This is called the consistent modeling approach.

Under a pricing measure Q, the joint dynamics of stock price S_t and its instantaneous variance V_t under the Heston affine SVSJ model assumes the form

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= (r - \lambda m)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t^S + (e^{J^S} - 1)\,\mathrm{d}N_t, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\,\mathrm{d}t + \varepsilon\sqrt{V_t}\,\mathrm{d}W_t^V + J^V \mathrm{d}N_t, \end{aligned}$$

where W_t^S and W_t^V are a pair of correlated standard Brownian motions with $\mathrm{d}W_t^S \mathrm{d}W_t^V = \rho\,\mathrm{d}t$, and N_t is a Poisson process with constant intensity λ that is independent of the two Brownian motions.

- J^S and J^V denote the random jump sizes of the log price and variance, respectively.
- These random jump sizes are assumed to be independent of W_t^S, W_t^V and N_t.

Joint moment generating function of the Heston SVSJ model

Let $X_t = \ln S_t$. The joint moment generating function (mgf) of X_t and V_t is defined to be

$$E[\exp(\phi X_T + bV_T + \gamma)],$$

where ϕ, b and γ are constant parameters.

Let $U(X_t, V_t, t)$ denote the non-discounted time-t value of a contingent claim with the terminal payoff function: $U_T(X_T, V_T)$, where T is the maturity date. Let $\tau = T - t$, $U(X, V, \tau)$ is governed by the following partial integro-differential equation (PIDE):

$$\begin{aligned}\frac{\partial U}{\partial \tau} = &\left(r - m\lambda - \frac{V}{2}\right)\frac{\partial U}{\partial X} + \kappa(\theta - V)\frac{\partial U}{\partial V} \\ &+ \frac{V}{2}\frac{\partial^2 U}{\partial X^2} + \frac{\varepsilon^2 V}{2}\frac{\partial^2 U}{\partial V^2} + \rho\varepsilon V\frac{\partial^2 U}{\partial X \partial V} \\ &+ \lambda E\left[U(X + J^S, V + J^V, \tau) - U(X, V, \tau)\right].\end{aligned}$$

The joint moment generating function (mgf) can be regarded as the time-t forward value of the contingent claim with the terminal payoff: $\exp(\phi X_T + bV_T + \gamma)$, so that the mgf also satisfies the PIDE.

Thanks to the affine structure in the SVSJ model, $U(X, V, \tau)$ admits an analytic solution of the following form:

$$U(X, V, \tau) = \exp\left(\phi X + B(\Theta; \tau, \mathbf{q})V + \Gamma(\Theta; \tau, \mathbf{q}) + \Lambda(\Theta; \tau, \mathbf{q})\right).$$

Here, $\mathbf{q} = (\phi \ b \ \gamma)^T$ and we use Θ to indicate the dependence of these parameter functions on the model parameters in the SVSJ model.

The parameter functions $B(\Theta;\tau,\mathbf{q})$, $\Gamma(\Theta;\tau,\mathbf{q})$ and $\Lambda(\Theta;\tau,\mathbf{q})$ satisfy the following Riccati system of ordinary differential equations:

$$\begin{aligned}\frac{\partial B}{\partial \tau} &= -\frac{1}{2}(\phi-\phi^2)-(\kappa-\rho\varepsilon\phi)B+\frac{\varepsilon^2}{2}B^2\\ \frac{\partial \Gamma}{\partial \tau} &= r\phi+\kappa\theta B\\ \frac{\partial \Lambda}{\partial \tau} &= \lambda\big(E[\exp(\phi J^S+BJ^V)-1]-m\phi\big)\end{aligned}$$

with the initial conditions: $B(0)=b$, $\Gamma(0)=\gamma$ and $\Lambda(0)=0$.

Canonical jump distributions

Suppose we assume that $J^V\sim\exp(1/\eta)$ and J^S follows

$$J^S|J^V\sim \text{Normal}(\nu+\rho_J J^V,\delta^2),$$

which is the Gaussian distribution with mean $\nu+\rho_J J^V$ and variance δ^2, we obtain

$$m=E[e^{J^S}-1]=\frac{e^{\nu+\delta^2/2}}{1-\eta\rho_J}-1,$$

provided that $\eta\rho_J<1$.

Parameter functions in mgf

Under the above assumptions on J^S and J^V, the parameter functions can be found to be

$$
\begin{aligned}
B(\Theta;\tau,\mathbf{q}) &= \frac{b(\xi_- e^{-\zeta\tau} + \xi_+) - (\phi-\phi^2)(1-e^{-\zeta\tau})}{(\xi_+ + \varepsilon^2 b)e^{-\zeta\tau} + \xi_- - \varepsilon^2 b}, \\
\Gamma(\Theta;\tau,\mathbf{q}) &= r\phi\tau + \gamma - \frac{\kappa\theta}{\varepsilon^2}\left[\xi_+\tau + 2\ln\frac{(\xi_+ + \varepsilon^2 b)e^{-\zeta\tau} + \xi_- - \varepsilon^2 b}{2\zeta}\right], \\
\Lambda(\Theta;\tau,\mathbf{q}) &= -\lambda(m\phi+1)\tau + \lambda e^{\phi\nu+\delta^2\phi^2/2} \\
&\quad \left[\frac{k_2}{k_4}\tau - \frac{1}{\zeta}\left(\frac{k_1}{k_3} - \frac{k_2}{k_4}\right)\ln\frac{k_3 e^{-\zeta\tau} + k_4}{k_3 + k_4}\right],
\end{aligned}
$$

