Solutions to Quiz version 2

Problem 1. (25 points) Compute the order of the given element in a group.
(a). $4, 5, 6$ in $\mathbb{Z}_8$.
(b). i, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ in $\mathbb{C}^*$.

Answer: (a). The order of 4 is 2; the order of 5 is 8; the order of 6 is 4.
(b). The order of i is 4; the roder of $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ is 8.

Problem 2. (25 points) Determine whether the given set of invertible 2×2 matrices is a subgroup of $GL(2, \mathbb{R})$ (just answer "Yes" or "No", no reasons needed)
(1). The set of 2×2 upper triangular matrices with positive numbers on the diagonal. **Yes**
(2). The set of 2×2 upper triangular matrices with diagonal entries equal to 1. **Yes**
(3). The set of 2×2 diagonal matrices with diagonal entries greater than 1. **No**
(4). The set of 2×2 diagonal matrices with determinant 3. **No**
(5). The set of 2×2 diagonal matrices with determinant 1. **Yes**

Problem 3. (25 points) Let $\sigma \in S_8$ be of the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 7 & a & b & 8 & 1 & 5 \end{pmatrix}.$$

Suppose σ is an even permutation,
(1). Find a and b. (2). Decompose σ as a product of disjoint cycles.
(3). Compute the order of σ. (4). Comupte σ^{2015}.

Answer: (1). $a = 2, b = 4$. (2). $\sigma = (1, 3, 7)(2, 6, 8, 5, 4)$.
(3). The order of σ is 15.
(4). $2015 = 144 \cdot 15 + 5$, so $\sigma^{2015} = \sigma^5 = (1, 3, 7)^5 (2, 6, 8, 5, 4)^5 = (1, 3, 7)^5 = (1, 3, 7)^5 = (1, 7, 3)$

Problem 4. (20 points) (1). Suppose G is an abelian group, prove that $H = \{a \in G \mid a^2 = e\}$ is a subgroup of G.
(2). Find an example of a group G such that $H = \{a \in G \mid a^2 = e\}$ is NOT a subgroup of G, and give reasons.

Answer: (1) Since $e^2 = e$, so $e \in H$. If $a, b \in H$, the $(ab)^2 = abab = aabb = a^2b^2 = e$, where the second $=$ follows from the condition G is abelian. So $ab \in H$. This proves H is closed under the binary operation. If $a \in H$, then $a^2 = e$, so $a^{-1} = a \in H$. This proves H is a subgroup.
(b). The easiest example is $G = S_3$. Then

$$H = \{e, (1,2), (2,3), (1,3)\}$$

which is not a subgroup of S_3, as $(1,3)(1,2) = (1,2,3) \notin H$, it is not closed. Other possible answer is $GL(2, \mathbb{R})$ or $GL(n, \mathbb{R})$ for $n \geq 2$. You need to show H is not closed by explicitly giving an example. For example

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in H, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \notin H.$$

Problem 5. (5 points) Let G be a finite group such that the order $|G|$ is an odd number. Suppose $a \in G$ satisfies $a^4 = e$, prove that $a = e$.

Answer: Because the order of a is a divisor of $|G|$, so the order of a is an odd number since it is a divisor of odd number $|G|$. This together with $a^4 = e$ imply that the order of a is 1 or 3. If the order of a is 1, then $a = e$, we are done. If the order of a is 3, then $a^3 = e$, since $a^4 = e$, so $a = e$, contradiction. This completes the proof.

A slight different proof: Since $a^4 = e$, so the order of a is $1, 2$ or 4. The order of a is divisor of $|G|$ which is odd, so the order of a is odd. Therefore the order of a is 1, so $a = e$.