Math 4991, Lecture on May 4, 2020

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Today's Plan.

(1). About Project "Meromorphic Functions and Trigonometric Functions"

(2). About Project "Weierstrass Elliptic Functions"

Purpose of this project: To study trigonometry functions using complex analysis.

Relations of e^z and trigonometry functions:

$$e^{iz} = \cos\ z + i \sin\ z$$

$$\sin\ z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos\ z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\operatorname{cotan} z = \frac{\cos z}{\sin z} = \frac{(e^{iz} + e^{-iz})i}{e^{iz} - e^{-iz}} = \frac{(e^{2iz} + 1)i}{e^{2iz} - 1} \tag{1}$$

Because $e^{2iz} - 1 = 0$ precisely when $z \in \pi\mathbb{Z}$, we see that the set of poles of $\operatorname{cotan} z$ is $\pi\mathbb{Z}$, all poles are simple.

We will normalize the variable z to consider

$$\operatorname{cotan} \pi z.$$

It is easy to see that $\operatorname{cotan} \pi z$ is an odd function and has period 1, that is,

$$\operatorname{cotan} \pi(-z) = -\operatorname{cotan} \pi z, \quad \operatorname{cotan} \pi(z+1) = \operatorname{cotan} \pi z.$$

$\mathbb{Z}$ are the poles of $\operatorname{cotan} \pi z$.

Problem 1.

Prove the formula:

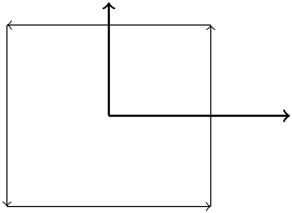
$$\pi \operatorname{cotan} \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} \tag{2}$$

Which is equivalent to

$$\pi \operatorname{cotan} \pi z = \frac{1}{z} + \sum_{n \in \mathbb{Z}, n \neq 0}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right). \tag{3}$$

Idea of Proof.

For $w \notin \mathbb{Z}$, let N be a positive integer, let C_N be the boundary of the rectangle with vertices $-(N+\frac{1}{2})-Ni$, $(N+\frac{1}{2})-Ni$, $(N+\frac{1}{2})+Ni$ and $-(N+\frac{1}{2})+Ni$ counter-clockwisely



Let $w \in \mathbb{C} - \mathbb{Z}$ and w is enclosed by C_N. By the residue theorem, we have

$$\frac{1}{2\pi i}\int_{C_N} \frac{\cot\pi z}{z-w}dz = \mathrm{res}_{z=w}\frac{\cot\pi z}{z-w} + \sum_{k=-N}^{N} \mathrm{res}_{z=k}\frac{\cot\pi z}{z-w} \tag{4}$$

Problem 1.1. Prove that the right hand side of (4) is

$$\operatorname{cotan} \pi w + \sum_{k=-N}^{N} \frac{1}{\pi(k-w)} = \operatorname{cotan} \pi w - \frac{1}{\pi w} - \sum_{k=1}^{N} \frac{2w}{\pi(w^2-k^2)}$$

So (6) implies that

$$\operatorname{cotan} \pi w = \frac{1}{\pi w} + \sum_{k=1}^{N} \frac{2w}{\pi(w^2-k^2)} + \frac{1}{2\pi i} \int_{C_N} \frac{\operatorname{cotan} \pi z}{z-w} dz. \qquad (5)$$

Recall

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \dots$$
$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \dots$$

$$\operatorname{cotan} \pi z = \frac{\cos \pi z}{\sin \pi z} = \frac{1 + \dots}{\pi z + \dots} = \frac{1}{\pi z} + \text{regular terms}$$

$$\operatorname{res}_{z=0} \frac{\operatorname{cotan} \pi z}{z - w} = \frac{1}{\pi(0 - w)}$$

Near $z = n$,

$$\operatorname{cotan} \pi(z - n) = \frac{1}{\pi(z - n)} + \text{regular terms}$$

So

$$\operatorname{res}_{z=n} \frac{\operatorname{cotan} \pi(z - n)}{z - w} = \frac{1}{\pi(n - w)}$$