with $\mathbf{q} = (\phi\ b\ \gamma)^T$ and

$$
\begin{aligned}
\zeta &= \sqrt{(\kappa - \rho\varepsilon\phi)^2 + \varepsilon^2(\phi-\phi^2)}, \\
\xi_\pm &= \zeta \mp (\kappa - \rho\varepsilon\phi), \\
k_1 &= \xi_+ + \varepsilon^2 b, \\
k_2 &= \xi_- - \varepsilon^2 b, \\
k_3 &= (1 - \phi\rho_J\eta)k_1 - \eta(\phi - \phi^2 + \xi_- b), \\
k_4 &= (1 - \phi\rho_J\eta)k_2 - \eta[\xi_+ b - (\phi - \phi^2)].
\end{aligned}
$$

Dynamics of S_t under Q follows:

$$\begin{aligned}\frac{dS_t}{S_{t-}} &= (r-\lambda\mu)dt + \sqrt{v_t}\, dW_t^1 + (e^{J^S}-1)dN_t,\\ dv_t &= \eta(\theta - v_t)dt + \sigma_v\sqrt{v_t}\, dW_t^2 + J^v\, dN_t.\end{aligned}$$

From Ito's lemma, we obtain

$$d(\ln S_t) = \left(r - \lambda\mu - \frac{v_t}{2}\right) dt + \sqrt{v_t}\, dW_t^1 + J^S\, dN_t.$$

We deduce that

$$\begin{aligned}\text{VIX}_{t,\tau}^2 &= \frac{2}{\tau}\mathbb{E}_t^Q\left[\int_t^{t+\tau} \frac{dS_t}{S_u} - d(\ln(S_u))\right] \times 100^2,\\ &= \frac{2}{\tau}\mathbb{E}_t^Q\left[\int_t^{t+\tau} \frac{v_u}{2}\, du + (e^{J^S} - 1 - J^S)dN_u\right] \times 100^2,\\ &= \left\{\frac{1}{\tau}\mathbb{E}_t^Q\left[\int_t^{t+\tau} v_u\, du\right] + \frac{2}{\tau}\mathbb{E}_t^Q\left[\int_t^{t+\tau} (e^{J^S} - 1 - J^S)dN_u\right]\right\} \times 100^2,\\ &= \left\{\frac{1}{\tau}\mathbb{E}_t^Q\left[\int_t^{t+\tau} v_u\, du\right] + 2\lambda[\mu - (\mu_S + \rho_J\mu_v)]\right\} \times 100^2.\end{aligned}$$

From the dynamics of v_t, we obtain

$$d(e^{\eta t} v_t) = e^{\eta t}(\eta\theta \, dt + \sigma_v \sqrt{v_t} \, dW_t^2) + e^{\eta t} J^v \, dN_t.$$

Solving the stochastic differential equation:

$$v_u = e^{-\eta(u-t)} v_t + \theta[1 - e^{-\eta(u-t)}] + \int_t^u \sigma_v e^{-\eta(s-u)} \sqrt{v_s} \, dW_s^2 + \int_t^u e^{-\eta(s-u)} J^v \, dN_s.$$

Taking expectations on both sides, we obtain

$$\begin{aligned}
\mathbb{E}_t^Q[v_u] &= \mathbb{E}^Q[v_u | v_t], \\
&= e^{-\eta(u-t)} v_t + \theta[1 - e^{-\eta(u-t)}] + \int_t^u e^{-\eta(s-u)} \mathbb{E}^Q[J^v] \mathbb{E}^Q[dN_s] \\
&= e^{-\eta(u-t)} v_t + \theta[1 - e^{-\eta(u-t)}] + \int_t^u e^{-\eta(s-u)} \mu_v \lambda \, ds \\
&= e^{-\eta(u-t)} v_t + \theta[1 - e^{-\eta(u-t)}] + \frac{\lambda \mu_v}{\eta}[1 - e^{-\eta(u-t)}] \\
&= e^{-\eta(u-t)} v_t + \left(\theta + \frac{\lambda \mu_v}{\eta}\right) [1 - e^{-\eta(u-t)}].
\end{aligned}$$

VIX^2 as linear function of v

Putting all relations together, we obtain

$$
\begin{aligned}
& \text{VIX}^2_{t,\tau} \\
= & \left\{ \frac{1}{\tau} \mathbb{E}^Q_t \left[\int_t^{t+\tau} v_u \, \mathrm{d}u \right] + 2\lambda(\mu - (\mu_S + \rho_J \mu_v)) \right\} \times 100^2, \\
= & \left\{ \frac{1}{\tau} \int_t^{t+\tau} \mathbb{E}^Q_t[v_u] \, \mathrm{d}u + 2\lambda[\mu - (\mu_S + \rho_J \mu_v)] \right\} \times 100^2, \\
= & \left\{ \frac{1 - e^{-\eta\tau}}{\eta\tau} v_t + \left(1 - \frac{1 - e^{-\eta\tau}}{\eta\tau}\right) \left(\theta + \frac{\lambda\mu_v}{\eta}\right) + 2\lambda[\mu - (\mu_S + \rho_J \mu_v)] \right\} \times 100^2.
\end{aligned}
$$

