Advanced Numerical Methods

Solution to Homework One

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1. When the asset pays *continuous dividend yield* at the rate q, the expected rate of return of the asset is $r-q$ under the risk neutral measure (see Chap 3 of Kwok's text for justification). Under the continuous Geometric Brownian process model, the logarithm of the asset price ratio over Δt interval is normally distributed with mean $\left(r-q-\frac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2\Delta t$. Accordingly, the mean and variance of $\frac{S_{t+\Delta t}}{S_t}$ are $e^{(r-q)\Delta t}$ and $e^{2(r-q)\Delta t}(e^{\sigma^2\Delta t}-1)$. By equating the mean and variance of the discrete binomial model and the continuous Geometric Brownian process model, we obtain

$$pu+(1-p)d=e^{(r-q)\Delta t}$$
$$pu^2++(1-p)d^2=e^{2(r-1)\Delta t}e^{\sigma^2\Delta t}.$$

Also, we use the usual tree-symmetry condition: $u=1/d$. Solving the equations, we obtain

$$u=\frac{1}{d}=\frac{\widetilde{\sigma}^2+1+\sqrt{(\widetilde{\sigma}^2+1)^2-4R^2}}{2R},\quad p=\frac{R-d}{u-d},$$

where $R=e^{(r-q)\Delta t}$ and $\widetilde{\sigma}^2=R^2e^{\sigma^2\Delta t}$. As an analytic approximation to u and d up to order Δt accuracy, we take

$$u=e^{\sigma\sqrt{\Delta t}}\quad\text{and}\quad d=e^{-\sigma\sqrt{\Delta t}}.$$

The only change occurs in the binomial parameter p, where

$$p=\frac{e^{(r-q)\Delta t}-d}{u-d},$$

while u and d remain the same. The binomial pricing formula takes a similar form (discounted expectation of the terminal payoff):

$$V=[pV^{\Delta t}+(1-p)V^{\Delta t}]e^{-r\Delta t}.$$

The discount factor $e^{-r\Delta t}$ remains the same while the risk neutral probability of up move p is modified.

2. (a) With the usual notation

$$p=\frac{R-d}{u-d}\quad\text{and}\quad 1-p=\frac{u-R}{u-d}.$$

If $R<d$ or $R>u$, then one of the two probabilities is negative. This happens when

$$e^{(r-q)\Delta t}<e^{-\sigma\sqrt{\Delta t}}$$

or

$$e^{(r-q)\Delta t}>e^{\sigma\sqrt{\Delta t}}.$$

This in turn happens when $(q-r)\sqrt{\Delta t}>\sigma$ or $(r-q)\sqrt{\Delta t}>\sigma$. Hence negative probabilities occur when

$$\sigma<|(r-q)\sqrt{\Delta t}|.$$

This result places a restriction on the time step. More precisely, the time step cannot be chosen to be larger than $\sigma^2/(r-q)^2$. If σ happens to be small, then the restriction can be quite severe.

(b) We approximate $\ln \dfrac{S_{t+\Delta t}}{S_t}$ by ζ^a, where

$$\zeta^a = \begin{cases} v_1 & \text{with probability } 0.5 \\ v_2 & \text{same probability} \end{cases}.$$

Matching the mean and variance, we obtain

$$\begin{aligned} E[\zeta^a] &= \frac{v_1 + v_2}{2} = \left(r - q - \frac{\sigma^2}{2}\right)\Delta t \\ \mathrm{var}(\zeta^2) &= \frac{v_1^2 + v_2^2}{2} = \sigma^2 \Delta t \text{ [after dropping } O((\Delta t)^2) \text{ term].} \end{aligned}$$

Solving the equation [up to $O(\Delta t)$ accuracy], we obtain

$$v_1 = \left(r - q - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} \quad \text{and} \quad v_2 = \left(r - q - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}.$$

