Math 6170 C, Lecture on April 6, 2020

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Plan

(1) A Brief Review of Complex Analysis.

(2). VI. §2. Elliptic Functions

(3). VI. §3. Constructions of Elliptic Functions.

Review of Complex Analysis.

Let D be a **connected** open set in $\mathbb{C}$, a continuous complex valued $f(z)$ defined on D is called an **analytic function** if $f'(z)$ exists everywhere in D.

Recall $f'(z)$ is the complex derivative defined by

$$\lim_{\delta \to 0} \frac{f(z+\delta) - f(z)}{\delta}$$

Analytic functions have good properties that general smooth functions don't have.

Theorem 1. Let $f(z)$ be an analytic function on D, C be a simple counter-clockwise closed contour in D, if the domain enclosed by C is in D, then

$$\int_C f(z)dz = 0.$$

For C as above, a is in the domain enclosed by C, then

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = f(a).$$

Theorem 2. If $f(z)$ is an analytic function on D, if $|f(z)|$ has a local maximal at some point in D, then $f(z)$ is a constant function.

Theorem 3. The derivative $f^{(n)}(z)$ of arbitrary order n exists, and

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{1}{n!}f^{(n)}(a).$$

Theorem 4. If $f(z)$ is an analytic function on D, for every $a \in D$, the Taylor expansion at a

$$f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \ldots$$

converges absolutely to $f(z)$ uniformly on any closed disc $|z-a| \leq r$ inside D.

Theorem 5. If the zero points $\{a \mid f(a) = 0\}$ has a limit point in D, then $f(z) = 0$.

Corollary. Let D be a connected open subset in $\mathbb{C}$, the ring of analytic functions on D is an integral domain.

A **meromorphic function** on D is a map $f : D \to \mathbb{C} \cup \{\infty\}$ such that
(1). If $f(a) = \infty$, then there is an open neighborhood U of a such that

$$f|_U = \frac{h(z)}{(z-a)^n}$$

where $h(z)$ is an analytic function on U with $h(a) \neq 0$ and $n \in \mathbb{Z}_{>0}$.
(2). By (1), $D - f^{-1}(\infty)$ is an open set, $f(z)$ is analytic on $D - f^{-1}(\infty)$.

$a \in D$ is a **pole** of f if $f(a) = \infty$, the positive integer n in (1) is the order of the pole.

Let $f(z)$ be a non-zero meromporphic function on D, for any $a \in D$, $f(z)$ has a Laurent power series expansion at a

$$f(z) = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + \text{higher terms}$$

where $c_m \neq 0$.

If $m < 0$, then a is a pole of f of order $-m$.

The **residue** of f at a is defined by

$$\mathrm{res}_a(f) = c_{-1} = \text{coefficent of } (z-a)^{-1}.$$

Theorem 6 (Residue Theorem). Let $f(z)$ be a meromorhic function on a simply connected domain D, C be a simple counter-clockwise closed contour in D that doesn't contains any poles of f, then

$$\frac{1}{2\pi i}\int_C f(z)dz = \sum_{a\text{:poles of f } enclosed\ by\ C} \mathrm{res}_a f$$

A **lattice** in $\mathbb{C}$ is a free $\mathbb{Z}$-submodule $\Lambda \subset \mathbb{C}$ of rank two such that a basis of Λ is $\mathbb{R}$-linearly independent.

A lattice can be written as

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

where ω_1 and ω_2 are $\mathbb{R}$-linear independent.

If ω_1, ω_2 is a $\mathbb{Z}$-basis of a lattice Λ, and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$$

(this implies that $\det A = \pm 1$).
Then

$$a\omega_1 + b\omega_2, \quad c\omega_1 + d\omega_2$$

is also a $\mathbb{Z}$-basis of Λ.

Definition. An **elliptic function** (relative to the lattice Λ) is a meromorphic function $f(z)$ on $\mathbb{C}$ such that

$$f(z+\omega) = f(z)$$

for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$.

Constant functions are elliptic functions relative to any lattice.

We denote the space of elliptic functions for Λ by

$$\mathbb{C}(\Lambda).$$

$\mathbb{C}(\Lambda)$ is a field.

