Classification and asymptotics for the fractional Yamabe equation with an isolated singularity

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Abstract

Continuing our work [23] on the case $\sigma = 1/2$, we classify positive singular solutions of the fractional Yamabe equation on $\mathbb{R}^n \setminus \{0\}$ for $n \geq 2$, $0 < \sigma < 1$, and $n > 2\sigma$. The known radial symmetry of global singular solutions reduces the problem to a one-dimensional nonlocal equation or, through the Caffarelli–Silvestre extension of the fractional Laplacian, to a weighted elliptic PDE in two variables. Classical phase-plane methods do not apply to either formulation. Following the Hamiltonian approach in [23], we combine local bifurcation with compactness to prove that every positive bounded solution of the cylindrical trace equation is constant, a bubble, or periodic, and is uniquely determined, up to translation, by its Hamiltonian energy. We also show that the solution space of the linearized trace equation at each stationary or periodic profile is two-dimensional, and every positive local solution with a nonremovable isolated singularity is asymptotic to a unique global singular solution, with an algebraically decaying relative error.

Keywords: fractional Yamabe problem; Fowler solutions; isolated singularities; asymptotic behavior.

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1 Introduction

This paper continues our work on the boundary Yamabe equation [23], which corresponds to fractional order $\sigma = 1/2$. There we classified global singular solutions by their Hamiltonian energy and proved that every local solution with a nonremovable isolated singularity is asymptotic to a unique global model. Here we use the same strategy to extend all the results in [23] to the full range $\sigma \in (0, 1)$.

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We consider positive solutions of

$$(-\Delta)^\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}}, \qquad u > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{1.1}$$

where $n \geq 2$, $0 < \sigma < 1$, and $n > 2\sigma$. We define the fractional Laplacian by the principal-value integral

$$(-\Delta)^\sigma u(x) = \frac{2^{2\sigma}\sigma\Gamma(\frac{n}{2}+\sigma)}{\pi^{n/2}\Gamma(1-\sigma)} \lim_{\varepsilon \downarrow 0} \int_{|z-x|>\varepsilon} \frac{u(x)-u(z)}{|x-z|^{n+2\sigma}}\,\mathrm{d}z, \qquad x \neq 0.$$

This normalization gives the Fourier symbol $|\xi|^{2\sigma}$. We assume $u \in C^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ and

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2\sigma}}\,\mathrm{d}x < \infty. \tag{1.2}$$

Thus $(-\Delta)^\sigma u(x)$ is well defined for every $x \neq 0$, and (1.1) is understood pointwise.

The extension of the fractional Laplacian by Caffarelli–Silvestre [7] gives a local formulation of (1.1). The extension U of u satisfies

$$\begin{cases} \operatorname{div}(y^{1-2\sigma}\nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -d_\sigma \lim\limits_{y \downarrow 0} y^{1-2\sigma}\partial_y U = u^{\frac{n+2\sigma}{n-2\sigma}} & \text{on } \mathbb{R}^n \setminus \{0\}, \\ U(x,0) = u(x), \end{cases} \tag{1.3}$$

where $X = (x,y) \in \mathbb{R}^n \times \mathbb{R}$, $\mathbb{R}^{n+1}_+ = \{X : y > 0\}$, and $d_\sigma = 2^{2\sigma-1}\Gamma(\sigma)/\Gamma(1-\sigma)$. At $\sigma = 1/2$ the extension is harmonic and $d_\sigma = 1$. This is the scalar-flat boundary Yamabe problem on the half-space, whose geometric background includes the work of Escobar [13, 14] and Han–Li [20].

If u is bounded near the origin, the singularity is removable and u is a bubble. This classification is due to Li–Zhu [27] for $\sigma = 1/2$ and Jin–Li–Xiong [22] for $0 < \sigma < 1$. For a nonremovable singularity, the growth and symmetry results of Caffarelli–Jin–Sire–Xiong [6] give radial symmetry about the origin and two-sided bounds of order $|x|^{-(n-2\sigma)/2}$ near zero. These results identify the leading power of $|x|$. The remaining question is the classification of all such solutions. For the classical Yamabe equation, radial symmetry and the Emden–Fowler transformation reduce the corresponding problem to an ODE, see Caffarelli–Gidas–Spruck [5] and Korevaar–Mazzeo–Pacard–Schoen [26]. Frank–König [15] also used ODE methods to classify positive singular solutions of the critical biharmonic equation.

We now formulate (1.1) in the Emden–Fowler variables

$$t = -\log r, \qquad u(r) = K_\sigma^{\frac{n-2\sigma}{4\sigma}} r^{-\frac{n-2\sigma}{2}} w(t), \qquad K_\sigma = \frac{\Gamma(\frac{n}{2}+\sigma)}{\Gamma(\frac{n}{2}-\sigma)}. \tag{1.4}$$

DelaTorre–González [12] showed that the rescaled trace w satisfies

$$\mathcal{P}_\sigma w = K_\sigma w^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{R}. \tag{1.5}$$

The Fourier symbol of $\mathcal{P}_\sigma$ is given in (2.2). The estimates of Caffarelli–Jin–Sire–Xiong [6] give a bounded trace w for every global singular solution. Although w depends on just one variable, $\mathcal{P}_\sigma w(t)$ depends on the trace on the whole line. In the extension formulation, axial symmetry leaves two variables, the logarithmic radius and the angle with the vertical axis. The classical phase-plane argument therefore does not apply. DelaTorre–González [12] showed that the cylindrical equation nevertheless admits a conserved Hamiltonian $\mathcal{H}$, given by (2.7) and (2.8).

A positive nonconstant periodic trace is called a Fowler solution. DelaTorre–del Pino–González–Wei [11] constructed such solutions for $0 < \sigma < 1$. Jin–Xiong [24] constructed solutions of sufficiently large periods for $0 < \sigma < n/2$. Cruz-Blázquez–DelaTorre–Ruiz [10] proved that global singular solutions have infinite Morse index and that every nonstationary singular profile crosses the stationary value. Andrade–DelaTorre–do Ó–Ratzkin–Wei [1] obtained a classification for $n \geq 3$ and σ sufficiently close to 1. For general σ, the existence of periodic profiles still leaves two questions. Must every nonstationary singular profile be periodic? Can two different periodic orbits have the same energy?

Two explicit energy values suggest the answer. The unique positive constant solution w_c has energy $H_c = \mathcal{H}(w_c) < 0$, whereas the bubble $w_b(t) = (\cosh t)^{-(n-2\sigma)/2}$ has energy zero. We prove that the periodic profiles fill the interval between these values, with exactly one orbit at each energy. As the energy increases from H_c to zero, the minimum decreases from w_c to zero and the maximum increases from w_c to the peak of the bubble. In the theorem below, κ_c is the positive linearized frequency at w_c defined by (2.6).

Theorem 1.1. *Let $n \geq 2$, $0 < \sigma < 1$, and $n > 2\sigma$. For $H \in \mathbb{R}$, the positive bounded solutions of* (1.5) *with Hamiltonian energy H are*

$$\begin{cases} \{w_c\}, & H = H_c, \\ \{w_H(\cdot + \tau) : \tau \in \mathbb{R}\}, & H_c < H < 0, \\ \{w_b(\cdot + \tau) : \tau \in \mathbb{R}\}, & H = 0, \\ \varnothing, & H \notin [H_c, 0]. \end{cases}$$

For $H_c < H < 0$, choose the translate w_H with $w_H(0) = \max w_H$, and let $T(H)$ be its least period. Then w_H is even and strictly decreasing on $(0, T(H)/2)$, with $M(H) = w_H(0)$ and $\varepsilon(H) = w_H(T(H)/2)$. For every integer $j \geq 0$, the map

$$H \longmapsto \Big(w_H(T(H)\,\cdot),\, T(H),\, \varepsilon(H),\, M(H) \Big)$$

is real analytic into $H^j(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^3$. The minimum $\varepsilon : (H_c, 0) \to (0, w_c)$ and maximum $M : (H_c, 0) \to (w_c, 1)$ are real-analytic diffeomorphisms, with $\varepsilon' < 0 < M'$. As $H \downarrow H_c$, $w_H \to w_c$ uniformly on $\mathbb{R}$ and $T(H) \to 2\pi/\kappa_c$; as $H \uparrow 0$, $w_H \to w_b$ smoothly on compact intervals and $T(H) \to \infty$.

Remark 1.2. *Let $0 < \sigma < 1$. Every positive solution u of* (1.1) *with a removable origin is a bubble of the form*

$$u(x) = (2^{2\sigma} K_\sigma)^{\frac{n-2\sigma}{4\sigma}} \left(\frac{\mu}{\mu^2 + |x - x_0|^2} \right)^{\frac{n-2\sigma}{2}}, \qquad \mu > 0, \quad x_0 \in \mathbb{R}^n. \tag{1.6}$$

If the origin is nonremovable, u is radial about zero, and by Theorem 1.1, u is either the homogeneous solution $K_\sigma^{\frac{n-2\sigma}{4\sigma}} w_c |x|^{-\frac{n-2\sigma}{2}}$ or a Fowler solution. In the latter case, $u(e^{-T}x) = e^{(n-2\sigma)T/2}u(x)$, where T is the least cylindrical period.

The proof follows our strategy for $\sigma = 1/2$ in [23]. The first step is to establish the two-dimensional linear theory of [23] for the weighted extension problem. For entire axial solutions with a variable Robin boundary coefficient in a suitable exponentially weighted class, two scalar Cauchy data parametrize the solution space, and the conserved Wronskian is nondegenerate. The main work here lies in the weighted regularity and quantitative spectral estimates. Since the weight $y^{1-2\sigma}$ vanishes or diverges at the boundary for $\sigma \neq 1/2$, we use the boundary estimates of Jin–Li–Xiong [22] in place of the Neumann estimates of McLean [30]. Related linear estimates under asymptotic assumptions on the potential, as well as Hamiltonian and Wronskian identities, have been studied by Ao–Chan–DelaTorre–Fontelos–González–Wei [2, 3].

We then apply the linear estimates to normalized differences of nonlinear solutions. At a nonstationary solution, the first variation of the Hamiltonian and the nondegeneracy of the Wronskian imply local uniqueness up to translation among solutions of the same energy. At the postive stationary solution, the first variation vanishes, and positivity of the second variation on the two-dimensional admissible linearized space implies that it is isolated in its energy level. The bifurcation theorem of Crandall–Rabinowitz [9] constructs a small periodic branch. The Cauchy coordinates show that this branch and its translates exhaust a neighborhood of v_c, while the positive quadratic term in the energy expansion gives exactly one periodic translation orbit at each energy sufficiently close to and above H_c.

Finally, the global classification follows by continuing this local family in energy. Compactness and local rigidity imply that every profile at an interior energy is periodic. Periodic continuation and compactness make the number of distinct translation orbits locally constant, so the single orbit near H_c determines the count throughout $(H_c, 0)$. We also obtain analytic parametrizations by the minimum and maximum, with $\varepsilon'(H) < 0 < M'(H)$. The estimate for even solutions in Lemma 2.12 first gives $M(H) < 1$ by comparison with the bubble and then rules out critical points of either extremal value, with the derivative signs determined by the endpoint limits.

The same two scalar data identify the linearized motions along a periodic orbit. Translation gives $\partial_t w_H$, which leaves the energy fixed, while differentiation of the centered family gives $\partial_H w_H$, whose energy variation is 1. These are the fractional counterparts of the translation and necksize variations studied by Mazzeo–Pacard [28] in the classical Yamabe problem. For fractional ground states with subcritical power nonlinearities, Frank–Lenzmann [16] proved uniqueness and nondegeneracy in one dimension, and Frank–Lenzmann–Silvestre [17] extended these results to arbitrary dimensions. Here the two variations span the space of polynomially growing solutions of the linearized trace equation at a periodic profile.

Theorem 1.3. *Let $0 < \sigma < 1$, $H \in [H_c, 0)$, and $w_{H_c} = w_c$. Among real functions $\phi \in C^2(\mathbb{R})$ satisfying $|\phi(t)| \leq C(1 + |t|)^j$ for some $C > 0$ and integer $j \geq 0$, the solution space of*

$$\mathcal{P}_\sigma \phi = \frac{n + 2\sigma}{n - 2\sigma} K_\sigma w_H^{\frac{4\sigma}{n-2\sigma}} \phi \tag{1.7}$$

is

$$\begin{cases} \operatorname{span}\{\partial_t w_H, \partial_H w_H\}, & H_c < H < 0, \\ \operatorname{span}\{\cos(\kappa_c t), \sin(\kappa_c t)\}, & H = H_c. \end{cases}$$

Here $\partial_H w_H$ is the derivative at fixed t. For $H_c < H < 0$, every such solution grows at most linearly.

Remark 1.4. *The conclusion of Theorem 1.3 also holds under the growth condition*

$$|\phi(t)| \le Ce^{\varepsilon|t|}, \qquad 0 < \varepsilon < \frac{n+2\sigma}{2},$$

provided $\mathcal{P}_\sigma\phi$ is interpreted as $d_\sigma \partial_\nu^\sigma \Phi$, where Φ is the cylindrical Poisson extension of ϕ; see Section 2.1. Indeed, the Poisson kernel in the proof decays at rate $(n+2\sigma)/2$, so Φ has the same exponential growth. The local energy estimates then place Φ in the two-sided weighted solution space used there, and the same argument applies. We assume polynomial growth in the statement to ensure $\phi \in \mathcal{S}'(\mathbb{R})$ a priori, so that the Fourier-multiplier definition of $\mathcal{P}_\sigma$ applies directly. An exponential growth bound alone does not guarantee that ϕ is a tempered distribution.

We finally return to a solution defined only near an isolated singularity. Mazzeo–Pollack–Uhlenbeck [29] studied the moduli of singular Yamabe metrics. For the fractional problem, see Chang–González [8] and González–Qing [19] on the conformal formulation, and González–Mazzeo–Sire [18] on singular metrics. Gluing constructions for isolated singularities and for higher-dimensional singular sets were given by Ao–DelaTorre–González–Wei [4] and Ao–Chan–DelaTorre–Fontelos–González–Wei [2], respectively. For the local problem, the question is which global profile describes a given singularity. This question was studied in the classical Yamabe problem by Caffarelli–Gidas–Spruck [5] and Korevaar–Mazzeo–Pacard–Schoen [26], and in the σ_k-Yamabe problem by Han–Li–Teixeira [21]. By the growth and asymptotic symmetry estimates of Caffarelli–Jin–Sire–Xiong [6], translating the cylindrical profile toward the singular end gives subsequential limits. Conservation of energy and Theorem 1.1 place all these limits on one orbit. In the periodic case this still allows different sequences of scales to select different phases. Theorem 1.3 identifies the two possible axial variations. Conservation of energy excludes the energy variation, and an orthogonality condition fixes the phase. At the stationary profile, positivity of the second variation of the Hamiltonian excludes both linearized modes. The resulting decay estimate gives convergence to a single translate with the relative error stated below.

Write

$$B_R^+ = \{X = (x, y) \in \mathbb{R}^{n+1} : |X| < R,\ y > 0\}, \qquad B_R' = \{x \in \mathbb{R}^n : |x| < R\}.$$

With $u = U(\cdot, 0)$, the local extension problem is

$$\begin{cases} \operatorname{div}(y^{1-2\sigma}\nabla U) = 0 & \text{in } B_R^+, \\ -d_\sigma \lim\limits_{y\downarrow 0} y^{1-2\sigma}\partial_y U = u^{\frac{n+2\sigma}{n-2\sigma}} & \text{on } B_R' \setminus \{0\}, \end{cases} \tag{1.8}$$

Theorem 1.5. *Let $n \geq 2$, $0 < \sigma < 1$, and $n > 2\sigma$. If a positive weak solution U of* (1.8) *has a nonremovable isolated singularity at zero, then there exist $\delta > 0$ and a unique Poisson extension U_∞ of a global solution of* (1.1)*, with cylindrical trace $w_H(\cdot + \tau)$ for some $H \in [H_c, 0)$ and $\tau \in \mathbb{R}$, such that*

$$U(X) = U_\infty(X)\Big(1 + O(|X|^\delta)\Big) \quad \text{as } X \to 0 \text{ in } \overline{B_R^+} \setminus \{0\}.$$

Section 2 contains the a priori estimates and the two-dimensional linear theory. In Section 3, we classify the Fowler family, parametrize it by energy, minimum, or maximum, and determine the solutions of the linearized trace equation. Section 4 proves the local asymptotic result. The appendices give the operator, spectral, and boundary estimates specific to general fractional orders. When an argument is the same as in the case $\sigma = 1/2$, we give only a sketch here and refer to our earlier paper [23] for full details.

2 A priori estimates and two-dimensional linear theory

We begin with the cylindrical formulation and the estimates obtained from [6].

2.1 Cylindrical formulation and stationary solutions

Write $X = \rho(\sin\theta\,\zeta, \cos\theta)$, where $\rho > 0$, $0 \leq \theta \leq \pi/2$, and $\zeta \in \mathbb{S}^{n-1}$. For an axially symmetric extension U, write

$$U(X) = K_\sigma^{\frac{n-2\sigma}{4\sigma}} \rho^{-\frac{n-2\sigma}{2}} v(-\log\rho, \theta).$$

Let

$$p = \frac{n-2\sigma}{2} > 0, \qquad \omega(\theta) = \sin^{n-1}\theta\, \cos^{1-2\sigma}\theta.$$

With $w(t) = v(t, \pi/2)$, equation (1.3) becomes

$$\begin{cases} \partial_t^2 v + \omega^{-1}\partial_\theta(\omega\partial_\theta v) - p^2 v = 0, & (t,\theta) \in \mathbb{R} \times (0, \pi/2), \\ \partial_\nu^\sigma v = (K_\sigma/d_\sigma) w^{\frac{n+2\sigma}{n-2\sigma}}, & t \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $\partial_\nu^\sigma v = \lim_{\theta\uparrow\pi/2} \omega(\theta)\partial_\theta v$. For $\sigma = 1/2$, (2.1) is the unweighted cylindrical equation we studied in [23]. A function on the upper hemisphere $\mathbb{S}^n_+$ is called axially symmetric if it is independent of ζ. Throughout, weak solutions of the cylindrical extension equations are understood to have finite local weighted energy and to satisfy the corresponding variational identity on the hemisphere, including the pole $\theta = 0$. Classical angular solutions are smooth at the pole. Finite local weighted energy means

$$\int_I \int_0^{\pi/2} ((\partial_t v)^2 + (\partial_\theta v)^2 + v^2)\, \omega \, \mathrm{d}\theta \, \mathrm{d}t < \infty \qquad \text{for } I \Subset \mathbb{R}.$$

The Dirichlet-to-Neumann identity is $\mathcal{P}_\sigma w = d_\sigma \partial_\nu^\sigma v$.

We take $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-it\xi} f(t)\,\mathrm{d}t$, with inverse transform $(2\pi)^{-1}\int_{\mathbb{R}} e^{it\xi}\widehat{f}(\xi)\,\mathrm{d}\xi$. By [12, Theorem 1.1], the cylindrical operator is the Fourier multiplier

$$\widehat{\mathcal{P}_\sigma f}(\xi) = \Theta_\sigma(\xi)\widehat{f}(\xi), \qquad \Theta_\sigma(\xi) = 2^{2\sigma}\frac{\left|\Gamma(\frac{n+2\sigma}{4} + \frac{i\xi}{2})\right|^2}{\left|\Gamma(\frac{n-2\sigma}{4} + \frac{i\xi}{2})\right|^2}. \tag{2.2}$$

The symbol is positive, even, and smooth, with

$$C^{-1}(1+\xi^2)^\sigma \le \Theta_\sigma(\xi) \le C(1+\xi^2)^\sigma, \qquad |\partial_\xi^j \Theta_\sigma(\xi)| \le C_j(1+|\xi|)^{2\sigma-j} \quad \text{for } j \ge 0, \tag{2.3}$$

and $\partial_\xi\Theta_\sigma(\xi) > 0$ for $\xi > 0$. Thus $\mathcal{P}_\sigma$ acts on $\mathcal{S}'(\mathbb{R})$ and is an isomorphism

$$\mathcal{P}_\sigma : H^r(\mathbb{R}) \to H^{r-2\sigma}(\mathbb{R}), \qquad r \in \mathbb{R}.$$

For a T-periodic function w, set $W(x) = w(Tx)$, so that W is 1-periodic. The corresponding operator on $\mathbb{R}/\mathbb{Z}$ satisfies

$$\mathcal{P}_{\sigma,T} e^{2\pi i j x} = \Theta_\sigma(2\pi j/T) e^{2\pi i j x}, \qquad j \in \mathbb{Z}. \tag{2.4}$$

For each fixed $T > 0$, this is an isomorphism $H^r(\mathbb{R}/\mathbb{Z}) \to H^{r-2\sigma}(\mathbb{R}/\mathbb{Z})$. For bounded w, equation (1.5) is understood in $\mathcal{S}'(\mathbb{R})$. Let G_σ be the convolution kernel of $\mathcal{P}_\sigma^{-1}$, characterized by $\widehat{G_\sigma}(\xi) = \Theta_\sigma(\xi)^{-1}$. As shown in Appendix A, $G_\sigma > 0$ and $G_\sigma \in L^1(\mathbb{R})$. Every positive bounded solution of (1.5) satisfies

$$w = K_\sigma G_\sigma * w^{\frac{n+2\sigma}{n-2\sigma}}. \tag{2.5}$$

Since $\mathcal{P}_\sigma 1 = \Theta_\sigma(0)$, the positive constant solution is

$$w_c = \left(\frac{\Theta_\sigma(0)}{K_\sigma}\right)^{\frac{n-2\sigma}{4\sigma}}.$$

The bubble is $w_b(t) = (\cosh t)^{-(n-2\sigma)/2}$. The linearized frequency $\kappa_c > 0$ is the unique positive root of

$$\Theta_\sigma(\kappa_c) = \frac{n+2\sigma}{n-2\sigma}\Theta_\sigma(0). \tag{2.6}$$

The cylindrical equation (2.1) has the Hamiltonian

$$\mathcal{H}(v, \partial_t v) = \frac{d_\sigma}{2}\int_0^{\pi/2} ((\partial_t v)^2 - (\partial_\theta v)^2 - p^2 v^2)\,\omega\,\mathrm{d}\theta + \frac{(n-2\sigma)K_\sigma}{2n} w^{\frac{2n}{n-2\sigma}}. \tag{2.7}$$

Write v_b, v_c, v_H for the cylindrical Poisson extensions of w_b, w_c, w_H, respectively, and define

$$\mathcal{H}(w) = \mathcal{H}(v, \partial_t v) \quad \text{when } v \text{ extends } w. \tag{2.8}$$

In particular, $\partial_t v_c = 0$.

Smoothness at the pole gives

$$\lim_{\theta\downarrow 0}\omega(\theta)v_\theta(t,\theta)v_t(t,\theta)=0.$$

By (2.1) and integration by parts,

$$\begin{aligned}\partial_t\mathcal{H}(v,\partial_t v)=d_\sigma\int_0^{\pi/2}\partial_t v\left(\partial_t^2 v+\omega^{-1}\partial_\theta(\omega\partial_\theta v)-p^2 v\right)\omega\,\mathrm{d}\theta\\ +\left(K_\sigma w^{\frac{n+2\sigma}{n-2\sigma}}-d_\sigma\partial_\nu^\sigma v\right)\partial_t w=0.\end{aligned}$$

Testing the angular equation for v_c by v_c gives

$$d_\sigma\int_0^{\pi/2}\left((\partial_\theta v_c)^2+p^2v_c^2\right)\omega\,\mathrm{d}\theta=K_\sigma w_c^{\frac{2n}{n-2\sigma}}.$$

Hence

$$\mathcal{H}(v_c)=-\frac{\sigma K_\sigma}{n}w_c^{\frac{2n}{n-2\sigma}}=H_c.$$

For v_b, the Poisson formula and a local energy estimate give

$$w_b(t)\to 0\quad\text{and}\quad\int_0^{\pi/2}\left((\partial_t v_b)^2+(\partial_\theta v_b)^2+v_b^2\right)\omega\,\mathrm{d}\theta\to 0\quad\text{as } t\to+\infty.$$

Thus $\mathcal{H}(v_b)=0$.

2.2 A priori trace and secant estimates

Positive bounded solutions of (1.5) are smooth by local regularity for the corresponding integral equation; see [24, Section 5]. The transformation (1.4) gives a positive radial solution of (1.1), and the Poisson extension [7] gives (2.1). The Kelvin transform preserves (1.1), and

$$u^{\#}(x)=|x|^{-(n-2\sigma)}u(x/|x|^2)=K_\sigma^{\frac{n-2\sigma}{4\sigma}}|x|^{-\frac{n-2\sigma}{2}}w(\log|x|).$$

The boundary normalization in [6] is obtained by replacing U with $d_\sigma^{-\frac{n-2\sigma}{4\sigma}}U$. Conversely, applying [6, Theorem 1.1] to a positive radial solution and its Kelvin transform gives a bounded cylindrical trace.

Lemma 2.1. *Let w be a positive bounded solution of* (1.5)*. If*

$$\liminf_{t\to+\infty}w(t)=0\quad\textit{or}\quad\liminf_{t\to-\infty}w(t)=0,$$

then $w=w_b(\cdot-t_0)$ for some $t_0\in\mathbb{R}$.

Proof. By Kelvin invariance, we may assume $\liminf_{t\to+\infty}w(t)=0$. Then u extends across 0 by [6, Propositions 3.3–3.4]. Radial symmetry and [22, Theorem 1.8] give, for some $\mu>0$,

$$u(x)=(2^{2\sigma}K_\sigma)^{\frac{n-2\sigma}{4\sigma}}\left(\frac{\mu}{\mu^2+|x|^2}\right)^{\frac{n-2\sigma}{2}},\qquad w(t)=w_b(t+\log\mu).$$

□

Proposition 2.2. *Every positive bounded solution w of* (1.5) *satisfies*

$$|\partial_t \log w| \le \frac{n-2\sigma}{2}, \qquad \sup_{\mathbb{R}} w \le w_*, \tag{2.9}$$

where $w_* = \left(\frac{\sqrt{\pi}\,\Gamma(\sigma+1)}{\Gamma(\sigma+\frac{1}{2})} \right)^{\frac{n-2\sigma}{4\sigma}}$.

Proof. If $\liminf w = 0$ at either end, Lemma 2.1 gives

$$w = w_b(\cdot - t_0), \qquad |\partial_t \log w| = p|\tanh(t-t_0)| \le p.$$

Otherwise, u and $u^\#$ have nonremovable singularities. By [6, Theorem 1.3],

$$\partial_r u \le 0, \qquad 0 \ge \frac{r\partial_r u^\#(r)}{u^\#(r)} = -(n-2\sigma) - \frac{r^{-1}\partial_r u(r^{-1})}{u(r^{-1})}.$$

Hence

$$-(n-2\sigma) \le \frac{r\partial_r u}{u} \le 0, \qquad \partial_t \log w = -p - \frac{r\partial_r u}{u} \in [-p, p]\,.$$

Thus $w(s) \ge w(t)e^{-p|s-t|}$ for $s, t \in \mathbb{R}$. By (2.5),

$$w(t) = K_\sigma \int_{\mathbb{R}} G_\sigma(t-s) w(s)^{\frac{n+2\sigma}{n-2\sigma}}\,\mathrm{d}s \ge K_\sigma w(t)^{\frac{n+2\sigma}{n-2\sigma}} \int_{\mathbb{R}} G_\sigma(s) e^{-\frac{n+2\sigma}{2}|s|}\,\mathrm{d}s.$$

The moment identity (A.3) gives

$$w(t)^{\frac{4\sigma}{n-2\sigma}} \le \left(K_\sigma \int_{\mathbb{R}} G_\sigma(s) e^{-\frac{n+2\sigma}{2}|s|}\,\mathrm{d}s \right)^{-1} = w_*^{\frac{4\sigma}{n-2\sigma}}.$$

□

For two solutions w_1, w_2, we have $\mathcal{P}_\sigma(w_1 - w_2) = d_\sigma b(w_1 - w_2)$, where

$$b(t) = \frac{(n+2\sigma)K_\sigma}{(n-2\sigma)d_\sigma} \int_0^1 [(1-s)w_2(t) + sw_1(t)]^{\frac{4\sigma}{n-2\sigma}}\,\mathrm{d}s. \tag{2.10}$$

Lemma 2.3. *Let w_1, w_2 be bounded solutions of* (1.5)*, each positive or identically zero. Then the coefficient b satisfies*

$$0 \le b \le b_* := \frac{(n+2\sigma)K_\sigma}{(n-2\sigma)d_\sigma} w_*^{\frac{4\sigma}{n-2\sigma}}, \qquad |\partial_t b| \le 2\sigma b. \tag{2.11}$$

Moreover, local uniform convergence of (w_1, w_2) implies local uniform convergence of b.