Recall that

$$v_1 = \ln u \text{ so that } u = e^{\left(r-q-\frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}}$$

and

$$v_2 = \ln d \text{ so that } d = e^{\left(r-q-\frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}$$

As a check, we consider

$$v_1^2 + v_2^2 = 2\left[\left(r - q - \frac{\sigma^2}{2}\right)\Delta t\right]^2 + 2\sigma^2\Delta t$$

so that

$$\frac{v_1^2 + v_2^2}{2} = \sigma^2 \Delta t + O((\Delta t)^2).$$

Remark: In the usual set of binomial parameters, we take $v_1 = -v_2 = \sigma\sqrt{\Delta t}$. The drift rate in the dynamics of $\ln \dfrac{S_{t+\Delta t}}{S_t}$ is reflected in taking different probability values for the up jump and down jump. Here, we set $p = 0.5$, the drift rate is reflected in adding the drift movement $\left(r - q - \dfrac{\sigma^2}{2}\right)\Delta t$ over the time interval Δt.

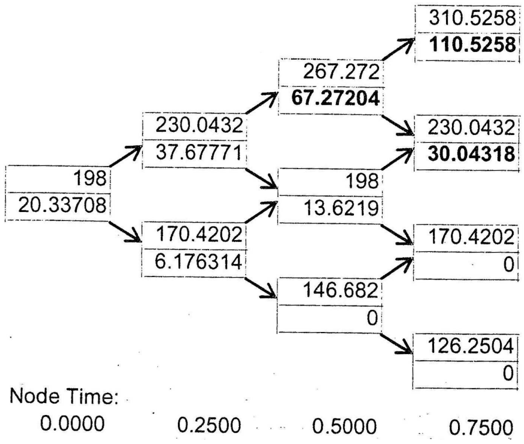
3. In this case, $F_0 = 198, X = 200, r = 0.08, \sigma = 0.3, T = 0.75$, and $\Delta t = 0.25$. Also

$$\begin{aligned} u &= e^{0.3\sqrt{0.25}} = 1.1618 \\ d &= \frac{1}{u} = 0.8607 \\ R &= 1 \\ p &= \frac{R - d}{u - d} = 0.4626 \\ 1 - p &= 0.5373. \end{aligned}$$

The output is shown in the Figure below. The calculated price of the option is 20.34 cents.

Growth factor per step, $R = 1.0000$
Probability of up move, $p = 0.4626$

Up step size, $u = 1.1618$
Down step size, $d = 0.8607$



The bold numbers represent the payoff from early exercise of the American futures option.

4. A binomial tree cannot be applied in a straightforward manner. This is an example of what is known as a history-dependent option. The payoff depends on the path followed by the stock price as well as its final value. The option cannot be simply valued by starting at the end of the tree and working backward since the payoff at the final branches is not known unambiguously. An efficient approach is the Forward Shooting Grid technique.

5. Suppose a dividend equal to D is paid during a certain time interval. If S is the stock price at the beginning of the time interval, it will be either $Su - D$ or $Sd - D$ at the end of the time interval. At the end of the next interval, it will be one of $(Su - D)u, (Su - D)d, (Sd - D)u$ and $(Sd - D)d$. Since $(Su - D)d$ does not equal $(Sd - D)u$, the tree does not recombine. If S is equal to the stock price less the present value of future dividends, this problem is avoided.

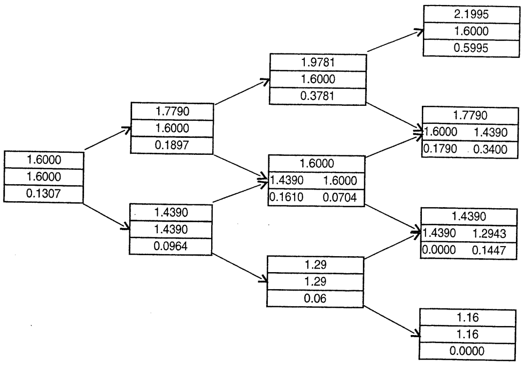
6. In this case $S_0 = 1.6, r = 0.05, r_f = 0.08, \sigma = 0.15, T = 1.5, \Delta t = 0.5$. This means that