Definition. Let $\Lambda \subset \mathbb{C}$ be a lattice, a **fundamental parallelogram** is a set of the form

$$D = \{a + t_1\omega_1 + t_2\omega_2 \mid 0 \leq t_1, t_2 < 1\}$$

where ω_1, ω_2 is a $\mathbb{Z}$-basis for Λ and $a \in \mathbb{C}$.

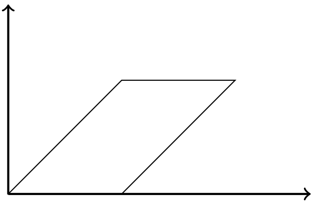


Figure: The domain $D = \{t_1 + t_2(1+i) \mid 0 \leq t_1, t_2 \leq 1\}$.

A fundamental parallelogram is a set of representative of the quotient group $\mathbb{C}/\Lambda$,

that is, for every $z \in \mathbb{C}$, there is unique $r \in D$ such that $z - r \in \Lambda$.

One dim analog: $\mathbb{Z} \subset \mathbb{R}$, $[0, 1)$ is a fundamental domain of $\mathbb{Z}$, for every $x \in \mathbb{R}$, there is a unique $r \in [0, 1)$ such that $x - r \in \mathbb{Z}$.

For a non-zero meromorphic function f on D, $a \in D$, we have Laurent series expansion at a:

$$f(z) = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + \text{higher terms}$$

with $c_m \neq 0$.
We define $\mathrm{ord}_a(f) = m$

a is a pole of f iff $\mathrm{ord}_a(f) < 0$. a is a zero of f iff $\mathrm{ord}_a(f) > 0$.

For $f \in \mathbb{C}(\Lambda)$, $a \in \mathbb{C}$, $\omega \in \Lambda$, we have

$$\mathrm{res}_a f = \mathrm{res}_{a+\omega} f$$

$$\mathrm{ord}_a f = \mathrm{ord}_{a+\omega} f$$

So $\sum_{w \in D} \mathrm{res}_w(f)$ are the same for all fundamental domain D, we denote it by

$$\sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(f)$$

Theorem VI 2.2.

Let $f \in \mathbb{C}(\Lambda), f \neq 0$, then

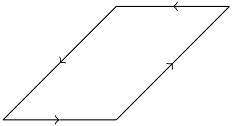
(a) $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(f) = 0$

(b) $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) = 0$

(c) $\sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f)\, w \in \Lambda$.

Proof.

Let D be a fundamental parallelogram of the period lattice Λ such that the four boundary edges of D contains no zeros nor poles.



Let C denote the contour that is the boundary of D oriented counter-clock wisely

Consider integral

$$I_a = \frac{1}{2\pi i} \int_C f(z) dz$$

By Residue Theorem

$$I_a = \sum_{w \in \mathbb{C}/\Lambda} \mathrm{res}_w(f)$$

C is a union of four contours C_1, C_2, C_3, C_4, where C_1, C_3 are parallel but oriented in the opposite direction, similar relations for C_2, C_4.
By periodicity of f,

$$\int_{C_1} f(z)dz + \int_{C_3} f(z)dz = 0, \quad \int_{C_2} f(z)dz + \int_{C_4} f(z)dz = 0$$

This proves (a).

For (b), we consider the integral,

$$I_b = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

By Residue Theorem, it is equal to the sum of residues of f'/f at the poles in D.

Recall that a is a pole f'/f iff a is a zero or pole of f. And

$$\mathrm{res}_a(\frac{f'}{f}) = \mathrm{ord}_a(f)$$

So I_b is the left hand side of (b).

On the other hand,

$$I_b = \frac{1}{2\pi i}\left(\int_{C_1}\frac{f'(z)}{f(z)}dz + \int_{C_2}\frac{f'(z)}{f(z)}dz + \int_{C_3}\frac{f'(z)}{f(z)}dz + \int_{C_4}\frac{f'(z)}{f(z)}dz\right)$$

Since the values of f'/f are equal on C_1 and C_3, but orientations on C_1 and C_3 are opposite, so $\int_{C_1} + \int_{C_3} = 0$. Similarly $\int_{C_2} + \int_{C_4} = 0$. So $I_b = 0$.

