Homework No.5 for Math 3121

Due Date: Nov. 29

Problem 1. (no reasons needed). Which of the following rings are integral domains, which of them are fields?
$\mathbb{Z}$, $\mathbb{Z}_{22}$, $\mathbb{Z}_{17}$, $\mathbb{Z}_{100}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Problem 2. Determine if each of the following maps is a ring homomorphism (no reasons needed)
(1). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(x) = -x$.
(2). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(x) = x^2$.
(3). $\phi : \mathbb{Z} \times \mathbb{Z}$ given by $\phi((a, b)) = b$.
(4). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(a + bi) = a - bi$.
(5). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix}$.
(6). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.
(7). $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\phi(x, y) = x$.

Problem 3. Prove that $R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b, \in \mathbb{R}\}$ is a subring of $M_2(\mathbb{R})$. Find a ring homomorphism $\Phi : R \to \mathbb{R}$ that is onto.

Problem 4. Let R be a commutative ring with unity 1. An element $a \in R$ is called to be **nilpotent** if $a^n = 0$ for some positive integer n.
(1). Prove that if a, b are nilpotent, then so is $a + b$.
(2). Prove that H defined as

$$H = \{1 - a \mid a \in R \text{ is nilpotent } \}$$

is a group under the multiplication.
(3). Suppose R is finite with $|R| = N$, prove that, if $a \in R$ is nilpotent, then

$$(1 - a)^N = 1.$$