Proof. If $w_1 = w_2 = 0$, then $b = 0$. Otherwise, differentiating (2.10) and applying Proposition 2.2 gives

$$0 \le b \le \frac{(n+2\sigma)K_\sigma}{(n-2\sigma)d_\sigma} w_*^{\frac{4\sigma}{n-2\sigma}} = b_*,$$

and

$$\begin{aligned}|b'| &\le \frac{4\sigma(n+2\sigma)K_\sigma}{(n-2\sigma)^2 d_\sigma}\int_0^1 [(1-s)w_2+sw_1]^{\frac{6\sigma-n}{n-2\sigma}}[(1-s)|w_2'|+s|w_1'|]\,\mathrm{d}s\\ &\le 2\sigma\frac{(n+2\sigma)K_\sigma}{(n-2\sigma)d_\sigma}\int_0^1 [(1-s)w_2+sw_1]^{\frac{4\sigma}{n-2\sigma}}\,\mathrm{d}s\\ &= 2\sigma b.\end{aligned}$$

The convergence of b follows from (2.10) and the uniform continuity of $s \mapsto s^{\frac{4\sigma}{n-2\sigma}}$ on $[0, w_*]$. □

Proposition 2.4. *Every positive radial solution of* (1.1) *has a cylindrical Poisson extension v satisfying*

$$0 \le v(t,\theta) \le \|w\|_\infty \le w_* \qquad \text{for } t \in \mathbb{R},\ 0 \le \theta \le \pi/2.$$

Conversely, every positive bounded solution of (2.1) *agrees with the Poisson extension of its trace and has finite local weighted energy.*

Proof. The Poisson formula [7], the angular maximum principle, and Proposition 2.2 give $0 \le v(t,\theta) \le \|w\|_\infty \le w_*$. Conversely, let $\widetilde{v}$ be the Poisson extension of $w(t) = v(t,\pi/2)$. The same comparison gives

$$0 \le \widetilde{v} \le \|w\|_\infty, \qquad \Phi := v - \widetilde{v}, \qquad \Phi(t,\pi/2) = 0.$$

Choose $\chi_R \in C_c^\infty(\mathbb{R})$ with $\chi_R = 1$ on $[-R,R]$, $\operatorname{supp}\chi_R \subset [-2R,2R]$, and $|\chi_R'| \le C/R$. Testing the interior equation by $\chi_R^2\Phi$ gives

$$\int_{-R}^{R}\int_0^{\pi/2}\left(\Phi_t^2+\Phi_\theta^2+p^2\Phi^2\right)\omega\,\mathrm{d}\theta\,\mathrm{d}t \le CR^{-2}\int_{-2R}^{2R}\int_0^{\pi/2}\Phi^2\,\omega\,\mathrm{d}\theta\,\mathrm{d}t \le C/R \longrightarrow 0.$$

Hence $v = \widetilde{v}$. Finite local weighted energy follows from the Poisson estimates in [7]. □

2.3 Robin spectral gap and the principal eigenspace

For an open interval $J \subset \mathbb{R}$, the Robin coefficients in Lemma 2.3 lead to the linear problem

$$\begin{cases} -\partial_t^2\Phi - \omega^{-1}\partial_\theta(\omega\partial_\theta\Phi) + p^2\Phi = f & (t,\theta) \in J\times(0,\pi/2),\\ \partial_\nu^\sigma\Phi = b(t)\Phi(t,\pi/2) & t \in J,\ \theta = \pi/2. \end{cases} \tag{2.12}$$

Let $L^2_\omega = L^2((0,\pi/2),\omega\,\mathrm{d}\theta)$ and $H^1_\omega = H^1((0,\pi/2),\omega\,\mathrm{d}\theta)$, with

$$\langle f,g\rangle_\omega = \int_0^{\pi/2} fg\,\omega\,\mathrm{d}\theta, \qquad \|f\|^2_{H^1_\omega} = \int_0^{\pi/2}(|\partial_\theta f|^2 + p^2|f|^2)\,\omega\,\mathrm{d}\theta.$$

For $b \in \mathbb{R}$, define

$$Q_b(f,g) = \int_0^{\pi/2}(\partial_\theta f\,\partial_\theta g + p^2 fg)\,\omega\,\mathrm{d}\theta - bf(\pi/2)g(\pi/2), \qquad f,g \in H^1_\omega.$$

For $b = 0$, $Q_0(f,f) = \|f\|^2_{H^1_\omega}$. Define $A_b : H^1_\omega \longrightarrow (H^1_\omega)^*$ by

$$\langle A_b f, g\rangle = Q_b(f,g).$$

We also write A_b for the self-adjoint operator associated with Q_b in L^2_ω. For $b = b(t)$, write

$$\mathcal{L}_b\Phi = -\partial_t^2\Phi + A_{b(t)}\Phi.$$

Then (2.12) is $\mathcal{L}_b\Phi = f$ on J. We first study the angular spectrum for fixed b. These estimates will give weighted solvability and two-dimensionality for $b = b(t)$. They also enter the boundary comparison in Lemma 2.12.

The weighted trace inequality [22, Section 2.1] gives, for every $\varepsilon > 0$,

$$|f(\pi/2)|^2 \le \varepsilon\|f\|^2_{H^1_\omega} + C_\varepsilon\|f\|^2_{L^2_\omega}. \tag{2.13}$$

By (2.13) and Cauchy–Schwarz,

$$\int_{\pi/2-\delta}^{\pi/2} |f|^2\,\omega\,\mathrm{d}\theta \le C(\delta^{2-2\sigma} + \delta^2)\|f\|^2_{H^1_\omega}.$$

Rellich compactness away from the equator gives $H^1_\omega \Subset L^2_\omega$. By (2.13), the traces also converge along any H^1_ω-bounded sequence converging in L^2_ω.

By (2.13),

$$Q_b(f,f) \ge \tfrac{1}{2}\|f\|^2_{H^1_\omega} - C_b\|f\|^2_{L^2_\omega}.$$

Thus Q_b is closed and symmetric, and A_b is self-adjoint in L^2_ω. The eigenvalue equation $A_b f = \lambda f$, with regularity at the pole, is

$$\begin{cases} -\omega^{-1}(\omega f')' + p^2 f = \lambda f, \\ \partial_\theta f(0) = 0, \qquad \partial_\nu^\sigma f = b f(\pi/2). \end{cases} \tag{2.14}$$

For $b \ge 0$, write

$$A_b e_j(b) = \lambda_j(b) e_j(b), \qquad \|e_j(b)\|_{L^2_\omega} = 1 \quad \text{for } j \ge 0,$$

with the eigenvalues in increasing order and $e_0(b) > 0$. Replacing the last boundary condition by $f(\pi/2) = 0$ or $\partial_\nu^\sigma f = 0$ gives the Dirichlet and Neumann problems, respectively. Denote these eigenvalues by λ_j^D and λ_j^N.

For $\lambda \notin \{\lambda_j^D : j \ge 0\}$, let f_λ solve

$$\begin{cases} -\omega^{-1}(\omega f_\lambda')' + p^2 f_\lambda = \lambda f_\lambda, \\ \partial_\theta f_\lambda(0) = 0, \qquad f_\lambda(\pi/2) = 1. \end{cases}$$

The angular Dirichlet-to-Neumann map is

$$F_\sigma(\lambda) := d_\sigma \partial_\nu^\sigma f_\lambda.$$

The explicit calculation in [2, proof of Proposition 3.4] gives

$$F_\sigma(\lambda) = 2^{2\sigma}\frac{\Gamma(\frac{n+2\sigma}{4}+\frac{\sqrt{\lambda}}{2})\Gamma(\frac{n+2\sigma}{4}-\frac{\sqrt{\lambda}}{2})}{\Gamma(\frac{n-2\sigma}{4}+\frac{\sqrt{\lambda}}{2})\Gamma(\frac{n-2\sigma}{4}-\frac{\sqrt{\lambda}}{2})}. \tag{2.15}$$

Set

$$k = \frac{n+2\sigma}{2}.$$

The following lemma identifies k^2 as the first Dirichlet eigenvalue.

Lemma 2.5. *The Neumann and Dirichlet eigenvalues are, respectively,*

$$\begin{aligned} &\lambda_j^N = (p+2j)^2, \qquad &&\lambda_j^D = (k+2j)^2, \\ &\lambda_j^N < \lambda_j^D < \lambda_{j+1}^N \qquad &&\textit{for } j \geq 0. \end{aligned} \tag{2.16}$$

For every $b \geq 0$,

$$\lambda_0(b) \leq p^2 < k^2 < \lambda_1(b).$$

The first eigenpair depends real analytically on b, with $e_0(b)$ valued in H^1_ω. Moreover,

$$\lambda_* := \min_{0\leq b\leq b_*} \lambda_1(b) > k^2.$$

For $z < k^2$, the function F_σ is strictly decreasing and strictly concave:

$$F_\sigma'(z) = -d_\sigma \|f_z\|^2_{L^2_\omega} < 0, \qquad F_\sigma''(z) = -2d_\sigma\langle f_z, \partial_z f_z\rangle_\omega < 0. \tag{2.17}$$

Proof. By the definition of f_λ, $F_\sigma(\lambda) = 0$ is equivalent to the Neumann condition. At a Dirichlet eigenvalue, the normalization $f_\lambda(\pi/2) = 1$ fails. The conormal derivative of a nonzero Dirichlet eigenfunction cannot vanish by ODE uniqueness, so F_σ has a pole there. Thus (2.15) gives, for $j \geq 0$,

$$\lambda_j^N = (p+2j)^2 < \lambda_j^D = (k+2j)^2 < \lambda_{j+1}^N = (p+2(j+1))^2.$$

The Dirichlet form domain $\mathcal{D} = \{f \in H^1_\omega : f(\pi/2) = 0\}$ has codimension one in H^1_ω, and $Q_b = Q_0$ on $\mathcal{D}$. The min–max principle therefore gives

$$\lambda_j(b) \leq \lambda_j^D \leq \lambda_{j+1}(b).$$

If a Dirichlet and a Robin eigenvalue coincide, their eigenfunctions are proportional by uniqueness of the regular solution at the pole. A nonzero Robin eigenfunction would then satisfy

$$f(\pi/2) = \partial_\nu^\sigma f = 0 \quad \Longrightarrow \quad f \equiv 0,$$

by uniqueness for the angular equation, a contradiction. Hence

$$\lambda_j(b) < \lambda_j^D < \lambda_{j+1}(b),$$

$$\lambda_0(b) = \min_{0\neq f\in H^1_\omega} \frac{Q_b(f,f)}{\|f\|^2_{L^2_\omega}} \leq Q_b(e_0(0), e_0(0)) = p^2 - b\, e_0(0,\pi/2)^2 \leq p^2 < k^2 < \lambda_1(b).$$

The forms Q_b are closed and lower bounded on the common domain H^1_ω, depend analytically on b, and the associated operators have compact resolvent by $H^1_\omega \Subset L^2_\omega$. Since $\lambda_0(b)$ is simple, Kato's theorem [25, Chapter VII, Section 4] gives the analytic dependence of the first eigenpair.

Since $b \mapsto \lambda_1(b)$ is continuous on $[0, b_*]$ and $\lambda_1(b) > k^2$, we obtain $\lambda_* > k^2$.

For $z, y < k^2$, Green's identity gives

$$F_\sigma(z) - F_\sigma(y) = d_\sigma(y - z)\langle f_z, f_y\rangle_\omega,$$

and hence, letting $y \to z$,

$$F'_\sigma(z) = -d_\sigma \|f_z\|^2_{L^2_\omega} < 0.$$

Differentiating the angular equation (2.14) gives

$$\partial_z f_z \in \mathcal{D}, \qquad Q_0(\partial_z f_z, \psi) - z\langle \partial_z f_z, \psi\rangle_\omega = \langle f_z, \psi\rangle_\omega \quad \text{for } \psi \in \mathcal{D}.$$

Taking $\psi = \partial_z f_z$ and using $\lambda^D_0 = k^2$,

$$\begin{aligned} F''_\sigma(z) &= -2d_\sigma\langle f_z, \partial_z f_z\rangle_\omega \\ &= -2d_\sigma\Big(Q_0(\partial_z f_z, \partial_z f_z) - z\|\partial_z f_z\|^2_{L^2_\omega}\Big) \\ &\le -2d_\sigma(k^2 - z)\|\partial_z f_z\|^2_{L^2_\omega} < 0. \end{aligned}$$

□

To control the variation of the principal eigenfunction, we use the Robin condition $F_\sigma(\lambda_0(b)) = d_\sigma b$. Euler's product formula for Γ [31, Section 5.8] gives

$$\Lambda(\lambda) := -\frac{F'_\sigma(\lambda)}{F_\sigma(\lambda)} = \sum_{j\ge 0}\left(\frac{1}{\lambda^N_j - \lambda} - \frac{1}{\lambda^D_j - \lambda}\right).$$

For $\lambda < p^2$, every summand is positive, so $\Lambda(\lambda) > 0$. Moreover, $\Lambda(\lambda) \to +\infty$ as $\lambda \uparrow p^2$; we set $1/\Lambda(p^2) := 0$ by continuity.

Lemma 2.6. *For $b \ge 0$, write $\lambda_j = \lambda_j(b)$, $e_j = e_j(b)$ and $\lambda = \lambda_0(b)$. Then*

$$\sum_{j\ge 1}\frac{(e_j(\pi/2))^2}{\lambda_j - k^2} = \frac{(e_0(\pi/2))^2}{k^2 - \lambda}, \quad \partial_b e_0 = \sum_{j\ge 1}\frac{e_0(\pi/2)e_j(\pi/2)}{\lambda_j - \lambda}e_j.$$

Moreover,

$$\partial_b\lambda_0(b) = -(e_0(\pi/2))^2, \qquad b(e_0(\pi/2))^2 = \frac{1}{\Lambda(\lambda)}.$$

The series for $\partial_b e_0$ converges in H^1_ω. At $b = 0$, the last identity is understood by continuity.

Proof. Fix $b \geq 0$ and choose $d > -\lambda$. The positive inner product $Q_b(f,g) + d\langle f, g\rangle_\omega$ induces a norm equivalent to $\|\cdot\|_{H^1_\omega}$, and $\{e_j/\sqrt{\lambda_j + d}\}_{j\geq 0}$ is orthonormal for this inner product. The trace $f \mapsto f(\pi/2)$ is bounded by (2.13). Bessel's inequality therefore gives

$$\sum_{j\geq 0} \frac{(e_j(\pi/2))^2}{\lambda_j + d} \leq C < \infty.$$

For $z < k^2$, Green's identity gives

$$Q_b(f_z, \psi) - z\langle f_z, \psi\rangle_\omega = \left(\frac{F_\sigma(z)}{d_\sigma} - b\right)\psi(\pi/2) \qquad (\psi \in H^1_\omega).$$

Taking $\psi = e_j$, we obtain

$$(\lambda_j - z)\langle f_z, e_j\rangle_\omega = \left(\frac{F_\sigma(z)}{d_\sigma} - b\right) e_j(\pi/2).$$

If $z \notin \operatorname{spec}(A_b)$, these are the coefficients of the spectral expansion of f_z. Since $f_z \in H^1_\omega$, the expansion converges in the form norm, which is equivalent to the H^1_ω-norm. Taking its trace and using $f_z(\pi/2) = 1$ yields

$$\sum_{j\geq 0} \frac{(e_j(\pi/2))^2}{\lambda_j - z} = \frac{d_\sigma}{F_\sigma(z) - d_\sigma b}.$$

By Lemma 2.5, $\lambda < k^2 < \lambda_1(b)$; in particular, $k^2 \notin \operatorname{spec}(A_b)$. For z sufficiently close to k^2,

$$\frac{1}{|\lambda_j - z|} \leq \frac{C}{\lambda_j + d} \qquad (j \geq 0).$$

The Bessel estimate above therefore permits passage to the limit $z \uparrow k^2$ in the series. Since F_σ has a pole at $k^2 = \lambda_0^D$, the right-hand side tends to zero. Thus

$$\frac{(e_0(\pi/2))^2}{\lambda - k^2} + \sum_{j\geq 1} \frac{(e_j(\pi/2))^2}{\lambda_j - k^2} = 0,$$

which proves the first identity.

The analytic dependence in Lemma 2.5 allows us to differentiate $Q_b(e_0, \psi) = \lambda\langle e_0, \psi\rangle_\omega$, for fixed $\psi \in H^1_\omega$, and $\langle e_0, e_0\rangle_\omega = 1$. Since $\partial_b Q_b(f,g) = -f(\pi/2)g(\pi/2)$, this gives

$$\begin{aligned} Q_b(\partial_b e_0, \psi) - \lambda\langle \partial_b e_0, \psi\rangle_\omega &= e_0(\pi/2)\psi(\pi/2) + \partial_b\lambda_0 \langle e_0, \psi\rangle_\omega, \\ \langle \partial_b e_0, e_0\rangle_\omega &= 0. \end{aligned}$$

Taking $\psi = e_0$ and $\psi = e_j$, $j \geq 1$, respectively, yields

$$\partial_b\lambda_0 = -(e_0(\pi/2))^2, \qquad \langle \partial_b e_0, e_j\rangle_\omega = \frac{e_0(\pi/2)e_j(\pi/2)}{\lambda_j - \lambda}.$$

Together with $\langle \partial_b e_0, e_0\rangle_\omega = 0$, these identities give the expansion in L^2_ω. To prove convergence in H^1_ω, use $\lambda_j \geq \lambda_1(b) > k^2 > \lambda$ to obtain

$$\frac{\lambda_j + d}{(\lambda_j - \lambda)^2} \leq \frac{C}{\lambda_j + d} \qquad (j \geq 1).$$

Consequently,

$$\begin{aligned}\sum_{j\geq 1}(\lambda_j + d)|\langle \partial_b e_0, e_j\rangle_\omega|^2 &= (e_0(\pi/2))^2 \sum_{j\geq 1} \frac{(\lambda_j + d)(e_j(\pi/2))^2}{(\lambda_j - \lambda)^2} \\ &\leq C(e_0(\pi/2))^2 \sum_{j\geq 1} \frac{(e_j(\pi/2))^2}{\lambda_j + d} \\ &< \infty.\end{aligned}$$

Thus the expansion converges in the form norm, hence in H^1_ω.

Finally, for $b > 0$, differentiating the Robin condition gives

$$-F'_\sigma(\lambda)(e_0(\pi/2))^2 = d_\sigma, \qquad b(e_0(\pi/2))^2 = -\frac{F_\sigma(\lambda)}{F'_\sigma(\lambda)} = \frac{1}{\Lambda(\lambda)}.$$

At $b = 0$, $\lambda_0(0) = p^2$. Continuity of $e_0(b, \pi/2)$ and the definition of $1/\Lambda(p^2)$ give the last identity at this endpoint. □

2.4 Two-dimensionality of the linear equation

We now let $b = b(t)$ and prove weighted solvability and two-dimensionality of the homogeneous solution space. Let $E_\omega = H^1_\omega \times L^2_\omega$.

Lemma 2.7. *Let $J_1 \Subset J_2$ be bounded open intervals and $C_0 > 0$. For every $b \in W^{1,\infty}(J_2)$ with $\|b\|_{W^{1,\infty}(J_2)} \leq C_0$, each weak solution Φ of $\mathcal{L}_b\Phi = 0$ on J_2 satisfies*

$$\|(\Phi, \partial_t\Phi)\|_{C(\overline{J_1};E_\omega)} \leq C\|\Phi\|_{L^2(J_2;L^2_\omega)}.$$

Here C depends on n, σ, J_1, J_2, C_0. If Φ_j is a sequence of such solutions, with coefficients b_j, and

$$\sup_j \|\Phi_j\|_{L^2(J_2;L^2_\omega)} < \infty,$$

then a subsequence of $(\Phi_j, \partial_t\Phi_j)$ converges in $C(\overline{J_1}; E_\omega)$.

Proof. We first prove a local energy estimate. Suppose Ψ satisfies the first equation in (2.12) with $f = 0$ and

$$\partial^\sigma_\nu \Psi = b\Psi + g \qquad \text{on } J_2,$$

where $g \in L^2(J_2)$. For $J \Subset J_2$, testing the interior equation by $\chi^2\Psi$ with $\chi \in C^\infty_c(J_2)$, $\chi = 1$ on J, and using (2.13) gives

$$\|(\Psi, \Psi_t)\|_{L^2(J;E_\omega)} \leq C\Big(\|\Psi\|_{L^2(J_2;L^2_\omega)} + \|g\|_{L^2(J_2)}\Big). \tag{2.18}$$

Fix $J \Subset J' \Subset J'' \Subset J_2$. Applying (2.18) to Φ with $g = 0$ and using the trace estimate yields

$$\|(\Phi, \Phi_t)\|_{L^2(J'';E_\omega)} + \|\Phi(\cdot, \pi/2)\|_{L^2(J'')} \le C\|\Phi\|_{L^2(J_2;L^2_\omega)}.$$

The difference quotient $\delta_h\Phi$ satisfies

$$\partial_\nu^\sigma \delta_h\Phi = b(t+h)\delta_h\Phi(t, \pi/2) + \delta_h b(t)\Phi(t, \pi/2).$$

Since $\|\delta_h b\|_\infty \le C_0$, the preceding bounds give

$$\|\delta_h\Phi\|_{L^2(J';L^2_\omega)} + \|\delta_h b\,\Phi(\cdot, \pi/2)\|_{L^2(J')} \le C\|\Phi\|_{L^2(J_2;L^2_\omega)}.$$

Applying (2.18) to $\delta_h\Phi$ with coefficient $b(\cdot + h)$, uniformly in small h, gives

$$\|(\delta_h\Phi, \partial_t\delta_h\Phi)\|_{L^2(J;E_\omega)} \le C\|\Phi\|_{L^2(J_2;L^2_\omega)}.$$

Letting $h \to 0$,

$$\|(\Phi, \Phi_t)\|_{H^1(J;E_\omega)} \le C\|\Phi\|_{L^2(J_2;L^2_\omega)} \qquad (J \Subset J_2).$$

The embedding $H^1(J; E_\omega) \hookrightarrow C(\overline{J}; E_\omega)$ gives the first assertion.

For compactness, take $J_1 \Subset J \Subset J_2$. The $W^{1,\infty}$-bound on b_j, the H^1_t-estimate and $H^1_\omega \Subset L^2_\omega$ give, by Arzelà–Ascoli, a subsequence such that

$$\|b_i - b_j\|_{C(\overline{J})} + \|\Phi_i - \Phi_j\|_{C(\overline{J};L^2_\omega)} \longrightarrow 0.$$

For $\Psi = \Phi_i - \Phi_j$,

$$\partial_\nu^\sigma \Psi = b_i\Psi + (b_i - b_j)\Phi_j.$$

By (2.18) and the uniform trace bound for Φ_j,

$$\|(\Psi, \Psi_t)\|_{L^2(J_1;E_\omega)} \le C\Big(\|\Psi\|_{L^2(J;L^2_\omega)} + \|b_i - b_j\|_{C(\overline{J})}\Big) \longrightarrow 0. \tag{2.19}$$

Interpolation and the uniform $H^1(J_1; E_\omega)$-bound give

$$\|(\Psi, \Psi_t)\|^2_{C(\overline{J_1};E_\omega)} \le C\|(\Psi, \Psi_t)\|_{L^2(J_1;E_\omega)}\|(\Psi, \Psi_t)\|_{H^1(J_1;E_\omega)} \longrightarrow 0.$$

□

For the weighted estimates below, let $b \in W^{1,\infty}(\mathbb{R})$ satisfy (2.11). We decompose the solution into the principal mode and its orthogonal complement. Write

$$e(t) = e_0(b(t)), \qquad \lambda(t) = \lambda_0(b(t)), \qquad A_\perp = A_{b(t)}|_{e(t)^\perp}.$$

Let $\Pi(t)$ be the orthogonal projection onto $e(t)^\perp$, and let ∇_t be the covariant derivative on this moving subspace:

$$\Pi(t) = I - e(t) \otimes e(t), \qquad \nabla_t = \Pi(t)\partial_t. \tag{2.20}$$

Since $b \in W^{1,\infty}$ and $b \mapsto e_0(b)$ is analytic, $e, e' \in L^\infty(\mathbb{R}; H^1_\omega)$. For a weak solution of $\mathcal{L}_b\Phi = f$, write

$$\Phi = q_1 e + \Phi_\perp, \qquad \Phi_t = q_2 e + \Psi_\perp, \qquad \Phi_\perp, \Psi_\perp \perp e(t), \tag{2.21}$$

where

$$q_\Phi(t) = (q_1(t), q_2(t)), \qquad q_1(t) = \langle e(t), \Phi(t)\rangle_\omega, \quad q_2(t) = \langle e(t), \partial_t\Phi(t)\rangle_\omega.$$

Testing with $e(t)$ shows that $q_\Phi \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^2)$; we use its continuous representative.

We use the weights e^{-2kt}, e^{2kt} and

$$\varpi_{t_0}(t) = e^{-2k|t-t_0|}.$$

For any of these weights, define

$$\|\Phi\|^2_\varpi = \int_{\mathbb{R}} \varpi(t)\Big(\|\Phi(t)\|^2_{H^1_\omega} + \|\partial_t\Phi(t)\|^2_{L^2_\omega}\Big)\,\mathrm{d}t.$$

The two-sided weighted class contains differences of bounded nonlinear solutions and polynomially growing solutions of the linearized equation, with their respective Robin coefficients. Define

$$\mathcal{C}_b = \{\Phi :\ \mathcal{L}_b\Phi = 0,\ \|\Phi\|_{\varpi_0} < \infty\}. \tag{2.22}$$

We shall prove that $\Phi \mapsto q_\Phi(0)$ identifies $\mathcal{C}_b$ with $\mathbb{R}^2$.