This proves (b)

For(c), we consider the contour integral

$$I_c \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz.$$

By Residue Theorem, I_c is the left hand side of (c).

On the other hand side,

$$
\begin{aligned}
I_c \quad &= \frac{1}{2\pi i}\left(\int_{C_1} \frac{zf'(z)}{f(z)} dz + \int_{C_3} \frac{zf'(z)}{f(z)} dz\right) \\
&+ \frac{1}{2\pi i}\left(\int_{C_2} \frac{zf'(z)}{f(z)} dz + \int_{C_4} \frac{zf'(z)}{f(z)} dz\right)
\end{aligned}
$$

$$\frac{1}{2\pi i}\left(\int_{C_1}\frac{zf'(z)}{f(z)}dz+\int_{C_3}\frac{zf'(z)}{f(z)}dz\right)=-\omega_2\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}dz$$

$$\frac{1}{2\pi i}\left(\int_{C_2}\frac{zf'(z)}{f(z)}dz+\int_{C_4}\frac{zf'(z)}{f(z)}dz\right)=-\omega_1\frac{1}{2\pi i}\int_{C_2}\frac{f'(z)}{f(z)}dz$$

Claim:

$$\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}dz \in \mathbb{Z}$$

$$\frac{1}{2\pi i}\int_{C_2}\frac{f'(z)}{f(z)}dz \in \mathbb{Z}$$

Proof of Claim.

Since $f(z)$ has no zeros nor poles on C_1, it has no zeros nor poles in an simply connected open neighborhood U of C_1. There exists an analytic function $h(z)$ on U such that $h'(z) = \frac{f'(z)}{f(z)}$ and $h(a+\omega_1) - h(a) \in 2\pi i\mathbb{Z}$ in fact $h(z) = \log f(z)$ (a branch of $\log f(z)$)

$$\frac{1}{2\pi i}\int_{C_1} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}(h(a+\omega_1) - h(a)) \in \mathbb{Z}.$$

VI. § 3. Construction of Elliptic Functions

The Weierstrass elliptic function $\wp(z)$ for a lattice Λ is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

we notice that on any compact disk defined by $|z| \leq R$, all but possibly finitely many $\omega \in \Lambda$ satisfies $|\omega| > 2R$. For such ω, one has

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{2\omega z - z^2}{\omega^2 (\omega - z)^2}\right| = \left|\frac{z\left(2 - \frac{z}{\omega}\right)}{\omega^3 \left(1 - \frac{z}{\omega}\right)^2}\right| \leq \frac{10R}{|\omega|^3}$$

This implies that the series converges uniformly on $|z| \leq R$, so we have a meromorphic function on $\mathbb{C}$ with poles on the lattice Λ (See the next page).

Theorem. If a series of analytic functions on a domain D

$$f_1(z) + f_2(z) + \dots$$

converges uniformly, then the limit $S(z)$ is also an analytic function on D. And

$$f_1'(z) + f_2'(z) + \dots$$

also converges on D and the convergence is uniform on every compact subsets in D, the limit is $S'(z)$.

By the above theorem, $\wp(z)$ is a meromorphic function on $\mathbb{C}$.

And we have

$$\wp'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}$$

has periods Λ, so we have

$$\wp(z+\omega) - \wp(z) = C$$

is a constant, put $z = -\frac{\omega}{2}$, we see that $\wp(\frac{\omega}{2}) - \wp(-\frac{\omega}{2}) = C$, it is obvious that $\wp(z)$ is even function, so $C = 0$. This proves $\wp(z)$ is an elliptic function with period Λ.

Theorem.

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z)\right)^3 - g_2\wp(z) - g_3 \tag{1}$$

where

$$g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}$$

and

$$g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6}.$$

Theorem.

The field $\mathcal{M}(\Lambda)$ is generated by $\wp(z)$ and $\wp'(z)$ over $\mathbb{C}$ subject to the relation

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z)\right)^3 - g_2\wp(z) - g_3$$

End