Analytic Pricing of the Third Generation Discrete Variance Derivatives

Yue Kuen KWOK

Department of Mathematics
Hong Kong University of Science and Technology
Hong Kong

* This is a joint work with Wendong Zheng.

Agenda

1. Stochastic volatility models

   Asset price, $S_t : dS_t = \mu S_t \; dt + \sqrt{V_t} S_t \; dZ_t$

   Instantaneous variance, $V_t : dV_t = \kappa V_t^{\alpha}(\theta - V_t)dt + \epsilon V^{\gamma} dW_t$,

   $\alpha = \{0, 1\}$, $\gamma = \{1/2, 1, 3/2\}$; $dZ_t dW_t = \rho dt$

   - Affine stochastic volatility model with simultaneous jumps
   - 3/2 - model

   Joint moment generating functions / characteristic functions.

2. Analytic pricing of the third generation exotic variance products

   - Gamma swaps: $E_Q\left[\sum_{k=1}^{N} \frac{S_{t_k}}{S_{t_0}} \left(\ln \frac{S_{t_k}}{S_{t_{k-1}}}\right)^2\right]$
   - Corridor swaps: $E_Q\left[\sum_{k=1}^{N} \left(\ln \frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 \mathbf{1}_{\{L < S_{t_{k-1}} \le U\}}\right]$
   - Timer options: Knock-out contingent on achieving variance budgets, subject to a mandated maximum expiration date.

Key results

- We manage to derive closed form pricing formulas for discrete barrier-style variance derivatives (corridor or knock-out feature) under stochastic volatility models.

- By virtue of the nice exponential affine structure of the Heston model, instead of deriving analytic approximation formulas for discrete variance swaps from those of the continuous counterparts, we obtain analytic pricing formulas for the continuous variance swaps by taking the time interval between successive monitoring dates to be zero. The direct solution of the continuous variance swap model appears to be intractable.

- Analytic results for the joint characteristic function of the triple: $(\ln S_t, V_t, \int_0^t V_u \, du)$ of the 3/2-model are obtained.

Gamma swaps on weighted discrete realized variance

Gamma swaps allow investors to acquire variance exposure proportional to the underlying level. Given a tenor structure $\{t_0, t_1, \cdots, t_N\}$, the terminal payoff of the gamma swap is defined by

$$\frac{A}{N}\sum_{k=1}^{N}\frac{S_{t_k}}{S_{t_0}}\left(\ln\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 - K.$$

where A is the annualized factor.

The share gamma of a derivative with value function $V_t(S_t, t)$ is defined to be $S_t\dfrac{\partial^2 V_t}{\partial S_t^2}$. Unlike the vanilla swap which provides constant cash gamma exposure, the gamma swap provides constant share gamma exposure.

Corridor variance swaps

The underlying price must fall inside a specified corridor $(L, U]$ ($L \geq 0$, $U < \infty$) in order for its squared return to be included in the floating leg of the corridor variance swap.

Suppose the corridor is monitored on the underlying price at the old time level t_{k-1} for the k^{th} squared log return, the floating leg with the corridor $(L, U]$ is given by

$$\frac{A}{N} \sum_{k=1}^{N} \left(\ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 \mathbf{1}_{\{L < S_{t_{k-1}} \leq U\}}.$$

Downside-variance swaps

The payoffs of downside-variance swaps and vanilla variance swaps are sufficient to span all different payoffs of various corridor variance swaps. An investor seeking crash protection may buy the downside-variance swap since it can provide almost the same level of crash protection as the vanilla variance swap but at a lower premium.

Goal: to find the fair strike price of a downside-variance swap with an upper bound U whose payoff at maturity T is given by

$$\frac{A}{N}\sum_{k=1}^{N}\left(\ln\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 \mathbf{1}_{\{S_{t_{k-1}}\leq U\}} - K.$$

An alternative definition is given by

$$\frac{A}{N}\sum_{k=1}^{N}\left(\ln\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 \mathbf{1}_{\{S_{t_k}\leq U\}} - K.$$

Finite-maturity discrete timer options

- The price of a vanilla option is determined by the level of implied volatility quoted in the market. However, the level of implied volatility is often higher than the realized volatility, reflecting the risk premium due to the uncertainty of the future asset price movement.
- For a finite-maturity timer call option, the buyer of the option has the right to purchase the underlying asset at the preset strike price at the first time when a pre-specified variance budget is fully consumed by the accumulated realized variance of the price process of the underlying asset or on the mandated preset expiration date, whichever comes earlier.
- We fix volatility and allow maturity to float. This would resolve the volatility misspecification risk. As revealed by empirical studies, 80% of the three-month calls that have matured in-the-money were overpriced.

Variance budget and mandated expiration date

At the initiation of the timer option, the investor specifies an expected investment horizon T_0 and a target volatility σ_0 to define a variance budget

$$B = \sigma_0^2 T_0.$$

Let τ_B be the first time in the tenor of monitoring dates at which the discrete realized variance exceeds the variance budget B, namely,

$$\tau_B = \min \left\{ j \,\middle|\, \sum_{i=1}^{j} \left(\ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \geq B \right\} \Delta.$$

Here, Δ is the time interval between consecutive monitoring dates.

The price of a finite-maturity discrete timer call option can be decomposed into a portfolio of timerlets.

Constant-elasticity-of-variance process for the instantaneous variance V_t

$$dV_t = \kappa V_t^{\alpha}(\theta - V_t)dt + \epsilon V_t^{\gamma} dZ_t$$

The class of stochastic volatility models that use a constant-elasticity-of-variance (CEV) process for the instantaneous variance exhibit nice analytical tractability when the CEV parameter γ takes a few specific values $(0, 1/2, 1, 3/2)$.

- Heston's model corresponds to $\gamma = 1/2$. Though exhibiting the best analytical tractability, it leads to downward sloping volatility of variance smiles, contradicting with empirical findings from market data.

- The choice of 3/2 fits best with the estimation of the CEV power in the instantaneous variance process using S&P 500 daily returns data. It maintains a certain level of analytical tractability. It can capture the volatility skew evolution better than the Heston model.

Heston stochastic volatility model with simultaneous jumps (SVSJ)

Under a pricing measure Q, the joint dynamics of stock price S_t and its instantaneous variance V_t under the Heston affine SVSJ model assumes the form

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= (r - \lambda m)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t^S + (e^{J^S} - 1)\,\mathrm{d}N_t, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\,\mathrm{d}t + \varepsilon\sqrt{V_t}\,\mathrm{d}W_t^V + J^V\mathrm{d}N_t, \end{aligned}$$

where W_t^S and W_t^V are a pair of correlated standard Brownian motions with $\mathrm{d}W_t^S\mathrm{d}W_t^V = \rho\,\mathrm{d}t$, and N_t is a Poisson process with constant intensity λ that is independent of the two Brownian motions.

- J^S and J^V denote the random jump sizes of the log price and variance, respectively.
- These random jump sizes are assumed to be independent of W_t^S, W_t^V and N_t.

Joint moment generating function of the Heston SVSJ model

Let $X_t = \ln S_t$. The joint moment generating function of X_t and V_t is defined to be

$$E[\exp(\phi X_T + bV_T + \gamma)],$$

where ϕ, b and γ are constant parameters.

Let $U(X_t, V_t, t)$ denote the non-discounted time-t value of a contingent claim with the terminal payoff function: $U_T(X_T, V_T)$, where T is the maturity date. Let $\tau = T - t$, $U(X, V, \tau)$ is governed by the following partial integro-differential equation (PIDE):

$$\begin{aligned}\frac{\partial U}{\partial \tau} = \left(r - m\lambda - \frac{V}{2}\right)\frac{\partial U}{\partial X} + \kappa(\theta - V)\frac{\partial U}{\partial V} \\ + \frac{V}{2}\frac{\partial^2 U}{\partial X^2} + \frac{\varepsilon^2 V}{2}\frac{\partial^2 U}{\partial V^2} + \rho\varepsilon V\frac{\partial^2 U}{\partial X \partial V} \\ + \lambda E\Big[U(X + J^S, V + J^V, \tau) - U(X, V, \tau)\Big].\end{aligned}$$

The joint moment generating function (MGF) can be regarded as the time-t forward value of the contingent claim with the terminal payoff: $\exp(\phi X_T + bV_T + \gamma)$, so that the MGF also satisfies the PIDE.

Thanks to the affine structure in the SVSJ model, $U(X, V, \tau)$ admits an analytic solution of the following form:

$$U(X, V, \tau) = \exp\Big(\phi X + B(\Theta; \tau, \mathbf{q})V + \Gamma(\Theta; \tau, \mathbf{q}) + \Lambda(\Theta; \tau, \mathbf{q})\Big).$$

Here, $\mathbf{q} = (\phi \ \ b \ \ \gamma)^T$ and we use Θ to indicate the dependence of these parameter functions on the model parameters in the SVSJ model.

Riccati system of ordinary differential equations

The parameter functions $B(\Theta;\tau,\mathbf{q})$, $\Gamma(\Theta;\tau,\mathbf{q})$ and $\Lambda(\Theta;\tau,\mathbf{q})$ satisfy the following Riccati system of ordinary differential equations:

$$\frac{\partial B}{\partial \tau} = -\frac{1}{2}(\phi - \phi^2) - (\kappa - \rho\varepsilon\phi)B + \frac{\varepsilon^2}{2}B^2$$
$$\frac{\partial \Gamma}{\partial \tau} = r\phi + \kappa\theta B$$
$$\frac{\partial \Lambda}{\partial \tau} = \lambda\Big(E[\exp(\phi J^S + BJ^V) - 1] - m\phi\Big)$$

with the initial conditions: $B(0) = b$, $\Gamma(0) = \gamma$ and $\Lambda(0) = 0$.