$$\begin{aligned} u &= e^{0.15\sqrt{0.5}} = 1.1119 \\ d &= \frac{1}{u} = 0.8994 \\ R &= e^{(0.05-0.08)\times 0.5} = 0.9851 \\ p &= \frac{R-d}{u-d} = 0.4033 \\ 1-p &= 0.5967. \end{aligned}$$

The option pays off

$$S_T - S_{min}.$$

The tree is shown in the Figure below. At each node, the upper number is the exchange rate, the middle number(s) are the minimum exchange rate(s) so far, and the lower number(s) are the value(s) of the option. The value of the option today is found to be 0.1307.



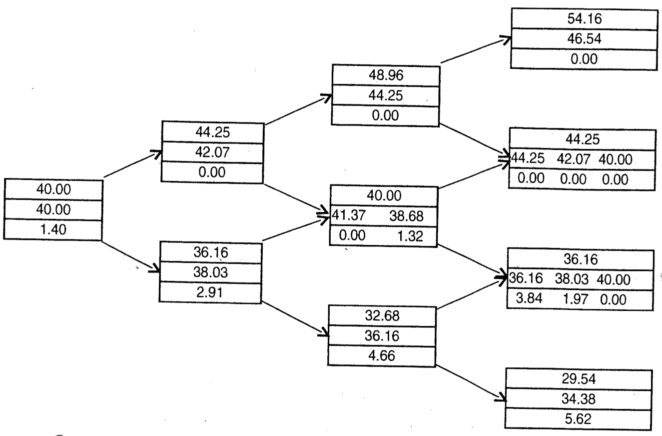
7. In this case, $S_0 = 40, X = 40, r = 0.01, \sigma = 0.35, T = 0.25, \Delta t = 0.08333$. This mean that

$$
\begin{aligned}
u &= e^{0.35\sqrt{0.08333}} = 1.1063 \\
d &= \frac{1}{u} = 0.9039 \\
R &= e^{0.1\times 0.08333} = 1.008368 \\
p &= \frac{R-d}{u-d} = 0.5161 \\
1-p &= 0.4839.
\end{aligned}
$$

The option pays off

$$40 - \overline{S}.$$

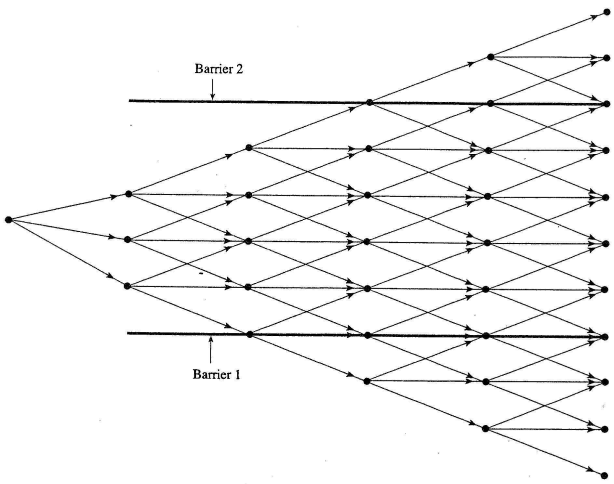
where $\overline{S}$ denotes the geometric average. The tree is shown in the Figure. At each node, the upper number is the stock price, the middle number(s) are the geometric average(s), and the lower number(s) are the value(s) of the option. The geometric averages are calculated using the first, the last and all intermediate stock prices on the path. The tree shows that the value of the option today is \$1.40.