Lemma 2.8. *Let $b \in W^{1,\infty}(\mathbb{R})$ satisfy* (2.11)*. There is $C = C(n, \sigma)$ with the following properties. For $t_0 \in \mathbb{R}$ and $f \in L^2(\varpi_{t_0}; L^2_\omega)$, every weak solution of $\mathcal{L}_b\Phi = f$ with $\|\Phi\|_{\varpi_{t_0}} < \infty$ satisfies*

$$\|\Phi\|_{\varpi_{t_0}} \le C\Big(\|f\|_{L^2(\varpi_{t_0}; L^2_\omega)} + |q_\Phi(t_0)|\Big). \tag{2.23}$$

For either choice $\varpi = e^{\pm 2kt}$ and $f \in L^2(\varpi; L^2_\omega)$, every weak solution of $\mathcal{L}_b\Phi = f$ with $\|\Phi\|_\varpi < \infty$ satisfies

$$\|\Phi\|_\varpi \le C\|f\|_{L^2(\varpi; L^2_\omega)}.$$

Proof. All differential identities below hold almost everywhere in t. Differentiating (2.21) and projecting onto $e(t)$ and $e(t)^\perp$ give

$$\begin{aligned} q_1' &= q_2 + \Big\langle e', \Phi_\perp\Big\rangle_\omega, & q_2' &= \lambda q_1 + \Big\langle e', \Psi_\perp\Big\rangle_\omega - \Big\langle e, f\Big\rangle_\omega, \\ \nabla_t\Phi_\perp &= \Psi_\perp - q_1 e', & \nabla_t\Psi_\perp &= A_\perp\Phi_\perp - q_2 e' - \Pi f. \end{aligned} \tag{2.24}$$

To estimate the principal coordinates, set

$$\ell = \max\{k, \sqrt{(-\lambda)_+}\}, \qquad \delta = \frac{\ell'}{2\ell}, \qquad a = \frac{\ell^2 + \lambda}{2\ell}, \qquad \alpha = (\delta^2 + a^2)^{1/2}. \tag{2.25}$$

For the weight $e^{-2\gamma t}$, $\gamma \in \mathbb{R}$, define

$$\begin{aligned} Q_{\shortparallel}^{\gamma} &= (\gamma-\delta)\ell q_1^2 + (\gamma+\delta)\ell^{-1}q_2^2 - 2aq_1q_2, \\ Q_{\perp}^{\gamma} &= Q_b(\Phi_\perp,\Phi_\perp) - \gamma^2\|\Phi_\perp\|_{L^2_\omega}^2 + \|\Psi_\perp - \gamma\Phi_\perp\|_{L^2_\omega}^2, \\ D_\gamma(\Phi,\Phi_t) &= Q_{\shortparallel}^{\gamma} + Q_{\perp}^{\gamma} - (q_1+\ell^{-1}q_2)\langle e', \ell\Phi_\perp + \Psi_\perp\rangle_\omega. \end{aligned} \tag{2.26}$$

The corresponding flux is

$$E = \langle \Phi_\perp, \Psi_\perp\rangle_\omega - \tfrac{1}{2}(\ell q_1^2 + \ell^{-1}q_2^2). \tag{2.27}$$

By the weak product rule and (2.24), $E \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and

$$\begin{aligned} \partial_t E = {} & \langle \Psi_\perp - q_1 e', \Psi_\perp\rangle_\omega + Q_b(\Phi_\perp,\Phi_\perp) \\ & - q_2\langle \Phi_\perp, e'\rangle_\omega - \langle \Phi_\perp, f\rangle_\omega - \ell q_1 q_1' - \ell^{-1}q_2q_2' - \tfrac{1}{2}\ell' q_1^2 + \frac{\ell'}{2\ell^2}q_2^2. \end{aligned}$$

Substituting the equations for q_1', q_2' and collecting the terms in $\partial_t E - 2\gamma E$, we obtain

$$e^{2\gamma t}\partial_t(e^{-2\gamma t}E) = D_\gamma(\Phi,\Phi_t) - \langle f, \Phi_\perp - \ell^{-1}q_2 e\rangle_\omega. \tag{2.28}$$

We now take $\gamma = k$, write $Q_{\shortparallel} = Q_{\shortparallel}^k$, $Q_\perp = Q_\perp^k$, and use the weight $\varpi = e^{-2kt}$. The matrix of $Q_{\shortparallel}$ in the variables $(\sqrt{\ell}q_1, q_2/\sqrt{\ell})$ is

$$\begin{pmatrix} k-\delta & -a \\ -a & k+\delta \end{pmatrix}$$

and has eigenvalues $k-\alpha$ and $k+\alpha$. By Lemma 2.6,

$$|\lambda'| \le \frac{2\sigma}{\Lambda(\lambda)}, \qquad \delta = 0 \quad \text{if } \lambda \ge -k^2, \qquad |\delta| \le \frac{\sigma}{2(-\lambda)\Lambda(\lambda)} \quad \text{if } \lambda < -k^2.$$

Thus ℓ is Lipschitz. By Lemma B.3,

$$\alpha \le k - c, \qquad c = c(n,\sigma) > 0. \tag{2.29}$$

Choose $k < \beta < \sqrt{\lambda_*}$, with $\beta = 7/4$ if $n = 2$ and $\sigma < 1/2$. By Lemmas 2.5 and B.2, $A_\perp \ge \beta^2$ uniformly in t, and by the trace estimate,

$$Q_b(\Phi_\perp,\Phi_\perp) \asymp \|\Phi_\perp\|^2_{H^1_\omega} \tag{2.30}$$

uniformly in t. Hence

$$Q_{\shortparallel} \ge (k-\alpha)(\ell q_1^2 + \ell^{-1}q_2^2) \ge c(q_1^2+q_2^2),$$

since ℓ is bounded above and below. Moreover,

$$Q_\perp = Q_b(\Phi_\perp,\Phi_\perp) + \|\Psi_\perp\|^2_{L^2_\omega} - 2k\langle \Phi_\perp, \Psi_\perp\rangle_\omega \ge (1-k/\beta)(Q_b(\Phi_\perp,\Phi_\perp) + \|\Psi_\perp\|^2_{L^2_\omega}).$$

Combining this with (2.30) and the boundedness of e, e' in H^1_ω, we obtain

$$Q_{\shortparallel} + Q_\perp \geq c(\|\Phi\|^2_{H^1_\omega} + \|\Phi_t\|^2_{L^2_\omega}).$$

It remains to control the coupling term. By Cauchy–Schwarz,

$$\begin{aligned}(q_1 + \ell^{-1}q_2)^2 &\leq \frac{Q_{\shortparallel}}{\ell(k^2-\alpha^2)}(1,1)\begin{pmatrix} k+\delta & a \\ a & k-\delta\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \\ &= \frac{2(k+a)}{\ell(k^2-\alpha^2)}Q_{\shortparallel}.\end{aligned}$$

Since

$$\ell\Phi_\perp + \Psi_\perp = (\ell+k)\Phi_\perp + (\Psi_\perp - k\Phi_\perp),$$

we have

$$|\langle e', \ell\Phi_\perp + \Psi_\perp\rangle_\omega|^2 \leq \ell R Q_\perp,$$

where

$$R = \ell^{-1}\|e'\|^2_{L^2_\omega} + \frac{(\ell+k)^2}{\ell}\langle e', (A_\perp - k^2)^{-1}e'\rangle_\omega.$$

Write $\lambda_j = \lambda_j(b(t))$ and $e_j = e_j(b(t))$. Using $e' = b'\partial_b e_0$ and Lemma 2.6,

$$R = (b')^2(e(t,\pi/2))^2\sum_{j\geq 1}\frac{(e_j(\pi/2))^2(\lambda_j + \ell^2 + 2k\ell)}{\ell(\lambda_j - k^2)(\lambda_j - \lambda)^2}.$$

For $s \geq k^2$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{s+\ell^2+2k\ell}{(s-\lambda)^2} = -\frac{s+\lambda+2\ell^2+4k\ell}{(s-\lambda)^3} < 0.$$

By Lemma 2.6 and $|b'| \leq 2\sigma b$,

$$R \leq \frac{(b')^2 e(t,\pi/2)^4}{\ell(k^2-\lambda)}\frac{\beta^2+\ell^2+2k\ell}{(\beta^2-\lambda)^2} \leq \frac{4\sigma^2}{\Lambda(\lambda)^2}\frac{\beta^2+\ell^2+2k\ell}{\ell(k^2-\lambda)(\beta^2-\lambda)^2}.$$

The comparison at $s = k^2$ likewise gives

$$R \leq \frac{4\sigma^2(\ell+k)^2}{\Lambda(\lambda)^2\ell(k^2-\lambda)^3}.$$

By Lemma B.3,

$$\frac{(k+a)R}{2(k^2-\alpha^2)} \leq (1-c)^2.$$

Therefore

$$\begin{aligned}|(q_1+\ell^{-1}q_2)\langle e', \ell\Phi_\perp + \Psi_\perp\rangle_\omega| &\leq \left(\frac{2(k+a)R}{k^2-\alpha^2}\right)^{1/2}(Q_{\shortparallel}Q_\perp)^{1/2} \\ &\leq (1-c)(Q_{\shortparallel}+Q_\perp).\end{aligned} \tag{2.31}$$

Combining (2.28), the coercivity of $Q_{\shortparallel} + Q_{\perp}$, and (2.31), we obtain

$$\partial_t(\varpi E) \geq c\varpi(\|\Phi\|_{H^1_\omega}^2 + \|\Phi_t\|_{L^2_\omega}^2) - C\varpi\|f\|_{L^2_\omega}(\|\Phi\|_{H^1_\omega} + \|\Phi_t\|_{L^2_\omega}).$$

To integrate the flux identity, use a cutoff $\chi_L(t) = \chi(t/L)$, where $\chi \in C_c^\infty((-2,2))$, $0 \leq \chi \leq 1$, and $\chi = 1$ on $[-1,1]$. Then

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \chi_L' \varpi \left[\langle \Phi_\perp, \Psi_\perp \rangle_\omega - \tfrac{1}{2}(\ell q_1^2 + \ell^{-1} q_2^2) \right] \mathrm{d}t \right| \\
&\leq \frac{C}{L} \int_{|t| \geq L} \varpi(\|\Phi\|_{H^1_\omega}^2 + \|\Phi_t\|_{L^2_\omega}^2)\, \mathrm{d}t \\
&\to 0 \qquad \text{as } L \to \infty,
\end{aligned}$$

because $\|\Phi\|_\varpi < \infty$. Integrating (2.28) and reversing time gives

$$\|\Phi\|_\varpi^2 \leq C\|f\|_{L^2(\varpi; L^2_\omega)} \|\Phi\|_\varpi, \qquad \varpi = e^{\pm 2kt}.$$

For ϖ_{t_0}, reverse time on $(-\infty, t_0)$. The endpoint terms add to

$$\begin{aligned}
&\left[-\langle \Phi_\perp, \Psi_\perp \rangle_\omega + \tfrac{1}{2}(\ell q_1^2 + \ell^{-1} q_2^2) \right]_{t=t_0} + \left[\langle \Phi_\perp, \Psi_\perp \rangle_\omega + \tfrac{1}{2}(\ell q_1^2 + \ell^{-1} q_2^2) \right]_{t=t_0} \\
&= \ell(t_0) q_1(t_0)^2 + \ell(t_0)^{-1} q_2(t_0)^2 \\
&\leq C |q_\Phi(t_0)|^2,
\end{aligned}$$

where we used the boundedness of ℓ. Hence

$$\|\Phi\|_{\varpi_{t_0}}^2 \leq C|q_\Phi(t_0)|^2 + C\|f\|_{L^2(\varpi_{t_0}; L^2_\omega)} \|\Phi\|_{\varpi_{t_0}}.$$

Young's inequality gives (2.23) for almost every t_0. Since

$$e^{-2k|s-t_0|} \varpi_{t_0}(t) \leq \varpi_s(t) \leq e^{2k|s-t_0|} \varpi_{t_0}(t),$$

both $\|\Phi\|_{\varpi_{t_0}}$ and $\|f\|_{L^2(\varpi_{t_0}; L^2_\omega)}$ are continuous in t_0. Together with the continuity of q_Φ, this gives (2.23) for every t_0.

Finally, for the principal system

$$q_1' = q_2, \qquad q_2' = \lambda q_1,$$

(2.29) gives

$$|\partial_t(\ell q_1^2 + \ell^{-1} q_2^2)| \leq 2\alpha(\ell q_1^2 + \ell^{-1} q_2^2) \leq 2(k-c)(\ell q_1^2 + \ell^{-1} q_2^2).$$

Since $\ell q_1^2 + \ell^{-1} q_2^2 \asymp q_1^2 + q_2^2$, Gronwall's inequality in both time directions gives, for the fundamental matrix $M(t,s)$,

$$\|M(t,s)\|_{\mathcal{L}(\mathbb{R}^2)} \leq Ce^{(k-c)|t-s|}. \tag{2.32}$$

□

Lemma 2.9. *Let $b \in W^{1,\infty}(\mathbb{R})$ satisfy* (2.11). *For $t_0 \in \mathbb{R}$, $\varpi \in \{e^{-2kt}, e^{2kt}, \varpi_{t_0}\}$ and $f \in L^2(\varpi; L^2_\omega)$, the problem*

$$\mathcal{L}_b \Phi = f, \qquad \|\Phi\|_\varpi < \infty,$$

with the additional condition $q_\Phi(t_0) = \xi$ for any prescribed $\xi \in \mathbb{R}^2$ when $\varpi = \varpi_{t_0}$, has a unique weak solution. In particular, $\{q_\Phi(0) : \Phi \in \mathcal{C}_b\} = \mathbb{R}^2$.

Proof. We use (2.20)–(2.21). Translating t, we may assume $t_0 = 0$. We construct $q = (q_1, q_2), \Phi_\perp, \Psi_\perp$ by continuation.

In (2.24), multiply the couplings through e' by $0 \le \tau \le 1$ and allow independent forcing terms:

$$\begin{aligned} q_1' - q_2 &= \tau \langle e', \Phi_\perp \rangle_\omega + f_1, & q_2' - \lambda q_1 &= \tau \langle e', \Psi_\perp \rangle_\omega + f_2, \\ \nabla_t \Phi_\perp - \Psi_\perp &= -\tau q_1 e' + f_{1,\perp}, & \nabla_t \Psi_\perp - A_\perp \Phi_\perp &= -\tau q_2 e' + f_{2,\perp}. \end{aligned} \tag{2.33}$$

Here $f_{1,\perp}, f_{2,\perp} \perp e(t)$ in L^2_ω, and

$$f_1, f_2 \in L^2(\varpi; \mathbb{R}), \qquad f_{1,\perp}, f_{2,\perp} \in L^2(\varpi; L^2_\omega).$$

For $Z = (q, \Phi_\perp, \Psi_\perp)$, define

$$\|Z\|_\varpi^2 = \int_{\mathbb{R}} \varpi \Big(|q|^2 + \|\Phi_\perp\|_{H^1_\omega}^2 + \|\Psi_\perp\|_{L^2_\omega}^2 \Big) \, \mathrm{d}t.$$

We seek solutions with $\|Z\|_\varpi < \infty$, with the fourth equation understood in $(H^1_\omega \cap e(t)^\perp)^*$; the initial condition $q(0) = \xi$ is imposed only for $\varpi = \varpi_0$.

At $\tau = 0$, the system decouples. The first two equations form the principal system

$$q' = \begin{pmatrix} 0 & 1 \\ \lambda(t) & 0 \end{pmatrix} q + (f_1, f_2).$$

Let $M(t,s)$ be its fundamental matrix. By (2.32),

$$\|M(t,s)\|_{\mathcal{L}(\mathbb{R}^2)} \le C e^{(k-c)|t-s|}.$$

Hence the solution of the principal system is

$$q(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} M(t,s)(f_1, f_2)(s) \, \mathrm{d}s, & \varpi = e^{-2kt}, \\ \displaystyle -\int_{t}^{\infty} M(t,s)(f_1, f_2)(s) \, \mathrm{d}s, & \varpi = e^{2kt}, \\ \displaystyle M(t,0)\xi + \int_0^t M(t,s)(f_1, f_2)(s) \, \mathrm{d}s, & \varpi = \varpi_0. \end{cases}$$

On the respective integration regions,

$$\frac{\varpi(t)^{1/2}}{\varpi(s)^{1/2}} \|M(t,s)\| \le C e^{-c|t-s|}.$$

By Young's inequality,

$$\|q\|_{L^2(\varpi)} \le \begin{cases} C\|(f_1,f_2)\|_{L^2(\varpi)}, & \varpi = e^{\pm 2kt}, \\ C\Big(|\xi| + \|(f_1,f_2)\|_{L^2(\varpi_0)}\Big), & \varpi = \varpi_0. \end{cases}$$

For the last two equations at $\tau = 0$, we use the space

$$V_\varpi = \Big\{\psi \in L^2(\varpi; H^1_\omega) : \psi \perp e(t), \quad \nabla_t \psi \in L^2(\varpi; L^2_\omega)\Big\}.$$

For $\psi \in V_\varpi$, the equations $\nabla_t \Phi_\perp = \Psi_\perp + f_{1,\perp}$ and $\nabla_t \Psi_\perp = A_\perp \Phi_\perp + f_{2,\perp}$ give the weak form

$$\int_{\mathbb{R}} \Big[\langle \nabla_t \Phi_\perp, \nabla_t(\varpi\psi)\rangle_\omega + \varpi Q_b(\Phi_\perp,\psi)\Big]\,\mathrm{d}t = \int_{\mathbb{R}} \Big[\langle f_{1,\perp}, \nabla_t(\varpi\psi)\rangle_\omega - \varpi\langle f_{2,\perp},\psi\rangle_\omega\Big]\,\mathrm{d}t.$$

Since $|\varpi'| \le 2k\varpi$ and $Q_b(\psi,\psi) \ge \beta^2 \|\psi\|^2_{L^2_\omega}$ with $k < \beta < \sqrt{\lambda_*}$,

$$\begin{aligned} &\int_{\mathbb{R}} \Big[\langle \nabla_t\psi, \nabla_t(\varpi\psi)\rangle_\omega + \varpi Q_b(\psi,\psi)\Big]\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \varpi\Big(\|\nabla_t\psi\|^2_{L^2_\omega} + Q_b(\psi,\psi)\Big)\,\mathrm{d}t + \int_{\mathbb{R}} \varpi'\langle \nabla_t\psi,\psi\rangle_\omega\,\mathrm{d}t \\ &\ge (1 - k/\beta)\int_{\mathbb{R}} \varpi\Big(\|\nabla_t\psi\|^2_{L^2_\omega} + Q_b(\psi,\psi)\Big)\,\mathrm{d}t. \end{aligned}$$

Thus the form is coercive on V_ϖ. Since $\nabla_t(\varpi\psi) = \varpi\nabla_t\psi + \varpi'\psi$ and $|\varpi'| \le 2k\varpi$, the right-hand side is bounded in V_ϖ^* by $C(\|f_{1,\perp}\|_{L^2(\varpi;L^2_\omega)} + \|f_{2,\perp}\|_{L^2(\varpi;L^2_\omega)})$. Lax–Milgram gives a unique $\Phi_\perp \in V_\varpi$. Setting

$$\Psi_\perp = \nabla_t \Phi_\perp - f_{1,\perp}$$

then gives $\nabla_t \Psi_\perp = A_\perp \Phi_\perp + f_{2,\perp}$ in the distributional sense, and

$$\|\Phi_\perp\|_{L^2(\varpi;H^1_\omega)} + \|\nabla_t\Phi_\perp\|_{L^2(\varpi;L^2_\omega)} + \|\Psi_\perp\|_{L^2(\varpi;L^2_\omega)} \le C\sum_{i=1}^{2} \|f_{i,\perp}\|_{L^2(\varpi;L^2_\omega)}.$$

Combining this with the estimate for q, we obtain

$$\|Z\|_\varpi \le \begin{cases} C\|(f_1,f_2,f_{1,\perp},f_{2,\perp})\|_{L^2(\varpi;\mathbb{R}^2\times(L^2_\omega)^2)}, & \varpi = e^{\pm 2kt}, \\ C\Big(|\xi| + \|(f_1,f_2,f_{1,\perp},f_{2,\perp})\|_{L^2(\varpi_0;\mathbb{R}^2\times(L^2_\omega)^2)}\Big), & \varpi = \varpi_0. \end{cases} \tag{2.34}$$

This proves existence for $\tau = 0$. Applying (2.34) to differences gives uniqueness.

For general $0 \le \tau \le 1$, use $\ell, E, Q_{\shortparallel}, Q_\perp$ from the proof of Lemma 2.8. The flux identity (2.28) has coupling term bounded by

$$\Big|\tau(q_1 + \ell^{-1}q_2)\langle e', \ell\Phi_\perp + \Psi_\perp\rangle_\omega\Big| \le (1-c)(Q_{\shortparallel} + Q_\perp).$$

For the source terms, Cauchy–Schwarz gives

$$\begin{aligned} &\int_{\mathbb{R}} \varpi\,\Big|\langle f_{1,\perp},\Psi_\perp\rangle_\omega + \langle \Phi_\perp, f_{2,\perp}\rangle_\omega - \ell q_1 f_1 - \ell^{-1} q_2 f_2\Big|\,\mathrm{d}t \\ &\qquad \le C\|(f_1,f_2,f_{1,\perp},f_{2,\perp})\|_{L^2(\varpi;\mathbb{R}^2\times(L^2_\omega)^2)}\|Z\|_\varpi. \end{aligned}$$

Integrating (2.28) with the cutoff χ_L from Lemma 2.8 and letting $L \to \infty$ gives (2.34) for every solution, with C independent of τ.

Suppose (2.33) is solvable at τ. Given $Z = (q, \Phi_\perp, \Psi_\perp)$, solve the system at τ with data

$$\begin{aligned} &f_1 + h\langle e', \Phi_\perp\rangle_\omega, && f_2 + h\langle e', \Psi_\perp\rangle_\omega, \\ &f_{1,\perp} - hq_1 e', && f_{2,\perp} - hq_2 e', \end{aligned}$$

and $q(0) = \xi$ when $\varpi = \varpi_0$. Since $e'(t) \perp e(t)$ and $\|e'(t)\|_{L^2_\omega} \le C$, (2.34) makes this map a contraction in $\|\cdot\|_\varpi$ for $0 < h < (2C)^{-1}$ and $\tau + h \le 1$. Its fixed point solves (2.33) at $\tau + h$. The first two equations give convergence of the principal coordinates in H^1_{loc}, so $q(0) = \xi$ is preserved. Since the step size is independent of τ, finitely many steps give solvability at $\tau = 1$.

At $\tau = 1$, choose

$$f_1 = f_{1,\perp} = 0, \qquad f_2 = -\langle e(t), f\rangle_\omega, \qquad f_{2,\perp} = -\Pi f, \qquad \Phi = q_1 e(t) + \Phi_\perp.$$

The system (2.33) then gives

$$\partial_t \Phi = \Big(q_1' - \langle e'(t), \Phi_\perp\rangle_\omega\Big) e(t) + q_1 e'(t) + \nabla_t \Phi_\perp = q_2 e(t) + \Psi_\perp,$$

and

$$\begin{aligned} \partial_t^2 \Phi &= \Big(q_2' - \langle e'(t), \Psi_\perp\rangle_\omega\Big) e(t) + q_2 e'(t) + \nabla_t \Psi_\perp \\ &= \lambda(t) q_1 e(t) + A_\perp \Phi_\perp - \langle e(t), f\rangle_\omega e(t) - \Pi f \\ &= A_{b(t)} \Phi - f. \end{aligned}$$

Therefore

$$\mathcal{L}_b \Phi = f, \qquad \|\Phi\|_\varpi \le C\|Z\|_\varpi < \infty,$$

and $q_\Phi(0) = q(0) = \xi$ when $\varpi = \varpi_0$. For two solutions Φ_1, Φ_2 with the same data, Lemma 2.8 gives $\|\Phi_1 - \Phi_2\|_\varpi = 0$, hence $\Phi_1 = \Phi_2$. For every $\xi \in \mathbb{R}^2$, the construction with $f = 0$ and $\varpi = \varpi_0$ gives

$$\Phi \in \mathcal{C}_b, \qquad q_\Phi(0) = \xi.$$

□

For $\Phi \in \mathcal{C}_b$, the weights ϖ_{t_0} are equivalent:

$$e^{-2k|t_0|}\varpi_0 \le \varpi_{t_0} \le e^{2k|t_0|}\varpi_0.$$

By Lemma 2.7 and (2.23),

$$\begin{aligned} \|(\Phi, \Phi_t)\|_{C([t_0-R, t_0+R]; E_\omega)} &\le C(n, \sigma, R)\|\Phi\|_{L^2((t_0-R-1, t_0+R+1); L^2_\omega)} \\ &\le C(n, \sigma, R)\|\Phi\|_{\varpi_{t_0}} \\ &\le C(n, \sigma, R)|q_\Phi(t_0)|. \end{aligned} \tag{2.35}$$

Corollary 2.10. *Let $b \in W^{1,\infty}(\mathbb{R})$ satisfy* (2.11)*. For every $t_0 \in \mathbb{R}$, the map $\Phi \mapsto q_\Phi(t_0)$ is an isomorphism from $\mathcal{C}_b$ onto $\mathbb{R}^2$. In particular,* $\dim \mathcal{C}_b = 2$.

Proof. Injectivity follows from (2.23), and surjectivity from Lemma 2.9. □

On the two-dimensional space $\mathcal{C}_b$, define the extension Wronskian (cf. [3, Lemma 5.3]) by

$$\Omega_t(\Phi_1, \Phi_2) = d_\sigma\Big(\langle \Phi_1, \partial_t \Phi_2\rangle_\omega - \langle \partial_t \Phi_1, \Phi_2\rangle_\omega\Big).$$

Proposition 2.11. *The form Ω_t is independent of t and nondegenerate on $\mathcal{C}_b$.*

Denote the resulting time-independent form by Ω.

Proof. Take $\mathcal{L}_b\Phi_1 = 0$ and $\mathcal{L}_b\Phi_2 = F$. Green's identity, understood in distributions, gives

$$\partial_t \Omega_t(\Phi_1, \Phi_2) = -d_\sigma \langle \Phi_1, F\rangle_\omega. \tag{2.36}$$

Taking $F = 0$ gives $\partial_t \Omega_t = 0$.

For nondegeneracy, take $0 \neq \Phi_1 \in \mathcal{C}_b$. Choose $R > 0$ and $\chi \in C_c^\infty((-R, R))$ such that

$$\chi \geq 0, \qquad \int_{\mathbb{R}} \chi(t) \|\Phi_1(t)\|_{L^2_\omega}^2 \, \mathrm{d}t > 0.$$

Since χ is compactly supported and $\Phi_1 \in \mathcal{C}_b$, we have $\chi\Phi_1 \in L^2(e^{\pm 2kt}; L^2_\omega)$. By Lemma 2.9, there exist $\Phi_\pm$ with

$$\mathcal{L}_b\Phi_\pm = \chi\Phi_1, \qquad \|\Phi_+\|_{e^{-2kt}} + \|\Phi_-\|_{e^{2kt}} < \infty.$$

On $t < -R$, $e^{-2k|t|}e^{-2kt} = 1$, and on $t > R$, $e^{-2k|t|}e^{2kt} = 1$. Weighted Cauchy–Schwarz therefore gives

$$\int_{-\infty}^{-R} |\Omega_t(\Phi_1, \Phi_+)| \, \mathrm{d}t + \int_R^\infty |\Omega_t(\Phi_1, \Phi_-)| \, \mathrm{d}t \leq C\|\Phi_1\|_{\varpi_0}\Big(\|\Phi_+\|_{e^{-2kt}} + \|\Phi_-\|_{e^{2kt}}\Big) < \infty.$$

On each half-line outside $(-R, R)$, (2.36) gives $\partial_t \Omega_t = 0$. The integrability above therefore implies

$$\Omega_t(\Phi_1, \Phi_+) = 0 \quad \text{for } t < -R, \qquad \Omega_t(\Phi_1, \Phi_-) = 0 \quad \text{for } t > R.$$

Let $\Phi_2 = \Phi_- - \Phi_+$. Then

$$\mathcal{L}_b\Phi_2 = 0, \qquad \|\Phi_2\|_{\varpi_0} \leq \|\Phi_-\|_{e^{2kt}} + \|\Phi_+\|_{e^{-2kt}} < \infty,$$

so $\Phi_2 \in \mathcal{C}_b$. For $t > R$,

$$\Omega_t(\Phi_1, \Phi_2) = -\Omega_t(\Phi_1, \Phi_+).$$

Since $\Omega_s(\Phi_1, \Phi_+) = 0$ for $s < -R$ and

$$\partial_s \Omega_s(\Phi_1, \Phi_+) = -d_\sigma \langle \Phi_1, \chi\Phi_1\rangle_\omega,$$

we obtain

$$\Omega_t(\Phi_1, \Phi_+) = -d_\sigma \int_{\mathbb{R}} \chi(t) \|\Phi_1(t)\|_{L^2_\omega}^2 \, \mathrm{d}t.$$

Thus

$$\Omega(\Phi_1, \Phi_2) = d_\sigma \int_{\mathbb{R}} \chi(t) \|\Phi_1(t)\|_{L^2_\omega}^2 \, \mathrm{d}t > 0,$$

proving that Ω is nondegenerate on $\mathcal{C}_b$. □

Let v be a positive bounded entire solution of (2.1), and let b be its linearized Robin coefficient, given by (2.10) with $w_1 = w_2 = v(\cdot, \pi/2)$. By Lemma 3.3, $v_t \in \mathcal{C}_b$ and

$$D\mathcal{H}(v, v_t)[\Phi, \Phi_t] = \Omega(v_t, \Phi) \qquad (\Phi \in \mathcal{C}_b).$$

If $v_t \not\equiv 0$, then $\Omega(v_t, v_t) = 0$ and $\Omega(v_t, \cdot)$ is a nonzero linear functional on the two-dimensional space $\mathcal{C}_b$. Hence

$$\Big\{\Phi \in \mathcal{C}_b : D\mathcal{H}(v, v_t)[\Phi, \Phi_t] = 0\Big\} = \operatorname{span}\{v_t\}.$$

2.5 Boundary observation for even solutions

We conclude the linear theory with a boundary observation for even solutions. It will be used to parametrize the Fowler family by its minimum and maximum. Set

$$b_\sigma = \frac{(n + 2\sigma)K_\sigma}{(n - 2\sigma)d_\sigma}.$$

Since $w_b(0) = 1$, Lemma 2.3 gives $b_\sigma \le b_*$.

