Solutions to Homework 3 and 4

Problem 1, Homework 3. Let $\sigma \in S_8$ be of the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 3 & 2 & 1 & 5 & 8 & 7 \end{pmatrix}.$$

(1) Computer σ^2. (2). Decompose σ as a product of disjoint cycles. (3). Compute the order of σ. (4). Compute σ^{-1}.

Answer: (1) $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 3 & 4 & 6 & 1 & 7 & 8 \end{pmatrix}$

(2) $\sigma = (1,6,5)(2,4)(7,8)$. (3) The order of σ is 6.

(4) $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 & 6 & 1 & 8 & 7 \end{pmatrix}$.

Problem 2, Homework 3. Let $\sigma \in S_8$ be of the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 6 & 3 & 2 & a & b & 1 & 7 \end{pmatrix}.$$

Suppose σ is an odd permutation,

(1). Find a and b. (2). Decompose σ as a product of disjoint cycles.

(3). Compute the order of σ. (4). Decompose σ^{-1} as a product of disjoint cycles. (5). Comupte σ^{2017}.

Answer: (1). $a = 4$, $b = 5$. (2). $\sigma = (1,8,7)(2,6,5,4)$. (3) The order of σ is 12. (4) $\sigma = (1,7,8)(2,4,5,6)$. (5). $2017 = 12 \cdot 168 + 1$, $\sigma^{2017} = (\sigma^{12})^{168}\sigma^1 = \sigma$.

Problem 3, Homework 3. Which of the following is a coset of the subgroup $H = \{e, (12)\}$ in S_3?

(1). $B_1 = \{(123), (132), e\}$.

(2). $B_2 = \{(123), (12)\}$.

(3). $B_3 = \{(123), (13)\}$.

(4). $B_4 = \{(123), (132)\}$.
(5). $B_5 = \{e, (132)\}$.

Answer: Every coset of H should have 2 elements, so B_1 is not a coset of H. (2). No. A quick way to see this is that B_2 intersects with H but no equal to H. (3). $B_3 = (13)H$ is a coset of H. (4). No. (5). No. B_5 intersects with H, but is not equal to H.

Problem 4, Homework 3. Let G be an abelian group, prove that $H = \{a \in G \mid a^3 = e\}$ is a subgroup of G.

Proof. Since $e^3 = e$, so $e \in H$. If $a, b \in H$, then $a^3 = b^3 = e$. $(ab)^3 = a^3b^3 = ee = e$, where the first "=" follows from the assumption G is abelian, so $ab \in H$. This proves H is closed. If $a \in H$, $a^3 = e$, taking inverse both sides, we get $(a^{-1})^3 = e$, so $a^{-1} \in H$. This proves H is a subgroup.

Problem 1, Homework 4. Determine if the following maps are homomorphisms of groups (No reasons needed).
(1). $\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(a) = 2018a$
(2). $\Phi : \mathbb{R}^* \to \mathbb{R}^*, \quad \Phi(a) = 2018a$
(3). $\Phi : \mathbb{R}^* \to \mathbb{R}^*, \quad \Phi(a) = a^{2018}$
(4). $\Phi : GL(n, \mathbb{R}) \to \mathbb{R}^*, \quad \Phi(A) = Det(A)^{10}$.
(5). $\Phi : \mathbb{R} \to \mathbb{R}^*, \quad \Phi(a) = 10^a$.
(6). $\Phi : \mathbb{R}^* \to \mathbb{R}, \quad \Phi(a) = 10^a$.
(7). $\Phi : S_5 \to S_5, \quad \Phi(\sigma) = \sigma^{120}$.

Answer: (1) Yes. (2) No. (3) Yes. (4) Yes. (5) Yes. (6) No.
(7) Yes (because $|S_5| = 5! = 120$, so $\sigma^{120} = e$ for all σ).

Problem 2, Homework 4. Find a homomorphism $\Phi : \mathbb{R}^* \to \mathbb{R}$ such that

$\Phi(2) = 3$.

Answer: $\Phi(a) = 3\,\log_2|a|$. Other answers are possible, also any solution that contains expression $\log_2 a$ is wrong, as $\log_s a$ is NOT defined for $a < 0$.

Problem 3, Homework 4. Let G be a group, H_1 and H_2 be finite subgroups of G. Suppose that $|H_1|$ and $|H_2|$ are relatively prime, prove that $H_1 \cap H_2$ has only one element (hint: use the Lagrange Theorem).

Method 1. It can be proved that $H_1 \cap H_2$ is a subgroup of G, so it is also a subgroup of H_1 and H_2. By Lagrange Theorem, $|H_1 \cap H_2|$ is a divisor of $|H_1|$ and $|H_2|$, so $|H_1 \cap H_2|$ is a common divisor of $|H_1|$ and $|H_2|$. The assumption that $|H_1|$ and $|H_2|$ are relatively prime implies that the only common divisor of $|H_1|$ and $|H_2|$ is 1. Therefore $|H_1 \cap H_2| = 1$. This means $H_1 \cap H_2 = \{e\}$.

Method 2. If $a \in H_1 \cap H_2$, then $a \in H_1$, so the order of a is a divisor of $|H_1|$. Similarly, the order of a is a divisor of $|H_2|$. So the order of a is a common divisor of $|H_1|$ and $|H_2|$. Because $|H_1|$ and $|H_2|$ are relatively prime, 1 is the only common divisor of $|H_1|$ and $|H_2|$, so the order of a is 1, so $a = e$. This proves $H_1 \cap H_2 = \{e\}$.

Problem 4, Homework 4. Let G, G' be finite groups. Suppose that $|G|$ and $|G'|$ are relatively prime. Prove that a homomorphism $\Phi : G \to G'$ must be trivial, i.e., $\Phi(a) = e'$ for all $a \in G$ (hint: Use the Lagrange Theorem).

Proof. Consider the image $\Phi(G)$, it is a subgroup of G' (See Theorem 13.12 (3)). By Lagrange Theorem, $|\Phi(G)|$ is a divisor of $|G'|$. By Homomorphism theorem (Theorem 14.11), $|G|/|H| = |\Phi(G)|$, where $H = Ker(\Phi)$, this implies that $|\Phi(G)|$ is a divisor of $|G|$. So $|\Phi(G)|$ is a common divisor of $|G'|$ and $|G|$. Because $|G|$ and $|G'|$ are relatively prime, so $|\Phi(G)| = 1$, so $\Phi(G) = \{e'\}$. This proves $\Phi(a) = e'$ for all $a \in G$.