In summary, we obtain

$$\text{VIX}^2_{t,\tau} = (a v_t + b) \times 100^2,$$

where the time dependent coefficient functions a and b are given by

$$
\begin{aligned}
a &= \frac{1 - e^{-\eta\tau}}{\eta\tau}, \\
b &= \left(1 - \frac{1 - e^{-\eta\tau}}{\eta\tau}\right) \left(\theta + \frac{\lambda\mu_v}{\eta}\right) + 2\lambda[\mu - (\mu_S + \rho_J \mu_v)].
\end{aligned}
$$

VIX futures

The time-t price of the T-maturity VIX futures is given by

$$F(V_t, t) = E_t^Q[\text{VIX}_T] = E_t^Q[\sqrt{aV_T + b}].$$

We write $X = aV_T + b$. It is desirable to have analytic closed forms of the moment generating function (mgf) and cumulant generating function (cgf) of X. Let $f(z; \tau, V_t)$ denote the mgf of V_T, where $\tau = T - t$ is the time to maturity. We are able to obtain $f(z; \tau, V_t)$ in analytic form:

$$f(z; \tau, V_t) = E_t^Q\left[e^{zV_T}\right] = e^{B(z;\tau)V_t + \Gamma(z;\tau) + \Lambda(z;\tau)}, \quad \text{Re } z < \alpha_+,$$

where

$$\begin{aligned}
B(z;\tau) &= \frac{2\kappa z}{\sigma_V^2(1 - e^{\kappa\tau})z + 2\kappa e^{\kappa\tau}} \\
\Gamma(z;\tau) &= -\frac{2\kappa\theta}{\sigma_V^2}\log\left(1 + \frac{\sigma_V^2 z}{2\kappa}\left(e^{-\kappa\tau} - 1\right)\right) \\
\Lambda(z;\tau) &= \frac{2\lambda\mu_V}{2\kappa\mu_V - \sigma_V^2}\log\left(1 + \frac{z(\sigma_V^2 - 2\kappa\mu_V)}{2\kappa(1 - \mu_V z)}(e^{-\kappa\tau} - 1)\right),
\end{aligned}$$

and α_+ is determined by requiring the arguments of the above logarithm terms to be greater than zero.

The corresponding mgf of X is seen to be

$$E\left[e^{zX}\right] = e^{bz}E\left[e^{azV_T}\right] = e^{bz}f(az;\tau,V_t), \quad \text{Re } z < \alpha_+;$$

and the cgf of X is given by

$$\begin{aligned}\kappa(z) &= \log E\left[e^{zX}\right] = bz + \log f(az;\tau,V_t) \\ &= bz + B(az;\tau)V_t + \Gamma(az;\tau) + \Lambda(az;\tau), \quad \text{Re } z < \alpha_+.\end{aligned}$$

It can be shown that

$$\begin{aligned}E[\sqrt{X}] &= \frac{1}{4\sqrt{\pi}i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\kappa(z)}}{z^{3/2}}\,\mathrm{d}z \\ &= \frac{1}{4\sqrt{\pi}i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\kappa(z)-\frac{3}{2}\log z}\,\mathrm{d}z, \quad \gamma \in (0,\alpha_+).\end{aligned}$$

The Bromwich integral can be evaluated using (i) direct numerical integration, (ii) analytic approximation using the saddlepoint approximation formulas (Kwok and Zheng, 2018).

The VIX call option price is $C(V_t, t) = e^{-r(T-t)} E_t^Q\left[(\sqrt{X} - K)^+\right]$. We consider

$$\begin{aligned} E\left[(\sqrt{X} - K)^+\right] &= \int_{K^2}^{\infty} (\sqrt{x} - K) p(x)\,\mathrm{d}x \\ &= \int_{K^2}^{\infty} (\sqrt{x} - K) \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\kappa(z) - zx}\,\mathrm{d}z\,\mathrm{d}x, \\ &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\kappa(z)} \left(\int_{K^2}^{\infty} \sqrt{x} e^{-zx}\,\mathrm{d}x - \int_{K^2}^{\infty} K e^{-zx}\,\mathrm{d}x \right) \mathrm{d}z. \end{aligned}$$

The Bromwich integral representation of $E[(\sqrt{X} - K)^+]$ is given by

$$E\left[(\sqrt{X} - K)^+\right] = \frac{1}{4\sqrt{\pi} i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\kappa(z)}\left[1 - \mathit{erf}(\sqrt{z}K)\right]}{z^{3/2}}\,\mathrm{d}z, \quad \gamma \in (0, \alpha_+),$$

where $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t$.

The discrete realized variance of the asset price process S_t converges to the quadratic variation of the log price process $\ln S_t$ in probability. Given a tenor structure $\{t_0, t_1, \cdots, t_N\}$, let Δt_k be the monitoring time interval $[t_{k-1}, t_k]$. With $\Delta t = \max\limits_{1 \leq k \leq N} \Delta t_k \to 0$, we have

$$\lim_{\Delta t \to 0} \sum_{k=1}^{N} \left(\ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 \stackrel{P}{=} [\ln S_t, \ln S_t]_T$$

A variance swap is a swap contract between two parties who agree to exchange the realized variance over certain accrual period for a fixed strike at some future date.