Lemma 2.12. *Suppose $0 < \sigma < 1$ and $b = b(t) \in W^{1,\infty}(\mathbb{R})$ is even, $0 \le b \le b_\sigma$, $b(0) > 0$, and $|b'| \le 2\sigma b$. If Φ is a nonzero even homogeneous solution of* (2.12) *with at most polynomial growth in E_ω, then*

$$\operatorname{sign} \Phi(0, \pi/2) = \operatorname{sign}\langle \Phi(0), e_0(b(0))\rangle_\omega.$$

In particular, $\Phi(0, \pi/2) \neq 0$.

Proof. Let $N = n/2$. Since $|b'| \le 2\sigma b$ and $b(0) > 0$, Gronwall's inequality gives $b(t) > 0$ for every t. By Lemma C.4, the form D_N in (2.26) is positive definite and satisfies

$$\sigma b(t)|\Phi(t, \pi/2)|^2 \le C_n D_N(\Phi(t), \Phi_t(t)).$$

Evenness gives $\Phi_t(0) = 0$, and polynomial growth makes $e^{-2Nt}E(t) \to 0$ as $t \to \infty$. Integrating (2.28) with $f = 0$ gives

$$\int_0^\infty e^{-2Nt} D_N(\Phi(t), \Phi_t(t))\, \mathrm{d}t = \frac{\ell(0)}{2} q_1(0)^2. \tag{2.37}$$

If $q_1(0) = 0$, positivity forces $\Phi = 0$. We may therefore divide by $q_1(0)$ and assume $q_1(0) = 1$.

Let $b_0 = b(0)$, $\mu = \lambda_0(b_0)$, $\ell_0 = \ell(0)$, $E_j = e_j(b_0, \pi/2)$, and $r_j = \sqrt{\lambda_j(b_0)}$ for $j \ge 1$. For $h_j(t) = \langle \Phi(t), e_j(b_0)\rangle_\omega$, the equation and the Neumann Green function on the half-line give

$$\begin{aligned} -h_j'' + r_j^2 h_j &= (b - b_0)E_j \Phi(t, \pi/2), \\ h_j(0) &= \frac{E_j}{r_j} \int_0^\infty e^{-r_j t}(b(t) - b_0)\Phi(t, \pi/2)\, \mathrm{d}t. \end{aligned}$$

Here $h_j'(0) = 0$, and polynomial growth excludes an exponentially growing homogeneous term. The bound $|b'| \le 2\sigma b$ implies

$$\frac{(b(t)-b_0)^2}{b(t)b_0} \le e^{2\sigma t} - 2 + e^{-2\sigma t} \qquad (t \ge 0).$$

Since $r_j > k$,

$$\int_0^\infty e^{-2(r_j-N)t}(e^{2\sigma t} - 2 + e^{-2\sigma t})\,\mathrm{d}t = \frac{\sigma^2}{(r_j-k)(r_j-N)(r_j-p)}.$$

Cauchy–Schwarz and (2.37) thus give

$$|h_j(0)|^2 \le \frac{C_n \ell_0 b_0 \sigma E_j^2}{2r_j^2(r_j-k)(r_j-N)(r_j-p)}.$$

By trace continuity and Lemma 2.6,

$$\Phi(0,\pi/2) = E_0 + \sum_{j\ge1} E_j h_j(0), \qquad \sum_{j\ge1} \frac{E_j^2}{r_j^2-k^2} = \frac{E_0^2}{k^2-\mu}.$$

A second application of Cauchy–Schwarz gives

$$\frac{|\Phi(0,\pi/2)-E_0|^2}{E_0^2} \le \frac{C_n\ell_0}{2(k^2-\mu)}\,\sigma b_0 \sum_{j\ge1} \frac{E_j^2(r_j+k)}{r_j^2(r_j-N)(r_j-p)} = R_{\mathrm{bd}} < 1, \tag{2.38}$$

where the last inequality is Lemma C.5. Hence $\Phi(0,\pi/2) > 0$. Restoring the factor $q_1(0)$ proves the assertion. □

3 Hamiltonian structure and the global Fowler family

We classify bounded positive entire solutions of (2.1) up to translation. For $0 < \sigma < 1$, define

$$\mathcal{S} = \{0\} \cup \{v : \ v \text{ is a positive bounded entire solution of (2.1)}\}.$$

We give $\mathcal{S}$ the topology of local convergence of $(v, \partial_t v)$ in $C(I; E_\omega)$ for every compact interval $I \subset \mathbb{R}$. Define

$$\mathcal{S}_H = \{v \in \mathcal{S} : \mathcal{H}(v) = H\} \qquad \text{for } H \in \mathbb{R},$$

where $\mathcal{H}(v)$ denotes the conserved value in (2.7). We study the translation orbits in each $\mathcal{S}_H$. Local energy rigidity, periodic continuation, and compactness give the global classification. We then prove symmetry, endpoint limits, and the classification of the linearized solutions.

3.1 Local coordinates and Hamiltonian level sets

Two fixed projections of the Cauchy data give local bi-Lipschitz embeddings of $\mathcal{S}$ into $\mathbb{R}^2$. We then show that near a nonstationary solution, its energy level consists of translates. Recall from Section 2 that $w(t) = v(t, \pi/2)$, with corresponding notation for indexed solutions.

For a function f whose Cauchy pair belongs to E_ω, write

$$\mathrm{ev}_0(f) = \Big(f(0,\cdot), \partial_t f(0,\cdot)\Big).$$

For $v \in \mathcal{S}$, define the linearized Robin coefficient and the associated weighted linear solution space by

$$b_v = \frac{(n+2\sigma)K_\sigma}{(n-2\sigma)d_\sigma} w^{\frac{4\sigma}{n-2\sigma}}, \qquad \mathcal{C}_v := \mathcal{C}_{b_v}.$$

The coefficient b_v is the diagonal case of the secant coefficient in Lemma 2.3, with both traces equal to w. Thus b_v satisfies (2.11), and Corollary 2.10 gives $\dim \mathcal{C}_v = 2$.

Lemma 3.1. *The set $\mathcal{S}$ is compact, and $(t, v) \mapsto v(\cdot + t)$ is continuous. The map $v \mapsto \mathrm{ev}_0(v)$ is a homeomorphism from $\mathcal{S}$ onto its image in E_ω.*

Proof. For $v \in \mathcal{S}$, Propositions 2.2, 2.4 and Lemma 2.3 give

$$0 \le v \le w_*, \qquad \left\| \frac{K_\sigma}{d_\sigma} w^{\frac{4\sigma}{n-2\sigma}} \right\|_{W^{1,\infty}(\mathbb{R})} \le C.$$

Lemma 2.7 gives, for $\tau \in \mathbb{R}$,

$$\|(v, v_t)\|_{C([\tau-1,\tau+1];E_\omega)} \le C\|v\|_{L^2((\tau-2,\tau+2);L^2_\omega)} \le C.$$

For $v_j \in \mathcal{S}$, Lemma 2.7 and a diagonal argument give a subsequence such that

$$(v_j, v_{j,t}) \longrightarrow (v, v_t) \qquad \text{in } C(I; E_\omega), \quad I \Subset \mathbb{R}.$$

The trace estimate also gives $w_j \to w$ in $C(I)$.

Since $G_\sigma \in L^1$ and $0 \le w_j \le w_*$, dominated convergence in (2.5) gives

$$w(t) = K_\sigma \int_{\mathbb{R}} G_\sigma(t-s) w(s)^{\frac{n+2\sigma}{n-2\sigma}} \, \mathrm{d}s.$$

Since $G_\sigma > 0$, either $w \equiv 0$ or $w > 0$. Proposition 2.4 identifies v with the Poisson extension of w. Thus $v \in \mathcal{S}$, and $\mathcal{S}$ is compact.

For $v_1, v_2 \in \mathcal{S}$, set $\Phi = v_1 - v_2$, and let b be their secant coefficient. The uniform bound above implies $\Phi \in \mathcal{C}_b$. By (2.35),

$$\|(\Phi, \Phi_t)\|_{C(I;E_\omega)} \le C_I |q_\Phi(0)| \le C_I \|\mathrm{ev}_0(\Phi)\|_{E_\omega}.$$

This estimate proves injectivity of ev_0 and continuity of its inverse on the image.

Continuity of translations follows from local uniform convergence of the Cauchy pairs. □

We identify $\mathcal{S}$ with its image under $v \mapsto \mathrm{ev}_0(v)$. To obtain two scalar coordinates, for any L^2_ω-normalized eigenfunction e, let

$$\pi_e(f,g) = \Big(\langle e, f\rangle_\omega, \langle e, g\rangle_\omega\Big) \qquad \text{for } (f,g) \in E_\omega.$$

Fix $v_0 \in \mathcal{S}$, set $e_* = e_0(b_{v_0}(0))$, and define

$$c(v) = \pi_{e_*}(\mathrm{ev}_0(v)). \tag{3.1}$$

Proposition 3.2. *Fix $v_0 \in \mathcal{S}$ and the coordinate map c from* (3.1)*. There is a neighborhood $\mathcal{U}$ of v_0 in $\mathcal{S}$ such that $c : \mathcal{U} \to c(\mathcal{U})$ is bi-Lipschitz with respect to the E_ω norm. For every bounded interval $I \subset \mathbb{R}$ and $v_1, v_2 \in \mathcal{U}$,*

$$\|\mathrm{ev}_0(v_1) - \mathrm{ev}_0(v_2)\|_{E_\omega} + \|(v_1 - v_2, \partial_t v_1 - \partial_t v_2)\|_{C(I;E_\omega)} \le C_I |c(v_1) - c(v_2)|. \tag{3.2}$$

Moreover, $\Phi \mapsto \pi_{e_*}(\mathrm{ev}_0(\Phi))$ is an isomorphism from $\mathcal{C}_{v_0}$ onto $\mathbb{R}^2$.

Proof. Let $\Phi = v_1 - v_2$, let b be the secant coefficient, and set $e = e_0(b(0))$. By Lemmas 2.3, 2.5 and 3.1,

$$\Phi \in \mathcal{C}_b, \qquad q_\Phi(0) = \pi_e(\mathrm{ev}_0(\Phi)), \qquad e \to e_* \text{ in } H^1_\omega \quad \text{as } v_1, v_2 \to v_0.$$

By (2.35),

$$\|\mathrm{ev}_0(\Phi)\|_{E_\omega} \le C|q_\Phi(0)| \le C|c(v_1) - c(v_2)| + C\|e - e_*\|_{L^2_\omega}\|\mathrm{ev}_0(\Phi)\|_{E_\omega}.$$

Here $c(v_1) - c(v_2) = \pi_{e_*}(\mathrm{ev}_0(\Phi))$. Since $b(0) \to b_{v_0}(0)$ and $b \mapsto e_0(b)$ is continuous, choose $\mathcal{U}$ so that

$$C\|e - e_*\|_{L^2_\omega} \le \frac{1}{2} \qquad \text{for all } v_1, v_2 \in \mathcal{U}.$$

Then

$$\begin{aligned} \|\mathrm{ev}_0(\Phi)\|_{E_\omega} + |q_\Phi(0)| &\le C|c(v_1) - c(v_2)|, \\ |c(v_1) - c(v_2)| &\le \|\pi_{e_*}\|_{\mathcal{L}(E_\omega,\mathbb{R}^2)}\|\mathrm{ev}_0(\Phi)\|_{E_\omega}, \\ \|(\Phi, \Phi_t)\|_{C(I;E_\omega)} &\le C_I|q_\Phi(0)| \le C_I|c(v_1) - c(v_2)|. \end{aligned}$$

This proves (3.2) and the bi-Lipschitz bounds.

Finally, $\pi_{e_*}(\mathrm{ev}_0(\Phi)) = q_\Phi(0)$ for $\Phi \in \mathcal{C}_{v_0}$, so Corollary 2.10 gives the asserted isomorphism. □

We now turn to energy levels; for the classical Hamiltonian approach, see [29, Sections 2 and 7]. The Hamiltonian (2.7) extends to the following C^1 function on E_ω:

$$\mathcal{H}(f,g) = \frac{d_\sigma}{2}\|g\|^2_{L^2_\omega} - \frac{d_\sigma}{2}Q_0(f,f) + \frac{(n-2\sigma)K_\sigma}{2n}|f(\pi/2)|^{\frac{2n}{n-2\sigma}}.$$

The t-independent solutions in $\mathcal{S}$ are precisely 0 and v_c, since their traces are constant solutions of (1.5). Their energies are

$$\mathcal{H}(0) = 0, \qquad \mathcal{H}(v_c) = H_c < 0. \tag{3.3}$$

Section 2 also gives $\mathcal{H}(v_b) = 0$. We next relate the energy differential to translations.

Lemma 3.3. *Let $v \in \mathcal{S}$. Then $(v, v_t) \in C^1(\mathbb{R}; E_\omega)$, $v_t \in \mathcal{C}_v$, and*

$$D\mathcal{H}(\mathrm{ev}_0(v))[\mathrm{ev}_0(\Phi)] = \Omega(v_t, \Phi) \qquad \textit{for } \Phi \in \mathcal{C}_v.$$

If $v_t \not\equiv 0$, this functional is nonzero and

$$\{\Phi \in \mathcal{C}_v : D\mathcal{H}(\mathrm{ev}_0(v))[\mathrm{ev}_0(\Phi)] = 0\} = \mathrm{span}\{v_t\}. \tag{3.4}$$

Proof. Fix $v_0 \in \mathcal{S}$ and use the coordinates from Proposition 3.2. For $v \in \mathcal{S}$, write

$$q(t) = c(v(\cdot + t)), \qquad q = (q_1, q_2).$$

Then

$$q_1(t) = \langle e_*, v(t)\rangle_\omega, \quad q_2(t) = \langle e_*, v_t(t)\rangle_\omega, \qquad \partial_t q_1 = q_2.$$

Direct calculation gives

$$Q_0(e_*, v(t)) = \lambda_0(b_{v_0}(0))q_1(t) + b_{v_0}(0)e_*(\pi/2)w(t). \tag{3.5}$$

Testing the nonlinear equation (2.1) by χe_*, where $\chi \in C_c^1(\mathbb{R})$, gives

$$0 = \int_{\mathbb{R}} \Big[q_2\chi' + \chi\Big(Q_0(e_*, v) - \frac{K_\sigma}{d_\sigma} e_*(\pi/2)w^{\frac{n+2\sigma}{n-2\sigma}}\Big)\Big]\,\mathrm{d}t.$$

Hence, in distributions,

$$\partial_t q_2 = Q_0(e_*, v) - \frac{K_\sigma}{d_\sigma} e_*(\pi/2)w^{\frac{n+2\sigma}{n-2\sigma}}.$$

Substituting (3.5) then gives

$$\begin{cases} \partial_t q_1 = q_2, \\ \partial_t q_2 = \lambda_0(b_{v_0}(0))q_1 - e_*(\pi/2)\Big(\dfrac{K_\sigma}{d_\sigma} w^{\frac{n+2\sigma}{n-2\sigma}} - b_{v_0}(0)w\Big). \end{cases} \tag{3.6}$$

In particular, $q \in C^1(\mathbb{R}; \mathbb{R}^2)$. The uniform bound in Lemma 3.1 and (3.6) give $\sup_t |q'(t)| \le C$.

Now take $v_0 = v$ and set $\delta_h v = (v(\cdot + h) - v)/h$. For small $|h|$, $v(\cdot + h) \in \mathcal{U}$. Let b_h be the secant coefficient of $v(\cdot + h)$ and v. By (2.23), (3.2) and Lemma 2.3,

$$\delta_h v \in \mathcal{C}_{b_h}, \qquad b_h \to b_v \text{ locally uniformly},$$

and

$$\|\delta_h v\|_{\varpi_0} \le C\|\mathrm{ev}_0(\delta_h v)\|_{E_\omega} \le C\frac{|q(h) - q(0)|}{|h|} \le C.$$

By the compactness in Lemma 2.7, a subsequence of $(\delta_h v, \partial_t \delta_h v)$ converges in $C(I; E_\omega)$. Since $\delta_h v \to v_t$ in distributions, the limit is (v_t, v_{tt}). Hence the whole family converges:

$$(\delta_h v, \partial_t \delta_h v) \longrightarrow (v_t, v_{tt}) \quad \text{in } C(I; E_\omega), \qquad I \Subset \mathbb{R}.$$

Passing to the weak equation and using Fatou's lemma gives

$$\mathcal{L}_{b_v} v_t = 0, \qquad \|v_t\|_{\varpi_0} \le \liminf_{h\to 0} \|\delta_h v\|_{\varpi_0} \le C.$$

Thus $v_t \in \mathcal{C}_v$ and $(v, v_t) \in C^1(\mathbb{R}; E_\omega)$.

For $\Phi \in \mathcal{C}_v$, the nonlinear equation (2.1) at $t = 0$ gives

$$Q_0(v(0), \Phi(0)) - \frac{K_\sigma}{d_\sigma} w(0)^{\frac{n+2\sigma}{n-2\sigma}} \Phi(0, \pi/2) = \langle v_{tt}(0), \Phi(0)\rangle_\omega.$$

Therefore

$$\begin{aligned} D\mathcal{H}(\mathrm{ev}_0(v))[\mathrm{ev}_0(\Phi)] &= d_\sigma[\langle v_t(0), \Phi_t(0)\rangle_\omega - Q_0(v(0), \Phi(0))] + K_\sigma w(0)^{\frac{n+2\sigma}{n-2\sigma}} \Phi(0, \pi/2) \\ &= d_\sigma[\langle v_t(0), \Phi_t(0)\rangle_\omega - \langle v_{tt}(0), \Phi(0)\rangle_\omega] \\ &= \Omega(v_t, \Phi). \end{aligned}$$

If $v_t \not\equiv 0$, Proposition 2.11 shows that the functional is nonzero. Since Ω is nondegenerate on $\mathcal{C}_v$, $\Omega(v_t, v_t) = 0$ and $\dim \mathcal{C}_v = 2$, its kernel is $\mathrm{span}\{v_t\}$. This gives (3.4). □

The kernel identity in Lemma 3.3 gives the following local description of an energy level.

Proposition 3.4. *For every $v \in \mathcal{S}$ with $\partial_t v \not\equiv 0$, there are a neighborhood $\mathcal{U}$ of v and an $\varepsilon > 0$ such that*

$$\mathcal{U} \cap \mathcal{S}_{\mathcal{H}(v)} = \mathcal{U} \cap \{v(\cdot + \tau) : |\tau| < \varepsilon\}.$$

Proof. Use coordinates at v from Proposition 3.2. The coordinate velocity of the translation orbit at v is $a = \pi_{e_*}(\mathrm{ev}_0(v_t))$. Since $v_t \neq 0$ and $\pi_{e_*} \circ \mathrm{ev}_0$ is an isomorphism from $\mathcal{C}_v$ onto $\mathbb{R}^2$, we have $a \neq 0$. In this coordinate neighborhood take the transverse section

$$\Sigma = \{y : a \cdot [c(y) - c(v)] = 0\}.$$

By (3.6), $q'(t)$ depends continuously on $y(\cdot + t)$, and $q'(0) = a$ at $y = v$. Hence, after shrinking the neighborhood and choosing $\varepsilon > 0$ small, we have for $y \in \mathcal{U}$ and $|t| \le \varepsilon$

$$\partial_t\{a \cdot [c(y(\cdot + t)) - c(v)]\} = a \cdot q'(t) \ge \tfrac{1}{2}|a|^2.$$

With ε fixed, continuity of c allows us to shrink $\mathcal{U}$ further so that

$$|a \cdot [c(y) - c(v)]| < \tfrac{1}{4}\varepsilon |a|^2,$$

and integrating the preceding derivative bound gives

$$a \cdot [c(y(\cdot - \varepsilon)) - c(v)] < 0 < a \cdot [c(y(\cdot + \varepsilon)) - c(v)].$$

Strict monotonicity and the intermediate value theorem give a unique $|t| < \varepsilon$ with $y(\cdot + t) \in \Sigma$.

Suppose now that $y_j \in \Sigma \setminus \{v\}$, $y_j \to v$, and $\mathcal{H}(y_j) = \mathcal{H}(v)$. Let b_j be the secant coefficient of y_j and v, and set

$$\Phi_j = \frac{y_j - v}{\|\mathrm{ev}_0(y_j - v)\|_{E_\omega}}.$$

By Lemmas 2.3 and 2.8, we have

$$\|\Phi_j\|_{\varpi_0} \le C, \qquad \|\mathrm{ev}_0(\Phi_j)\|_{E_\omega} = 1, \qquad b_j \to b_v \text{ locally uniformly.}$$

Then by Lemma 2.7, after passing to a subsequence,

$$(\Phi_j, \Phi_{j,t}) \to (\Phi, \Phi_t) \quad \text{in } C(I; E_\omega) \qquad (I \Subset \mathbb{R}).$$

Passing to the weak equation and using Fatou's lemma gives

$$\mathcal{L}_{b_v}\Phi = 0, \qquad \|\Phi\|_{\varpi_0}^2 \le \liminf_j \|\Phi_j\|_{\varpi_0}^2 \le C, \qquad \|\mathrm{ev}_0(\Phi)\|_{E_\omega} = 1.$$

Since $\mathcal{H}$ is C^1 and $y_j \in \Sigma$,

$$0 = \frac{\mathcal{H}(y_j) - \mathcal{H}(v)}{\|\mathrm{ev}_0(y_j - v)\|_{E_\omega}} = D\mathcal{H}(\mathrm{ev}_0(v))[\mathrm{ev}_0(\Phi_j)] + o(1),$$

and

$$a \cdot \pi_{e_*}(\mathrm{ev}_0(\Phi_j)) = 0.$$

Letting $j \to \infty$ gives

$$D\mathcal{H}(\mathrm{ev}_0(v))[\mathrm{ev}_0(\Phi)] = 0, \qquad a \cdot \pi_{e_*}(\mathrm{ev}_0(\Phi)) = 0.$$

By (3.4), $\Phi \in \mathrm{span}\{v_t\}$. Since

$$a \cdot \pi_{e_*}(\mathrm{ev}_0(v_t)) = a \cdot a = |a|^2 \neq 0,$$

the condition $a \cdot \pi_{e_*}(\mathrm{ev}_0(\Phi)) = 0$ forces $\Phi = 0$, contradicting $\|\mathrm{ev}_0(\Phi)\|_{E_\omega} = 1$. Therefore there is a neighborhood of v such that

$$\Sigma \cap \mathcal{S}_{\mathcal{H}(v)} = \{v\}.$$

Finally, take $y \in \mathcal{U} \cap \mathcal{S}_{\mathcal{H}(v)}$. By the first part, there is a unique $|t| < \varepsilon$ with $y(\cdot + t) \in \Sigma$. Since $\mathcal{H}$ is translation invariant, $y(\cdot + t) \in \mathcal{S}_{\mathcal{H}(v)}$, so $y(\cdot + t) = v$. Thus $y = v(\cdot - t)$. After shrinking $\mathcal{U}$ if necessary,

$$\mathcal{U} \cap \mathcal{S}_{\mathcal{H}(v)} = \mathcal{U} \cap \{v(\cdot + \tau) : |\tau| < \varepsilon\}.$$

□

3.2 Periodic continuation and bifurcation at the cylinder

For a positive T-periodic trace w, set $W(x) = w(Tx)$, so that $W(x+1) = W(x) > 0$. With $\mathcal{P}_{\sigma,T}$ from (2.4), the periodic problem is

$$\mathcal{F}(T, W) = \mathcal{P}_{\sigma,T} W - K_\sigma W^{\frac{n+2\sigma}{n-2\sigma}} = 0.$$

Bifurcation at $T_0 = 2\pi/\kappa_c$ gives periodic solutions whose translates, together with v_c, exhaust a neighborhood of v_c. We then prove that near every nonstationary periodic solution, the principal coordinates parametrize an analytic family of periodic solutions, with analytic least period and nonvanishing energy differential.

For the cylindrical Poisson extension v of $w(t) = W(t/T)$, write $v_{T,W}(x,\theta) = v(Tx,\theta)$. Fix $r_0 > \frac{5}{2}$ throughout this subsection; this controls the first two time derivatives of the periodic extensions.

Let $g_\xi = f_{-\xi^2}$ for $\xi \in \mathbb{R}$. This is well defined because $-\xi^2 \le 0 < k^2 = \lambda_0^D$. Then

$$\begin{cases} -\omega^{-1}(\omega g_\xi')' + (p^2 + \xi^2) g_\xi = 0, \\ g_\xi'(0) = 0, \qquad g_\xi(\pi/2) = 1. \end{cases} \tag{3.7}$$

The periodic extension has the Fourier series

$$v_{T,W}(x,\theta) = \sum_{j\in\mathbb{Z}} \widehat{W}_j g_{2\pi j/T}(\theta) e^{2\pi i j x}. \tag{3.8}$$

Lemma 3.5. *For $T > 0$ and $W \in H^{r_0}(\mathbb{R}/\mathbb{Z})$ with $\min W > 0$, the maps $\mathcal{F}$ and $(T, W) \mapsto (v_{T,W}, T^{-1}\partial_x v_{T,W})$ are real analytic, with values in $H^{r_0-2\sigma}$ and $C(\mathbb{R}/\mathbb{Z}; E_\omega)$, respectively.*

Proof. Fix $T_* > 0$. By Lemma A.1, the maps $T \mapsto \widetilde{\Theta}_\sigma(2\pi j/T)$ are holomorphic on a common complex neighborhood of T_* and satisfy, for some $\rho > 0$,

$$\sup_{|T-T_*|<2\rho} |\widetilde{\Theta}_\sigma(2\pi j/T)| \le C(1+j^2)^\sigma \qquad \text{for } j \in \mathbb{Z}.$$

On this disk, extend the Fourier multiplier (2.4) by

$$\mathcal{P}_{\sigma,T} e^{2\pi i j x} = \widetilde{\Theta}_\sigma(2\pi j/T) e^{2\pi i j x}.$$

Cauchy's estimate for each multiplier gives, in $\mathcal{L}(H^{r_0}, H^{r_0-2\sigma})$,

$$\|\partial_T^\ell \mathcal{P}_{\sigma,T_*}\| \le C\ell! \rho^{-\ell} \qquad \text{for } \ell \ge 0.$$

Thus $T \mapsto \mathcal{P}_{\sigma,T}$ is analytic in operator norm for $|T - T_*| < \rho$. The power map is analytic on $\{W \in H^{r_0} : \min W > 0\}$, since H^{r_0} is a Banach algebra. Hence $\mathcal{F}$ is analytic.

For the extension, set

$$B_j(T) = \Big(g_{2\pi j/T}, i(2\pi j/T) g_{2\pi j/T}\Big).$$

By (3.8),

$$(v_{T,W}, T^{-1}\partial_x v_{T,W}) = \sum_{j\in\mathbb{Z}} \widehat{W}_j B_j(T) e^{2\pi i j x}.$$

By Lemma A.1, on a common complex disk, $T \mapsto B_j(T)$ is holomorphic and

$$\|B_j(T)\|_{E_\omega} \le C(1+j^2)^{1/2}.$$

On a smaller disk,

$$\begin{aligned}\left\| \sum_{|j|>N} \widehat{W}_j B_j(T) e^{2\pi i j x} \right\|_{C_x(E_\omega)} &\le C \sum_{|j|>N} |\widehat{W}_j|(1+j^2)^{1/2} \\ &\le C\|W\|_{H^{r_0}} \left[\sum_{|j|>N} (1+j^2)^{1-r_0} \right]^{1/2} \\ &\to 0.\end{aligned}$$

Since $r_0 > 3/2$, the series converges locally uniformly in $C(\mathbb{R}/\mathbb{Z}; E_\omega)$. The sum is holomorphic in T and linear in W, proving the assertion. □

Lemma 3.6. *For $\ell = 0, 1, 2$, the map*

$$(T, W) \mapsto T^{-\ell}\partial_x^\ell v_{T,W}$$

is continuous from $\mathbb{R}_+ \times H^{r_0}$ into $C(\mathbb{R}/\mathbb{Z} \times [0, \pi/2])$. More generally, for a compact $I \Subset \mathbb{R}_+$ and $r > \ell + 1/2$,

$$\|T^{-\ell}\partial_x^\ell v_{T,W}\|_{C(\mathbb{R}/\mathbb{Z}\times[0,\pi/2])} \le C\|W\|_{H^r}, \qquad T \in I. \tag{3.9}$$

Here $C = C(n, \sigma, I, \ell, r)$.