Canonical jump distributions

Suppose we assume that $J^V \sim \exp(1/\eta)$ and J^S follows

$$J^S | J^V \sim \text{Normal}(\nu + \rho_J J^V, \delta^2),$$

which is the Gaussian distribution with mean $\nu + \rho_J J^V$ and variance δ^2, we obtain

$$m = E[e^{J^S} - 1] = \frac{e^{\nu + \delta^2/2}}{1 - \eta\rho_J} - 1,$$

provided that $\eta\rho_J < 1$.

Under the above assumptions on J^S and J^V, the parameter functions can be found to be

$$
\begin{aligned}
B(\Theta;\tau,\mathbf{q}) &= \frac{b(\xi_- e^{-\zeta\tau}+\xi_+)-(\phi-\phi^2)(1-e^{-\zeta\tau})}{(\xi_+ +\varepsilon^2 b)e^{-\zeta\tau}+\xi_- -\varepsilon^2 b},\\
\Gamma(\Theta;\tau,\mathbf{q}) &= r\phi\tau+\gamma-\frac{\kappa\theta}{\varepsilon^2}\left[\xi_+\tau+2\ln\frac{(\xi_+ +\varepsilon^2 b)e^{-\zeta\tau}+\xi_- -\varepsilon^2 b}{2\zeta}\right],\\
\Lambda(\Theta;\tau,\mathbf{q}) &= -\lambda(m\phi+1)\tau+\lambda e^{\phi\nu+\delta^2\phi^2/2}\\
&\quad \left[\frac{k_2}{k_4}\tau-\frac{1}{\zeta}\left(\frac{k_1}{k_3}-\frac{k_2}{k_4}\right)\ln\frac{k_3 e^{-\zeta\tau}+k_4}{k_3+k_4}\right],
\end{aligned}
$$

with $\mathbf{q}=(\phi\ b\ \gamma)^T$ and

$$
\begin{aligned}
\zeta &= \sqrt{(\kappa-\rho\varepsilon\phi)^2+\varepsilon^2(\phi-\phi^2)},\\
\xi_\pm &= \zeta\mp(\kappa-\rho\varepsilon\phi),\\
k_1 &= \xi_+ +\varepsilon^2 b,\\
k_2 &= \xi_- -\varepsilon^2 b,\\
k_3 &= (1-\phi\rho_J\eta)k_1-\eta(\phi-\phi^2+\xi_- b),\\
k_4 &= (1-\phi\rho_J\eta)k_2-\eta[\xi_+ b-(\phi-\phi^2)].
\end{aligned}
$$

The 3/2 model

Consider the 3/2 stochastic volatility model specified as follows:

$$\frac{dS_t}{S_t} = (r-q)dt + \sqrt{V_t}\Big(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\Big),$$
$$dV_t = V_t(\theta_t - \kappa V_t)dt + \varepsilon V_t^{3/2} dW_t^2,$$

where W_t^1 and W_t^2 are two independent Brownian motions.

- The speed of mean reversion is now κV_t, which is linear in V_t. The mean reversion is faster when the instantaneous variance is higher.

- ε cannot be interpreted as the same volatility of variance in the Heston model.

Partial Fourier transform of the triple joint density function under 3/2 **model**

Write $X_t = \ln S_t$ and quadratic variation by $I_t = \int_0^t V_s \, \mathrm{d}s$. The integrated variance I_t is used as a proxy for the discrete realized variance used in the barrier condition in a timer option. Let $G(t, x, y, v; t', x', y', v')$ be the joint transition density of the triple (X, I, V) from state (x, y, v) at time t to state (x', y', v') at a later time t'.

The joint transition density G satisfies the following three-dimensional Kolmogorov backward equation:

$$-\frac{\partial G}{\partial t} = \left(r - q - \frac{v}{2}\right)\frac{\partial G}{\partial x} + \frac{v}{2}\frac{\partial^2 G}{\partial x^2} + v\frac{\partial G}{\partial y} + v(\theta_t - \kappa v)\frac{\partial G}{\partial v} + \frac{\varepsilon^2 v^3}{2}\frac{\partial^2 G}{\partial v^2} + \rho\varepsilon v^2 \frac{\partial^2 G}{\partial x \partial v},$$

with the terminal condition:

$$G(t', x, y, v; t', x', y', v') = \delta(x - x')\delta(y - y')\delta(v - v'),$$

where $\delta(\cdot)$ is the Dirac delta function.

We define the *generalized partial Fourier transform* of G by $\check{G}$ as follows:

$$\check{G}(t,x,y,v;t',\omega,\eta,v') = \int_{-\infty}^{\infty}\int_0^{\infty} e^{\mathrm{i}\omega x'+\mathrm{i}\eta y'} G(t,x,y,v;t',x',y',v')\mathrm{d}y'\mathrm{d}x',$$

where the transform variables ω and η can be complex numbers. The partial transform $\check{G}$ also solves the above PDE with the terminal condition being $\check{G}(t',x,y,v;t',\omega,\eta,v') = e^{\mathrm{i}\omega x+\mathrm{i}\eta y}\delta(v-v')$.

Note that $\check{G}$ admits the following solution form:

$$\check{G}(t,x,y,v;t',\omega,\eta,v') = e^{\mathrm{i}\omega x+\mathrm{i}\eta y} g(t,v;t',\omega,\eta,v'),$$

where g satisfies the following PDE:

$$-\frac{\partial g}{\partial t} = \left[\mathrm{i}\omega\left(r-q-\frac{v}{2}\right)-\omega^2\frac{v}{2}+\mathrm{i}\eta v\right]g+[v(\theta_t-\kappa v)+\mathrm{i}\omega\rho\varepsilon v^2]\frac{\partial g}{\partial v}+\frac{\varepsilon^2 v^3}{2}\frac{\partial^2 g}{\partial v^2},$$

with the terminal condition:

$$g(t',v;t',\omega,\eta,v') = \delta(v-v').$$

The double generalized Fourier transform on the log-asset and integrated variance pair reduces the three-dimensional governing equation to a one-dimensional equation. We manage to obtain

$$
\begin{aligned}
& g(t, v; t', \omega, \eta, v') \\
= \; & e^{a(t'-t)} \frac{A_t}{C_t} \exp\left(-\frac{A_t v + v'}{C_t v v'}\right) (v')^{-2} \left(\frac{A_t v}{v'}\right)^{\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^2}} I_{2c}\left(\frac{2}{C_t}\sqrt{\frac{A_t}{v v'}}\right),
\end{aligned}
$$

where I_{2c} is the modified Bessel function of order $2c$,

$$
\begin{aligned}
& a = \mathrm{i}\omega(r-q), \quad \tilde{\kappa} = \kappa - \mathrm{i}\omega\rho\varepsilon, \quad A_t = e^{\int_t^{t'} \theta_s \, \mathrm{d}s}, \\
& C_t = \frac{\varepsilon^2}{2}\int_t^{t'} e^{\int_t^s \theta_{s'} \mathrm{d}s'} \mathrm{d}s, \quad c = \sqrt{\left(\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^2}\right)^2 + \frac{\mathrm{i}\omega + \omega^2 - 2\mathrm{i}\eta}{\varepsilon^2}}.
\end{aligned}
$$

Note that c is in general complex and the numerical valuation of a modified Bessel function of complex order may pose some challenge.

Solution procedure

It is well known that the reciprocal of the 3/2 process is a CIR process. Indeed, if we define $U_t = \dfrac{1}{V_t}$, then U_t is governed by

$$dU_t = [(\kappa + \epsilon^2) - \theta_t U_t]dt - \epsilon\sqrt{U_t}\ dW_t.$$

For any $t' > t$, $U_{t'}$ follows a non-central chi-square distribution conditional on U_t.

The corresponding (conditional) density function is given by

$$p_U(U_{t'}|U_t) = \frac{A_t}{C_t}\exp\left(-\frac{A_t U_{t'} + U_t}{C_t}\right)\left(\frac{A_t U_{t'}}{U_t}\right)^{\frac{1}{2}+\frac{\kappa}{\epsilon^2}} I_{1+\frac{2\kappa}{\epsilon^2}}\left(\frac{2}{C_t}\sqrt{A_t U_{t'} U_t}\right).$$

Fair strike of the gamma swap under the SVSJ model

Let $X_k = \ln S_{t_k}$. The expectation of a typical term in the floating leg of the gamma swap is given by

$$\begin{aligned}
E\left[\frac{S_{t_k}}{S_{t_0}}\left(\ln\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2\right] &= e^{-X_0}E\left[e^{X_{t_k}-X_{t_{k-1}}}(X_{t_k}-X_{t_{k-1}})^2 e^{X_{t_{k-1}}}\right] \\
&= e^{-X_0}E\left[\frac{\partial^2}{\partial\phi^2}e^{\phi(X_{t_k}-X_{t_{k-1}})+X_{t_{k-1}}}\right]\Big|_{\phi=1} \\
&= e^{-X_0}\frac{\partial^2}{\partial\phi^2}E\left[E\left[e^{\phi X_{t_k}}\Big| X_{t_{k-1}}, V_{t_{k-1}}\right]e^{(1-\phi)X_{t_{k-1}}}\right]\Big|_{\phi=1} \\
&= \frac{\partial^2}{\partial\phi^2}e^{B(\Theta;\widetilde{\Delta}t_{k-1},\mathbf{q}_2)V_0+\Gamma(\Theta;\widetilde{\Delta}t_{k-1},\mathbf{q}_2)+\Lambda(\Theta;\widetilde{\Delta}t_{k-1},\mathbf{q}_2)}\Big|_{\phi=1},
\end{aligned}$$

where $\widetilde{\Delta}t_{k-1} = t_{k-1} - t_0$, $\Delta t_k = t_k - t_{k-1}$, $\mathbf{q}_1 = (\phi\ 0\ 0)^T$, and

$$\mathbf{q}_2 = \begin{pmatrix} 1 \\ B(\Theta;\Delta t_k,\mathbf{q}_1) \\ \Gamma(\Theta;\Delta t_k,\mathbf{q}_1)+\Lambda(\Theta;\Delta t_k,\mathbf{q}_1) \end{pmatrix}.$$

Note that $\mathbf{q}_2$ has dependence on ϕ through $\mathbf{q}_1$. There is no dependence on the initial price level S_{t_0}.