Gamma swaps allow investors to acquire variance exposure proportional to the underlying level. The terminal payoff of the gamma swap is defined by

$$\frac{A}{N}\sum_{k=1}^{N}\frac{S_{t_k}}{S_{t_0}}\left(\ln\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 - K.$$

where A is the annualized factor. Note that the damping factor $\frac{S_{t_k}}{S_{t_0}}$ is added.

The motivation of choosing the weight to be proportional to the underlying level is to impose damping on the large downside variance when the stock price falls close to zero. This serves to protect the swap seller from paying substantial amount in a catastrophic crash.

Let $X_k = \ln S_{t_k}$. The expectation of a typical term in the floating leg of the gamma swap is given by

$$\begin{aligned} E\left[\frac{S_{t_k}}{S_{t_0}}\left(\ln\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2\right] &= e^{-X_0}E\left[e^{X_{t_k}-X_{t_{k-1}}}(X_{t_k}-X_{t_{k-1}})^2 e^{X_{t_{k-1}}}\right] \\ &= e^{-X_0}E\left[\frac{\partial^2}{\partial\phi^2}e^{\phi(X_{t_k}-X_{t_{k-1}})+X_{t_{k-1}}}\right]\Big|_{\phi=1} \\ &= e^{-X_0}\frac{\partial^2}{\partial\phi^2}E\left[E\left[e^{\phi X_{t_k}}\Big|X_{t_{k-1}},V_{t_{k-1}}\right]e^{(1-\phi)X_{t_{k-1}}}\right]\Big|_{\phi=1} \\ &= \frac{\partial^2}{\partial\phi^2}e^{B(\Theta;\widetilde{\Delta}t_{k-1},\mathbf{q}_2)V_0+\Gamma(\Theta;\widetilde{\Delta}t_{k-1},\mathbf{q}_2)+\Lambda(\Theta;\widetilde{\Delta}t_{k-1},\mathbf{q}_2)}\Big|_{\phi=1}, \end{aligned}$$

where $\widetilde{\Delta}t_{k-1} = t_{k-1} - t_0$, $\Delta t_k = t_k - t_{k-1}$, $\mathbf{q}_1 = (\phi\ 0\ 0)^T$, and

$$\mathbf{q}_2 = \begin{pmatrix} 1 \\ B(\Theta;\Delta t_k,\mathbf{q}_1) \\ \Gamma(\Theta;\Delta t_k,\mathbf{q}_1)+\Lambda(\Theta;\Delta t_k,\mathbf{q}_1) \end{pmatrix}.$$

Note that $\mathbf{q}_2$ has dependence on ϕ through $\mathbf{q}_1$. There is no dependence on the initial price level S_{t_0}.

Two-step expectation calculations

After taking the inner expectation according to the tower rule of conditional expectation, we obtain

$$\begin{aligned} & E[e^{\phi X_{t_k}} | X_{t_{k-1}}, V_{t_{k-1}}] \\ = \quad & e^{\phi X_{t_{k-1}} + B(\Theta;\Delta t_k, \mathbf{q}_1) V_{t_{k-1}} + \Gamma(\Theta;\Delta t_k, \mathbf{q}_1) + \Lambda(\Theta;\Delta t_k, \mathbf{q}_1)}, \end{aligned}$$

which remains to be in the exponential affine form. Note that $e^{(1-\phi)X_{t_{k-1}}}$ is later combined with $e^{\phi X_{t_{k-1}}}$ to give $e^{X_{t_{k-1}}}$, so the first component of $\boldsymbol{q}_2$ is one.

The fair strike price of the gamma swap is then given by

$$K_\Gamma(T, N) = \frac{A}{N} \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2} e^{B(\Theta;\widetilde{\Delta} t_{k-1}, \mathbf{q}_2) V_0 + \Gamma(\Theta;\widetilde{\Delta} t_{k-1}, \mathbf{q}_2) + \Lambda(\Theta;\widetilde{\Delta} t_{k-1}, \mathbf{q}_2)} \Big|_{\phi=1}.$$

We take the asymptotic limit $N \to \infty$ of $K_\Gamma(T, N)$ to deduce the closed form formula for the *fair strike of the continuously monitored gamma swap*.

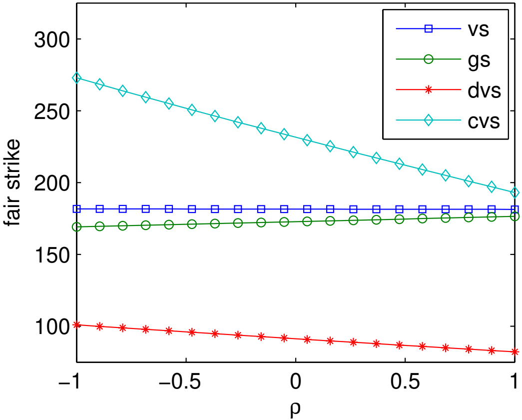
$$K_\Gamma(T,\infty) = \frac{1}{T}\left[\left(V_0 - \frac{\kappa\theta}{\kappa-\rho\varepsilon} - C_2\right)\frac{e^{(r-d-\kappa+\rho\varepsilon)T}-1}{r-d-\kappa+\rho\varepsilon} + \left(\frac{\kappa\theta}{\kappa-\rho\varepsilon} + C_1 + C_2\right)\frac{e^{(r-d)T}-1}{r-d}\right],$$