Proof. Fix a compact interval $I \Subset \mathbb{R}_+$ and $\ell \in \{0, 1, 2\}$. For trigonometric polynomials W,

$$T^{-\ell}\partial_x^\ell v_{T,W} = \sum_{j\in\mathbb{Z}} (2\pi i j/T)^\ell \widehat{W}_j g_{2\pi j/T} e^{2\pi i j x}.$$

We first estimate the angular factors. By (3.7) and the maximum principle, $0 < g_\xi \le 1$ for real ξ. For $\xi, \xi' \in \mathbb{R}$, let $h = g_\xi - g_{\xi'}$. Then

$$\begin{cases} -\omega^{-1}(\omega h')' + (p^2+\xi^2)h = (\xi'^2 - \xi^2) g_{\xi'}, \\ h'(0) = 0, \qquad h(\pi/2) = 0. \end{cases}$$

By the maximum principle,

$$\|g_\xi - g_{\xi'}\|_{C([0,\pi/2])} \le \frac{|\xi^2 - \xi'^2|}{p^2+\xi^2} \|g_{\xi'}\|_{C([0,\pi/2])}.$$

Taking $\xi = 2\pi j/T$ and $\xi' = 2\pi j/T'$, for $j \neq 0$,

$$\begin{aligned}\|g_{2\pi j/T} - g_{2\pi j/T'}\|_{C([0,\pi/2])} &\leq \frac{(2\pi j)^2|T^{-2} - (T')^{-2}|}{p^2 + (2\pi j/T)^2} \\ &\leq T^2|T^{-2} - (T')^{-2}| \\ &\leq C_I|T - T'|.\end{aligned}$$

For $j = 0$ the difference vanishes. Hence, for every $j \in \mathbb{Z}$, the map $T \mapsto g_{2\pi j/T}$ is Lipschitz on I with values in $C([0, \pi/2])$, with a Lipschitz constant independent of j.

For $r > \ell + 1/2$,

$$\begin{aligned}\|T^{-\ell}\partial_x^\ell v_{T,W}\|_\infty &\leq C_I \sum_j (1 + |j|)^\ell |\widehat{W}_j| \\ &\leq C_I \Big[\sum_j (1 + j^2)^{\ell - r}\Big]^{1/2} \|W\|_{H^r} \\ &\leq C\|W\|_{H^r}.\end{aligned}$$

Together with $|T^{-\ell} - (T')^{-\ell}| \leq C_I|T - T'|$ and the angular Lipschitz bound, we obtain

$$\Big\|(2\pi ij/T)^\ell g_{2\pi j/T} - (2\pi ij/T')^\ell g_{2\pi j/T'}\Big\|_\infty \leq C(1 + j^2)^{\ell/2}|T - T'|.$$

Hence, by linearity in W and summation,

$$\begin{aligned}&\|T^{-\ell}\partial_x^\ell v_{T,W} - (T')^{-\ell}\partial_x^\ell v_{T',W'}\|_\infty \\ &\quad\leq C\sum_j (1 + j^2)^{\ell/2}\Big(|\widehat{W}_j - \widehat{W}'_j| + |T - T'|\,|\widehat{W}'_j|\Big) \\ &\quad\leq C\Big(\|W - W'\|_{H^r} + |T - T'|\|W'\|_{H^r}\Big).\end{aligned}$$

Fourier truncation extends both estimates to all $W, W' \in H^r$. In particular, (3.9) holds. Since $r_0 > \ell + 1/2$ for $\ell = 0, 1, 2$, taking $r = r_0$ proves continuity. □

Lemma 3.7. *At a positive solution W of the periodic equation, the linearization*

$$L_{T,W}\phi = \mathcal{P}_{\sigma,T}\phi - \frac{(n + 2\sigma)K_\sigma}{n - 2\sigma} W^{\frac{4\sigma}{n-2\sigma}}\phi$$

is Fredholm of index zero from H^{r_0} to $H^{r_0 - 2\sigma}$, and

$$\operatorname{Im} L_{T,W} = H^{r_0-2\sigma}(\mathbb{R}/\mathbb{Z}) \cap (\ker L_{T,W})^{\perp_{L^2(\mathbb{R}/\mathbb{Z})}}.$$

Proof. The operator $\mathcal{P}_{\sigma,T} : H^{r_0} \to H^{r_0-2\sigma}$ is an isomorphism by (2.4), and

$$\mathcal{P}_{\sigma,T}^{-1}\Big[W^{\frac{4\sigma}{n-2\sigma}} \cdot\Big] : H^{r_0} \longrightarrow H^{r_0+2\sigma} \Subset H^{r_0}.$$

Thus $L_{T,W}$ is Fredholm of index zero. Self-adjointness and elliptic regularity give

$$\operatorname{Im} L_{T,W} = H^{r_0-2\sigma} \cap (\ker L_{T,W})^{\perp_{L^2(\mathbb{R}/\mathbb{Z})}}.$$

□

Lemma 3.8. *A real-analytic branch of positive even periodic solutions bifurcates from w_c at $T_0 = 2\pi/\kappa_c$. It can be parametrized by $|a| < a_0$ as $(T(a), W_a)$, where*

$$T(a) = T_0 + O(a^2), \qquad W_a = w_c + a\cos(2\pi x) + O_{H^{r_0}}(a^2). \tag{3.10}$$

For $a \neq 0$, W_a has least period one. Moreover, $T(-a) = T(a)$ and $W_{-a}(x) = W_a(x + 1/2)$.

Proof. We work on the real even subspaces $H^{r_0}_{\text{even}}$ and $H^{r_0-2\sigma}_{\text{even}}$. For every $T > 0$, $\mathcal{F}(T, w_c) = 0$. The eigenvalues of

$$L_{T,w_c} = D_W\mathcal{F}(T, w_c)$$

are

$$\mu_j(T) = \Theta_\sigma(2\pi j/T) - \frac{n + 2\sigma}{n - 2\sigma}\Theta_\sigma(0), \qquad j = 0, 1, \dots.$$

By strict monotonicity of Θ_σ,

$$\mu_1(T_0) = 0, \qquad \mu_j(T_0) \neq 0 \text{ for } j \neq 1, \qquad \mu_1'(T_0) = -\frac{2\pi}{T_0^2}\Theta_\sigma'(\kappa_c) \neq 0.$$

Let $\phi(x) = \cos(2\pi x)$, with orthogonality taken in $L^2(0, 1)$. Restricting to the even subspace preserves the Fredholm index and the range formula. Hence, in the even subspace,

$$\ker L_{T_0,w_c} = \operatorname{span}\{\phi\}, \qquad \operatorname{Im} L_{T_0,w_c} = H^{r_0-2\sigma}_{\text{even}} \cap \phi^\perp.$$

The simple-eigenvalue bifurcation theorem [9, Theorem 1.7] gives a local branch containing all nearby nonconstant even solutions. Since $\mathcal{F}$ is analytic by Lemma 3.5, the implicit function theorem in that proof gives an analytic branch. Normalize it by

$$2\int_0^1 (W_a - w_c)\phi \,\mathrm{d}x = a, \qquad T(0) = T_0, \qquad W_a = w_c + a\phi + O_{H^{r_0}}(a^2).$$

Translation by $1/2$ preserves evenness and the equation, and

$$2\int_0^1 (W_a(x + 1/2) - w_c)\phi(x) \,\mathrm{d}x = -a.$$

Uniqueness with this normalization gives

$$W_{-a}(x) = W_a(x + 1/2), \qquad T(-a) = T(a).$$

Since T is analytic and even, $T(a) = T_0 + O(a^2)$.

For $a \neq 0$, $\widehat{W_a}(1) = a/2 \neq 0$. Any period s satisfies $e^{2\pi i s}\widehat{W_a}(1) = \widehat{W_a}(1)$, so $s \in \mathbb{Z}$ and the least positive period is 1.

For small a, $W_a \to w_c$ uniformly, hence $W_a > 0$. By Proposition 2.4,

$$v_a(t, \theta) = v_{T(a),W_a}(t/T(a), \theta) \in \mathcal{S}.$$

It is even. Since W_a has least period 1, both $t \mapsto W_a(t/T(a))$ and v_a have least period $T(a)$. □

Proposition 3.9. *A neighborhood of v_c in $\mathcal{S}$ consists of the cylinder and one periodic translation orbit at each nearby energy:*

$$\mathcal{U}_c = \{v_c\} \cup \{v_H(\cdot + \tau) : H_c < H < H_c + \delta,\ \tau \in \mathbb{R}\}, \qquad \mathcal{H}(v_H) = H.$$

Let $T(H)$ denote the least period of v_H. Then

$$\|w_H - w_c\|_\infty \longrightarrow 0, \qquad T(H) \longrightarrow 2\pi/\kappa_c \qquad \text{as } H \downarrow H_c.$$

Proof. At the cylinder, the linearized Robin coefficient is constant:

$$b_{v_c} = \frac{(n+2\sigma)K_\sigma}{(n-2\sigma)d_\sigma} w_c^{\frac{4\sigma}{n-2\sigma}} = \frac{n+2\sigma}{(n-2\sigma)d_\sigma}\Theta_\sigma(0).$$

Let $v_a(t,\theta) = v_{T(a),W_a}(t/T(a),\theta)$ be the even branch from Lemma 3.8, and set $e_c = e_0(b_{v_c})$. Since $g_{\kappa_c} > 0$ satisfies the Robin condition at b_{v_c},

$$\lambda_0(b_{v_c}) = -\kappa_c^2, \qquad g_{\kappa_c} = \frac{e_c}{e_c(\pi/2)}, \qquad \partial_a v_a(t,\theta)|_{a=0} = \frac{e_c(\theta)}{e_c(\pi/2)}\cos(\kappa_c t).$$

Use coordinates with $e_* = e_c$. For $\tau \in \mathbb{R}/(2\pi\mathbb{Z})$, define

$$P(a\cos\tau, a\sin\tau) = c\Big(v_a(\cdot + T(a)\tau/(2\pi))\Big).$$

The identities $T(-a) = T(a)$ and $W_{-a}(x) = W_a(x+1/2)$ make this well defined. By Lemma 3.5,

$$\begin{aligned} P(a\cos\tau, a\sin\tau) &= c(v_c) + \frac{a}{e_c(\pi/2)}(\cos\tau, -\kappa_c\sin\tau) + R(a,\tau), \\ |R| + |\partial_\tau R| + |a\partial_a R| &\le Ca^2. \end{aligned} \tag{3.11}$$

For $a \neq 0$,

$$DP(a\cos\tau, a\sin\tau) = \frac{\operatorname{diag}(1,-\kappa_c)}{e_c(\pi/2)} + \begin{pmatrix}\partial_a R & a^{-1}\partial_\tau R\end{pmatrix}\begin{pmatrix}\cos\tau & \sin\tau \\ -\sin\tau & \cos\tau\end{pmatrix},$$

and

$$\left|DP(a\cos\tau, a\sin\tau) - \frac{\operatorname{diag}(1,-\kappa_c)}{e_c(\pi/2)}\right| \le |\partial_a R| + |a|^{-1}|\partial_\tau R| \le C|a| \longrightarrow 0.$$

Hence P extends to a C^1 map on the disk $B_{a_1}(0)$, with

$$DP(0) = \frac{\operatorname{diag}(1,-\kappa_c)}{e_c(\pi/2)}, \qquad \det DP(0) = -\frac{\kappa_c}{e_c(\pi/2)^2} \neq 0.$$

By the inverse function theorem, after decreasing a_1, $P(B_{a_1}(0))$ is an open neighborhood of $c(v_c)$. Proposition 3.2 makes c a homeomorphism onto its image. Its injectivity and the definition of P give

$$c^{-1}(P(B_{a_1}(0))) = \{v_c\} \cup \{v_a(\cdot + t) : 0 < a < a_1,\ t \in \mathbb{R}\}.$$

Thus this set is a neighborhood of v_c in $\mathcal{S}$.

At the cylinder,

$$D\mathcal{H}(\mathrm{ev}_0(v_c)) = 0, \qquad \mathcal{C}_{v_c} = \mathrm{span}\{e_c \cos(\kappa_c t), e_c \sin(\kappa_c t)\}.$$

For $\Phi = e_c(A_1 \cos \kappa_c t + A_2 \sin \kappa_c t)$,

$$\begin{aligned} \tfrac{1}{2} D^2\mathcal{H}(\mathrm{ev}_0(v_c))[\mathrm{ev}_0(\Phi), \mathrm{ev}_0(\Phi)] &= \frac{d_\sigma}{2}(\|\Phi_t(0)\|^2_{L^2_\omega} - Q_{b_{v_c}}(\Phi(0), \Phi(0))) \\ &= \frac{d_\sigma \kappa_c^2}{2}(A_1^2 + A_2^2). \end{aligned} \tag{3.12}$$

Thus

$$\begin{aligned} \mathcal{H}(v_a) &= H_c + \frac{d_\sigma \kappa_c^2}{2e_c(\pi/2)^2} a^2 + O(a^3), \\ \frac{\mathrm{d}}{\mathrm{d}a}\mathcal{H}(v_a) &= \frac{d_\sigma \kappa_c^2}{e_c(\pi/2)^2} a + O(a^2) > 0 \qquad \text{for } 0 < a < a_1. \end{aligned}$$

After shrinking a_1, the map $a \mapsto \mathcal{H}(v_a)$ is a real-analytic bijection from $(0, a_1)$ onto $(H_c, H_c + \delta)$, where $\delta = \mathcal{H}(v_{a_1}) - H_c$. Denote its inverse by $a(H)$ and set $v_H = v_{a(H)}$, $T(H) = T(a(H))$. Since $a(H) \to 0$ as $H \downarrow H_c$, (3.10) gives both limits. □

Lemma 3.10. *For all $H > H_c$ sufficiently close to H_c, the orbit in Proposition 3.9 has a unique representative v_H with $w_H(0) = \max w_H$. It is even in t and satisfies*

$$\partial_t v_H(t, \theta) < 0 \qquad \textit{for } 0 < t < T(H)/2,\ 0 \le \theta \le \pi/2.$$

Proof. Let $\kappa_a = 2\pi/T(a)$. By (3.10) and (3.9),

$$v_a(t, \theta) = v_c(\theta) + a g_{\kappa_a}(\theta) \cos(\kappa_a t) + R_a(t, \theta), \qquad \|\partial_t^2 R_a\|_\infty \le C a^2.$$

Here R_a is the extension of the remainder in (3.10); (3.9) with $\ell = 2$ gives the stated bound.

Evenness and periodicity give

$$\partial_t R_a(0, \theta) = \partial_t R_a(T(a)/2, \theta) = 0,$$

and hence

$$|\partial_t R_a(t, \theta)| \le C a^2 \min\{t, T(a)/2 - t\}.$$

Since $g_\xi(\theta)$ is continuous in (ξ, θ), $g_\xi(\pi/2) = 1$, and κ_a stays in a compact subset of $(0, \infty)$, there is $c > 0$ such that

$$\min_{0 \le \theta \le \pi/2} g_{\kappa_a}(\theta) \ge c.$$

Also, since $\kappa_a \to \kappa_c > 0$, there is $c > 0$ such that

$$\kappa_a \sin(\kappa_a t) \ge c \min\{t, T(a)/2 - t\} \qquad \text{for } 0 \le t \le T(a)/2.$$

For $0 < t < T(a)/2$ and $0 \le \theta \le \pi/2$,

$$-\partial_t v_a = a\kappa_a g_{\kappa_a} \sin(\kappa_a t) - \partial_t R_a \ge a(c - Ca) \min\{t, T(a)/2 - t\} > 0$$

when $a > 0$ is small. Since $a(H) \to 0$, this applies to v_H. By evenness, $\partial_t v_H(0, \theta) = 0$ and $\partial_t v_H(T(H)/2, \theta) = 0$. Since $\partial_t v_H < 0$ on $0 < t < T(H)/2$, the trace decreases strictly from its maximum at $t = 0$ to its minimum at $t = T(H)/2$. Symmetry and periodicity therefore give a unique maximum in each period, at $t = 0$. □

At a nonconstant periodic solution, translation gives $D_W \mathcal{F}(T, W)\partial_x W = 0$. We retain T as an unknown and prescribe the two principal coordinates.

Proposition 3.11. *Let $v_0 \in \mathcal{S}$ be periodic with $\partial_t v_0 \neq 0$. There is a neighborhood $\mathcal{U}$ of v_0 in $\mathcal{S}$ consisting entirely of periodic solutions, and c identifies $\mathcal{U}$ with an open subset of $\mathbb{R}^2$. For $v \in \mathcal{U}$, let $T(v)$ be its least period. Then, for every integer $j \ge 0$, the map*

$$v \longmapsto \Big(T(v),\ \big[x \mapsto v(T(v)x, \pi/2)\big] \Big)$$

is real analytic from $\mathcal{U}$ into $\mathbb{R} \times H^j(\mathbb{R}/\mathbb{Z})$. Moreover, $v \mapsto \mathcal{H}(v)$ is real analytic with nonzero differential, and $v \mapsto \mathrm{ev}_0(v)$ is real analytic.

Proof. Let T_0 be the least period of v_0, $W_0(x) = v_0(T_0 x, \pi/2)$, and $e_* = e_0(b_{v_0}(0))$. Self-adjointness and commutation with ∂_x give

$$\langle \mathcal{P}_{\sigma,T} W, W_x \rangle = -\langle \mathcal{P}_{\sigma,T} W, W_x \rangle = 0, \qquad \int_0^1 W^{\frac{n+2\sigma}{n-2\sigma}} W_x \, \mathrm{d}x = 0.$$

Hence

$$\langle \mathcal{F}(T, W), W_x \rangle_{L^2} = 0. \tag{3.13}$$

Let $\Pi_{\partial_x W_0}$ project onto $(\partial_x W_0)^\perp$. We adjoin the principal coordinates to the projected equation and define

$$G(T, W) = \Big(\Pi_{\partial_x W_0} \mathcal{F}(T, W), \pi_{e_*}(v_{T,W}(0), T^{-1} \partial_x v_{T,W}(0)) \Big).$$

By Lemmas 3.5 and 3.7, G is analytic into $(H^{r_0 - 2\sigma} \cap (\partial_x W_0)^\perp) \times \mathbb{R}^2$. Projecting away one range direction adds 1 to the index; adjoining the variable T adds 1, and prescribing two coordinates subtracts 2. Hence

$$\operatorname{ind} DG(T_0, W_0) = 0.$$

We show that $\ker DG(T_0, W_0) = \{0\}$. Take $(\delta T, \delta W) \in \ker DG(T_0, W_0)$. The first component of DG and the derivative of (3.13) give, since $\mathcal{F}(T_0, W_0) = 0$,

$$L_{T_0, W_0} \delta W + \delta T \, \partial_T \mathcal{P}_{\sigma, T_0} W_0 \in \operatorname{span}\{\partial_x W_0\} \cap \{\partial_x W_0\}^\perp = \{0\}.$$

Set

$$T_s = T_0 + s\delta T, \qquad W_s = W_0 + s\delta W, \qquad v_s(t, \theta) = v_{T_s, W_s}(t/T_s, \theta).$$

Differentiation at fixed t gives

$$\Phi(t,\theta) = \partial_s v_s(t,\theta)\big|_{s=0} = D_{(T,W)} v_{T,W}(t/T_0,\theta)[\delta T, \delta W] - \frac{\delta T}{T_0} t\, \partial_t v_0(t,\theta), \tag{3.14}$$

where $D_{(T,W)}$ is evaluated at (T_0, W_0) with x fixed. Differentiating the boundary equation for v_s gives

$$d_\sigma\Big(\partial_\nu^\sigma \Phi - b_{v_0}\Phi(t,\pi/2)\Big) = \Big(L_{T_0,W_0}\delta W + \delta T\, \partial_T \mathcal{P}_{\sigma,T_0} W_0\Big)(t/T_0) = 0.$$

Thus $\mathcal{L}_{b_{v_0}}\Phi = 0$.

By Lemma 3.5, the fixed-x variation in (3.14) has a bounded periodic Cauchy pair, while $(v_{0,t}, v_{0,tt})$ is also bounded and periodic. Using the second component of DG,

$$\|(\Phi, \Phi_t)(t)\|_{E_\omega} \le C(1+|t|), \qquad \pi_{e_*}(\mathrm{ev}_0(\Phi)) = 0.$$

Since $\varpi_0(t) = e^{-2k|t|}$, this linear bound implies $\Phi \in \mathcal{C}_{v_0}$. By Proposition 3.2, $\Phi = 0$. Differentiating $v_s(t+T_s) = v_s(t)$ at $s=0$ gives

$$\Phi(t+T_0) - \Phi(t) = -\delta T\, \partial_t v_0(t), \tag{3.15}$$

so $\delta T = 0$. Then

$$\Phi(t,\theta) = D_{(T,W)} v_{T,W}(t/T_0,\theta)[0,\delta W],$$

and evaluating at $\theta = \pi/2$ gives $\delta W(x) = \Phi(T_0 x, \pi/2) = 0$. Hence $\ker DG(T_0,W_0) = \{0\}$. Since $\mathrm{ind}\, DG(T_0,W_0) = 0$, $DG(T_0,W_0)$ is an isomorphism.

The analytic implicit function theorem gives

$$G(T(q), W(q)) = (0,q) \qquad (q \text{ near } c(v_0)).$$

For such q,

$$\min W(q) \ge \tfrac{1}{2}\min W_0 > 0, \qquad \langle \partial_x W_0, \partial_x W(q)\rangle \ge \tfrac{1}{2}\|\partial_x W_0\|_{L^2}^2 > 0.$$

By (3.13),

$$\mathcal{F}(T(q), W(q)) \in \mathrm{span}\{\partial_x W_0\} \cap \{\partial_x W(q)\}^\perp = \{0\}.$$

Proposition 2.4 gives

$$v_q(t,\theta) = v_{T(q),W(q)}(t/T(q),\theta) \in \mathcal{S}, \qquad c(v_q) = q.$$

After shrinking the q-neighborhood, $W(q)$ has least period one. Otherwise, take $q_j \to c(v_0)$ such that $W(q_j)$ has period $1/\ell_j$, $\ell_j \ge 2$. Since $W(q_j) \to W_0$ in C^1, bounded ℓ_j give a smaller period of W_0, while $\ell_j \to \infty$ and the uniform C^1 bound force W_0 to be constant. Both contradict its least period. Thus $T(q)$ is the least physical period. By injectivity in Proposition 3.2, $c(v) = c(v_{c(v)})$ implies $v = v_{c(v)}$, so the coordinate image is open.

For $j = 1, 2$, differentiation at fixed physical time, using (3.14) and (3.15), gives

$$\mathcal{L}_{b_{v_q}} \partial_{q_j} v_q = 0, \qquad \|\partial_{q_j}(v_q, \partial_t v_q)(t)\|_{E_\omega} \le C_q(1+|t|).$$

Since $c(v_q) = q$,

$$\pi_{e*}\mathrm{ev}_0 D_q v_q = I_{\mathbb{R}^2}.$$

Thus $\partial_{q_1} v_q, \partial_{q_2} v_q$ are independent elements of $\mathcal{C}_{v_q}$, and $\dim \mathcal{C}_{v_q} = 2$ gives

$$\mathcal{C}_{v_q} = \mathrm{span}\{\partial_{q_1} v_q, \partial_{q_2} v_q\}.$$

By the chain rule,

$$\partial_{q_j} \mathcal{H}(v_q) = D\mathcal{H}(\mathrm{ev}_0(v_q))[\mathrm{ev}_0(\partial_{q_j} v_q)].$$

If both derivatives vanished, $D\mathcal{H}$ would vanish on $\mathrm{ev}_0(\mathcal{C}_{v_q})$, contrary to Lemma 3.3. Hence $D_q\mathcal{H}(v_q) \neq 0$.

Analyticity of $\mathrm{ev}_0(v_q)$ follows from Lemma 3.5, and positivity of the trace gives analyticity of $\mathcal{H}(v_q)$. The vector field in (3.6) is analytic in q, so the local translation action is analytic. The same argument in H^j, $j > r_0$, gives the stated higher regularity by elliptic regularity and uniqueness in H^{r_0}. □

3.3 The global Fowler family

We classify the translation orbits in $\mathcal{S}_H$, then prove symmetry, monotonicity, and the endpoint limits of the periodic profiles.

Proposition 3.12. *We have*

$$\mathcal{H}(\mathcal{S}) = [H_c, 0], \qquad \mathcal{S}_{H_c} = \{v_c\}, \qquad \mathcal{S}_0 = \{0\} \cup \{v_b(\cdot + \tau) : \tau \in \mathbb{R}\}.$$

For each $H \in (H_c, 0)$, there is a periodic $v \in \mathcal{S}_H$ with $\partial_t v \not\equiv 0$, and

$$\mathcal{S}_H = \{v(\cdot + \tau) : \tau \in \mathbb{R}\}.$$

Proof. Let $v \in \mathcal{S}$ be nonstationary and nonperiodic, and write $H = \mathcal{H}(v)$. We first examine the possible limits of translates $v(\cdot + t_j)$ as $t_j \to +\infty$ or $t_j \to -\infty$. By Lemma 3.1, after passing to a subsequence we may assume $v(\cdot + t_j) \to y$ in $\mathcal{S}$. Conservation of the Hamiltonian gives $\mathcal{H}(y) = H$. If $y_t \not\equiv 0$, then Proposition 3.4 would give

$$v(\cdot + t_j) = y(\cdot + \tau_j), \qquad v(\cdot + t_j - \tau_j) = y, \qquad \sup_j |\tau_j| < \infty.$$

Choosing i, j with $|t_j - t_i| > 2\sup_j |\tau_j|$ would then yield

$$v(\cdot + (t_j - \tau_j) - (t_i - \tau_i)) = v, \qquad (t_j - \tau_j) - (t_i - \tau_i) \neq 0,$$

contradicting the nonperiodicity of v. Thus every such subsequential limit is stationary and equals 0 or v_c.

If $H \notin \{0, H_c\}$, this is impossible because 0 and v_c have energies 0 and H_c, respectively. Thus $H \in \{0, H_c\}$. If $H = H_c$, the above limit would have to be v_c. Then $v(\cdot + t_j) \in \mathcal{U}_c$ for large j; since v is nonperiodic, $v(\cdot + t_j) \neq v_c$, and Proposition 3.9 would make v periodic, a

contradiction. Therefore $H = 0$. Now every limit of translates is stationary with energy 0, hence equals 0. Thus $w(t) \to 0$ as $t \to \pm\infty$, and Lemma 2.1 gives

$$v = v_b(\cdot + \tau)$$

for some $\tau \in \mathbb{R}$.

The Hamiltonian attains its extrema on the compact set $\mathcal{S}$. For a nonstationary periodic v, Proposition 3.11 gives an open coordinate neighborhood on which $D_q\mathcal{H}(v_q) \neq 0$; such a point is not an extremum. The remaining solutions are 0, v_c and bubbles, with energies 0, H_c and 0, respectively, by (3.3) and $\mathcal{H}(v_b) = 0$. Hence

$$\mathcal{H}(\mathcal{S}) \subset [H_c, 0], \qquad \mathcal{S}_{H_c} = \{v_c\}, \qquad \mathcal{S}_0 = \{0\} \cup \{v_b(\cdot + \tau) : \tau \in \mathbb{R}\}. \tag{3.16}$$

Every solution with $H_c < \mathcal{H}(v) < 0$ is therefore nonstationary and periodic.

Let $N(H)$ count the translation orbits in $\mathcal{S}_H$, with $N(H) = 0$ if $\mathcal{S}_H = \varnothing$. We prove $N(H) = 1$ on $(H_c, 0)$. Near an orbit of energy H_0, Proposition 3.11 gives an analytic curve y_H with $\mathcal{H}(y_H) = H$ and least period $T(H)$. For each $\tau_0 \in [0, T(H_0)]$, choose coordinates c near $y_{H_0}(\cdot + \tau_0)$. At (H_0, τ_0),

$$\begin{aligned} \partial_H \mathcal{H}(y_H(\cdot + \tau)) = 1, &\qquad \partial_\tau \mathcal{H}(y_H(\cdot + \tau)) = 0, \\ \partial_\tau c(y_H(\cdot + \tau)) \neq 0, &\quad \operatorname{rank} D_{(H,\tau)} c(y_H(\cdot + \tau)) = 2. \end{aligned}$$

The inverse function theorem, compactness of the phase circle and the least-period condition give an open neighborhood $\mathcal{U}$ of this orbit and $\delta > 0$ such that

$$\mathcal{U} \cap \mathcal{S}_H = \{y_H(\cdot + \tau) : \tau \in \mathbb{R}\} \qquad \text{for } |H - H_0| < \delta.$$

We next prove that $\mathcal{S}_H$ consists of finitely many translation orbits. By the local uniqueness just established, each $v \in \mathcal{S}_H$ has an open neighborhood U_v in $\mathcal{S}$ such that $U_v \cap \mathcal{S}_H$ is contained in a single translation orbit. Since $\mathcal{S}_H$ is compact by Lemma 3.1 and the continuity of $\mathcal{H}$, finitely many such neighborhoods cover $\mathcal{S}_H$. Hence $N(H) < \infty$.