Remark: In general, the number of possible geometric average values at the nodes that are n time steps from the tip of the binomial tree can be $C_1^n, C_2^n, \cdots, C_n^n$ where C_k^n is the binomial coefficient (number of ways of choosing k objects from n objects). For example, how many paths that lead to the second upper node at maturity? Out of the 3 time steps, we choose one step to move down and the other two to move up. The number of paths is C_1^3. For a n-step binomial tree, the total number of possible averaging values is 2^n.

8. Suppose that there are two horizontal barriers, H_1 and H_2, with $H_1 > H_2$ and that the underlying stock price follows geometric Brownian motion. In a trinomial tree, there are three possible movements in the asset's price at each node: up by a proportional amount u; stay the same; and down by a proportional amount d where $d = 1/u$. We can always choose u so that the nodes lie on both barriers. The condition that must be satisfied by u is
$$H_2 = H_1 u^N$$
or
$$\ln H_2 = \ln H_1 + N \ln u$$
for some integer N.

Tree with nodes lying on each of two barriers is shown in the Figure. It may occur that the initial asset price may not lie on any one of these horizontal rows. In this case, it may be necessary to adjust the branching in the first time step (see the Figure).



9. Applying the distributive rule, we have
$$\begin{aligned} & \max(\min(P_{cont}, K), X - S_j^n) \\ = \; & \min(\max(P_{cont}, X - S_j^n), \max(K, X - S_j^n)), \end{aligned}$$
which gives the same dynamic programming procedure derived from the perspective of the issuer.

Financial interpretation

The issuer's calling right enforces a non-called American put to have value below K. When P_{cont} is above K, the American put is called. The holder can take the maximum of K (receiving the cash K) or the exercise payoff $X - S_j^n$. When P_{cont} is below K, the holder can still choose the maximum of $X - S_j^n$ and P_{cont} as in a non-callable American put option.

10. Unlike the derivation in the lecture note, we now keep all the terms that are $O((\Delta t)^2)$. From the second equation, we obtain
$$v^2 = \left(r - \frac{\sigma^2}{2}\right) \Delta t^2 + \sigma^2 \Delta t.$$
Once v is obtained, by substituting into the first equation, we obtain
$$p = \frac{1}{2}\left[1 + \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta t}{\sqrt{\sigma^2 \Delta t + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2}}\right].$$

11. Consider the system of equations for p_1, p_2 and p_3:

$$\begin{pmatrix} 1 & 1 & 1 \\ u & 1 & d \\ u^2 & 1 & d^2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ R \\ W \end{pmatrix}.$$

Eliminating p_2 from the equations, we obtain

$$\begin{aligned} (u-1)p_1 + (d-1)p_3 &= R - 1 \\ (u^2-1)p_1 + (d^2-1)p_3 &= W - 1. \end{aligned}$$

Solving for p_1 and p_3 gives

$$p_1 = \frac{(W-R)u - (R-1)}{(u-1)(u^2-1)} \quad \text{and} \quad p_3 = \frac{(W-R)u^2 - (R-1)u^3}{(u-1)(u^2-1)}.$$

When $\lambda = 1$, the parameter u agrees with that of the Cox-Rubinstein-Ross binomial scheme. We expect to have

$$p_1 + p_3 = 1 + O(\Delta t),$$

or equivalently,

$$p_2 = O(\Delta t).$$

12. The largest and the smallest asset price at the extreme nodes at expiry are S_0e^{nu} and S_0e^{-nu}, respectively. With respect to $\ln S$, the width of the interval between the largest value of $\ln S$ and the smallest value of $\ln S$ is given by $(\ln S_0 + n \ln u) - (\ln S_0 - n \ln u) = 2n \ln u = 2n\sigma\sqrt{\Delta t}$. Let n denote the total number of time steps in the trinomial tree. Since $n\Delta t = T =$ life of the option, which is a finite quantity, the width of the interval $= 2\sqrt{n}\sigma\sqrt{T} \sim \sqrt{n}$.

