Solutions to Homework 5.

Problem 1. (no reasons needed). Which of the following rings are integral domains, which of them are fields?
$\mathbb{Z}$, $\mathbb{Z}_{22}$, $\mathbb{Z}_{17}$, $\mathbb{Z}_{100}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Answer: Notice that 17 is a prime, while $22, 100$ are not primes.
So $\mathbb{Z}$, $\mathbb{Z}_{17}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are integral domains. And $\mathbb{Z}_{17}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields

Problem 2. Determine if each of the following maps is a ring homomorphism (no reasons needed)
(1). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(x) = -x$. **No.**
(2). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(x) = x^2$. **No**
(3). $\phi : \mathbb{Z} \times \mathbb{Z}$ given by $\phi((a, b)) = b$. **Yes**
(4). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(a + bi) = a - bi$. **Yes**
(5). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix}$. **Yes.**
(6). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. **Yes**
(7). $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\phi(x, y) = x$. **Yes**

Problem 3. Prove that $R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b, \in \mathbb{R}\}$ is a subring of $M_2(\mathbb{R})$. Find a ring homomorphism $\Phi : R \to \mathbb{R}$ that is onto.

Proof. If $\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \in R$, $\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \in R$, then

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ 0 & a_1 - a_2 \end{pmatrix} \in R$$

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 a_2 \\ 0 & a_1 a_2 \end{pmatrix} \in R.$$

So R is closed under substraction and multiplication. So it is a subring.
$\Phi : R \to \mathbb{R}$ given by

$$\Phi \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = a$$

is a ring homomorphism and onto.

Problem 4. Let R be a commutative ring with unity 1. An element $a \in R$ is called to be **nilpotent** if $a^n = 0$ for some positive integer n.
(1). Prove that if a, b are nilpotent, then so is $a + b$.
(2). Prove that H defined as

$$H = \{1 - a \mid a \in R \text{ is nilpotent } \}$$

is a group under the multiplication.
(3). Suppose R is finite with $|R| = N$, prove that, if $a \in R$ is nilpotent, then

$$(1 - a)^N = 1.$$

Proof. (1). Since a, b are nilpotent, so $a^m = 0$ and $b^n = 0$ for some positive integers m, n. Then $(a + b)^{m+n} = \sum_{j=0}^{m+n} \binom{m+n}{j} a^j b^{m+n-j}$. If $j \geq m$, $a^j = 0$; for $0 \leq j < m$, $m + n - j > n$, so $b^{m+n-j} = 0$. This proves every term $a^j b^{m+n-j} = 0$. So $(a + b)^{m+n} = 0$. So $a + b$ is nilpotent.
(2). First, $1 = 1 + 0 \in H$. We then prove H is closed. $(1 - a)(1 - b) = 1 - (a + b - ab)$. By (1), $a + b$ is nilpotent, since ab is nilpotent (prove it!), so $a + b - ab$ is nilpotent by (1). So $1 - (a + b - ab) \in H$. This proves H is closed. For a nilpotent, so $a^n = 0$ for some positive integer n. Then $(1-a)(1+a+\cdots+a^{n-1}) = 1-a^n = 1$. Since $-(a+\cdots+a^{n-1})$ is nilpotent, so $1+a+\cdots+a^{n-1} \in H$. So every element in H has an inverse in H. This proves H is a group under the multiplication.
(3). Set $S = \{a \in R \mid a \text{ is nilpotent}\}$. By (1), it is easy to see that S is a subgroup of $(R, +)$. So $|S|$ is a divisor of $|R| = N$, that is, $N = k|S|$. We have a bijection $T : S \to H, T(a) = 1-a$. So $|H| = |S|$. So $N = k|S| = k|H|$. By a corollary of Lagrange Theorem, $(1 - a)^{|H|} = 1$. So

$$(1 - a)^N = (1 - a)^{k|H|} = ((1 - a)^{|H|})^k = 1^k = 1.$$