Homework for Math 4991 about Elliptic Functions

Deadline: April 9.

Problem 1. Let $f(z)$ be an periodic meromorphic function on $\mathbb{C}$ with period 1, i.e., $f(z+1) = f(z)$ for all $z \in \mathbb{C}$. Let n be a positive integer, we define a function $T_n f$ by

$$(T_n f)(z) = \sum_{k=0}^{n-1} f(\frac{z}{n} + \frac{k}{n}).$$

(1) Prove that $T_n f$ is a periodic function with period 1.
(2) Prove that $T_m T_n f = T_{mn} f$.
(3) Let

$$f(z) = \sum_{k \in \mathbb{Z}} \frac{1}{(z+k)^N}$$

where $N \geq 2$ is positive integer. Prove that f is an eigenfunction of operator T_n, i.e., $T_n f = \lambda_n f$. Find the eigenvalue λ_n.

Problem 2. Let $f(z)$ be an elliptic function with the lattice of periods $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, for a positive integer n, let $T_n f$ be

$$(T_n f)(z) = \sum_{k_1,k_2=0}^{n-1} f(\frac{z}{n} + \frac{k_1}{n}\omega_1 + \frac{k_2}{n}\omega_2).$$

(1) Prove that $T_n f$ is an elliptic function with the lattice of periods L.
(2) Let

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

be the Weierstrass function, prove that $\wp(z)$ is an eigenfunction of T_n, what is the eigenvalue?
(3) Prove that $\wp'(z)$ is also an eigenfucntion of T_n, find the eigenvalue.

Problem 3. Let f be an elliptic function with period lattice $L = \mathbb{Z} + \mathbb{Z}i$. Prove that for every $a = m + ni$, where m, n are integers, $f(az)$ is an elliptic function with period lattice $L = \mathbb{Z} + \mathbb{Z}i$.