The distinct orbits are compact and disjoint. Using the local description above, choose pairwise disjoint neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_{N(H)}$ of these orbits and $\delta > 0$ such that $\mathcal{U}_i \cap \mathcal{S}_{H'}$ is a single translation orbit for $|H' - H| < \delta$. No other orbit occurs at nearby energies. Otherwise, there would be $H_j \to H$ and

$$v_j \in \mathcal{S}_{H_j} \setminus \bigcup_{i=1}^{N(H)} \mathcal{U}_i.$$

Compactness gives, along a subsequence, $v_j \to v \in \mathcal{S}_H \subset \bigcup_{i=1}^{N(H)} \mathcal{U}_i$. The union is open, so $v_j \in \bigcup_i \mathcal{U}_i$ for large j, a contradiction. Thus N is locally constant on $(H_c, 0)$.

If $\mathcal{S}_H = \varnothing$, then $\operatorname{dist}(H, \mathcal{H}(\mathcal{S})) > 0$, since $\mathcal{H}(\mathcal{S})$ is compact. Finally, a sequence with

$$H_j \downarrow H_c, \qquad v_j \in \mathcal{S}_{H_j} \setminus \mathcal{U}_c$$

would have a subsequence converging to $v_c \in \mathcal{U}_c$, a contradiction. Proposition 3.9 gives $N(H) = 1$ near H_c. Connectedness of $(H_c, 0)$ gives $N(H) = 1$ throughout. □

Lemma 3.13. *Let $L > 0$ and $b \in L^\infty(0, L)$ with $0 \le b \le C_0$. Suppose a nonnegative, nonzero weak solution of the homogeneous equation* (2.12) *on $(0, L) \times (0, \pi/2)$ satisfies $\Phi(0, \theta) = 0$, and $\Phi, \partial_t \Phi$ are continuous on the closed strip. Then*

$$\partial_t \Phi(0, \theta) > 0 \qquad \text{for } 0 \le \theta \le \pi/2.$$

Proof. The Harnack inequality [6, Proposition 2.1] up to the boundary gives, in cylindrical coordinates,

$$\Phi > 0 \qquad \text{for } 0 < t < L,\ 0 \le \theta \le \pi/2.$$

Away from the equator the equation is uniformly elliptic on the hemisphere, including its pole $\theta = 0$. The ordinary Hopf lemma gives

$$\Phi_t(0, \theta) > 0 \quad \text{for } 0 \le \theta < \pi/2.$$

It remains to treat $\theta = \pi/2$. Set $h(t) = \sinh(pt)$. Then $h(0) = 0$, $h'(0) = p > 0$, $h'' - p^2 h = 0$, and $\partial_\nu^\sigma h = 0$. We shall prove $\Phi \ge \varepsilon h$ near $(0, \pi/2)$ for some $\varepsilon > 0$.

Let $r = \pi/2 - \theta$ and

$$\widetilde{\omega}(r) = \cos^{n-1} r \sin^{1-2\sigma} r, \qquad Q_R = (0, R)_t \times (0, R)_r.$$

For $\psi(t, R) = 0$, Cauchy–Schwarz gives

$$\int_0^R |\psi(t, 0)|^2 \, \mathrm{d}t \le \left(\int_0^R \widetilde{\omega}^{-1} \, \mathrm{d}r \right) \int_{Q_R} |\psi_r|^2 \, \widetilde{\omega} \, \mathrm{d}r \, \mathrm{d}t \le C_{n,\sigma} R^{2\sigma} \int_{Q_R} |\psi_r|^2 \, \widetilde{\omega} \, \mathrm{d}r \, \mathrm{d}t.$$

Choose

$$R < \min\{\pi/8, L\}, \qquad C_0 C_{n,\sigma} R^{2\sigma} < 1/2.$$

The positivity of Φ gives the first minimum below, while the Hopf lemma at $(t, r) = (0, R)$ gives the second limit:

$$\min_{0 \le r \le R} \frac{\Phi(R, \pi/2 - r)}{h(R)} > 0, \qquad \lim_{t \downarrow 0} \frac{\Phi(t, \pi/2 - R)}{h(t)} = \frac{\Phi_t(0, \pi/2 - R)}{p} > 0.$$

Take

$$\varepsilon = \frac{1}{2} \min \left\{ \min_{0 \le r \le R} \frac{\Phi(R, \pi/2 - r)}{h(R)}, \inf_{0 < t \le R} \frac{\Phi(t, \pi/2 - R)}{h(t)} \right\} > 0.$$

Let $f(t, r) = \Phi(t, \pi/2 - r) - \varepsilon h(t)$. It satisfies the homogeneous interior equation on Q_R, vanishes on $t = 0$, is nonnegative on $t = R$ and $r = R$, and satisfies

$$-\lim_{r \downarrow 0} \widetilde{\omega} f_r - bf = \varepsilon b h \ge 0 \quad \text{on } r = 0.$$

Thus $f^- = \max\{-f, 0\}$ vanishes on $t = 0$, $t = R$ and $r = R$. Testing by f^- and using the trace inequality gives

$$\begin{aligned}
&\int_{Q_R} \left(|\nabla f^-|^2 + p^2|f^-|^2\right) \widetilde{\omega}\,\mathrm{d}r\,\mathrm{d}t \\
&= \int_0^R b(t)|f^-(t,0)|^2\,\mathrm{d}t - \varepsilon\int_0^R b(t)h(t)f^-(t,0)\,\mathrm{d}t \\
&\le C_0\int_0^R |f^-(t,0)|^2\,\mathrm{d}t \\
&\le C_0C_{n,\sigma}R^{2\sigma}\int_{Q_R}|f_r^-|^2\,\widetilde{\omega}\,\mathrm{d}r\,\mathrm{d}t \le \frac{1}{2}\int_{Q_R}|\nabla f^-|^2\,\widetilde{\omega}\,\mathrm{d}r\,\mathrm{d}t.
\end{aligned}$$

Hence $f \ge 0$ on $\overline{Q_R}$. Therefore

$$\Phi_t(0,\pi/2) = \lim_{t\downarrow 0}\frac{\Phi(t,\pi/2)}{t} \ge \varepsilon\lim_{t\downarrow 0}\frac{h(t)}{t} = \varepsilon p > 0.$$

□

Lemma 3.14. *For $H \in (H_c, 0)$ and $v \in \mathcal{S}_H$, with least period $T = T(H)$,*

$$\#\{\tau \in [0,T) : v(\tau + t) = v(\tau - t) \text{ for all } t \in \mathbb{R}\} = 2.$$

These two centers differ by $T/2$, and each centered representative can be selected locally real analytically as a function of H.

Proof. Since reflection preserves the energy, Proposition 3.12 gives

$$v(-\cdot) = v(\cdot + t_*)$$

for some $t_* \in \mathbb{R}$. For $\tau \in \mathbb{R}$, the condition $v(\tau + t) = v(\tau - t)$ for all t is equivalent, after setting $s = t - \tau$, to $v(\cdot + 2\tau) = v(\cdot + t_*)$, hence to

$$2\tau = t_* \pmod T.$$

Thus the reflection centers are exactly

$$\tau = t_*/2, \qquad \tau = t_*/2 + T/2 \pmod T,$$

so there are exactly two modulo T, and they differ by $T/2$.

Now translate one center to zero, so that v is even. Recall that $e_* = e_0(b_v(0))$ and use the coordinates $c = \pi_{e_*} \circ \mathrm{ev}_0$ on a small neighborhood of v. Intersect this neighborhood with its reflection. Writing $c(y) = (c_1(y), c_2(y))$, we have

$$c(y(-\cdot)) = (c_1(y), -c_2(y)).$$

By injectivity of c, a nearby y is even exactly when

$$c_2(y) = \langle e_*, y_t(0)\rangle_\omega = 0.$$

Indeed, $c_2(y) = 0$ implies $c(y(-\cdot)) = c(y)$ and hence $y(-\cdot) = y$.

Since v is even, $v_t(0) = 0$. Since $v_t \in \mathcal{C}_v \setminus \{0\}$, Proposition 3.2 gives

$$0 \neq \pi_{e_*}(\mathrm{ev}_0(v_t)) = (0, \langle e_*, v_{tt}(0)\rangle_\omega).$$

Choose an analytic energy curve y_H through v with $\mathcal{H}(y_H) = H$. Such a curve follows from Proposition 3.11 and the implicit function theorem, since the energy differential is nonzero. The analytic implicit function theorem applied to

$$(H, \tau) \longmapsto \langle e_*, \partial_t y_H(\tau)\rangle_\omega$$

gives an analytic $\tau(H)$ near $\mathcal{H}(v)$ such that

$$\langle e_*, \partial_t y_H(\tau(H))\rangle_\omega = 0, \qquad \tau(\mathcal{H}(v)) = 0.$$

Thus $y_H(\cdot + \tau(H))$ is even. Since $T(H) = T(y_H)$ is analytic by Proposition 3.11, the other centered representative

$$y_H(\cdot + \tau(H) + T(H)/2)$$

is also locally real analytic in H. □

Proposition 3.15. *For each $H \in (H_c, 0)$ there is a unique representative $v_H \in \mathcal{S}_H$ with $w_H(0) = \max w_H$, where $w_H(t) = v_H(t, \pi/2)$. This representative is even in t and*

$$\partial_t v_H(t, \theta) < 0 \qquad \textit{for } 0 < t < T(H)/2,\ 0 \le \theta \le \pi/2.$$

The map

$$H \longmapsto \Big(T(H), w_H(T(H)\,\cdot)\Big) \quad \textit{in } \mathbb{R} \times H^j(\mathbb{R}/\mathbb{Z}) \quad \textit{for } j = 0, 1, 2, \ldots$$

is locally real analytic.

Proof. Let H_* be the supremum of those $h \in (H_c, 0)$ for which an even decreasing representative exists at every energy $H \in (H_c, h)$. Lemma 3.10 gives $H_* > H_c$. Suppose $H_* < 0$. For $H_j \uparrow H_*$, compactness and Propositions 3.12 and 3.11 give a subsequence with

$$v_{H_j} \to v_*, \qquad \mathcal{H}(v_*) = H_*, \qquad T(H_j) \to T_* := T(H_*).$$

Since each v_{H_j} is even and $T(H_j)$-periodic, the limit v_* is even and T_*-periodic. It is nonstationary because $\mathcal{H}(v_*) = H_* \in (H_c, 0)$.

By Lemma 3.6,

$$\mathcal{L}_{b_{v_*}}(-v_{*,t}) = 0, \quad -v_{*,t} \ge 0, \quad -v_{*,t} \not\equiv 0 \quad \text{for } 0 < t < T_*/2, \quad v_{*,t}(0) = v_{*,t}(T_*/2) = 0.$$

By the boundary Harnack inequality [6, Proposition 2.1] applied to $-v_{*,t}$, we have $-v_{*,t} > 0$ on $0 < t < T_*/2$. Applying Lemma 3.13 to $-v_{*,t}$ at $t = 0$, and to its reflection at $t = T_*/2$, gives

$$-v_{*,t} > 0 \quad \text{for } 0 < t < T_*/2, \qquad \min_\theta[-v_{*,tt}(0, \theta)] > 0, \qquad \min_\theta v_{*,tt}(T_*/2, \theta) > 0.$$

Choose the nearby even family through v_* by Lemma 3.14. For $\ell = 1, 2$,

$$\partial_t^\ell v_H(T(H)\,\cdot, \cdot) \longrightarrow \partial_t^\ell v_*(T_*\,\cdot, \cdot) \quad \text{in } C([0, 1/2] \times [0, \pi/2]).$$

For some $c > 0$, $0 < \eta < T_*/4$, and all nearby H,

$$-v_{H,tt}(t, \theta) \geq c, \qquad v_{H,tt}(T(H)/2 - t, \theta) \geq c \quad \text{for } 0 \leq t \leq \eta,\ 0 \leq \theta \leq \pi/2.$$

Since $v_{H,t} = 0$ at both endpoints,

$$\begin{aligned} -v_{H,t}(t, \theta) &= \int_0^t -v_{H,tt}(s, \theta)\,\mathrm{d}s \geq ct && \text{for } 0 < t \leq \eta, \\ -v_{H,t}(t, \theta) &= \int_t^{T(H)/2} v_{H,tt}(s, \theta)\,\mathrm{d}s \geq c(T(H)/2 - t) && \text{for } T(H)/2 - \eta \leq t < T(H)/2. \end{aligned}$$

On the middle interval, after decreasing c,

$$\min_{\substack{\eta/2 \leq t \leq T_*/2 - \eta/2 \\ 0 \leq \theta \leq \pi/2}} (-v_{*,t}(t, \theta)) \geq 2c, \qquad \min_{\substack{\eta \leq t \leq T(H)/2 - \eta \\ 0 \leq \theta \leq \pi/2}} (-v_{H,t}(t, \theta)) \geq c.$$

Thus the required strict decrease persists for H on both sides of H_*. In particular, for some $\varepsilon > 0$, every $H \in (H_c, H_* + \varepsilon)$ has an even decreasing representative, contradicting the definition of H_*. Hence $H_* = 0$.

The representative with its maximum at 0 is strictly decreasing on $(0, T(H)/2)$, so its trace has a unique maximum in each least period. Since every other representative is a translate, this normalization determines v_H uniquely. The maximum can be selected analytically by Lemma 3.14; the period and rescaled profile are analytic by Proposition 3.11. □

With this normalization, we determine the endpoint limits and the ranges of the maximum and minimum. For a nonconstant T-periodic trace, integration of (1.5) gives

$$\begin{aligned} &\Theta_\sigma(0) \int_0^T w\,\mathrm{d}t = K_\sigma \int_0^T w^{\frac{n+2\sigma}{n-2\sigma}}\,\mathrm{d}t, \qquad \Theta_\sigma(0) = K_\sigma w_c^{\frac{4\sigma}{n-2\sigma}}, \\ &\int_0^T w \left(w^{\frac{4\sigma}{n-2\sigma}} - w_c^{\frac{4\sigma}{n-2\sigma}} \right) \mathrm{d}t = 0, \qquad \min w < w_c < \max w. \end{aligned} \tag{3.17}$$

Indeed, if $w \geq w_c$ on $[0, T]$, then the integrand in the second identity is nonnegative and not identically zero, so the integral cannot vanish; the case $w \leq w_c$ is similar.

Proposition 3.16. *For the family normalized by* $w_H(0) = \max w_H$*,*

$$\begin{aligned} w_H &\longrightarrow w_c \text{ uniformly on } \mathbb{R}, & T(H) &\longrightarrow 2\pi/\kappa_c && \text{as } H \downarrow H_c, \\ w_H &\longrightarrow w_b \text{ in } C^\infty_{\mathrm{loc}}(\mathbb{R}), & T(H) &\longrightarrow \infty && \text{as } H \uparrow 0, \\ \min w_H &\longrightarrow 0, & \max w_H &\longrightarrow 1 && \text{as } H \uparrow 0. \end{aligned}$$

Proof. Let $H_j \downarrow H_c$ and $t_j \in \mathbb{R}$. By Lemma 3.1 and Proposition 3.12, a subsequence satisfies

$$\begin{aligned}(v_{H_j}, \partial_t v_{H_j})(t_j + \cdot) &\longrightarrow (v_*, \partial_t v_*) \quad \text{in } C(I; E_\omega), \qquad I \Subset \mathbb{R},\\ \mathcal{H}(v_*) &= H_c,\\ v_* &= v_c.\end{aligned}$$

If the uniform convergence $w_H \to w_c$ failed, there would exist $\varepsilon > 0$, $H_j \downarrow H_c$ and t_j with $|w_{H_j}(t_j) - w_c| \geq \varepsilon$. The subsequence argument above would then give a limit v_* with $v_*(0, \pi/2) \neq w_c$, contradicting $v_* = v_c$. Hence $w_H \to w_c$ uniformly on $\mathbb{R}$, and Proposition 3.9 gives $T(H) \to 2\pi/\kappa_c$.

Let $H_j \uparrow 0$. By compactness, take $v_{H_j} \to v_*$ locally. By (3.17) and conservation,

$$\mathcal{H}(v_*) = 0, \qquad v_*(0, \pi/2) = \max_{t \in \mathbb{R}} v_*(t, \pi/2) \geq w_c.$$

The zero-energy classification gives $v_* = v_b(\cdot + \tau)$; the maximum at zero gives $\tau = 0$. Thus every subsequential limit is v_b, and compactness gives convergence of the whole family. By local boundary regularity,

$$w_H \longrightarrow w_b \quad \text{in } C^\infty_{\mathrm{loc}}(\mathbb{R}), \qquad \max w_H = w_H(0) \longrightarrow w_b(0) = 1.$$

For each $R > 0$,

$$0 \leq \limsup_{H \uparrow 0} \min w_H \leq \lim_{H \uparrow 0} w_H(R) = w_b(R).$$

Let $R \to \infty$. Then $\min w_H \to 0$. Finally, by Proposition 2.2 and monotonicity on $[0, T(H)/2]$,

$$\log \frac{\max w_H}{\min w_H} = \int_0^{T(H)/2} -\frac{\partial_t w_H}{w_H} \, \mathrm{d}t \leq \frac{n - 2\sigma}{4} T(H).$$

Since $\max w_H \to 1$ and $\min w_H \to 0$, $T(H) \to \infty$. □

3.4 Necksize and maximum

We now apply Lemma 2.12 to show that the minimum and maximum each determine the Fowler profile up to translation.

Proposition 3.17. *The maximum and minimum $M(H) = w_H(0)$ and $\varepsilon(H) = w_H(T(H)/2)$ are real-analytic diffeomorphisms from $(H_c, 0)$ onto $(w_c, 1)$ and $(0, w_c)$, respectively, with $M'(H) > 0$ and $\varepsilon'(H) < 0$.*

Proof. Write $T = T(H)$. By Proposition 3.15, M and ε are analytic. Since

$$1 = K_\sigma \int_{\mathbb{R}} G_\sigma(t) w_b(t)^{\frac{n+2\sigma}{n-2\sigma}} \, \mathrm{d}t < K_\sigma \int_{\mathbb{R}} G_\sigma(t) \, \mathrm{d}t = \frac{K_\sigma}{\Theta_\sigma(0)},$$

we have $w_c < 1$.

Suppose $M(H) = 1$, and let b be the secant coefficient of v_H and v_b in (2.10). Since $0 < w_H, w_b \le 1$ and $w_H(0) = w_b(0) = 1$,

$$b(-t) = b(t), \qquad 0 < b(t) \le b_\sigma, \qquad |b'| \le 2\sigma b.$$

The last bound is Lemma 2.3. The difference $\Phi = v_H - v_b$ satisfies $\mathcal{L}_b\Phi = 0$, is even, has polynomially growing Cauchy pair, and $\Phi(0, \pi/2) = 0$. Since $\mathcal{H}(v_H) = H < 0 = \mathcal{H}(v_b)$ and $\Phi \not\equiv 0$, Lemma 2.12 gives $\Phi(0, \pi/2) \neq 0$, a contradiction. Thus $M(H) \neq 1$ for every $H \in (H_c, 0)$. Since M is continuous and $M(H_c+) = w_c < 1$, we have $M(H) < 1$ on $(H_c, 0)$.

Let v be either v_H or its half-period translate $v_H(\cdot + T/2)$, and write $b = b_v$. In both cases $\partial_H v \in \mathcal{C}_b$ is even, nonzero, and has polynomially growing Cauchy pair; moreover

$$D\mathcal{H}(\mathrm{ev}_0(v))[\mathrm{ev}_0(\partial_H v)] = 1.$$

Iterating (3.15) with parameter H gives at most linear growth in either centering.

Since $0 < w \le M(H) < 1$, the formula for b_v and Lemma 2.3 give, in both centerings,

$$b(-t) = b(t), \qquad 0 < b \le b_\sigma, \qquad |b'| \le 2\sigma b.$$

Applying Lemma 2.12 to $\partial_H v$ in the two centerings gives $\partial_H w_H(0) \neq 0$ and $\partial_H w_H(T/2) \neq 0$. Since

$$M' = \partial_H w_H(0), \qquad \varepsilon' = \partial_H w_H(T/2) + \tfrac{1}{2}T'\partial_t w_H(T/2) = \partial_H w_H(T/2),$$

we have $M' \neq 0$ and $\varepsilon' \neq 0$.

Since M' and ε' are continuous and nowhere zero on $(H_c, 0)$, each has a fixed sign. By Proposition 3.16,

$$M(H_c+) = w_c < 1 = M(0-), \qquad \varepsilon(H_c+) = w_c > 0 = \varepsilon(0-).$$

Hence $M' > 0$ and $\varepsilon' < 0$. The analytic inverse function theorem gives both diffeomorphisms. □

Proof of Theorem 1.1. The assertions follow from Propositions 3.12, 3.15, 3.16 and 3.17. □

3.5 Solutions of the linearized equation

Differentiation of the periodic family with respect to translation and energy gives the two solutions in Theorem 1.3.

Proof of Theorem 1.3. For $H_c < H < 0$, iterating (3.15) with parameter H gives

$$\|(\partial_t v_H, \partial_t^2 v_H)(t)\|_{E_\omega} \le C_H, \qquad \|(\partial_H v_H, \partial_t \partial_H v_H)(t)\|_{E_\omega} \le C_H(1 + |t|).$$

Both derivatives belong to $\mathcal{C}_{v_H}$, and

$$D\mathcal{H}(\mathrm{ev}_0(v_H))[\mathrm{ev}_0(\partial_t v_H)] = 0, \qquad D\mathcal{H}(\mathrm{ev}_0(v_H))[\mathrm{ev}_0(\partial_H v_H)] = 1.$$

Since $\partial_t v_H \neq 0$, Corollary 2.10 gives

$$\mathcal{C}_{v_H} = \operatorname{span}\{\partial_t v_H, \partial_H v_H\}. \tag{3.18}$$

At $H = H_c$, Proposition 3.9 gives

$$\mathcal{C}_{v_c} = \operatorname{span}\{e_c \cos(\kappa_c t), e_c \sin(\kappa_c t)\}.$$

For a scalar solution ϕ with the stated polynomial bound, take the cylindrical Poisson extension [7],

$$\Phi(t,\theta) = \int_{\mathbb{R}} \mathcal{K}_\sigma(t-s,\theta)\phi(s)\,\mathrm{d}s,$$
$$\mathcal{K}_\sigma(s,\theta) = \frac{\Gamma(\frac{n+2\sigma}{2})(\cos\theta)^{2\sigma}}{\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{S}^{n-1}} [2(\cosh s - \sin\theta\,\zeta_1)]^{-\frac{n+2\sigma}{2}}\,\mathrm{d}S(\zeta), \qquad 0 \le \theta < \pi/2.$$

The kernel bounds are

$$\int_{\mathbb{R}} \mathcal{K}_\sigma(s,\theta)\,\mathrm{d}s \le 1, \qquad 0 \le \mathcal{K}_\sigma(s,\theta) \le Ce^{-\frac{n+2\sigma}{2}|s|} \quad \text{for } |s| \ge 1.$$

Hence

$$\sup_{\theta<\pi/2} \int_{\mathbb{R}} \mathcal{K}_\sigma(s,\theta)(1+|s|)^j\,\mathrm{d}s \le C_j, \qquad \sup_{\theta\le\pi/2} |\Phi(t,\theta)| \le C(1+|t|)^j.$$

By the Poisson formula and the Dirichlet-to-Neumann identity,

$$\Phi(t,\pi/2) = \phi(t), \qquad \mathcal{L}_{b_{v_H}}\Phi = 0.$$

The extension has finite local weighted energy. Lemma 2.7 on translated intervals gives

$$\|(\Phi,\Phi_t)(t)\|_{E_\omega} \le C\|\Phi\|_{L^2((t-1,t+1);L^2_\omega)} \le C'(1+|t|)^j.$$

Consequently,

$$\|\Phi\|^2_{\varpi_0} \le C\int_{\mathbb{R}} e^{-2k|t|}(1+|t|)^{2j}\,\mathrm{d}t < \infty, \qquad \Phi \in \mathcal{C}_{v_H}.$$

Taking traces in the bases above gives the required decomposition.

Conversely, differentiated periodicity gives $|\partial_H w_H(s)| \le C(1+|s|)$ locally uniformly in H. The kernel moment bound permits differentiation under the integral:

$$\partial_t v_H = \mathcal{K}_\sigma * \partial_t w_H, \qquad \partial_H v_H = \mathcal{K}_\sigma * \partial_H w_H.$$

The Dirichlet-to-Neumann identity gives (1.7) for both traces. At the cylinder, $g_{\kappa_c} = e_c/e_c(\pi/2)$ gives the extensions of $\cos(\kappa_c t)$ and $\sin(\kappa_c t)$. All these scalar solutions have at most linear growth. □

4 Asymptotics of local isolated singularities

4.1 Local bounds and weighted estimates

We now turn to the local extension problem (1.8), where $u = U(\cdot, 0)$. A weak solution belongs to $H^1(y^{1-2\sigma}, B_r^+ \setminus \overline{B_\varepsilon^+})$ for all $0 < \varepsilon < r < R$. The conormal condition for U is

$$d_\sigma \int_{B_R^+} y^{1-2\sigma} \nabla U \cdot \nabla \psi \, \mathrm{d}X = \int_{B_R'} u^{\frac{n+2\sigma}{n-2\sigma}} \psi(x, 0) \, \mathrm{d}x$$

for smooth test functions compactly supported away from 0 and the curved boundary. Let U satisfy the hypotheses of Theorem 1.5. On the upper hemisphere, write

$$\begin{aligned} \mathbb{S}_+^n &= \{\eta \in \mathbb{S}^n : \eta_{n+1} > 0\}, \\ \mathrm{d}\mu &= \frac{\eta_{n+1}^{1-2\sigma}}{|\mathbb{S}^{n-1}|} \, \mathrm{d}S(\eta) = \frac{\omega(\theta)}{|\mathbb{S}^{n-1}|} \, \mathrm{d}\theta \, \mathrm{d}S(\zeta), \\ \Delta_\mu f &= \eta_{n+1}^{2\sigma-1} \operatorname{div}_{\mathbb{S}^n} \left(\eta_{n+1}^{1-2\sigma} \nabla_{\mathbb{S}^n} f \right). \end{aligned}$$

Use $\eta = (\sin\theta \, \zeta, \cos\theta)$ and identify $\zeta \in \mathbb{S}^{n-1}$ with $(\zeta, 0) \in \partial\mathbb{S}_+^n$. Axially symmetric functions are independent of ζ. For $z = \eta_{n+1}$, the outward conormal derivative is, as before,

$$\partial_\nu^\sigma f = -\lim_{z \downarrow 0} z^{1-2\sigma} \partial_z f.$$

Set $t_0 = -\log R$ and

$$V(t, \eta) = K_\sigma^{-\frac{n-2\sigma}{4\sigma}} e^{-\frac{n-2\sigma}{2} t} U(e^{-t}\eta), \qquad w(t, \zeta) = V(t, \zeta).$$

Direct substitution gives

$$\begin{cases} \partial_t^2 V + \Delta_\mu V - p^2 V = 0 & (t, \eta) \in (t_0, \infty) \times \mathbb{S}_+^n, \\ \partial_\nu^\sigma V = (K_\sigma / d_\sigma) w^{\frac{n+2\sigma}{n-2\sigma}} & (t, \zeta) \in (t_0, \infty) \times \mathbb{S}^{n-1}. \end{cases} \tag{4.1}$$

Apply [6, Theorems 1.1–1.2] to $d_\sigma^{-\frac{n-2\sigma}{4\sigma}} U$. The growth estimates and the Harnack inequality give

$$0 < c_0 \le V(t, \eta) \le C_0 \qquad \text{for } t \ge t_1, \ \eta \in \overline{\mathbb{S}_+^n}. \tag{4.2}$$

The asymptotic symmetry estimate in [6, Theorem 1.2], together with (4.2), gives

$$\| w(t, \cdot) - \overline{w}(t) \|_{L^\infty(\mathbb{S}^{n-1})} \le C e^{-t}, \qquad \overline{w}(t) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} w(t, \zeta) \, \mathrm{d}S(\zeta). \tag{4.3}$$

On rescaled half-annuli, [22, Proposition 2.6 and Theorem 2.15] and (4.2) give a uniform Hölder bound and

$$\| \partial_t w \|_{L^\infty([t_1, \infty) \times \mathbb{S}^{n-1})} \le C.$$

Tangential derivatives of the extension are bounded by [22, Proposition 2.13] on the same annuli.