Two-step expectation calculations

After taking the inner expectation according to the tower rule of conditional expectation, we obtain

$$\begin{aligned} & E[e^{\phi X_{t_k}} | X_{t_{k-1}}, V_{t_{k-1}}] \\ = \;& e^{\phi X_{t_{k-1}} + B(\Theta;\Delta t_k, \mathbf{q}_1) V_{t_{k-1}} + \Gamma(\Theta;\Delta t_k, \mathbf{q}_1) + \Lambda(\Theta;\Delta t_k, \mathbf{q}_1)}, \end{aligned}$$

which remains to be in exponential affine form. Note that $e^{(1-\phi)X_{t_{k-1}}}$ is later combined with $e^{\phi X_{t_{k-1}}}$ to give $e^{X_{t_{k-1}}}$, so the first component of $\boldsymbol{q}_2$ is one.

The fair strike price of the gamma swap is then given by

$$K_\Gamma(T,N) = \frac{A}{N} \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2} e^{B(\Theta;\widetilde{\Delta} t_{k-1}, \mathbf{q}_2) V_0 + \Gamma(\Theta;\widetilde{\Delta} t_{k-1}, \mathbf{q}_2) + \Lambda(\Theta;\widetilde{\Delta} t_{k-1}, \mathbf{q}_2)} \Big|_{\phi=1}.$$

Asymptotic price formula of the continuously sampled gamma swap

We take the asymptotic limit $N \to \infty$ of $K_\Gamma(T,N)$ to deduce the closed form formula for the *fair strike of the continuously monitored gamma swap*.

$$K_\Gamma(T,\infty) = \frac{1}{T}\left[\left(V_0 - \frac{\kappa\theta}{\kappa-\rho\varepsilon} - C_2\right)\frac{e^{(r-d-\kappa+\rho\varepsilon)T}-1}{r-d-\kappa+\rho\varepsilon} + \left(\frac{\kappa\theta}{\kappa-\rho\varepsilon} + C_1 + C_2\right)\frac{e^{(r-d)T}-1}{r-d}\right],$$

where

$$\begin{aligned} C_1 &= \frac{\lambda e^{\nu+\delta^2/2}}{1-\rho_J\eta}\left[\left(\nu+\delta^2+\frac{\rho_J\eta}{1-\rho_J\eta}\right)^2 + \delta^2 + \left(\frac{\rho_J\eta}{1-\rho_J\eta}\right)^2\right], \\ C_2 &= \frac{\lambda\eta e^{\nu+\delta^2/2}}{(1-\rho_J\eta)^2(\kappa-\rho\varepsilon)}. \end{aligned}$$

- The price formula does not exhibit linear rate of convergence in Δt nor the sign of the correction (convergence can be from above or below) to the continuously sampled counterpart. Also, the fair strike is not quite sensitive to monitoring frequency (percentage difference in values is less than 0.1% even under weekly monitoring).

Downside-variance swaps under the SVSJ model

The challenge in the evaluation of the nested expectation is how to deal with the barrier indicator term $\mathbf{1}_{\{S_{t_{k-1}} \leq U\}}$. The expectation calculation of a typical term gives

$$
\begin{aligned}
& E\left[\left(\ln \frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 \mathbf{1}_{\{S_{t_{k-1}} \leq U\}}\right] \\
&= E\left[E\left[\frac{\partial^2}{\partial \phi^2} e^{\phi(X_{t_k} - X_{t_{k-1}})} \middle| X_{t_{k-1}}, V_{t_{k-1}}\right] \mathbf{1}_{\{X_{t_{k-1}} \leq \ln U\}}\right]\Bigg|_{\phi=0} \\
&= E\left[\frac{\partial^2}{\partial \phi^2} e^{B(\Theta;\Delta t_k,\mathbf{q}_1)V_{t_{k-1}} + \Gamma(\Theta;\Delta t_k,\mathbf{q}_1) + \Lambda(\Theta;\Delta t_k,\mathbf{q}_1)} \mathbf{1}_{\{X_{t_{k-1}} \leq \ln U\}}\right]\Bigg|_{\phi=0} \\
&= \frac{\partial^2}{\partial \phi^2} E\left[e^{B(\Theta;\Delta t_k,\mathbf{q}_1)V_{t_{k-1}} + \Gamma(\Theta;\Delta t_k,\mathbf{q}_1) + \Lambda(\Theta;\Delta t_k,\mathbf{q}_1)} \mathbf{1}_{\{X_{t_{k-1}} \leq \ln U\}}\right]\Bigg|_{\phi=0}, \qquad \text{(A)}
\end{aligned}
$$

where $\mathbf{q}_1 = (\phi\ 0\ 0)^T$.

- For $k = 1$, X_0 and V_0 are known; so we have

$$
E\left[\left(\ln \frac{S_{t_1}}{S_{t_0}}\right)^2 \mathbf{1}_{\{S_{t_0} \leq U\}}\right] = \frac{\partial^2}{\partial \phi^2} e^{B(\Theta;\Delta t_1,\mathbf{q}_1)V_0 + \Gamma(\Theta;\Delta t_1,\mathbf{q}_1) + \Lambda(\Theta;\Delta t_1,\mathbf{q}_1)} \mathbf{1}_{\{X_0 \leq \ln U\}}\Bigg|_{\phi=0}.
$$

- For $k \geq 2$, the evaluation of expectation in formula (A) requires the representation of the indicator function $\mathbf{1}_{\{X_{t_{k-1}} \leq \ln U\}}$ to be expressed in terms of an inverse Fourier transform.

Generalized Fourier transform of the indicator function $\mathbf{1}_{\{X_{t_{k-1}} \leq u\}}$

We take the Fourier transform variable w to be complex and write $w = w_r + \mathrm{i}w_i$. We treat $\mathbf{1}_{\{X_{t_{k-1}} \leq u\}}$ as a function of u and consider its Fourier transform:

$$\int_{-\infty}^{\infty} \mathbf{1}_{\{X_{t_{k-1}} \leq u\}} e^{-\mathrm{i}uw}\, \mathrm{d}u = \int_{X_{t_{k-1}}}^{\infty} e^{-\mathrm{i}uw}\, \mathrm{d}u = \frac{e^{-\mathrm{i}X_{t_{k-1}}w}}{\mathrm{i}w}, \quad u = \ln U.$$

Provided that w_i is chosen to be sufficiently negative value, the above generalized Fourier transform exists. By taking the corresponding generalized inverse Fourier transform, we obtain

$$\mathbf{1}_{\{X_{t_{k-1}} \leq u\}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}uw} \frac{e^{-\mathrm{i}X_{t_{k-1}}w}}{\mathrm{i}w}\, \mathrm{d}w_r.$$

Formula (A) can be expressed in terms of a Fourier integral as follows:

$$
\begin{aligned}
& E\left[\left(\ln \frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 \mathbf{1}_{\{S_{t_{k-1}} \le U\}}\right] \\
= \; & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial \phi^2} E\left[e^{-\mathrm{i} w X_{t_{k-1}} + B(\Theta;\Delta t_k,\mathbf{q}_1) V_{t_{k-1}} + \Gamma(\Theta;\Delta t_k,\mathbf{q}_1) + \Lambda(\Theta;\Delta t_k,\mathbf{q}_1)} \right]\Bigg|_{\phi=0} \frac{e^{\mathrm{i} u w}}{\mathrm{i} w}\, \mathrm{d} w_r \\
= \; & \frac{e^{w_i (X_0 - u)}}{\pi} \int_0^{\infty} \mathrm{Re}\left(e^{-\mathrm{i} w_r (X_0 - u)} \frac{F_k(w_r + \mathrm{i} w_i)}{\mathrm{i} w_r - w_i} \right) \mathrm{d} w_r, \; k \ge 2,
\end{aligned}
$$

where $w = w_r + \mathrm{i} w_i$, $u = \ln U$, and

$$
F_k(w) = \frac{\partial^2}{\partial \phi^2} e^{B(\Theta;t_{k-1},\mathbf{q}_2)V_0 + \Gamma(\Theta;t_{k-1},\mathbf{q}_2) + \Lambda(\Theta;t_{k-1},\mathbf{q}_2)}\Big|_{\phi=0}, \; k \ge 2,
$$

with

$$
\mathbf{q}_2 = \begin{pmatrix} -\mathrm{i} w \\ B(\Theta;\Delta t_k,\mathbf{q}_1) \\ \Gamma(\Theta;\Delta t_k,\mathbf{q}_1) + \Lambda(\Theta;\Delta t_k,\mathbf{q}_1) \end{pmatrix}.
$$

The fair strike price of the discretely sampled downside-variance swap is then given by

$$K_D(T,N) = \frac{\partial^2}{\partial\phi^2} e^{B(\Theta;\Delta t_1,\mathbf{q}_1)V_0+\Gamma(\Theta;\Delta t_1,\mathbf{q}_1)+\Lambda(\Theta;\Delta t_1,\mathbf{q}_1)}\mathbf{1}_{\{X_0\le \ln U\}}\Big|_{\phi=0}$$
$$+\frac{e^{w_i(X_0-u)}}{\pi}\int_0^\infty \mathrm{Re}\left(e^{-\mathrm{i}w_r(X_0-u)}\frac{\sum_{k=2}^N F_k(w_r+\mathrm{i}w_i)}{\mathrm{i}w_r-w_i}\right)\mathrm{d}w_r.$$

The fair strike formula still involves a Fourier integral whose numerical evaluation can be effected by FFT calculations.