13. By equating the corresponding mean, variances and covariances (up to $O(\Delta\tau)$ accuracy), we have

$$\begin{aligned}
E[\zeta_1^a] &= v_1(p_1+p_2+p_3+p_4-p_5-p_6-p_7-p_8) = \left(r - \frac{\sigma_1^2}{2}\right)\Delta t && \text{(i)} \\
E[\zeta_2^a] &= v_2(p_1+p_2-p_3-p_4+p_5+p_6-p_7-p_8) = \left(r - \frac{\sigma_2^2}{2}\right)\Delta t && \text{(ii)} \\
E[\zeta_3^a] &= v_3(p_1-p_2+p_3-p_4+p_5-p_6+p_7-p_8) = \left(r - \frac{\sigma_3^2}{2}\right)\Delta t && \text{(iii)} \\
\mathrm{var}(\zeta_1^a) &= v_1^2(p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8) = \sigma_1^2\Delta t && \text{(iv)} \\
\mathrm{var}(\zeta_2^a) &= v_2^2(p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8) = \sigma_2^2\Delta t && \text{(v)} \\
\mathrm{var}(\zeta_3^a) &= v_3^2(p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8) = \sigma_3^2\Delta t && \text{(vi)} \\
E[\zeta_1^a\zeta_2^a] &= v_2^1(p_1+p_2-p_3-p_4-p_5-p_6+p_7+p_8) = \sigma_1\sigma_2\rho_{12}\Delta t && \text{(vii)} \\
E[\zeta_1^a\zeta_3^a] &= v_3^1(p_1-p_2-p_3-p_4-p_5-p_6+p_7+p_8) = \sigma_1\sigma_3\rho_{13}\Delta t && \text{(viii)} \\
E[\zeta_2^a\zeta_3^a] &= v_3^2(p_1-p_2-p_3+p_4+p_5-p_6-p_7+p_8) = \sigma_2\sigma_3\rho_{23}\Delta t && \text{(ix)}
\end{aligned}$$

Lastly, the sum of probabilities must be one so that

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 = 1. \qquad \text{(x)}$$

Recall that $v_1 = \lambda_1\sigma\sqrt{\Delta t}, v_2 = \lambda_2\sigma\sqrt{\Delta t}$ and $v_3 = \lambda_3\sigma\sqrt{\Delta t}$. In order that Eqs (iv), (v) and (vi) are consistent, we must set $\lambda_1 = \lambda_2 = \lambda_3$. Set the common value to be λ. These 3 equations then reduce to single equation:

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = \frac{1}{\lambda^2}.$$

There are 8 equations for the 9 unknowns. The solution to the probabilities values is obtained as:

$$
\begin{aligned}
p_1 &= \frac{1}{8}\left[\frac{1}{\lambda^2}+\frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r-\frac{\sigma_1^2}{2}}{\sigma_1}+\frac{r-\frac{\sigma_2^2}{2}}{\sigma_2}+\frac{r-\frac{\sigma_3^2}{2}}{\sigma_3}\right)\right. \\
&\qquad \left. +\frac{\rho_{12}+\rho_{13}+\rho_{23}}{\lambda^2}\right], \\
p_2 &= \frac{1}{8}\left[\frac{1}{\lambda^2}+\frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r-\frac{\sigma_1^2}{2}}{2}+\frac{r-\frac{\sigma_2^2}{2}}{2}-\frac{r-\frac{\sigma_3^2}{2}}{2}\right)\right. \\
&\qquad \left. +\frac{\rho_{12}-\rho_{13}-\rho_{23}}{\lambda^2}\right], \\
p_3 &= \frac{1}{8}\left[\frac{1}{\lambda^2}+\frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r-\frac{\sigma_1^2}{2}}{\sigma_1}-\frac{r-\frac{\sigma_2^2}{2}}{\sigma_2}+\frac{r-\frac{\sigma_3^2}{2}}{2}\right)\right. \\
&\qquad \left. +\frac{\rho_{13}-\rho_{12}-\rho_{23}}{\lambda^2}\right], \\
p_4 &= \frac{1}{8}\left[\frac{1}{\lambda^2}+\frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r-\frac{\sigma_1^2}{2}}{2}-\frac{r-\frac{\sigma_2^2}{2}}{2}-\frac{r-\frac{\sigma_3^2}{2}}{2}\right)\right. \\
&\qquad \left. +\frac{\rho_{23}-\rho_{13}-\rho_{12}}{\lambda^2}\right], \text{ etc.}
\end{aligned}
$$