For two solutions V_1, V_2, write $w_i(t,\zeta) = V_i(t,\zeta)$. If $0 < c \le w_i \le C$ and $|\partial_t w_i| \le C$, their difference satisfies

$$\partial_t^2(V_1 - V_2) + \Delta_\mu(V_1 - V_2) - p^2(V_1 - V_2) = 0, \qquad \partial_\nu^\sigma(V_1 - V_2) = b(w_1 - w_2),$$

where b is given pointwise by (2.10) and satisfies

$$|b| + |\partial_t b| \le C. \tag{4.4}$$

The coefficient $(K_\sigma/d_\sigma)w^{\frac{4\sigma}{n-2\sigma}}$ in the equation for V satisfies the same bounds.

The normalized inner product and bilinear form are

$$\langle f, g\rangle_\mu = \int_{\mathbb{S}^n_+} fg \,\mathrm{d}\mu, \qquad Q_0(f,g) = \int_{\mathbb{S}^n_+} (\nabla f \cdot \nabla g + p^2 fg)\,\mathrm{d}\mu.$$

Let L^2_μ, H^1_μ be the corresponding weighted spaces, with $\|f\|^2_{H^1_\mu} = Q_0(f,f)$, and set $E_\mu = H^1_\mu \times L^2_\mu$. For axially symmetric functions

$$\|f\|_{H^1_\mu} = \|f\|_{H^1_\omega}, \qquad \|f\|_{L^2_\mu} = \|f\|_{L^2_\omega}.$$

On $\{0 < \eta_{n+1} < 1/2\}$ use $z = \eta_{n+1}$ and $\zeta \in \mathbb{S}^{n-1}$. The weighted surface measure is comparable to $z^{1-2\sigma}\,\mathrm{d}z\,\mathrm{d}S(\zeta)$, and the metric coefficients are uniformly bounded. Integrating (2.13) over ζ gives, for every $\varepsilon > 0$,

$$\|f\|^2_{L^2(\mathbb{S}^{n-1})} \le \varepsilon \|f\|^2_{H^1_\mu} + C_\varepsilon \|f\|^2_{L^2_\mu}. \tag{4.5}$$

The same calculation also gives

$$\int_{\{0<z<r\}} |f|^2 \,\mathrm{d}\mu \le C r^{2-2\sigma} \|f\|^2_{L^2(\mathbb{S}^{n-1})} + C r^2 \|f\|^2_{H^1_\mu}.$$

Hence $H^1_\mu \Subset L^2_\mu$ by Rellich compactness away from the equator. For a bounded sequence f_j in H^1_μ converging in L^2_μ, (4.5) gives

$$\|f_i - f_j\|^2_{L^2(\mathbb{S}^{n-1})} \le \varepsilon \|f_i - f_j\|^2_{H^1_\mu} + C_\varepsilon \|f_i - f_j\|^2_{L^2_\mu},$$

so the trace map $H^1_\mu \to L^2(\mathbb{S}^{n-1})$ is compact.

For a bounded open interval J and a Robin coefficient $b(t,\zeta)$, the linear equation on $J \times \mathbb{S}^n_+$ is

$$\begin{cases} \partial_t^2 \Phi + \Delta_\mu \Phi - p^2 \Phi = 0 & (t,\eta) \in J \times \mathbb{S}^n_+, \\ \partial_\nu^\sigma \Phi = b(t,\zeta)\Phi(t,\zeta) & (t,\zeta) \in J \times \mathbb{S}^{n-1}. \end{cases} \tag{4.6}$$

Lemma 4.1. *Let $I \Subset J$ be bounded open intervals. Suppose Φ is a weak solution of* (4.6) *and $\|b\|_\infty + \|\partial_t b\|_\infty \le C_0$. Then*

$$\|\Phi\|_{H^1(I;H^1_\mu)} + \|\partial_t \Phi\|_{H^1(I;L^2_\mu)} \le C\|\Phi\|_{L^2(J;L^2_\mu)}, \qquad (4.7)$$
$$\|\Phi\|_{C^\alpha(\overline{I}\times\overline{\mathbb{S}^n_+})} \le C\|\Phi\|_{L^2(J;L^2_\mu)}$$

for some $\alpha \in (0,1)$, where C and α depend on n, σ, I, J, C_0. Also $(\Phi, \partial_t\Phi) \in C(\overline{I}; E_\mu)$. If a sequence of such solutions is bounded in $L^2(J; L^2_\mu)$ and its coefficients converge locally uniformly, a subsequence converges in $C(\overline{I}; E_\mu)$.

Proof. We use the argument of Lemma 2.7 on the full hemisphere, with (4.5) in place of (2.13).

Suppose Ψ satisfies the homogeneous interior equation and $\partial^\sigma_\nu \Psi = b\Psi + g$. Choose $\chi \in C^\infty_c(J)$ with $\chi = 1$ on I. Testing by $\chi^2\Psi$ and using (4.5) in place of (2.13) gives

$$\|(\Psi, \Psi_t)\|_{L^2(I;E_\mu)} \le C\Big(\|\Psi\|_{L^2(J;L^2_\mu)} + \|g\|_{L^2(J\times\mathbb{S}^{n-1})}\Big). \qquad (4.8)$$

Applying (4.8) to Φ and to its difference quotients, and using $H^1_\mu \Subset L^2_\mu$ together with the compactness of the trace map $H^1_\mu \to L^2(\mathbb{S}^{n-1})$ established above, yields (4.7) and the subsequence convergence exactly as in the proof of Lemma 2.7. The Hölder estimate follows from [22, Proposition 2.6(i),(iii)] on rescaled half-annuli, applied to Φ and $-\Phi$. □

Lemma 4.2. *The first eigenvalue of $-\Delta_\mu + p^2$ on $\mathbb{S}^n_+$ with Dirichlet condition on $\partial\mathbb{S}^n_+$ is at least k^2.*

Let Φ be a solution of (4.6) *with $J = \mathbb{R}$, satisfying, for some $C > 0$ and integer $j \ge 0$,*

$$\|(\Phi, \partial_t\Phi)(t)\|_{E_\mu} \le C(1 + |t|)^j \qquad (t \in \mathbb{R}).$$

Then Φ is uniquely determined by its trace on $\mathbb{R} \times \partial\mathbb{S}^n_+$.

If the trace on $\mathbb{R} \times \partial\mathbb{S}^n_+$ is independent of $\zeta \in \mathbb{S}^{n-1}$, then Φ is axially symmetric.

Proof. Let $h = \eta_{n+1}^{2\sigma}$. A direct calculation gives

$$-\Delta_\mu h = 2\sigma n h, \qquad 2\sigma n + p^2 = k^2.$$

In particular, $h > 0$ in $\mathbb{S}^n_+$ and $h = 0$ on $\partial\mathbb{S}^n_+$.

For $f \in C^\infty_c(\mathbb{S}^n_+ \setminus \partial\mathbb{S}^n_+)$, integration by parts using the equation for h gives

$$Q_0(f,f) - k^2\|f\|^2_{L^2_\mu} = \int_{\mathbb{S}^n_+} h^2 \left|\nabla\left(\frac{f}{h}\right)\right|^2 \mathrm{d}\mu \ge 0.$$

Since $C^\infty_c(\mathbb{S}^n_+ \setminus \partial\mathbb{S}^n_+)$ is dense in the Dirichlet form domain $H^1_{\mu,0}$, the inequality extends to $H^1_{\mu,0}$. Hence the first Dirichlet eigenvalue of $-\Delta_\mu + p^2$ is at least k^2.

Now let Φ solve (4.6) and have zero trace on $\mathbb{R} \times \partial\mathbb{S}^n_+$. For a.e. t, we have $\Phi(t) \in H^1_{\mu,0}$. By the spectral theorem and the compact embedding $H^1_{\mu,0} \Subset L^2_\mu$, the Dirichlet eigenfunctions form an orthonormal basis of L^2_μ. Write

$$\Phi(t) = \sum_{j \geq 1} d_j(t) e_j,$$

where e_j are the Dirichlet eigenfunctions with eigenvalues $\lambda_j^D \geq k^2 > 0$. Since e_j vanish on $\partial\mathbb{S}^n_+$, projecting the equation onto e_j eliminates the boundary term and gives $d_j'' = \lambda_j^D d_j$. Hence

$$d_j(t) = A_j e^{\sqrt{\lambda_j^D}\, t} + B_j e^{-\sqrt{\lambda_j^D}\, t}.$$

Each d_j has polynomial growth, whereas $\lambda_j^D \geq k^2 > 0$. Hence $A_j = B_j = 0$ for every mode, and completeness gives $\Phi \equiv 0$.

If the trace is independent of ζ, subtract the rotational average of Φ. The difference has zero trace and polynomial growth, so it vanishes by the preceding argument. Thus Φ is axially symmetric. □

4.2 The limiting Fowler orbit

We now study the forward asymptotics of the solution V of (4.1). The Hamiltonian on E_μ is

$$\mathcal{H}(f,g) = \frac{d_\sigma}{2}\|g\|^2_{L^2_\mu} - \frac{d_\sigma}{2} Q_0(f,f) + \frac{(n-2\sigma)K_\sigma}{2n|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} |f(\zeta)|^{\frac{2n}{n-2\sigma}}\, \mathrm{d}S(\zeta).$$

For axially symmetric functions this agrees with (2.7). By local regularity, testing (4.1) by $\partial_t V$ gives

$$\partial_t[\mathcal{H}(V(t), \partial_t V(t))] = d_\sigma\Big(\langle \partial_t^2 V, \partial_t V\rangle_\mu - Q_0(V, \partial_t V)\Big) + \frac{K_\sigma}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} w^{\frac{n+2\sigma}{n-2\sigma}} \partial_t w \, \mathrm{d}S(\zeta) = 0.$$

Write $\mathcal{H}(V) = H_\infty$ and set $v_{H_c} = v_c$. We also use $\mathrm{ev}_0(\Phi) = (\Phi(0,\cdot), \partial_t\Phi(0,\cdot))$ for Cauchy pairs in E_μ.

Proposition 4.3. *The conserved energy H_∞ belongs to $[H_c, 0)$, and the forward time translates of V approach the orbit $\{v_{H_\infty}(\cdot + \tau) : \tau \in \mathbb{R}\}$: for every bounded interval $I \subset \mathbb{R}$,*

$$\inf_{\tau \in \mathbb{R}} \Big\| (V, \partial_t V)(t + \cdot) - (v_{H_\infty}, \partial_t v_{H_\infty})(t + \cdot + \tau) \Big\|_{C(I; E_\mu)} \longrightarrow 0 \qquad \text{as } t \to +\infty. \tag{4.9}$$

Moreover, $c_0 \leq v_{H_\infty} \leq C_0$ on the closed cylinder.

Proof. Let $t_j \to \infty$. By (4.2), Lemma 4.1, and its Hölder estimate, a diagonal subsequence satisfies

$$\begin{aligned} V(t_j + \cdot, \cdot) &\longrightarrow V_* && \text{in } C(I \times \overline{\mathbb{S}^n_+}), \\ (V, V_t)(t_j + \cdot) &\longrightarrow (V_*, V_{*,t}) && \text{in } C(I; E_\mu), \qquad I \Subset \mathbb{R}, \\ c_0 \leq V_* &\leq C_0, && \sup_t \|(V_*, V_{*,t})(t)\|_{E_\mu} \leq C. \end{aligned}$$

Passing to the limit in the weak formulation shows that V_* solves (4.1). By (4.3),

$$\|w(t_j+\cdot,\cdot)-\overline{w}(t_j+\cdot)\|_{C(I\times\mathbb{S}^{n-1})}\le C_I e^{-t_j}\longrightarrow 0.$$

Thus the trace of V_* is independent of ζ. By Lemma 4.2, V_* is axially symmetric. Proposition 2.4 then identifies V_* with an element of $\mathcal{S}$, and

$$\mathcal{H}(V_*)=\lim_j \mathcal{H}(V(t_j+\cdot))=H_\infty.$$

Since $\inf V_*\ge c_0>0$, V_* is neither zero nor a bubble. By Theorem 1.1, $H_\infty\in[H_c,0)$ and $V_*=v_{H_\infty}(\cdot+\tau)$ for some τ; the bounds $c_0\le V_*\le C_0$ transfer to v_{H_∞}.

If (4.9) fails, choose $t_j\to\infty$ with the infimum at least $\varepsilon>0$ and take the subsequence above. Taking $s=\tau-t_j$ (modulo the period) in the infimum gives

$$\begin{aligned}\varepsilon&\le \inf_{s\in\mathbb{R}}\|(V,V_t)(t_j+\cdot)-(v_{H_\infty},\partial_t v_{H_\infty})(t_j+\cdot+s)\|_{C(I;E_\mu)}\\&\le\|(V,V_t)(t_j+\cdot)-(V_*,V_{*,t})\|_{C(I;E_\mu)}\\&\to 0,\end{aligned}$$

a contradiction. □

4.3 Convergence to a fixed Fowler model

We next estimate the distance to the limiting orbit and prove convergence of the phase. For the estimates below, write

$$\|f\|_I=\|(f,f_t)\|_{C(I;E_\mu)}.$$

Lemma 4.4. *Let V, H_∞ be as in Proposition 4.3. There are $\tau_\infty\in\mathbb{R}$ and $\delta=\delta(V)>0$ such that, for every bounded interval $I\subset\mathbb{R}$,*

$$\|V(t+\cdot)-v_{H_\infty}(t+\cdot+\tau_\infty)\|_I\le C_I e^{-\delta t} \tag{4.10}$$

for all sufficiently large t, with $C_I=C(V,I)$. If $H_\infty=H_c$, we take $\tau_\infty=0$.

Proof. We adapt the rescaling and phase-correction argument of [26, Section 4, Proposition 5]. For simplicity, write $H=H_\infty$ and $v=v_H$. Choose the comparison time scale and interval by

$$T=\begin{cases}T(H), & H>H_c,\\ 1, & H=H_c,\end{cases}\qquad J=[-2T,2T].$$

Let $d(t)$ be the distance to the limiting orbit, and let $D(t)$ be a nonincreasing envelope including the angular error e^{-t} from (4.3):

$$d(t)=\min_{\tau\in\mathbb{R}/T\mathbb{Z}}\|V(t+\cdot)-v(t+\cdot+\tau)\|_J,\qquad D(t)=\sup_{r\ge t}d(r)+e^{-t}.$$

By Proposition 4.3, $D(t)\to 0$. Choose a minimizing phase $\tau(t)$ for each t, taking $\tau(t)=0$ when $H=H_c$.

Step 1. Phase comparison and growth control. Assume $H > H_c$. For close phase representatives and $|I| \geq T$,

$$\|v(\cdot+\alpha) - v(\cdot+\beta) - (\alpha-\beta)\partial_t v(\cdot+\beta)\|_I \leq C|\alpha-\beta|^2,$$
$$\|\partial_t v(\cdot+\beta)\|_I = \|\partial_t v\|_{[0,T]} > 0.$$

Compactness of the phase circle and the least period give

$$c\,\mathrm{dist}_{\mathbb{R}/T\mathbb{Z}}(\alpha,\beta) \leq \|v(\cdot+\alpha) - v(\cdot+\beta)\|_I \leq C\,\mathrm{dist}_{\mathbb{R}/T\mathbb{Z}}(\alpha,\beta),$$

uniformly in the starting point of I.

For $|r| \leq T$, the overlap $(t+J)\cap(t+r+J)$ contains a full period. By the minimizing property of the phases,

$$\|v(\cdot+\tau(t)) - v(\cdot+\tau(t+r))\|_{(t+J)\cap(t+r+J)} \leq d(t) + d(t+r).$$

The lower translation bound gives

$$\mathrm{dist}_{\mathbb{R}/T\mathbb{Z}}(\tau(t+r),\tau(t)) \leq C(d(t)+d(t+r)) \quad \text{for } |r| \leq T,\ t, t+r \text{ sufficiently large.} \tag{4.11}$$

Iterating this estimate over steps of size at most T yields, for general r,

$$\mathrm{dist}_{\mathbb{R}/T\mathbb{Z}}(\tau(t+r),\tau(t)) \leq C(1+|r|)D(\min\{t,t+r\}).$$

Comparing with the minimizing phase at $t+r$ gives

$$\|V - v(\cdot+\tau(t))\|_{t+[r-T,r+T]} \leq C(1+|r|)D(\min\{t,t+r\}). \tag{4.12}$$

For $H = H_c$, the phase is zero and

$$\|V - v_c\|_{[t+r-1,t+r+1]} \leq D(t+r).$$

Step 2. Contraction and exponential decay. We claim that there are $L > 0$ and $t_* > t_0$ such that

$$D(t+L) \leq \tfrac{1}{2}D(t) \qquad \text{for } t \geq t_*. \tag{4.13}$$

Otherwise, there are $L_j, s_j \to \infty$ such that, with $\varepsilon_j = D(s_j)$,

$$D(s_j+L_j) > \tfrac{1}{2}\varepsilon_j, \qquad e^{-(s_j+L_j)} \leq e^{-L_j}\varepsilon_j.$$

Thus, for large j, $\sup_{r\geq s_j+L_j} d(r) > \varepsilon_j/3$, and we can choose

$$t_j \geq s_j + L_j, \qquad d(t_j) > \varepsilon_j/3.$$

For $H > H_c$, compactness of the phase circle allows us to pass to a subsequence with $t_j + \tau(t_j) \to \tau_*$ modulo T. For $H = H_c$, set $\tau_* = 0$. Let

$$v_j(t,\theta) = v(t_j + t + \tau(t_j), \theta), \qquad \Phi_j(t,\eta) = \frac{V(t_j+t,\eta) - v_j(t,\theta)}{\varepsilon_j}.$$

For every fixed bounded interval, $t_j + I \subset [s_j, \infty)$ for large j. By (4.12) and the bound for the stationary case,

$$\|\Phi_j\|_{[r-T,r+T]} \le C(1+|r|) \quad \text{if } H > H_c, \qquad \|\Phi_j\|_{[r-1,r+1]} \le C \quad \text{if } H = H_c,$$

for fixed r and large j. Also

$$\|\Phi_j\|_J = d(t_j)/\varepsilon_j > 1/3.$$

Let b_j be the secant coefficient of $w(t_j + \cdot, \zeta)$ and $v_j(\cdot, \pi/2)$. The equation for Φ_j is (4.6), and

$$\|b_j\|_\infty + \|\partial_t b_j\|_\infty \le C.$$

By the Hölder estimates and the convergence to the orbit,

$$\begin{aligned} w(t_j + \cdot, \cdot) &\to v(\cdot + \tau_*, \pi/2) && \text{in } C(I \times \mathbb{S}^{n-1}), \\ b_j &\to b_{v(\cdot+\tau_*)} && \text{locally uniformly.} \end{aligned}$$

The compactness in Lemma 4.1 gives a diagonal subsequence with

$$(\Phi_j, \Phi_{j,t}) \to (\Phi, \Phi_t) \quad \text{in } C(I; E_\mu), \qquad I \Subset \mathbb{R}.$$

Passing to the limit in (4.6) shows that Φ satisfies

$$\partial_t^2 \Phi + \Delta_\mu \Phi - p^2 \Phi = 0, \qquad \partial_\nu^\sigma \Phi = b_{v(\cdot+\tau_*)} \Phi.$$

Moreover,

$$\|\Phi\|_J \ge 1/3, \qquad \|(\Phi, \Phi_t)(t)\|_{E_\mu} \le \begin{cases} C(1+|t|), & H > H_c, \\ C, & H = H_c. \end{cases}$$

By (4.3),

$$\left\| \Phi_j(t,\zeta) - \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \Phi_j(t,\xi)\, \mathrm{d}S(\xi) \right\|_{L^\infty(I \times \mathbb{S}^{n-1})} \le C_I e^{-t_j}/\varepsilon_j \le C_I e^{-L_j} \to 0.$$

Thus Φ is axially symmetric by Lemma 4.2. The polynomial bound and $\varpi_0(t) = e^{-2k|t|}$ give

$$\Phi \in \mathcal{C}_{v(\cdot+\tau_*)}.$$

The local estimates and (4.2) give

$$\|\mathrm{ev}_0(\Phi_j)\|_{E_\mu} + \|\Phi_j(0)\|_{L^\infty(\mathbb{S}^{n-1})} \le C, \qquad \sup_{c_0 \le s \le C_0} \left| \frac{\mathrm{d}^3}{\mathrm{d}s^3} s^{\frac{2n}{n-2\sigma}} \right| + \int_{\mathbb{S}^{n-1}} |\Phi_j(0,\zeta)|^3 \, \mathrm{d}S \le C.$$

Since $\mathcal{H}(V) = \mathcal{H}(v_j) = H$, Taylor's formula gives

$$\begin{aligned} 0 &= \mathcal{H}(\mathrm{ev}_0(v_j) + \varepsilon_j \mathrm{ev}_0(\Phi_j)) - \mathcal{H}(\mathrm{ev}_0(v_j)) \\ &= \varepsilon_j D\mathcal{H}(\mathrm{ev}_0(v_j))[\mathrm{ev}_0(\Phi_j)] + \frac{\varepsilon_j^2}{2} D^2\mathcal{H}(\mathrm{ev}_0(v_j))[\mathrm{ev}_0(\Phi_j), \mathrm{ev}_0(\Phi_j)] + O(\varepsilon_j^3). \end{aligned}$$

For $H > H_c$, division by ε_j and passage to the limit give

$$D\mathcal{H}(\mathrm{ev}_0(v(\cdot+\tau_*)))[\mathrm{ev}_0(\Phi)] = 0, \qquad \Phi \in \mathrm{span}\{\partial_t v(\cdot+\tau_*)\}$$

by (3.4). Write $\Phi = a\partial_t v(\cdot+\tau_*)$. The minimizing property of $\tau(t_j)$ gives

$$\frac{d(t_j)}{\varepsilon_j} \le \left\| \Phi_j - \frac{v_j(\cdot + a\varepsilon_j) - v_j}{\varepsilon_j} \right\|_J \longrightarrow 0.$$

Here the limit follows from Taylor's formula for translations of v and the convergence of Φ_j, contradicting the lower bound $d(t_j)/\varepsilon_j > 1/3$.

For $H = H_c$, the angular equation gives

$$D\mathcal{H}(\mathrm{ev}_0(v_c))[\mathrm{ev}_0(\Phi_j)] = -d_\sigma Q_0(v_c, \Phi_j(0)) + \frac{K_\sigma}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} w_c^{\frac{n+2\sigma}{n-2\sigma}}\Phi_j(0,\zeta)\,\mathrm{d}S = 0.$$

Divide the energy identity by ε_j^2. Since $\Phi \in \mathcal{C}_{v_c}$, (3.12) gives

$$0 = \tfrac{1}{2}D^2\mathcal{H}(\mathrm{ev}_0(v_c))[\mathrm{ev}_0(\Phi),\mathrm{ev}_0(\Phi)] = \frac{d_\sigma}{2}\Big(\|\Phi_t(0)\|_{L^2_\omega}^2 + \kappa_c^2\|\Phi(0)\|_{L^2_\omega}^2\Big).$$

Thus $\Phi(0) = \Phi_t(0) = 0$, hence $q_\Phi(0) = 0$. By (2.23), $\Phi = 0$, again a contradiction. This proves (4.13).

For $t_* + jL \le t < t_* + (j+1)L$, iteration and monotonicity give $D(t) \le 2^{-j}D(t_*)$. Hence

$$D(t) \le Ce^{-\delta t}, \qquad \delta = \frac{\log 2}{L} > 0, \quad t \ge t_*. \tag{4.14}$$

Step 3. Convergence of the phase. By the minimizing property and (4.14),

$$\|V(t+\cdot) - v(t+\cdot+\tau(t))\|_J = d(t) \le D(t) \le Ce^{-\delta t}.$$

For $H > H_c$, let $t_j = t_* + jT$. Equations (4.11) and (4.14) give

$$\sum_{j\ge N} \mathrm{dist}_{\mathbb{R}/T\mathbb{Z}}(\tau(t_{j+1}),\tau(t_j)) \le C\sum_{j\ge N} e^{-\delta t_j} \le Ce^{-\delta t_N}.$$

Thus $(\tau(t_j))$ is Cauchy in $\mathbb{R}/T\mathbb{Z}$; denote its limit by τ_∞. For $t_j \le t < t_{j+1}$,

$$\begin{aligned}\mathrm{dist}_{\mathbb{R}/T\mathbb{Z}}(\tau(t),\tau_\infty) &\le \mathrm{dist}_{\mathbb{R}/T\mathbb{Z}}(\tau(t),\tau(t_j)) + \sum_{i\ge j}\mathrm{dist}_{\mathbb{R}/T\mathbb{Z}}(\tau(t_{i+1}),\tau(t_i))\\ &\le Ce^{-\delta t_j}\\ &\le Ce^{\delta T}e^{-\delta t}.\end{aligned}$$

Choose a real representative of τ_∞. By the exponential error bound and the Lipschitz continuity of translations of v,

$$\|V(t+\cdot) - v(t+\cdot+\tau_\infty)\|_{[-2T,2T]} \le Ce^{-\delta t}.$$

For $H = H_c$, this holds with $\tau_\infty = 0$.

For a bounded interval $I \subset \mathbb{R}$, choose finitely many $r_1,\dots,r_N$ such that $I \subset \bigcup_{k=1}^N (r_k + [-2T,2T])$. Applying the preceding estimate at $t + r_k$ gives (4.10) for large t. □

Proof of Theorem 1.5. Let H_∞ be given by Proposition 4.3, set $H = H_\infty$, $v = v_H$, and take τ_∞, δ from Lemma 4.4. By (4.4), Lemma 4.1 and (4.10),

$$\|V - v(\cdot + \tau_\infty)\|_{L^\infty([t-1,t+1]\times\overline{\mathbb{S}^n_+})} \le C\|V - v(\cdot + \tau_\infty)\|_{L^2((t-2,t+2);L^2_\mu)} \le Ce^{-\delta t}.$$

Set

$$U_\infty(X) = K_\sigma^{\frac{n-2\sigma}{4\sigma}} |X|^{-\frac{n-2\sigma}{2}} v(-\log|X| + \tau_\infty, \theta), \qquad X = |X|(\sin\theta\,\zeta, \cos\theta).$$

Since $v \ge c_0$ by Proposition 4.3,

$$\sup_{\substack{|X|=\rho \\ X\in\overline{\mathbb{R}^{n+1}_+}}} \left| \frac{U(X)}{U_\infty(X)} - 1 \right| \le C\rho^\delta, \qquad \rho = e^{-t}.$$

For uniqueness, set $v_\infty = v(\cdot + \tau_\infty)$. Let $\widetilde{U}_\infty$ be another extension with the same asymptotics, and let $\widetilde{v}$ be its cylindrical profile. The local estimates of [6] and the bounds $c_0 \le v, \widetilde{v} \le C_0$ give

$$\|v_\infty - \widetilde{v}\|_{L^\infty([t-2,t+2]\times\overline{\mathbb{S}^n_+})} \longrightarrow 0.$$

Lemma 4.1 then gives

$$\|(v_\infty - \widetilde{v}, \partial_t v_\infty - \partial_t \widetilde{v})(t)\|_{E_\mu} \longrightarrow 0,$$

hence $\mathcal{H}(v_\infty) = \mathcal{H}(\widetilde{v})$ by continuity of $\mathcal{H}$. By Theorem 1.1, v_∞ and $\widetilde{v}$ lie on the same orbit. If $H = H_c$, both equal v_c. If $H > H_c$, periodicity and the L^∞-convergence give $v_\infty = \widetilde{v}$. Thus $\widetilde{U}_\infty = U_\infty$. □

Corollary 4.5. *Let $u \ge 0$ on $\mathbb{R}^n$ satisfy* (1.2) *and $(-\Delta)^\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}}$ in $B'_R \setminus \{0\}$, with $u \in C^2_{\mathrm{loc}}(B'_R \setminus \{0\})$ and $u > 0$ in $B'_R \setminus \{0\}$. If zero is nonremovable, there are a unique $H_\infty \in [H_c, 0)$ and a unique global solution*

$$u_\infty(x) = K_\sigma^{\frac{n-2\sigma}{4\sigma}} |x|^{-\frac{n-2\sigma}{2}} w_{H_\infty}(-\log|x| + \tau_\infty)$$

such that

$$u(x) = u_\infty(x)\Big(1 + O(|x|^\delta)\Big) \qquad \text{as } x \to 0,$$

for some $\delta > 0$.