where

$$\begin{aligned} C_1 &= \frac{\lambda e^{\nu+\delta^2/2}}{1-\rho_J\eta}\left[\left(\nu+\delta^2+\frac{\rho_J\eta}{1-\rho_J\eta}\right)^2 + \delta^2 + \left(\frac{\rho_J\eta}{1-\rho_J\eta}\right)^2\right], \\ C_2 &= \frac{\lambda\eta e^{\nu+\delta^2/2}}{(1-\rho_J\eta)^2(\kappa-\rho\varepsilon)}. \end{aligned}$$

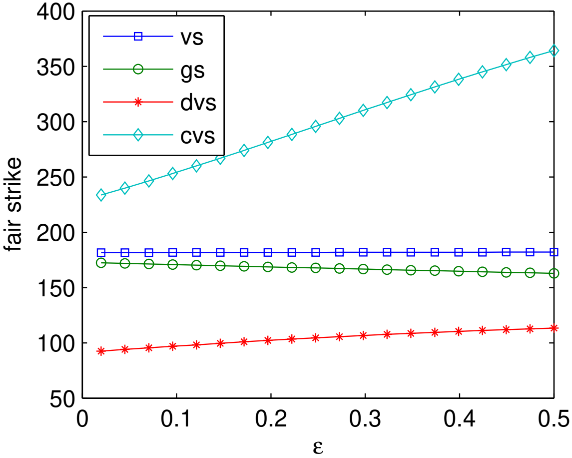
- The price formula does not exhibit linear rate of convergence in Δt nor the sign of the correction (convergence can be from above or below) to the continuously sampled counterpart. Also, the fair strike is not quite sensitive to monitoring frequency (percentage difference in values is less than 0.1% even under weekly monitoring).

Fair strike against correlation coefficient

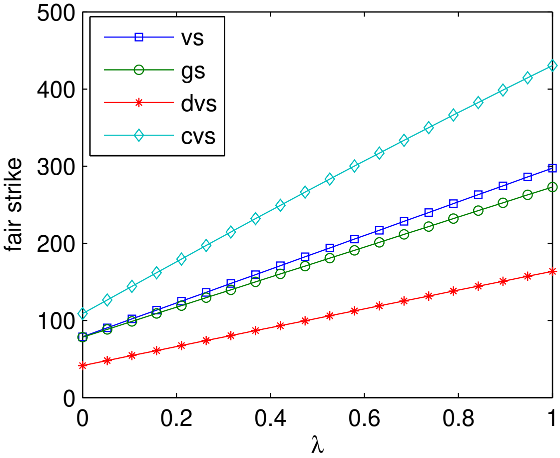


Since the corridor feature is sensitive to asset price, the fair strikes of conditional variance swaps are highly sensitive to correlation coefficient, ρ. The fair strike of the vanilla swap and gamma swap are almost insensitive to ρ.

Fair strike against volatility of variance



The fair strike prices of variance swaps and gamma swaps are almost insensitive to volatility of variance, ε.



The fair strike prices of the corridor variance swaps show the highest sensitivity to the jump intensity, λ.

Timer options - expiry floats with discrete realized variance

1. Product nature

   - Barrier options in the volatility space: knock-out depends on the discrete realized variance hitting the preset variance budget

2. Analytic pricing of timer options under stochastic volatility models

   - Decomposition into a portfolio of timerlets

   - Joint characteristic function of log-asset price and integrated variance

The investor specifies a maximum bound T on the option life and a target volatility σ_0 to define a *variance budget*

$$B = \sigma_0^2 T.$$

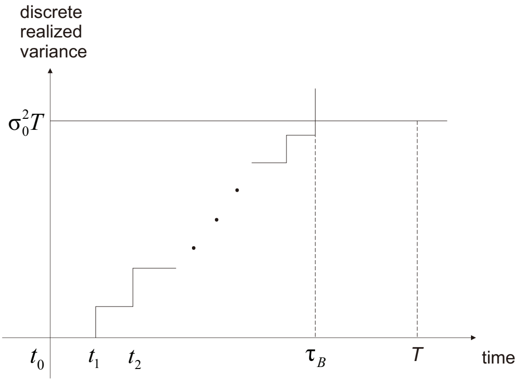
Let t_i, $i = 0, 1, 2, \ldots, N$, be the monitoring dates. Let τ_B be the random first hitting time in the tenor of monitoring dates at which the discrete realized variance exceeds the variance budget B, namely,

$$\tau_B = \min\left\{ j \,\middle|\, \sum_{i=1}^{j} \left(\ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \geq B \right\} \Delta.$$

Here, Δ is the uniform time interval between consecutive monitoring dates.

Termination date of a finite-maturity timer option $= \min(\tau_B, T)$, where T is the preset *mandated* expiration date.