14. If m is set equal to $\widehat{m}$, then the window Parisian feature reduces to the consecutive Parisian feature.

- We define a binary string $A = a_1, a_2, \cdots, aN_w$ of size N_w to represent the history of the asset price path falling inside or outside the knock-out region at the previous N_w consecutive monitoring instants prior to the current time. By notation, the value of a_p is et to be 1 if the asset price falls on or below the down barrier B at the p-th monitoring instant counting backward from the current time, and is set to be 0 if otherwise.
- There are altogether 2^{N_w} different strings to represent all possible breaching history of asset price paths at the previous N_w monitoring instants. The number of states that have to be recorded is $C_0^{N_2}+C_1^{N_w}+\cdots+C_{N-1}^{N_w}$, where $C_i^{N_w}$ denotes the combination of N_w denotes the combination of N_w strings taken i at a time. This is because the window Parisian option value becomes zero when the number of breaches reaches N, so those states with N or more "1" in the string are irrelevant.
- Let $V_{win}[m, j; A]$ denote the value of a window Parisian option at the (m, j)-th node, and with the asset price path history represented by the binary string A. The binary string A has to be modified according to the event of either breaching or no breaching at a monitoring instant.
- The corresponding numerical scheme can be succinctly represented by

$$
V_{win}[m-1, j; A] = \begin{cases}
\{p_u W_{win}[m, j+1; A] \\
+\, p_0 V_{min}[m, j; A] \\
+\, p_d V_{win}[m, j-1; A]\} e^{-r\Delta t} & \text{if } m\Delta t \neq t_\ell^* \\
\{p_u V_{win}[m, j+1; ; g_{win}(A, j+1)] \\
+\, p_0 V_{win}[m, j; g_{win}(A, j)] \\
+\, p_d V_{win}[m, j-1; g_{win}(A, j-1)]\} e^{-r\Delta t} & \text{if } m\Delta t = t_\ell^*
\end{cases}
$$

where

$$\begin{cases} 1_{a_1 a_2 \cdots a_{N_w-1}} & \text{if } x_j \le \ln B \\ 0_{a_1 a_2 \cdots a_{N_w-1}} & \text{if } x_j > \ln B \end{cases}.$$

As in the numerical procedure for the consecutive Parisian option, it is necessary to compute $V_{win}[m, j; A]$ for those states of A with $N-1$ or less "1" in the string before we move to a time level corresponding to a monitoring instant. Note that $V_{win}[m, j; A] = 0$ at a monitoring instant when the string A has N or more "1". Due to the higher level of path-dependence exhibited by the window feature, the operation counts of the window Parisian option calculations are about $C_0^{N_w} + C_1^{N_w} + \cdots + C_{N-1}^{N_w}$ times of those of plain vanilla option calculations.