- The fair strike price of the continuously sampled downside-variance swap can be deduced by taking $\Delta t \to 0$.

Linear rate of convergence of discrete realized variance with the number of sampling dates

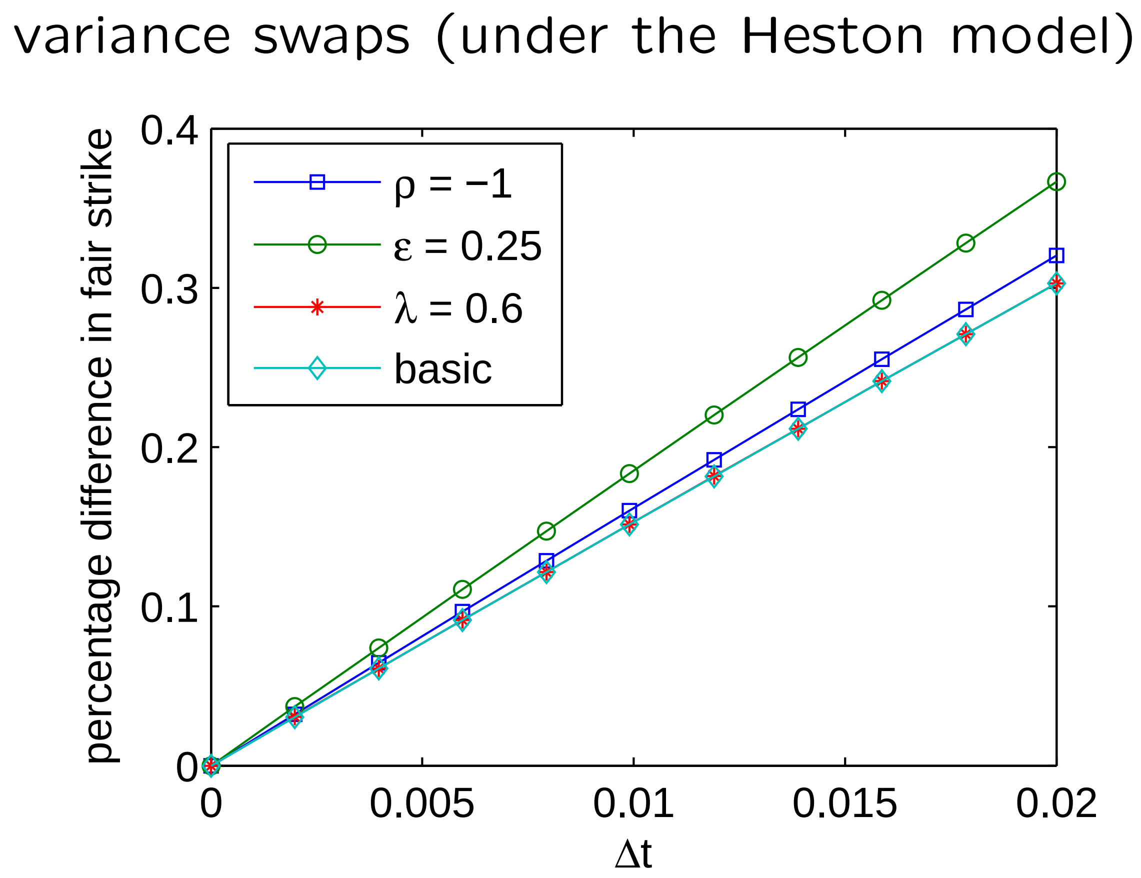
For vanilla variance swaps, Broadie and Jain (2008) show that the discrete variance strike $K^*_{var}(n)$ converges to the continuous variance strike K^*_{var} linearly with Δt ($\Delta t = \frac{T}{n}$). That is,

$$K^*_{var}(n) = K^*_{var} + O\left(\frac{1}{n}\right) \longrightarrow K^*_{var} \text{ as } n \to \infty.$$

Similar results can be extended to the more general stochastic volatility models (Bernard and Cui, 2012):

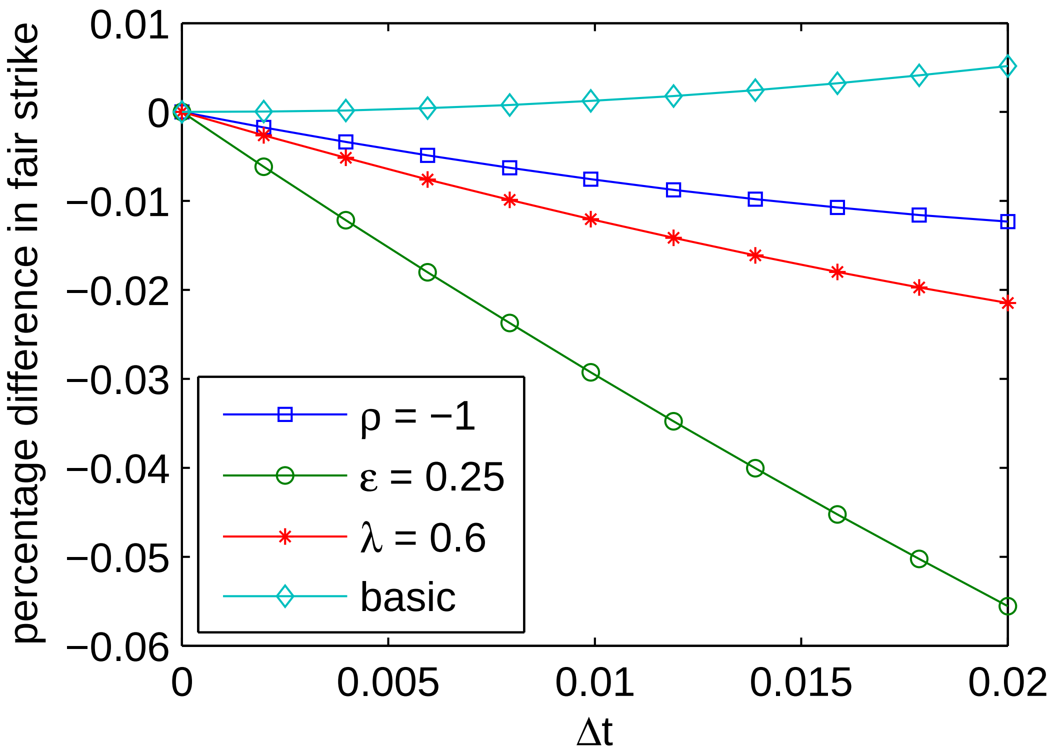
$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= r\,\mathrm{d}t + m(V_t)\left(\rho\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2\right) \\ \mathrm{d}V_t &= \mu(V_t)\mathrm{d}t + \sigma(V_t)\,\mathrm{d}W_t^1. \end{aligned}$$

Plot of the percentage difference in the fair strike prices of various discretely sampled generalized variance swaps against sampling time interval Δt (in units of year) under the Heston model.