Problem 1.2. To prove our formula (2), it is enough to prove

$$\lim_{N\to\infty}\int_{C_N}\frac{\cot\pi z}{z-w}dz=0 \tag{6}$$

the contour C_N can be decomposed as the following four contours:
$C_N(1)$: From $-(N+\frac{1}{2})-Ni$ to $(N+\frac{1}{2})-Ni$;
$C_N(2)$: From $(N+\frac{1}{2})-Ni$ to $(N+\frac{1}{2})+Ni$;
$C_N(3)$: From $(N+\frac{1}{2})+Ni$ to $-(N+\frac{1}{2})+Ni$;
$C_N(3)$: From $-(N+\frac{1}{2})+Ni$ to $-(N+\frac{1}{2})-Ni$.

By a change of variable,

$$\int_{C_N(1)} \frac{\operatorname{cotan} \pi z}{z - w} dz + \int_{C_N(3)} \frac{\operatorname{cotan} \pi z}{z - w} dz$$
$$= \int_{C_N(1)} \frac{2w \operatorname{cotan} \pi z}{z^2 - w^2} dz.$$

Prove that

$$\lim_{N \to \infty} \int_{C_N(1)} \frac{2w \operatorname{cotan} \pi z}{z^2 - w^2} dz = 0$$

Then prove

$$\int_{C_N(2)} \frac{\cot\pi z}{z-w}dz + \int_{C_N(4)} \frac{\cot\pi z}{z-w}dz$$
$$= \int_{C_N(2)} \left(\frac{\cot\pi z}{z-w} + \frac{\cot\pi z}{-z-w}\right) dz$$

Prove this has limit 0 as $N \to \infty$.

Problem 2.

The purpose of this Problem is to investigate if $\cotan \pi z$ is an eigenfunction of operators T_n. Recall that if $f(z)$ is a periodic meromorphic function on $\mathbb{C}$ with period 1, i.e., $f(z+1) = f(z)$ for all $z \in \mathbb{C}$. Let n be a positive integer, we define a function $T_n f$ by

$$(T_n f)(z) = \sum_{k=0}^{n-1} f(\frac{z}{n} + \frac{k}{n}).$$

Then $T_n f$ is a periodic function with period 1 (see Homework). And we have

$$T_m T_n f = T_{mn} f.$$

Problem 2.1 Prove that for every positive integer m,

$$(-1)^m \frac{1}{m!} \frac{d^m}{d^m z} \operatorname{cotan} \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^{m+1}}.$$

By Homework, $\frac{d^m}{d^m z} \operatorname{cotan} \pi z$ is an eigenfunction of operators T_n.

Problem 2.2. Is $\operatorname{cotan} \pi z$ is an eigenfunction of T_n for all n?

This is the **main problem** of this project.

Problem 3.

The purpose of this problem is to compute the values of Riemann zeta function at positic 2 and 4 and study the property of $\zeta(2m)$ ($m \in \mathbb{Z}_{\geq 1}$). The Riemann zete function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The series converges on $\mathrm{re}\, s > 1$ and has meromorphic continuation on $\mathbb{C}$.

Problem 3.1. Use the formula (4) to prove that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Problem 3.2. Prove that for every even positive integer $2m$, $\frac{\zeta(2m)}{\pi^{2m}}$ is a rational number.

Hint: by (1),

$$\pi\mathrm{cotan}\,\pi z = \frac{\pi \cos\ \pi z}{\sin\ \pi z} = \frac{(e^{2\pi i z} + 1)\pi i}{e^{2\pi i z} - 1} \tag{7}$$

by (2),

$$\pi\,\mathrm{cotan}\,\pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} \tag{8}$$

$$\frac{(e^{2iz}+1)\pi i}{e^{2iz}-1} = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2}$$

Consider Laurent expansion of both sides at $z=0$,

$$\frac{1}{z^2-n^2} = -\frac{1}{n^2}\frac{1}{1-\frac{z^2}{n^2}} = -\sum_{k=0}^{\infty}\frac{z^{2k}}{n^{2(k+1)}}$$

The right hand side is

$$\frac{1}{z} - \sum_{k=0}^{\infty} 2\zeta(2k+2)\, z^{2k+1}$$

Try to do something for the left side. Compare the coefficients.