15. The payoff of a floating strike lookback call is

$$\max_{\tau \in [0,T]} S_\tau - S_T,$$

where $\max_{\tau \in [0,T]} S_\tau$ denotes the realized *maximum* of the asset price over $[0, T]$. The corresponding grid function at the $(n, j)^{\text{th}}$ node with asset price $S_j^n = Su^j d^{n-j}$ is given by

$$g_{lookback}(k, j) = \max(k, j),$$

where k is the numbering index for the lookback state variable. The FSG algorithm is

$$\begin{aligned} V_{j,k}^{n-1} &= \left[p_u V^n_{j+1, g_{lookback}(k,j+1)} + p_0 V^n_{j, g_{lookback}(k,j)} \right. \\ &\quad \left. + \, p_d V^n_{j-1, g_{lookback}(k,j-1)} \right] e^{-r\Delta t}. \end{aligned}$$

To incorporate the American early exercise feature, we simply add the dynamic procedure at each node and for each number index:

$$\begin{aligned} V_{j,k}^{n-1} &= \max \left([p_u V^n_{j+1, g_{lookback}(k,j+1)} + p_0 V^n_{j, g_{lookback}(k,j)} \right. \\ &\quad \left. + \, p_d V^n_{j-1, g_{lookback}(k,j-1)}] e^{-r\Delta t}, Su^k d^{n-k} - Su^j d^{n-j} \right). \end{aligned}$$

16. The strike reset feature dictates the updated strike price at a prespecified reset date t_ℓ to be given by

$$X_\ell = \max(X_{\ell-1}, S(t_\ell)), \quad \ell = 1, 2, \cdots, m, \tag{A}$$

where X_0 is the original strike price and $S(t_\ell)$ is the asset price at t_ℓ.

- If we apply the backward induction procedure in a trinomial calculation for pricing the reset option, we encounter the difficulty in evaluating the terminal payoff since the strike price is not yet known. The difficulty arises because the strike price adopted in the payoff depends on realization of the asset price on the trinomial tree.
- Let m_ℓ denote the number of time steps counting from the top node of the trinomial tree to the ℓ-th reset dates is $2m_\ell + 2, \ell = 0, 1, \cdots, M$. Here, the 0-th reset date and the $(M+1)$-th reset date are taken to be the inception time and the expiration date, respectively. We have $(2m_\ell + 2)$ possible strike prices, since there are $(2m_\ell + 1$ possible asset values at the time level that is m_ℓ time steps from the top node of the trinomial tree, and the one additional possible strike price is the original strike price X set at initiation of the option contract.
- When we follow the backward induction procedure in the reset option calculation, we first compute the terminal payoff values for all possible strike prices ($2m_M + 2$ of them). Now, the augmented state vector at each lattice node in the FSG algorithm includes all possible strike prices. As we proceed backwards, in particular at a time level corresponding to a reset date, the vector of strike prices will be adjusted according to the rule stated in Equation (A).

- Let k denote the index relating to the logarithm of the strike price x_k (recall that $x_k = \ln S + k\Delta x$, where S is the asset value at the top of the trinomial tree), and write $V_{res}[m, j; k]$ as the numerical value of the reset option at the (m, j)-th node with (log) strike price x_k. Let the original strike price X be related to the index value k_0 by $x_{k_0} = \ln X = \ln S + k_0 \Delta x$.
- The construction of the FSG algorithm for pricing the reset call option gives

$$V_{res}[m-1, j; k] = \begin{cases} \{p_u V_{res}[m, j+1; k] \\ + p_0 V_{res}[m, j; k] \\ + p_d V_{res}[m, j-1; k]\} e^{-r\Delta t} & \text{if } m\Delta t \neq \widehat{t}_\ell \\ \{p_u V_{res}[m, j+1; g_{res}(k, j+1)] \\ + p_0 V_{res}[m, j; g_{res}(k, j)] \\ + p_d V_{res}[m, j-1; g_{res}(k, j-1)]\} e^{-r\Delta t} & \text{if } m\Delta t = \widehat{t}_\ell \end{cases},$$

  where the grid function is given by

$$g_{res}(k, j) = \min(k, j, k_0).$$

  At maturity (say, M_T time steps from the current time on the trinomial tree), the terminal payoff is given by

$$V_{res}[M_T, j; k] = \max(e^{x_j} - e^{x_k}, 0) \qquad \text{for } -M_T \leq j \leq M_T \text{ and } -m_M \leq k \leq m_M.$$