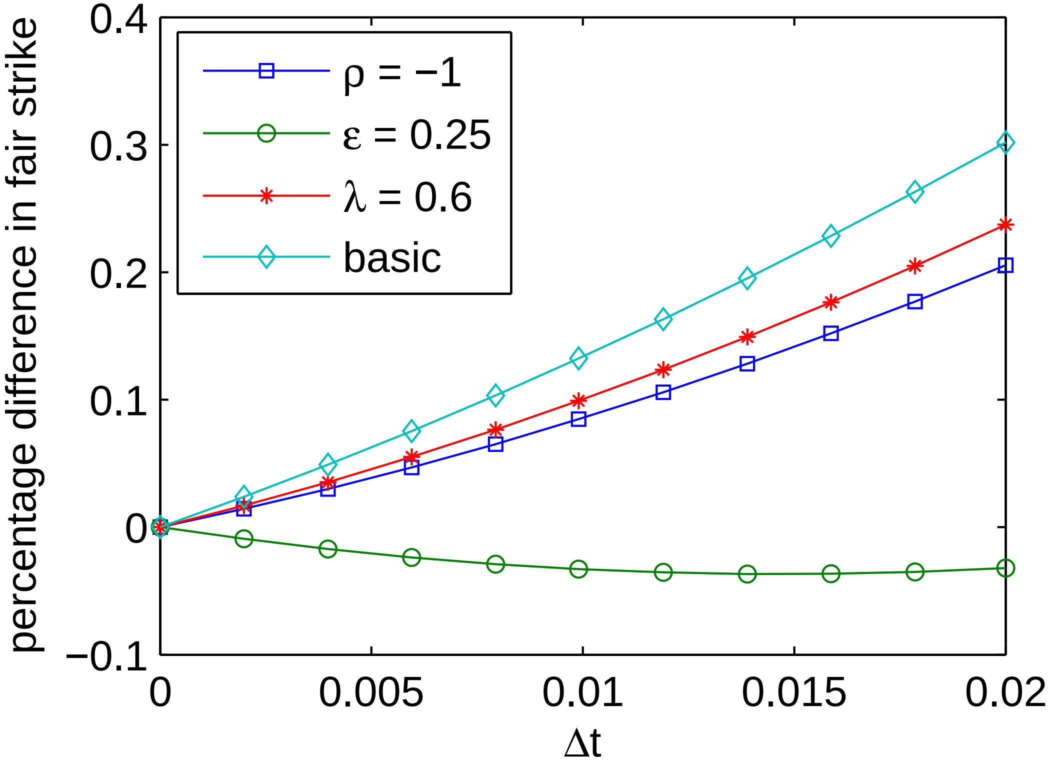
The vanilla variance swap exhibits a linear rate of convergence with respect to Δt.

gamma swaps (under the Heston model)



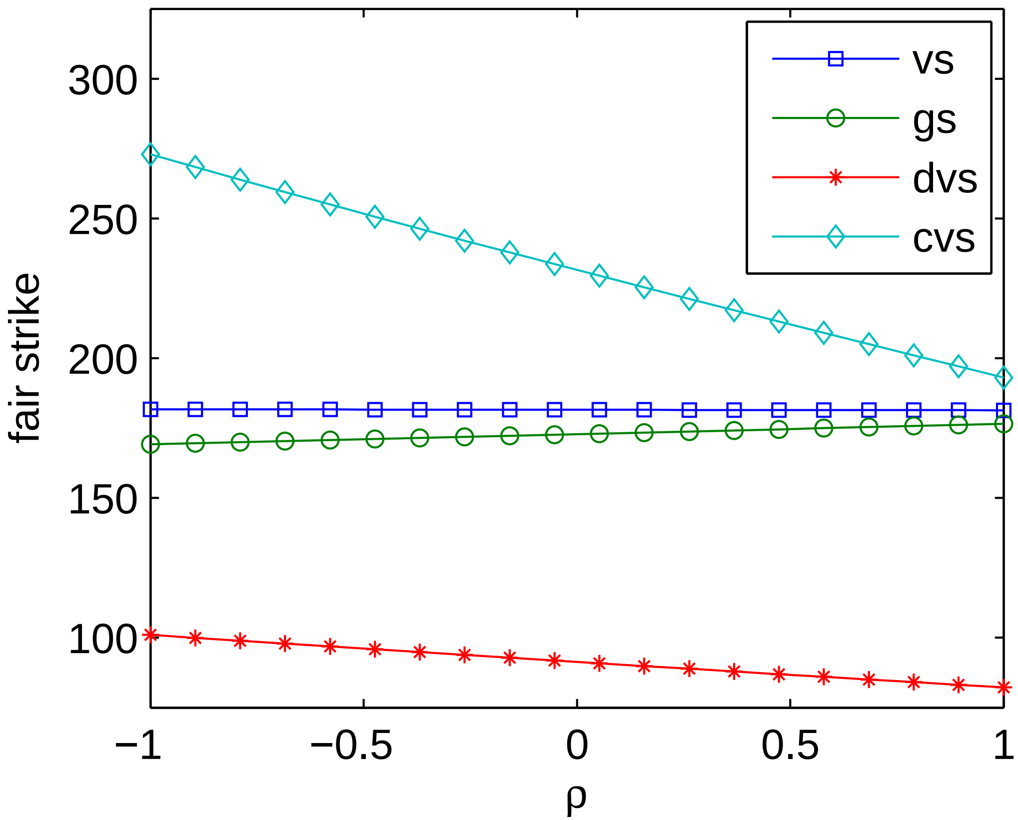
The convergence of the fair strike prices to the continuous limit can be from the above or below with vanishing width of the sampling interval. Since the relative percentage difference in fair strike between discrete gamma swap and its continuously monitored counterpart is small, so the pricing formula for the continuously monitored gamma swap provides a highly accurate approximation.

downside variance swaps (under the Heston model)



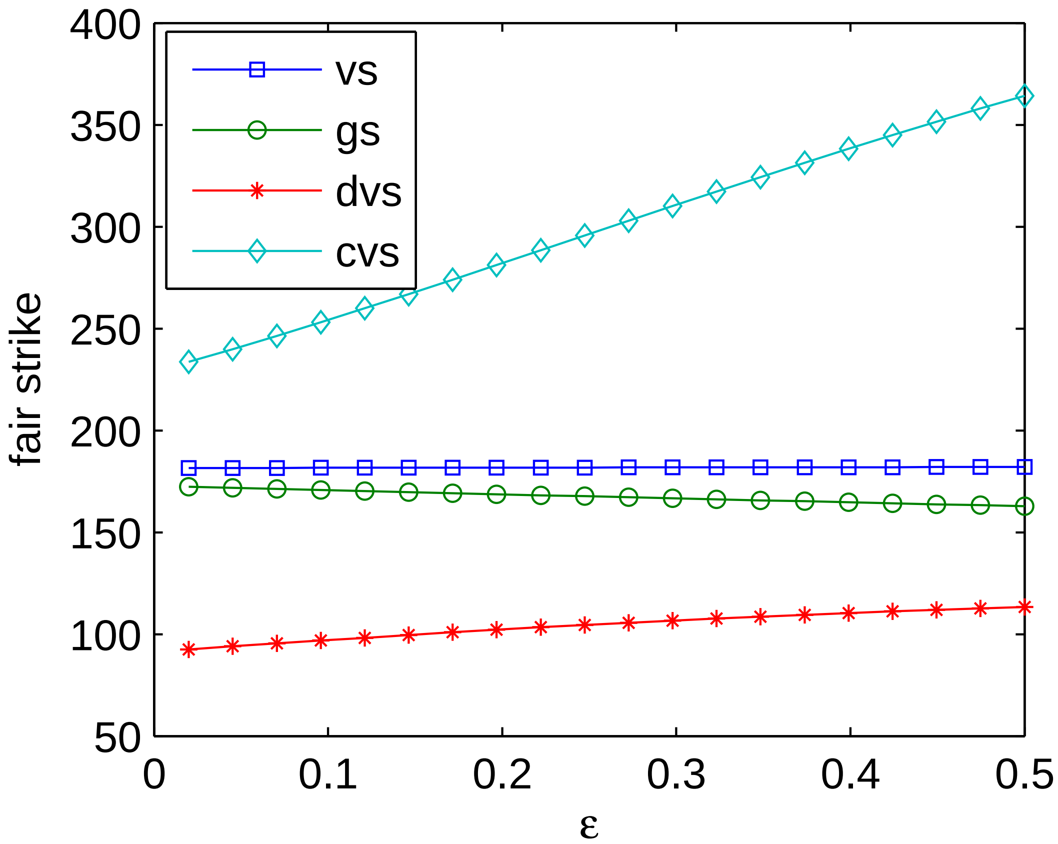
The convergence trend of the fair strike of the downside variance swaps can be nonlinear.

Fair strike against correlation coefficient (under the Heston model)



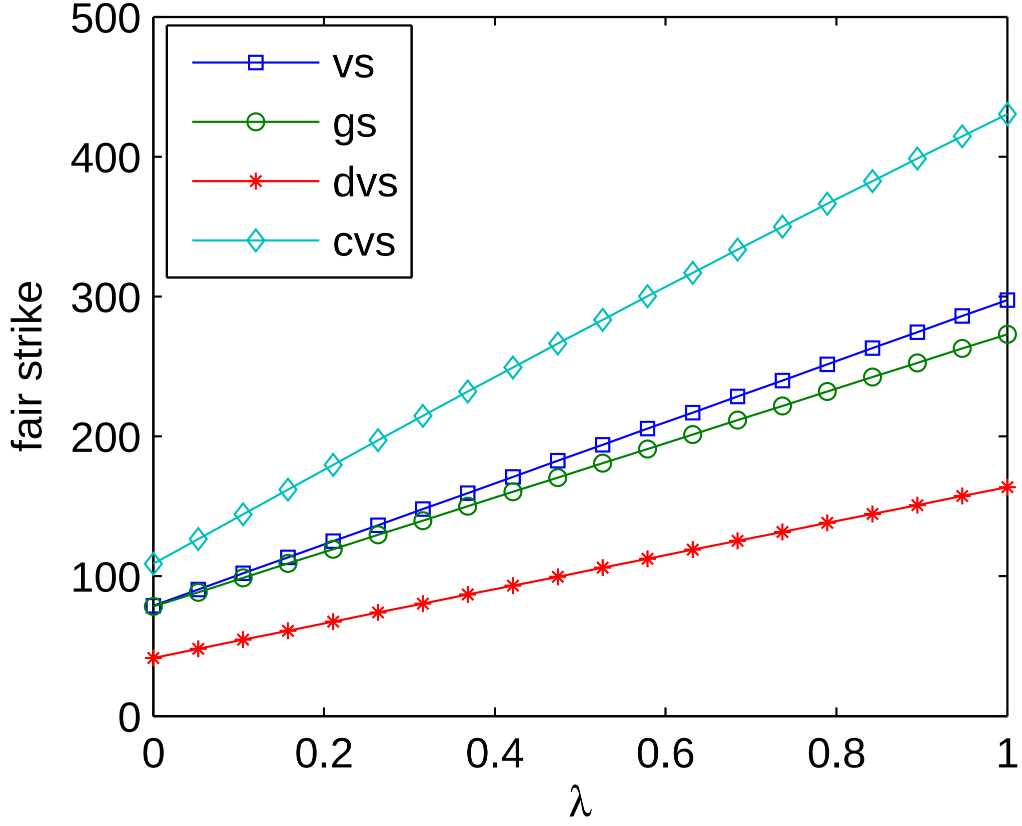
Since the corridor feature is sensitive to asset price, the fair strikes of conditional variance swaps are highly sensitive to correlation coefficient, ρ. The fair strike of the vanilla swap and gamma swap are almost insensitive to ρ.