Knock-out feature



Knocked out at τ_B since the variance budget has been breached. This occurs earlier than T.

Define the continuous integrated variance to be $I_t = \int_0^t v_s \, ds$. We use I_t as a proxy of the discrete realized variance for the monitoring of the first hitting time. We define τ_B to be

$$\tau_B = \min\left\{ j \middle| I_{t_j} \geq B \right\} \Delta.$$

This approximation does not introduce a noticeable error for daily monitored timer options. Note that

$$\begin{aligned} C_0(X_0, I_0, V_0) &= \mathbb{E}_0[e^{-r(T \wedge \tau_B)} \max(S_{T \wedge \tau_B} - K, 0)] \\ &= \mathbb{E}_0[e^{-rT} \max(S_T - K, 0) \mathbf{1}_{\{\tau_B > T\}} \\ &\quad + e^{-r\tau_B} \max(S_{\tau_B} - K, 0) \mathbf{1}_{\{\tau_B \leq T\}}], \end{aligned}$$

where K is the strike price and r is the constant interest rate.

The event $\{\tau_B > t\}$ is equivalent to $\{I_t < B\}$. Note that $\tau_B = t_{j+1}$ if and only if $I_{t_j} < B$ and $I_{t_{j+1}} \geq B$. Therefore, we have

$$\{\tau_B \leq T\} = \bigcup_{j=0}^{N-1} \{I_{t_j} < B, I_{t_{j+1}} \geq B\}.$$

A *finite-maturity discrete* timer call option can be decomposed into a portfolio of timerlets:

$$\begin{aligned} C_0 &= \mathbb{E}_0[\, e^{-rT} \max(S_T - K, 0)\mathbf{1}_{\{I_T < B\}}] \\ &\quad + \mathbb{E}_0\left[\sum_{j=0}^{N-1} e^{-rt_{j+1}} \Big(\max(S_{t_{j+1}} - K, 0)\mathbf{1}_{\{I_{t_j} < B\}} \right. \\ &\qquad \left. - \max(S_{t_{j+1}} - K, 0)\mathbf{1}_{\{I_{t_{j+1}} < B\}}\Big)\right]. \end{aligned}$$

The challenge is the modeling of the joint processes of $\{S_{t_{j+1}}, I_{t_j}\}$ and $\{S_{t_{j+1}}, I_{t_{j+1}}\}$.

Consider the stochastic volatility model specified as follows:

$$\frac{\mathrm{d}S_t}{S_t} = (r-q)\mathrm{d}t + \sqrt{v_t}\left(\rho\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2\right),$$
$$\mathrm{d}v_t = \alpha(v_t)\mathrm{d}t + \beta(v_t)\mathrm{d}W_t^2,$$

where W_t^1 and W_t^2 are two independent Brownian motions. The drift function $\alpha(v_t)$ and the volatility function $\beta(v_t)$ are measurable functions with respect to the natural filtration generated by the two correlated Brownian motions.

- For the 3/2-model, we choose

$$\alpha(v_t) = v_t(\theta_t - \kappa v_t) \quad \text{and} \quad \beta(v_t) = \epsilon v_t^{3/2}.$$

- For the Heston 1/2-model, we choose

$$\alpha(v_t) = \lambda(\overline{v} - v_t) \quad \text{and} \quad \beta(v_t) = \eta v_t^{1/2}.$$

Write log asset price $X_t = \ln S_t$ and integrated variance $I_t = \int_0^t v_s \, \mathrm{d}s$, where I_t is used as a proxy for the discrete realized variance used in the barrier condition in the timer option.

Let $G(t, x, y, v; t', x', y', v')$ be the joint transition density of the triple (X, I, V) from state (x, y, v) at time t to state (x', y', v') at a later time t'. The joint transition density G satisfies the following three-dimensional Kolmogorov backward equation:

$$\begin{aligned} -\frac{\partial G}{\partial t} &= \left(r - q - \frac{v}{2}\right)\frac{\partial G}{\partial x} + \frac{v}{2}\frac{\partial^2 G}{\partial x^2} + v\frac{\partial G}{\partial y} + \alpha(v)\frac{\partial G}{\partial v} \\ &\quad + \frac{\beta(v)^2}{2}\frac{\partial^2 G}{\partial v^2} + \rho\sqrt{v}\beta(v)\frac{\partial^2 G}{\partial x \partial v}, \end{aligned}$$

with the terminal condition:

$$G(t', x, y, v; t', x', y', v') = \delta(x - x')\delta(y - y')\delta(v - v'),$$

where $\delta(\cdot)$ is the Dirac delta function.

We define the *generalized partial Fourier transform* of G by $\check{G}$ as follows:

$$\check{G}(t, x, y, v; t', \omega, \eta, v') = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{\mathrm{i}\omega x' + \mathrm{i}\eta y'} G(t, x, y, v; t', x', y', v') \mathrm{d}y' \mathrm{d}x',$$

where the transform variables ω and η are complex variables.