Weierstrass Elliptic Functions

A **lattice** in the complex plane $\mathbb{C}$ is a subgroup Λ that can be written as

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

such that ω_1 and ω_2 are $\mathbb{R}$-linearly independent. An **elliptic function** for a lattice Λ is a meromorphic function $f(z)$ on $\mathbb{C}$ such that

$$f(z) = f(z + \omega)$$

for all $z \in \mathbb{C}$ and $\omega \in \Lambda$.

The Weierstrass elliptic function is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

We will write $\wp(z, \Lambda)$ for $\wp(z)$ when there is a need to emphasize the dependence on Λ. We have

$$\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$$

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{9}$$

where

$$g_2 = g_2(\Lambda) = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}$$

and

$$g_3 = g_3(\Lambda) = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6}.$$

Every elliptic function for Λ can is a rational function of $\wp(z, \Lambda)$ and $\wp'(z, \Lambda)$. The field $M(\Lambda)$ of elliptic functions for Λ is isomorphic to

$$\mathrm{Frac}\, \mathbb{C}[x, y]/\left(y^2 - (4x^3 - g_2(\Lambda)x - g_3(\Lambda))\right).$$

Problem 1. About Eigenfunctions of Operators T_n?

For every positive integer n, we introduce an operator T_n on the space $M(\Lambda)$ by

$$(T_n f)(z) = \sum_{k_1,k_2=0}^{n-1} f(\frac{z}{n} + \frac{k_1}{n}\omega_1 + \frac{k_2}{n}\omega_2).$$

The operators T_n's satisfies the relation

$$T_m T_n = T_{mn}.$$

It is easy to prove all the derivatives $\wp^{(k)}(z)$ $(k \geq 1)$ is an eigenfunction of T_n, in fact, we have

$$T_n \wp^{(k)}(z) = n^{k+2} \wp^{(k)}(z).$$

Question: Is $\wp(z)$ an eigenfunction of T_n?

$$T_n\wp(z) = \sum_{j,k=0}^{n-1} \wp(\frac{z}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2)$$

$$= \sum_{j,k=0}^{n-1} \left(\frac{1}{(\frac{z}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2)^2} + \sum_{\omega\in\Lambda-\{0\}} \left(\frac{1}{(\frac{z}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2 - \omega)^2} - \frac{1}{\omega^2} \right)\right)$$

$$
\begin{aligned}
&n^{-2} T_n \wp(z) \\
&= \sum_{j,k=0}^{n-1} \left(\frac{1}{(z + j\omega_1 + k\omega_2)^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z + j\omega_1 + k\omega_2 - n\omega)^2} - \frac{1}{n^2\omega^2} \right) \right. \\
&= \wp(z, \Lambda) + \sum_{j,k=0,\text{not both } 0}^{n-1} \wp(j\omega_1 + k\omega_2, n\Lambda)
\end{aligned}
$$

$$
\begin{aligned}
&T_n\wp(z) \\
&= n^2\wp(z,\Lambda) + n^2 \sum_{j,k=0,\text{not both } 0}^{n-1} \wp(j\omega_1 + k\omega_2, n\Lambda)
\end{aligned}
$$

One can prove that $\wp(z)$ is an eigenfunction of T_2 with eigenvalue 4.

Problem 2. Addition Formula of $\wp(z)$.

Prove that

$$\wp(z+w) = \frac{1}{4}\left(\frac{\wp'(z)-\wp'(w)}{\wp(z)-\wp(w)}\right)^2 - \wp(z) - \wp(w) \tag{10}$$

Hint: Fix w such that $\wp(w)$ and $\wp'(w)$ are finite and non-zero, we consider

$$F(z) = \wp(z+w) - \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2 + \wp(z) + \wp(w)$$

Step 1. Prove that $F(z)$ is an elliptic function with possible poles at $0, -w$ modulo Λ.
Step 2. Use Laurent expansion of F at $z = 0$ to show that 0 is not a pole and $F(0) = 0$.
Step 3. Use Laurent expansion of F at $z = -w$ to show that $-w$ is not a pole.

End