Fair strike against volatility of variance (under the Heston model)



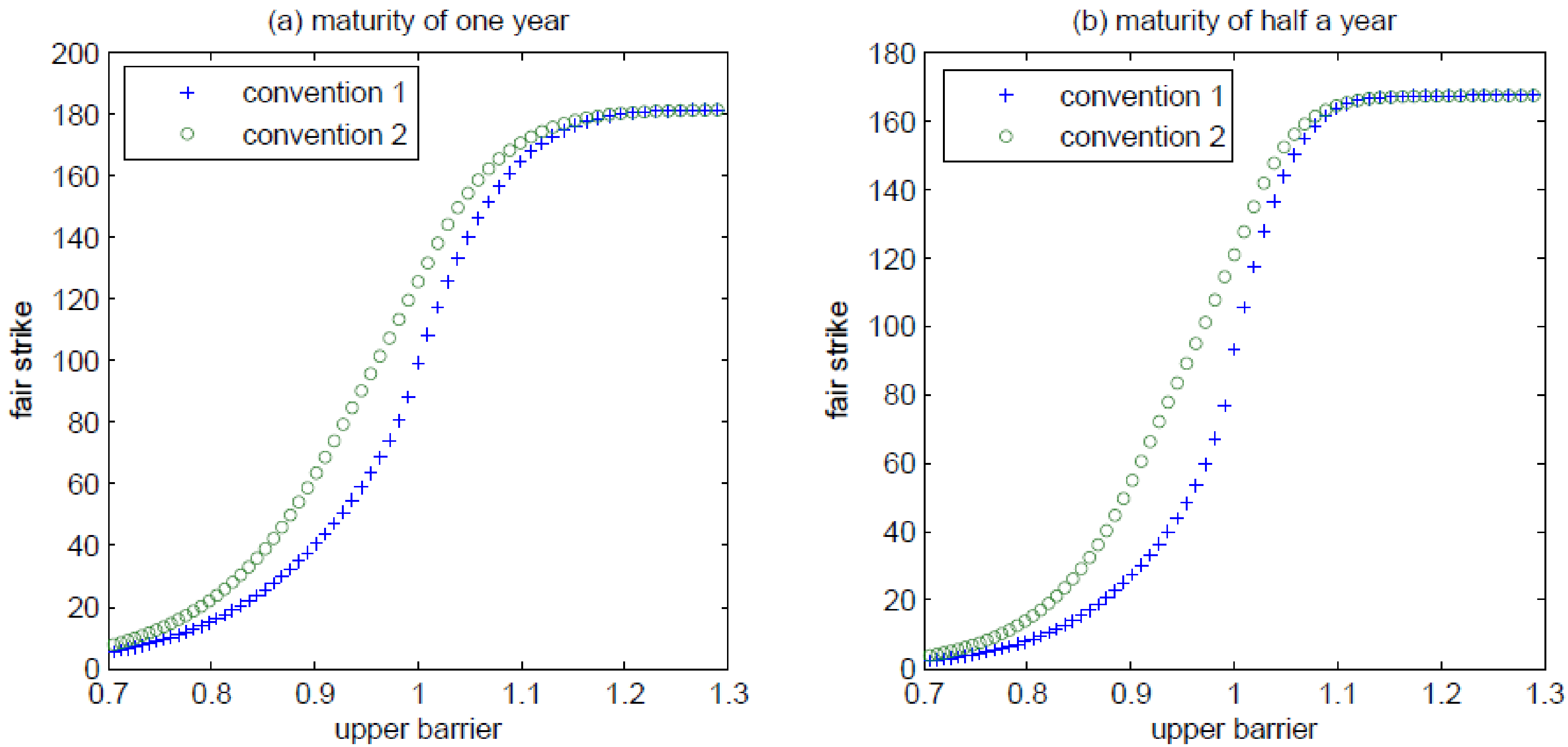
The fair strike prices of variance swaps and gamma swaps are almost insensitive to volatility of variance, ε.

Fair strike against jump intensity (under the Heston model)



The fair strike prices of the corridor variance swaps show the highest sensitivity to the jump intensity, λ.

Downside variance swaps – choice of the time level for monitoring the corridor feature



Comparison of the fair strike prices of the weekly sampled downside-variance swaps with varying values of the corridor's upper barrier when breaching of the corridor is monitored on the stock price at the old time level ("convention 1") or new time level ("convention 2").

Observations

- The difference in the fair strike prices of the two different types of downside-variance swaps, corresponding to the corridor's upper barrier U being monitored on the stock price at the old time level or new time level, can be quite substantial when the upper bound is below the current stock price S_0 (here, $S_0 = 1$).

- When U is close to S_0, the uncertainty that the upper barrier is breached at the subsequent sampling dates is relatively high. Therefore, the choice of the stock price either at the old or new time level that is used for monitoring becomes more significant when U is close to but below S_0.

Pricing of timer options under the 3/2-model

We use the quadratic variation I_t as a proxy of the discrete realized variance for the monitoring of the first hitting time. We define τ_B to be

$$\tau_B = \min\left\{ j \middle| I_{t_j} \geq B \right\} \Delta.$$

This approximation does not introduce noticeable error for daily monitored timer options. This is consistent with the observation that the difference in discrete realized variance and its continuous counterpart is very small.

$$\begin{aligned} C_0(X_0, I_0, V_0) &= \mathbb{E}_0[e^{-r(T\wedge\tau_B)} \max(S_{T\wedge\tau_B} - K, 0)] \\ &= \mathbb{E}_0[e^{-rT} \max(S_T - K, 0)\mathbf{1}_{\{\tau_B > T\}} \\ &\quad + e^{-r\tau_B} \max(S_{\tau_B} - K, 0)\mathbf{1}_{\{\tau_B \leq T\}}], \end{aligned}$$

where K is the strike price and r is the constant interest rate.

Decomposition into timerlets

The event $\{\tau_B > t\}$ is equivalent to $\{I_t < B\}$. Note that $\tau_B = t_{j+1}$ if and only if $I_{t_j} < B$ and $I_{t_{j+1}} \geq B$. Therefore, we have

$$\{\tau_B \leq T\} = \bigcup_{j=0}^{N-1} \{I_{t_j} < B, I_{t_{j+1}} \geq B\}.$$

The price of a finite-maturity discrete timer call option can be conveniently computed by decomposing it into a European call option conditional on $\tau_B > T$ and a portfolio of timerlets as follows

$$\begin{aligned} C_0 &= \mathbb{E}_0[\, e^{-rT} \max(S_T - K, 0) \mathbf{1}_{\{I_T < B\}}] \\ &+ \ \mathbb{E}_0\left[\sum_{j=0}^{N-1} e^{-rt_{j+1}} \Big(\max(S_{t_{j+1}} - K, 0) \mathbf{1}_{\{I_{t_j} < B\}} \right. \\ & \qquad \qquad \left. - \ \max(S_{t_{j+1}} - K, 0) \mathbf{1}_{\{I_{t_{j+1}} < B\}} \Big) \right]. \end{aligned}$$

Evaluation of the timelets

To evaluate the above series of expectations, we derive the explicit representation for the characteristic functions of (X_{t_j}, I_{t_j}) and $(X_{t_{j+1}}, I_{t_j})$. Note that

$$\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_j}+\mathrm{i}\eta I_{t_j}}] = e^{\mathrm{i}\omega X_0+\mathrm{i}\eta I_0} h(t_0, V_0; t_j, \omega, \eta),$$

where

$$\begin{aligned} h(t, v; t', \omega, \eta) &= \int_0^\infty g(t, V_t; t', \omega, \eta, v')\ dv' \\ &= e^{a(t'-t)} \frac{\Gamma(\tilde{\beta} - \tilde{\alpha})}{\Gamma(\tilde{\beta})} \left(\frac{1}{C_t v}\right)^{\tilde{\alpha}} M\left(\tilde{\alpha}, \tilde{\beta}, -\frac{1}{C_t v}\right), \end{aligned}$$

$$\tilde{\alpha} = -\frac{1}{2} - \frac{\tilde{\kappa}}{\varepsilon^2} + c, \quad \tilde{\beta} = 1 + 2c,$$

Γ is the gamma function, M is the confluent hypergeometric function of the first kind.