The partial transform $\check{G}$ solves the three-dimensional Kolmogorov equation with the terminal condition:

$$\check{G}(t', x, y, v; t', \omega, \eta, v') = e^{\mathrm{i}\omega x + \mathrm{i}\eta y} \delta(v - v').$$

Note that $\check{G}$ admits the following solution form:

$$\check{G}(t,x,y,v;t',\omega,\eta,v') = e^{\mathrm{i}\omega x + \mathrm{i}\eta y} g(t,v;t',\omega,\eta,v'),$$

where g satisfies the following one-dimensional partial differential equation:

$$-\frac{\partial g}{\partial t} = \left[\mathrm{i}\omega\left(r - q - \frac{v}{2}\right) - \omega^2 \frac{v}{2} + \mathrm{i}\eta v\right] g + [\alpha(v) + \mathrm{i}\omega\rho\sqrt{v}\beta(v)]\frac{\partial g}{\partial v} + \frac{\beta(v)^2}{2}\frac{\partial^2 g}{\partial v^2},$$

with the terminal condition:

$$g(t',v;t',\omega,\eta,v') = \delta(v - v').$$

The double generalized Fourier transform on the log-asset and integrated variance pair reduces the three-dimensional governing equation to a one-dimensional equation.

To evaluate the series of expectations in the portfolio of timerlets, we derive the explicit representation for the characteristic functions of (X_{t_j}, I_{t_j}) and $(X_{t_{j+1}}, I_{t_j})$.

The characteristic function of (X_{t_j}, I_{t_j}) is found to be

$$\begin{aligned}\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_j}+\mathrm{i}\eta I_{t_j}}] &= e^{\mathrm{i}\omega X_0+\mathrm{i}\eta I_0} h(t_0, V_0; t_j, \omega, \eta) \\ &= e^{i\omega X_0+i\eta I_0} \int_0^\infty g(t, v; t', \omega, \eta, v')\, \mathrm{d}v'.\end{aligned}$$

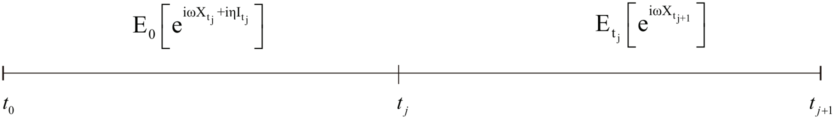
The expectation calculation

$$\mathbb{E}_0[e^{-rt_{j+1}} \max(S_{t_{j+1}} - K, 0)\mathbf{1}_{\{I_{t_j}<B\}}]$$

requires the joint characteristic function of I_t at t_j and S_t at t_{j+1}.

Iterated expectation

Working backward in time from t_{j+1} to t_j, we compute $E_{t_j}[e^{i\omega X_{t_{j+1}}}]$; and from t_j to t_0, we compute $E_0[e^{i\omega X_{t_j}+i\eta I_{t_j}}]$. This is done by setting $\eta = 0$ in $h(t_j, v'; t_{j+1}, \omega, 0)$ and integrating over v' from 0 to ∞.



By the two-step expectation calculation, we obtain

$$\begin{aligned} &\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_{j+1}}+\mathrm{i}\eta I_{t_j}}] \\ = \quad & e^{\mathrm{i}\omega X_0+\mathrm{i}\eta I_0}\int_0^\infty g(t_0, v_0; t_j, \omega, \eta, v')h(t_j, v'; t_{j+1}, \omega, 0)\,\mathrm{d}v'. \end{aligned}$$

Here, v' is the dummy variable for the instantaneous variance v_{t_j}.

Pricing of the timerlets involves the joint process of S_t and I_t (may or may not be at the same time point).

The Fourier transform of the terminal payoff $(S_{t_{j+1}} - K, 0)\mathbf{1}_{\{I_{t_j} < B\}}$ and $(S_{t_{j+1}} - K, 0)\mathbf{1}_{\{I_{t_{j+1}} < B\}}$ admit the same analytic representation

$$\hat{F}(\omega, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega x - \mathrm{i}\eta y}(e^x - K)^+ \mathbf{1}_{\{y<B\}} \, \mathrm{d}x\mathrm{d}y = \frac{K^{1-\mathrm{i}\omega} e^{-\mathrm{i}\eta B}}{(\mathrm{i}\omega + \omega^2)\mathrm{i}\eta},$$

where x stands for $\ln S_{t_{j+1}}$ and y stands for I_{t_j} or $I_{t_{j+1}}$.

We take $\omega = \omega_R + i\omega_I$ and $\eta = \eta_R + i\eta_I$, where the damping factors are chosen such that $\omega_I < -1$ and $\eta_I < 0$, to ensure the existence of the two-dimensional Fourier transform.

Analytic formula in terms of a two-dimensional integral

The finite-maturity discrete timer option price admits the following analytic formula in terms of a two-dimensional Fourier integral:

$$
\begin{aligned}
C_0 &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-rT}\hat{F}(\omega,\eta)\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_N}+\mathrm{i}\eta I_{t_N}}]\,\mathrm{d}\omega_R\mathrm{d}\eta_R \\
&\quad + \sum_{j=0}^{N-1}\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-rt_{j+1}} \\
&\quad \left(\hat{F}(\omega,\eta)\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_{j+1}}+\mathrm{i}\eta I_{t_j}}]-\hat{F}(\omega,\eta)\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_{j+1}}+\mathrm{i}\eta I_{t_{j+1}}}]\right)\,\mathrm{d}\omega_R\mathrm{d}\eta_R \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{F}(\omega,\eta)H(\omega,\eta)\,\mathrm{d}\omega_R\mathrm{d}\eta_R,
\end{aligned}
$$

where

$$
\begin{aligned}
H(\omega,\eta) &= e^{-rT}e^{i\omega X_0+i\eta I_0}h(t_0,V_0;t_N,\omega,\eta)+e^{\mathrm{i}\omega X_0+\mathrm{i}\eta I_0}\sum_{j=0}^{N-1}e^{-rt_{j+1}} \\
&\quad \left[\int_0^{\infty}g(t_0,V_0;t_j,\omega,\eta,v')h(t_j,v';t_{j+1},\omega,0)\,\mathrm{d}v'-h(t_0,V_0;t_{j+1},\omega,\eta)\right].
\end{aligned}
$$