The expectation calculation

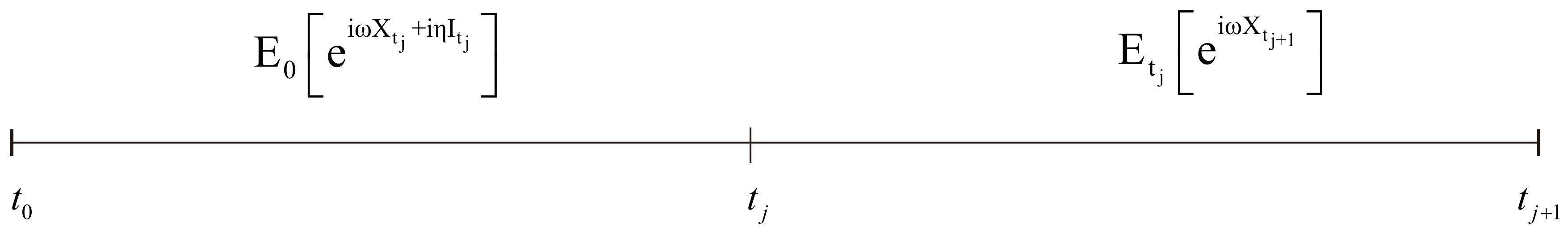
$$E_0\left[e^{-rt_{j+1}}\max(S_{t_{j+1}} - K, 0)\mathbf{1}_{\{I_{t_j}<B\}}\right]$$

requires the joint characteristic function of state variables across successive time instants t_j and t_{j+1}.

The bivariate joint characteristic function can be obtained as

$$\begin{aligned} & \mathbb{E}_0[e^{\mathrm{i}\omega X_{t_{j+1}}+\mathrm{i}\eta I_{t_j}}] \\ = \quad & e^{\mathrm{i}\omega X_0+\mathrm{i}\eta I_0}\int_0^\infty h(t_0, V_0; t_j, \omega, \eta, v')g(t_j, v'; t_{j+1}, \omega, 0)\ \mathrm{d}v'. \end{aligned}$$

This involves a two-step expectation calculation. Working backward in time from t_{j+1} to t_j, we first compute $E_{t_j}[e^{i\omega X_{t_{j+1}}}]$; next from t_j to t_0, we then compute $E_0[e^{i\omega X_{t_j}+i\eta I_{t_j}}]$. Here, we integrate over v', where v' is the dummy variable for the instantaneous variance V_{t_j}.



Discounted risk neutral expectation of payoff

We transform the integration of the product of the terminal payoff and transition density function from the real domain to the Fourier domain via Parseval's theorem.

One-dimensional Parseval theorem

$$< f, g > = \int_{-\infty}^{\infty} f(x)\overline{g}(x)\ dx = \frac{1}{2\pi} < \mathcal{F}_f(u), \mathcal{F}_g(u) >$$

The Fourier transform formulas of the payoff functions, $(S_{t_{j+1}} - K, 0)\mathbf{1}_{\{I_{t_j}<B\}}$ and $(S_{t_{j+1}} - K, 0)\mathbf{1}_{\{I_{t_{j+1}}<B\}}$, admit the same analytic representation:

$$\widehat{F}(\omega, \eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathrm{i}\omega x - \mathrm{i}\eta y}(e^x - K)^+\mathbf{1}_{\{y<B\}}\ \mathrm{d}x\mathrm{d}y = \frac{K^{1-\mathrm{i}\omega}e^{-\mathrm{i}\eta B}}{(\mathrm{i}\omega + \omega^2)\mathrm{i}\eta},$$

where x stands for $\ln S_{t_{j+1}}$ and y stands for I_{t_j} or $I_{t_{j+1}}$. We consider the generalized Fourier transform and take the transform variables ω and η to be complex. We write $\omega = \omega_R + i\omega_I$ and $\eta = \eta_R + i\eta_I$, where $\omega_I < -1$ and $\eta_I > 0$.

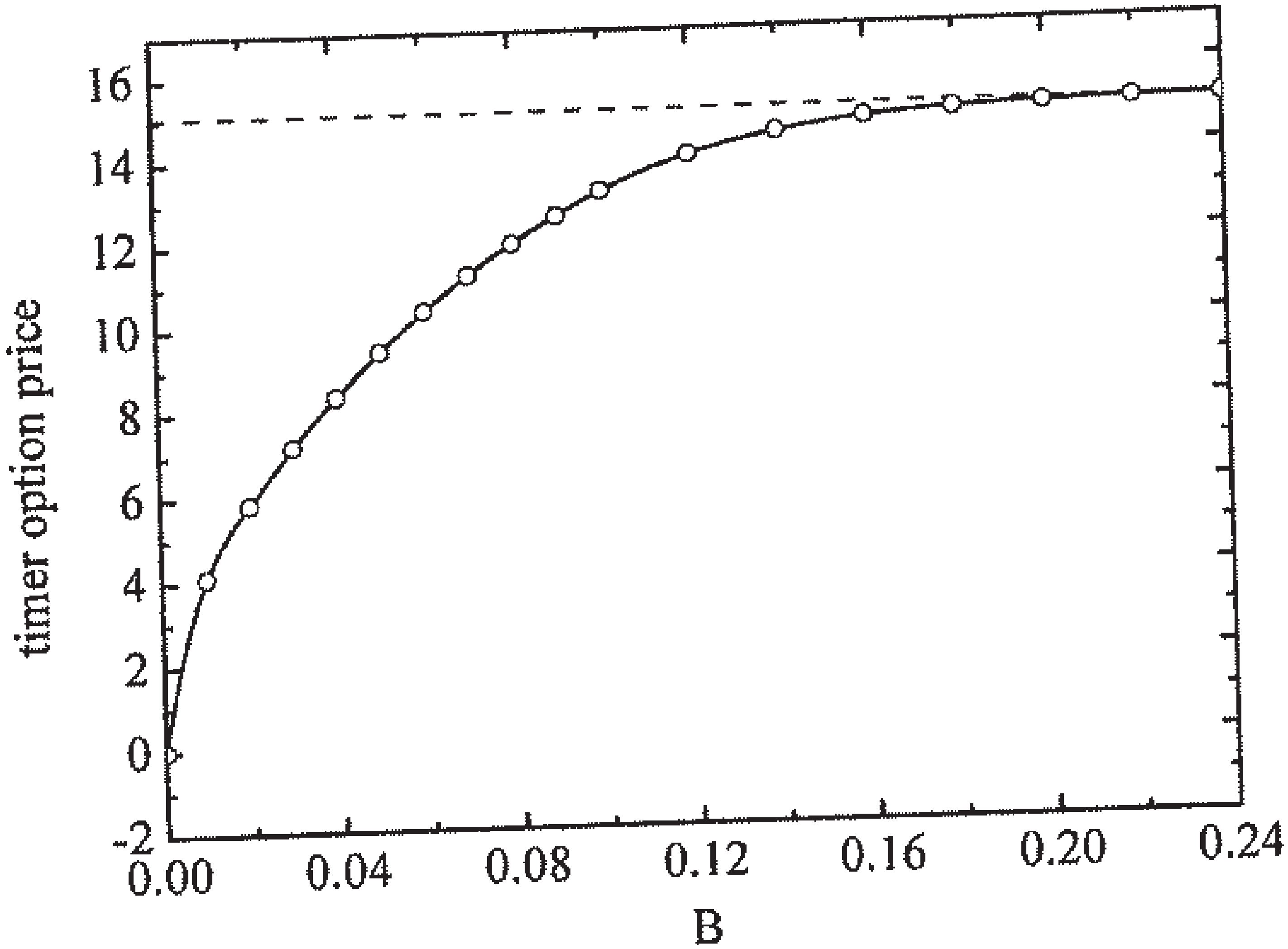
The finite-maturity discrete timer option price can be derived as follows

$$
\begin{aligned}
C_0 &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-rT}\widehat{F}(\omega,\eta)\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_N}+\mathrm{i}\eta I_{t_N}}]\ \mathrm{d}\omega_R \mathrm{d}\eta_R \\
&\quad + \sum_{j=0}^{N-1}\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-rt_{j+1}} \\
&\quad \left\{\widehat{F}(\omega,\eta)\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_{j+1}}+\mathrm{i}\eta I_{t_j}}] - \widehat{F}(\omega,\eta)\mathbb{E}_0[e^{\mathrm{i}\omega X_{t_{j+1}}+\mathrm{i}\eta I_{t_{j+1}}}]\right\}\ \mathrm{d}\omega_R \mathrm{d}\eta_R \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{F}(\omega,\eta)H(\omega,\eta)\,\mathrm{d}\omega_R \mathrm{d}\eta_R,
\end{aligned}
$$