Analytic expressions for g and h under the 3/2-model

For the 3/2-model, we manage to obtain

$$\begin{aligned} & g(t, v; t', v') \\ = \quad & e^{a(t'-t)} \frac{A_t}{C_t} \exp\left(-\frac{A_t v + v'}{C_t v v'}\right) (v')^{-2} \left(\frac{A_t v}{v'}\right)^{\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^2}} I_{2c}\left(\frac{2}{C_t}\sqrt{\frac{A_t}{v v'}}\right), \end{aligned}$$

where I_{2c} is the modified Bessel function of order $2c$,

$$a = \mathrm{i}\omega(r-q), \quad \tilde{\kappa} = \kappa - \mathrm{i}\omega\rho\varepsilon, \quad A_t = e^{\int_t^{t'} \theta_s \, \mathrm{d}s},$$

$$C_t = \frac{\varepsilon^2}{2} \int_t^{t'} e^{\int_t^s \theta_{s'} \, \mathrm{d}s'} \mathrm{d}s, \quad c = \sqrt{\left(\frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2}\right)^2 + \frac{\mathrm{i}\omega + \omega^2 - 2\mathrm{i}\eta}{\varepsilon^2}}.$$

$$\begin{aligned} h(t, v; t', \omega, \eta) &= \int_0^\infty g(t, v_t; t', \omega, \eta, v') \, dv' \\ &= e^{a(t'-t)} \frac{\Gamma(\tilde{\beta} - \tilde{\alpha})}{\Gamma(\tilde{\beta})} \left(\frac{1}{C_t v}\right)^{\tilde{\alpha}} M\left(\tilde{\alpha}, \tilde{\beta}, -\frac{1}{C_t v}\right), \end{aligned}$$

where $\tilde{\alpha} = -\frac{1}{2} - \frac{\tilde{\kappa}}{\varepsilon^2} + c$, $\tilde{\beta} = 1 + 2c$, Γ is the gamma function, M is the confluent hypergeometric function of the first kind.

Conclusion

- Volatility of S&P 500 index as fear gauze. Express VIX on S&P 500 index in terms of traded call and put options on S&P 500 index.

- Hedging volatility exposure using VIX derivatives. Trading volatility as an asset class. Pricing VIX futures and options under the SVSJ model.

- The analytical tractability of generalized swap products on discrete realized variance relies on the availability of the joint moment generating function of the SVSJ model, thanks to the affine structure of the stochastic volatility model. The analytic price formula of the continuously monitored counterpart can be deduced by taking the asymptotic limit of vanishing time interval between successive monitoring dates.

- The sensitivity of the fair strike price on sampling frequency is low for gamma swaps but it can be significant for variance swaps with the corridor feature.

- The general belief of linear rate of convergence of $1/N$, where N is the number of monitoring instants, is shown to be invalid for exotic swap products under the SVSJ models.

- The fair strike prices of the corridor type variance swaps can be highly sensitive to the contractual terms in the contracts and the model parameter values (like volatility of variance, correlation coefficient, etc).

Published papers

1. Pricing exotic swaps under the Heston models and 3/2-stochastic volatility models

   "Closed form pricing formulas for discretely sampled generalized variance swaps," *Mathematical Finance*, vol. 24(4), p.855-881 (2014).

   "Pricing exotic variance swaps under 3/2-stochastic volatility models," *Applied Mathematical Finance*, vol. 22(5), p.421-449 (2015).

2. Pricing options on discrete realized variance using numerical and analytic approximation methods

   "Saddlepoint approximation methods for pricing derivatives on discretely sampled realized variance," *Applied Mathematical Finance*, vol. 21(1), p.1-31 (2014).

Related works

"Fourier transform algorithms for pricing and hedging discretely sampled exotic variance products and volatility derivatives under additive processes," *Journal of Computational Finance*, vol. 18(2), p.3-30 (2014).

"Pricing options on discrete realized variance with partially exact and bounded approximation," *Quantitative Finance*, vol. 15(12), p.2011-2019 (2015).

"Numerical algorithms for pricing discrete variance and volatility derivatives under time-changed Levy-processes," *International Journal of Theoretical and Applied Finance*, vol. 19(2), 1650011 (29 pages) (2016).

3. Timer options (knock-out condition floats with realized variance)

   "Fast Hilbert transform algorithms for pricing discrete timer options under stochastic volatility models," *International Journal of Theoretical and Applied Finance*, vol. 18(7), 1550046 (26 pages) (2015).