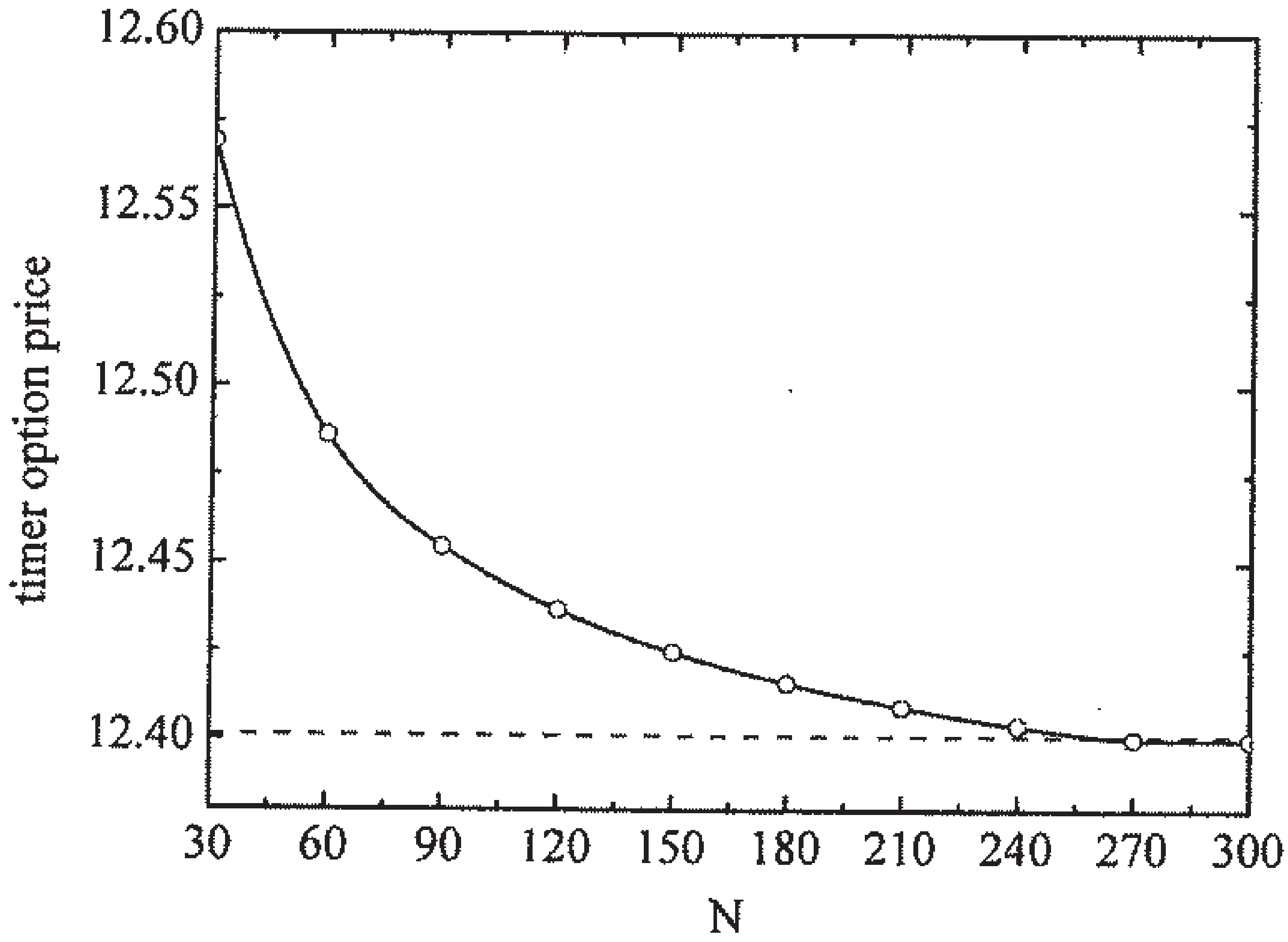
where

$$
\begin{aligned}
H(\omega,\eta) &= e^{-rT}e^{i\omega X_0+i\eta I_0}h(t_0,V_0;t_N,\omega,\eta) + e^{\mathrm{i}\omega X_0+\mathrm{i}\eta I_0}\sum_{j=0}^{N-1} e^{-rt_{j+1}} \\
&\quad \left[\int_0^{\infty} h(t_0,V_0;t_j,\omega,\eta,v')g(t_j,v';t_{j+1},\omega,0)\ \mathrm{d}v' - h(t_0,V_0;t_{j+1},\omega,\eta)\right].
\end{aligned}
$$

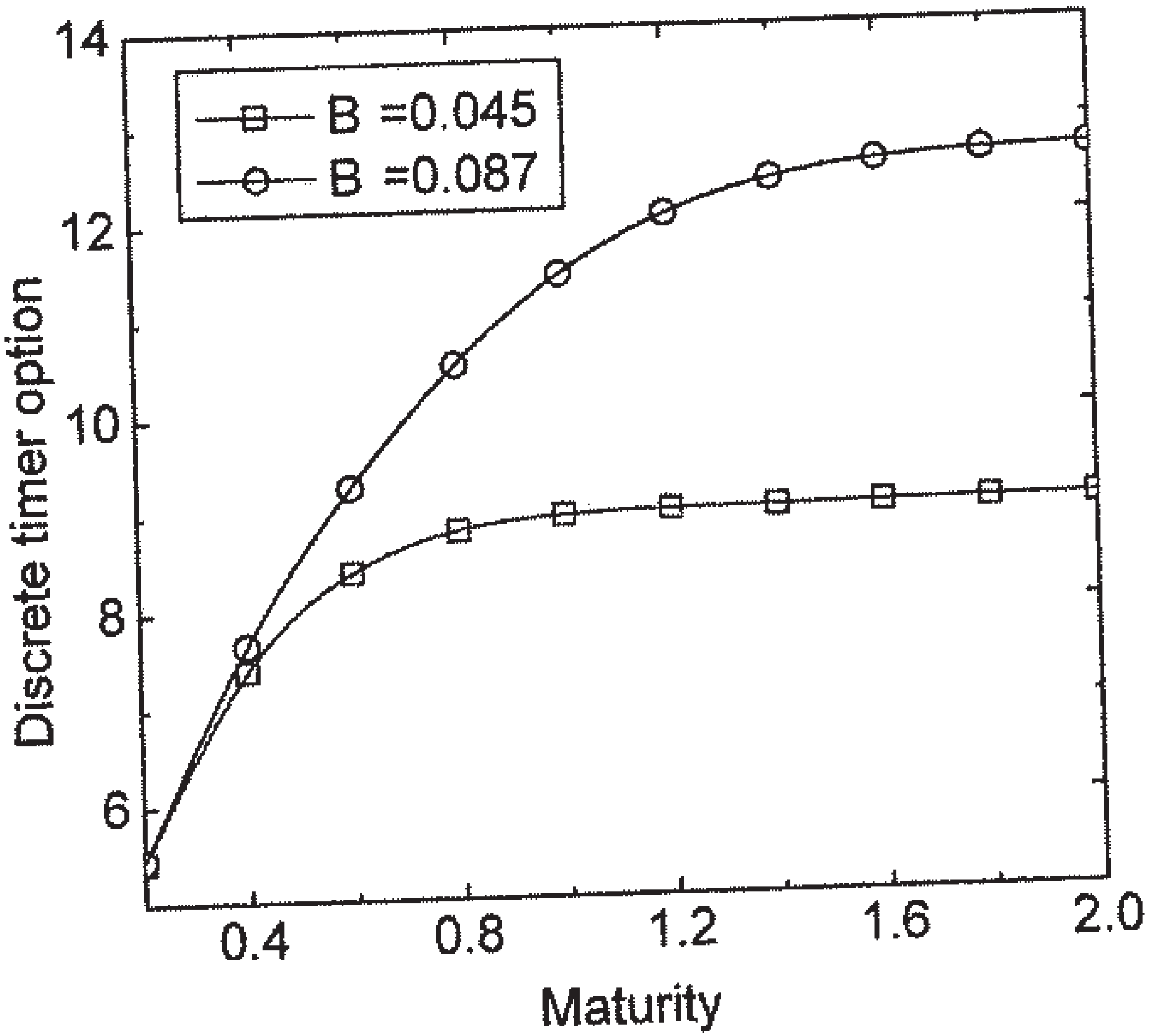
Computational challenge in the evaluation of the closed form pricing formula: triple integration is required where the integrand involves the modified Bessel functions of complex order.



Plot of the finite-maturity discrete timer call option prices against variance budget B. The discrete timer call option price reduces to the vanilla European call option when B is sufficiently large.



Plot of the finite-maturity discrete timer call option prices against number of monitoring instants N. The dashed line represents the finite-maturity timer call option price under continuous monitoring.



Plot of the finite-maturity discrete timer call option price versus maturity under two different values of variance budget.

Conclusion

Analytic procedures

- The analytical tractability relies on the availability of the joint moment generating function of the SVSJ model. The analytic price formula of the continuously monitored counterpart can be deduced by taking the asymptotic limit of vanishing time interval between successive monitoring dates. The analytic procedure can be applied to other higher moments swaps.

- By decomposing a timer option into a European option conditional on no knock-out and a portfolio of timerlets, we manage to price a finite-maturity timer option under the 3/2-model of stochastic volatility based on the explicit representation of the joint characteristic function of log asset price and its quadratic variation.

Pricing behaviors

- The sensitivity of the fair strike price on sampling frequency is low for gamma swaps but it can be significant for variance swaps with the corridor feature.

- The general belief of linear rate of convergence of $1/N$, where N is the number of monitoring instants, is shown to be invalid for exotic swap products under the SVSJ models.

- The fair strike prices of the corridor type variance swaps can be highly sensitive to the contractual terms in the contracts and the model parameter values (like volatility of variance, correlation coefficient, etc).

- The price of a timer option may depend sensibly on the choice of the variance budget and mandated expiration date.