Proof. Let U be the Poisson extension of u. By (1.2), the local extension estimates and the Dirichlet-to-Neumann identity [7], U solves (1.8) and has finite local energy away from zero. Theorem 1.5 gives a unique Poisson extension U_∞ of a global solution such that

$$U(X) = U_\infty(X)\Big(1 + O(|X|^\delta)\Big) \qquad (X \to 0).$$

Taking the boundary trace proves the corollary. □

A Auxiliary estimates for the cylindrical operator

A.1 Symbol and inverse kernel

The symbol. By [12, Theorem 1.1],

$$\Theta_\sigma(\xi) = 2^{2\sigma} \frac{\left|\Gamma\left(\frac{n+2\sigma}{4} + \frac{i\xi}{2}\right)\right|^2}{\left|\Gamma\left(\frac{n-2\sigma}{4} + \frac{i\xi}{2}\right)\right|^2}. \tag{A.1}$$

Recall that $p = (n - 2\sigma)/2$. Euler's product formula for Γ gives

$$\frac{\partial_\xi \Theta_\sigma(\xi)}{2\xi\Theta_\sigma(\xi)} = \sum_{j=0}^{\infty} \left(\frac{1}{(p+2j)^2 + \xi^2} - \frac{1}{(p+2\sigma+2j)^2+\xi^2} \right) > 0 \qquad (\xi > 0),$$

hence $\xi \mapsto \Theta_\sigma(\xi)$ is strictly increasing on $(0,\infty)$. Stirling's formula and its differentiated expansions give (2.3); see [31, Section 5.11]. These bounds imply the Sobolev isomorphisms in Section 2.

The inverse kernel. Applying $(-\Delta)^{-\sigma}$ to radial functions and using (1.4),

$$\mathcal{P}_\sigma^{-1} f = G_\sigma * f, \tag{A.2}$$

where

$$G_\sigma(t) = \frac{\Gamma(p)}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{S}^{n-1}} [2(\cosh t - \zeta_1)]^{-p} \, \mathrm{d}S(\zeta) = g_\sigma e^{-p|t|} \, {}_2F_1(p, 1-\sigma; p+\sigma; e^{-2|t|}),$$
$$g_\sigma = \frac{2^{1-2\sigma}\Gamma(p)}{\Gamma(\sigma)\Gamma(p+\sigma)}.$$

The powers of $r = e^{-t}$ and $\rho = e^{-\tau}$ cancel because $p\frac{n+2\sigma}{n-2\sigma} = p + 2\sigma$; the spherical hypergeometric integral formulas in [31, Sections 15.6 and 15.8] give the second formula. Near zero, $G_\sigma(t) \asymp |t|^{2\sigma-1}$ for $0 < \sigma < 1/2$, with a logarithmic singularity at $\sigma = 1/2$, and G_σ is bounded for $\sigma > 1/2$; at infinity $G_\sigma(t) \asymp e^{-p|t|}$. Hence

$$G_\sigma > 0, \qquad G_\sigma \in L^1(\mathbb{R}), \qquad \widehat{G}_\sigma = \Theta_\sigma^{-1}, \qquad \int_{\mathbb{R}} G_\sigma = \Theta_\sigma(0)^{-1}.$$

Multiplication by Θ_σ^{-1} in $\mathcal{S}'(\mathbb{R})$ gives (2.5); $G_\sigma \in L^1(\mathbb{R})$ gives continuity of the convolution.

A moment identity. To prove the bound in Proposition 2.2, set

$$I_\sigma = \int_{\mathbb{R}} G_\sigma(t) e^{-(p+2\sigma)|t|} \, \mathrm{d}t.$$

With $y = e^{-2t}$, termwise integration and Gauss's identity [31, Section 15.4] give

$$\begin{aligned} I_\sigma &= g_\sigma \int_0^1 y^{p+\sigma-1} \, {}_2F_1(p, 1-\sigma; p+\sigma; y) \, \mathrm{d}y \\ &= \frac{g_\sigma}{p+\sigma} \, {}_2F_1(p, 1-\sigma; p+\sigma+1; 1) \\ &= g_\sigma \frac{\Gamma(p+\sigma)\Gamma(2\sigma)}{\Gamma(\sigma+1)\Gamma(p+2\sigma)}. \end{aligned}$$

By the duplication formula [31, Section 5.5],

$$K_\sigma I_\sigma = \frac{2^{1-2\sigma}\Gamma(2\sigma)}{\Gamma(\sigma)\Gamma(\sigma+1)} = \frac{\Gamma(\sigma+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\sigma+1)} = w_*^{-\frac{4\sigma}{n-2\sigma}}. \tag{A.3}$$

Combining (A.3) with the logarithmic derivative estimate in Proposition 2.2 gives $w(t) \le w_*$.

A.2 Uniform estimates for complex periods

The analyticity of $T \mapsto \mathcal{P}_{\sigma,T}$ and of the periodic extension map in Lemma 3.5 relies on the following uniform bounds for the symbol and the angular modes in a complex neighborhood of a fixed period.

Lemma A.1. *Use the complexification $E_\omega^{\mathbb{C}}$. Fix $T_* > 0$. There are $\rho > 0$ and C such that, for every $j \in \mathbb{Z}$, the maps*

$$T \longmapsto \widetilde{\Theta}_\sigma(2\pi j/T), \qquad T \longmapsto \Big(g_{2\pi j/T},\, i(2\pi j/T) g_{2\pi j/T}\Big) \in E_\omega^{\mathbb{C}}$$

are holomorphic for $|T - T_*| < \rho$, and

$$\sup_{|T-T_*|<\rho} |\widetilde{\Theta}_\sigma(2\pi j/T)| \le C(1+j^2)^{\sigma},$$
$$\sup_{|T-T_*|<\rho} \Big\| \Big(g_{2\pi j/T},\, i(2\pi j/T) g_{2\pi j/T}\Big) \Big\|_{E_\omega^{\mathbb{C}}} \le C(1+j^2)^{1/2}.$$

Here $\widetilde{\Theta}_\sigma$ is the meromorphic continuation of Θ_σ to $\mathbb{C}$, and g_ξ is given by (3.7).

Proof. Write $\xi_j(T) = 2\pi j/T$ and $\langle j \rangle = (1+j^2)^{1/2}$. Define

$$B_j(T) = \Big(g_{\xi_j(T)},\, i\xi_j(T) g_{\xi_j(T)}\Big).$$

By (2.2), the meromorphic continuation of Θ_σ is given by

$$\widetilde{\Theta}_\sigma(z) = 2^{2\sigma} \frac{\Gamma(\frac{n+2\sigma}{4} + \frac{iz}{2})\Gamma(\frac{n+2\sigma}{4} - \frac{iz}{2})}{\Gamma(\frac{n-2\sigma}{4} + \frac{iz}{2})\Gamma(\frac{n-2\sigma}{4} - \frac{iz}{2})}.$$

For $0 < \sigma < 1$, its poles are the simple poles $z = \pm i(k+2r)$, $r = 0, 1, \ldots$.

Fix $T_* > 0$ and choose $0 < \rho < T_*/4$. For $|T - T_*| \le \rho$ and $j \neq 0$,

$$\operatorname{Re} T > 0, \qquad \operatorname{Re}(\xi_j(T)^2) \ge c|\xi_j(T)|^2, \qquad |\xi_j(T)| \asymp |j|.$$

The gamma factors are holomorphic on $\mathbb{C} \setminus i\mathbb{R}$, while $\xi_j(T) \in \mathbb{C} \setminus i\mathbb{R}$. Hence $T \mapsto \widetilde{\Theta}_\sigma(\xi_j(T))$ is holomorphic for $j \neq 0$; for $j = 0$, $\xi_0(T) = 0$ and $\widetilde{\Theta}_\sigma(0) = \Theta_\sigma(0)$ is constant.

By Stirling's formula, uniformly for $|j|$ large,

$$|\widetilde{\Theta}_\sigma(\xi_j(T))| \le C\langle j \rangle^{2\sigma}, \qquad |T - T_*| \le \rho.$$

Compactness gives the same bound for the remaining finitely many j. Thus

$$\sup_{|T-T_*|\le\rho}\sup_{j\in\mathbb{Z}}\langle j\rangle^{-2\sigma}|\widetilde{\Theta}_\sigma(\xi_j(T))|\le C.$$

For $j\neq 0$ and $|T-T_*|\le\rho$, set $\xi=\xi_j(T)$ and write $g_\xi=g_0+h_\xi$ with $h_\xi(\pi/2)=0$. By (2.16) and $\mathrm{Re}(\xi^2)\ge c|\xi|^2$, the zero-trace equation is coercive; it has a unique solution, and a local Neumann series gives holomorphic dependence on T in H^1_ω. Multiplying the equation for h_ξ by $\overline{h_\xi}$, integrating and taking real parts gives

$$\begin{aligned}(k^2+c|\xi|^2)\|h_\xi\|^2_{L^2_\omega}&\le\|h_\xi\|^2_{H^1_\omega}+\mathrm{Re}(\xi^2)\|h_\xi\|^2_{L^2_\omega}\\&=-\mathrm{Re}\biggl(\xi^2\int_0^{\pi/2}g_0\overline{h_\xi}\,\omega\,\mathrm{d}\theta\biggr)\\&\le|\xi|^2\|g_0\|_{L^2_\omega}\|h_\xi\|_{L^2_\omega}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|h_\xi\|_{L^2_\omega}&\le\frac{C|\xi|^2}{k^2+|\xi|^2},\\\|g_\xi\|_{L^2_\omega}&\le\|g_0\|_{L^2_\omega}+\|h_\xi\|_{L^2_\omega}\le C,\\\|g_\xi\|_{H^1_\omega}&\le\|g_0\|_{H^1_\omega}+C|\xi|\le C(1+|\xi|).\end{aligned}$$

For $j=0$, the factor $g_{\xi_0}=g_0$ is independent of T and satisfies the same bounds. Since $|\xi_j(T)|\asymp|j|$, we obtain

$$\sup_{|T-T_*|\le\rho}\|B_j(T)\|_{E^{\mathbb{C}}_\omega}\le C\langle j\rangle.$$

□

B Estimates for the weighted linear theory

Let

$$N=\frac{n}{2}$$

and

$$C_\sigma=\frac{n+2\sigma}{n-2\sigma}K_\sigma=d_\sigma b_\sigma=\frac{\Gamma(N+1+\sigma)}{\Gamma(N+1-\sigma)}.$$

We write $F_{n,\sigma}$ for (2.15) when the dependence on the dimension must be explicit.

Lemma B.1. *Let $n\ge 3$ and $\rho>0$. Then*

$$F_{n,\sigma}(-\rho)\ge\bigl(\rho+N^2-2N+3/5\bigr)^\sigma$$

whenever $\rho+N^2-2N+3/5>0$. In particular,

$$F_{n,\sigma}(-\rho)\ge(\rho-3/20)^\sigma,\qquad F_{6,\sigma}(-\rho)\ge(\rho+18/5)^\sigma$$

whenever the bases are positive.

Proof. We compare $f_{-\rho}$ with an explicit Bessel-type solution. Recall that $f_{-\rho}(\pi/2) = 1$. In the inward coordinate $x = \pi/2 - \theta$, set $Y(x) = \omega(\pi/2 - x)^{1/2} f_{-\rho}(\pi/2 - x)$. Direct differentiation gives

$$-\partial_x^2 Y + V_{n,\sigma}(x) Y = -\rho Y, \qquad V_{n,\sigma}(x) = \frac{(n-1)(n-3)}{4\cos^2 x} + \frac{\sigma^2 - 1/4}{\sin^2 x}.$$

The partial-fraction formula [31, Section 4.22] gives

$$\csc^2 x - x^{-2} = \sum_{j \geq 1} \left(\frac{1}{(j\pi - x)^2} + \frac{1}{(j\pi + x)^2} \right).$$

Each bracket is positive and strictly increasing for $0 < x < \pi/2$. Hence, using $\pi^2 < 10$,

$$0 < \csc^2 x - x^{-2} < 1 - \frac{4}{\pi^2} < \frac{3}{5}.$$

Writing

$$V_{n,\sigma} = \frac{(n-1)(n-3)}{4\cos^2 x} + \frac{\sigma^2 - 1/4}{x^2} + \left(\sigma^2 - \frac{1}{4}\right)(\csc^2 x - x^{-2}),$$

and using

$$\frac{(n-1)(n-3)}{4\cos^2 x} \geq N^2 - 2N + \frac{3}{4},$$
$$\left(\sigma^2 - \frac{1}{4}\right)(\csc^2 x - x^{-2}) \geq -\frac{3}{20},$$

we obtain

$$V_{n,\sigma}(x) \geq \frac{\sigma^2 - 1/4}{x^2} + N^2 - 2N + \frac{3}{5}.$$

Set $c_0 = N^2 - 2N + 3/5$ and assume $\rho_0 = \rho + c_0 > 0$. Define

$$B_{\rho_0}(x) = \frac{x^{1/2-\sigma}}{\Gamma(\sigma)} \int_0^\infty r^{\sigma-1} \exp\left(-r - \frac{\rho_0 x^2}{4r}\right) \mathrm{d}r.$$

Differentiation and integration by parts give

$$-\partial_x^2 B_{\rho_0} + (\sigma^2 - 1/4) x^{-2} B_{\rho_0} = -\rho_0 B_{\rho_0}.$$

As $x \downarrow 0$, $B_{\rho_0}(x) \sim x^{1/2-\sigma}$. If $h_{\rho_0} = x^{\sigma-1/2} B_{\rho_0}$, the change of variables $r = x^2 q$ in its derivative gives

$$-\lim_{x \downarrow 0} x^{1-2\sigma} \partial_x h_{\rho_0}(x) = \frac{\rho_0}{2\Gamma(\sigma)} \int_0^\infty q^{\sigma-2} e^{-\rho_0/(4q)} \,\mathrm{d}q = \frac{\rho_0^\sigma}{d_\sigma}.$$

Positivity of $f_{-\rho}$, hence of Y, follows from the angular maximum principle. In the inward coordinate, the boundary condition for $f_{-\rho}$ is

$$\lim_{x \downarrow 0} x^{1-2\sigma} \partial_\theta f_{-\rho}(\pi/2 - x) = F_{n,\sigma}(-\rho)/d_\sigma.$$

Since $\omega(\pi/2-x)^{1/2} = x^{1/2-\sigma}(1+O(x^2))$, the Wronskian $W = (\partial_x Y)B_{\rho_0} - Y\partial_x B_{\rho_0}$ satisfies

$$W(0) = -\frac{F_{n,\sigma}(-\rho) - \rho_0^\sigma}{d_\sigma}, \qquad \partial_x W = \Big(V_{n,\sigma} - \frac{\sigma^2 - 1/4}{x^2} - c_0\Big) Y B_{\rho_0} \geq 0.$$

At the pole $x = \pi/2$, one has $Y = O((\pi/2-x)^{(n-1)/2})$, so $\partial_x Y(\pi/2) \leq 0$, while $B_{\rho_0}(\pi/2) > 0$. Hence $W(\pi/2) \leq 0$. Since $\partial_x W \geq 0$, we obtain $W(0) \leq W(\pi/2) \leq 0$, which is $F_{n,\sigma}(-\rho) \geq \rho_0^\sigma$. □

Lemma B.2. *Let* $0 \leq b \leq b_*$.

(i) *If* $\lambda_0(b) = -\rho < 0$, *then*

$$\rho < \begin{cases} 4(N+1)^2 + 3/20, & n \geq 3, \\ 12, & n = 2. \end{cases} \tag{B.1}$$

(ii) *If* $n = 2$ *and* $0 < \sigma \leq 1/2$, *then* $\lambda_1(b) > 49/16$.

Proof. The Robin condition $F_\sigma(\lambda_0(b)) = d_\sigma b$ (see Lemma 2.5) and the assumption give

$$F_\sigma(-\rho) = d_\sigma b \leq d_\sigma b_*.$$

We first bound $d_\sigma b_*$. Let $\psi(x) = \frac{\partial_x \Gamma(x)}{\Gamma(x)}$. Since $\partial_x^2 \psi < 0$,

$$\log C_\sigma = \int_{-\sigma}^{\sigma} \psi(N+1+r)\,\mathrm{d}r \leq 2\sigma\psi(N+1).$$

Differentiating the integral for Γ and using Jensen's inequality,

$$\psi(x) = \frac{1}{\Gamma(x)} \int_0^\infty r^{x-1} e^{-r} \log r\,\mathrm{d}r < \log\left(\frac{1}{\Gamma(x)} \int_0^\infty r^x e^{-r}\,\mathrm{d}r\right) = \log x.$$

Moreover,

$$\partial_\sigma^2 \log \frac{\sqrt{\pi}\,\Gamma(\sigma+1)}{\Gamma(\sigma+\frac{1}{2})} = \partial_\sigma \psi(\sigma+1) - \partial_\sigma \psi(\sigma+\tfrac{1}{2}) < 0.$$

The gamma quotient equals 1 at $\sigma = 0$, and its logarithmic derivative there is $\log 4$. Since $d_\sigma b_* = C_\sigma \sqrt{\pi}\,\Gamma(\sigma+1)/\Gamma(\sigma+\frac{1}{2})$, we obtain

$$\begin{aligned} C_\sigma^{1/\sigma} &\leq e^{2\psi(N+1)} < (N+1)^2, \\ \left(\frac{\sqrt{\pi}\,\Gamma(\sigma+1)}{\Gamma(\sigma+\frac{1}{2})}\right)^{1/\sigma} &\leq 4, \\ (d_\sigma b_*)^{1/\sigma} &< 4(N+1)^2. \end{aligned} \tag{B.2}$$

For $n \geq 3$ and $\rho > 3/20$, Lemma B.1 and (B.2) give

$$(\rho - 3/20)^\sigma \leq F_\sigma(-\rho) \leq d_\sigma b_* < \Big(4(N+1)^2\Big)^\sigma,$$

which proves (B.1). The case $0 < \rho \le 3/20$ is immediate.

For $n = 2$, if $\lambda_0(b) = -\rho < 0$ then we claim that

$$0 < \rho = -\lambda_0 < 12 \qquad \text{for } 0 < \sigma < 1. \tag{B.3}$$

Indeed, the dimension-six comparison in Lemma B.1 and the recurrence formula for Γ give

$$F_{6,\sigma}(-\rho) \ge (\rho + 18/5)^{\sigma}, \qquad F_{6,\sigma}(-\rho) = F_{2,\sigma}(-\rho)\frac{\rho + (1+\sigma)^2}{\rho + (1-\sigma)^2}.$$

For $\rho \ge 4/3$, the function $g(x) = 2x/(\rho + x^2)$ is concave on $[0, 2]$, since $\partial_x^2 g(x) = 4x(x^2 - 3\rho)/(\rho + x^2)^3 \le 0$. Jensen's inequality on $[1-\sigma, 1+\sigma]$ gives

$$\log \frac{\rho + (1+\sigma)^2}{\rho + (1-\sigma)^2} = \int_{1-\sigma}^{1+\sigma} g(x)\,\mathrm{d}x \le \frac{4\sigma}{\rho + 1},$$

and hence

$$F_{2,\sigma}(-\rho)^{1/\sigma} \ge (\rho + 18/5)e^{-4/(\rho+1)}. \tag{B.4}$$

The preceding bounds for C_σ and the gamma quotient give

$$(d_\sigma b_*)^{1/\sigma} \le 4e^{2\psi(2)} = 4e^{2-2\gamma_E} < 4e < 11,$$

where γ_E is Euler's constant, $\gamma_E > 1/2$, and $e < 11/4$. At $\rho = 12$,

$$F_{2,\sigma}(-12)^{1/\sigma} \ge \frac{78}{5}e^{-4/13} > 11.$$

Since $\Lambda = -F'_{2,\sigma}/F_{2,\sigma} > 0$,

$$-\partial_\lambda F_{2,\sigma}(-\rho) = F_{2,\sigma}(-\rho)\Lambda(-\rho) > 0.$$

Together with $d_\sigma b \le d_\sigma b_*$, the strict comparison at $\rho = 12$ proves (B.3).

Second eigenvalue. For $n = 2$ and $0 < \sigma \le 1/2$, it suffices to show

$$F_\sigma((7/4)^2) > d_\sigma b_*.$$

Let $t = 3/4$, $x = 11/8$, and $y = 5/8$. The recurrence formula for Γ gives

$$\frac{F_\sigma((1+t)^2)}{4^\sigma} = \frac{t+\sigma}{t-\sigma}\frac{\Gamma(x+\sigma/2)}{\Gamma(x-\sigma/2)}\frac{\Gamma(y+\sigma/2)}{\Gamma(y-\sigma/2)}.$$

The logarithm is analytic for $|\sigma| < t$ and has only odd powers. Its linear coefficient is $2/t + \psi(x) + \psi(y)$. At every odd degree $\ell \ge 3$, the coefficient is

$$\frac{2}{\ell}\left[t^{-\ell} - \sum_{j\ge 0}(2x+2j)^{-\ell} - \sum_{j\ge 0}(2y+2j)^{-\ell}\right] > 0.$$

Indeed, for $\ell \geq 3$,

$$\left(\frac{t}{2x+2j}\right)^{\ell} \leq \left(\frac{t}{2x+2j}\right)^{3}, \qquad \left(\frac{t}{2y+2j}\right)^{\ell} \leq \left(\frac{t}{2y+2j}\right)^{3}.$$

Using $\sum_{j\geq 0}(q+2j)^{-3} \leq q^{-3} + 1/(4q^2)$ for $q > 0$ gives

$$\left(\frac{t}{2x}\right)^{3} + \frac{t^3}{4(2x)^2} + \left(\frac{t}{2y}\right)^{3} + \frac{t^3}{4(2y)^2} < \frac{1}{3}.$$

For the linear coefficient, the digamma series gives

$$\frac{2}{t} + \psi(x) + \psi(y) - 2\psi(2) = \frac{2}{t} - \sum_{j\geq 0} a_j, \qquad a_u = \frac{1}{u+x} + \frac{1}{u+y} - \frac{2}{u+2}.$$

The function a_u is positive and decreasing on $u \geq 0$, since $x, y < 2$. Hence

$$\sum_{j\geq 2} a_j \leq \int_1^{\infty} a_u \, \mathrm{d}u = \log \frac{576}{247} < \frac{6}{7},$$

and therefore

$$\frac{2}{t} + \psi(x) + \psi(y) - 2\psi(2) > \frac{13}{3} - \frac{8}{11} - \frac{8}{5} - \frac{8}{19} - \frac{8}{13} - \frac{6}{7} = \frac{32089}{285285} > 0.$$

Together with $\log C_\sigma \leq 2\sigma\psi(2)$ and (B.2), these coefficient bounds give

$$\log \frac{F_\sigma((7/4)^2)}{4^\sigma} > 2\sigma\psi(2) \geq \log \frac{d_\sigma b_*}{4^\sigma}.$$

The min–max principle gives $k^2 < \lambda_1(b) \leq (p+2)^2$. Green's identity gives

$$\partial_\lambda F_\sigma(\lambda) = -d_\sigma \|f_\lambda\|^2_{L^2_\omega} < 0 \qquad \text{for } k^2 < \lambda < (p+2)^2.$$

Since $k < 7/4 < p+2$ and $F_\sigma(\lambda_1(b)) = d_\sigma b$, the comparison $F_\sigma((7/4)^2) > d_\sigma b_*$ proves $\lambda_1(b) > 49/16$. For $b = 0$, $\lambda_1(0) = (p+2)^2$. □

We verify the uniform scalar bounds used in Lemma 2.8. For $0 \leq b \leq b_*$, write $\lambda = \lambda_0(b)$ and use ℓ, a from (2.25). Allow any real δ satisfying

$$\delta = 0 \quad \text{if } \lambda \geq -k^2, \qquad |\delta| \leq \frac{\sigma}{2(-\lambda)\Lambda(\lambda)} \quad \text{if } \lambda < -k^2.$$

We retain $\alpha = (\delta^2 + a^2)^{1/2}$. Fix $\beta > k$, with $\beta = 7/4$ when $n = 2$ and $\sigma < 1/2$. Let $R \geq 0$ satisfy the coupling bounds

$$\begin{aligned} R &\leq \frac{4\sigma^2(\ell+k)^2}{\Lambda(\lambda)^2 \ell (k^2-\lambda)^3}, \\ R &\leq \frac{4\sigma^2}{\Lambda(\lambda)^2} \frac{\beta^2 + \ell^2 + 2k\ell}{\ell(k^2-\lambda)(\beta^2-\lambda)^2} \quad \text{if } n = 2,\ \sigma < 1/2,\ \lambda < -k^2. \end{aligned} \tag{B.5}$$

At $\lambda = p^2$, these bounds are understood by continuity.

Lemma B.3. *Under these assumptions, there are constants $\delta_* > 0$ and $0 < \vartheta_* < 1$, depending only on n, σ, β, such that*

$$\min\{k - \alpha, \beta - k\} \geq \delta_*, \qquad \frac{(k+a)R}{2(k^2 - \alpha^2)} \leq \vartheta_*^2.$$

Proof. For $\alpha < k$, abbreviate the normalized coupling ratio by

$$\vartheta^2 = \frac{(k+a)R}{2(k^2 - \alpha^2)}.$$

We verify the lower bounds for Λ used when $\lambda < -k^2$ at the end of the proof.
Case $-k^2 \leq \lambda \leq p^2$. Here $\ell = k$, so $\delta = 0$ and

$$\alpha = a = \frac{k^2 + \lambda}{2k} \geq 0, \qquad k - \alpha = \frac{k^2 - \lambda}{2k}.$$

Writing $\delta_0 = p^2 - \lambda$, the $j = 0$ term in the series for Λ gives $\Lambda(\lambda)/\sigma \geq 2n/[\delta_0(k^2 - \lambda)]$ for $\lambda < p^2$. Substitution of the bound for R into the definition of ϑ cancels $k + a$ and gives

$$\vartheta^2 \leq \frac{4k^2\delta_0^2}{n^2(k^2 - \lambda)^2} \leq \left(\frac{n^2/4 + \sigma^2}{(n/2)(n/2 + \sigma)}\right)^2 < 1.$$

Here $\delta_0 \leq p^2 + k^2$, and the last strict inequality is $\sigma < n/2$. Also

$$k - \alpha \geq \frac{k^2 - p^2}{2k} = \frac{n\sigma}{k} > 0.$$

Case $\lambda = -\rho < -k^2$. If $n \geq 3$, or if $n = 2$ and $\sigma \geq \frac{1}{2}$, we first claim that

$$\frac{k\rho}{\sigma}\Lambda(-\rho) \geq \frac{5}{4}. \tag{B.6}$$

Set $r = \sqrt{\rho}/k \geq 1$. Then $\ell = \sqrt{\rho}$, $a = 0$, and $\alpha \leq \sigma/[2\rho\Lambda(-\rho)] \leq 2k/5$. The bound for R and the definition of ϑ imply

$$\vartheta^2 \leq \frac{G(r)}{k^2\rho^2\Lambda(-\rho)^2/\sigma^2 - 1/4}, \qquad G(r) = \frac{2r^3(r+1)^2}{(r^2+1)^3}.$$

Since $(r+1)^2 \leq 2(r^2+1)$ and $4r^3/(1+r^2)^2$ has its maximum at $r = \sqrt{3}$,

$$G(r) \leq \frac{4r^3}{(1+r^2)^2} \leq \frac{3\sqrt{3}}{4}.$$

Consequently

$$\vartheta^2 \leq \frac{4\sqrt{3}}{7} < 1, \qquad k - \alpha \geq \frac{3k}{5}.$$

If $n = 2$ and $0 < \sigma < 1/2$, set $\rho = -\lambda$. Lemma B.2 gives $\rho < 12$ and $\lambda_1(b) > \beta^2$, where $\beta = 7/4$. We also claim that $\rho\Lambda(-\rho)/\sigma > 1$. Thus $\alpha \le \sigma/[2\rho\Lambda(-\rho)] < 1/2$. The second bound in (B.5) gives

$$\vartheta^2 < \frac{2k\ell^3(\beta^2 + \ell^2 + 2k\ell)}{(k^2 - 1/4)(\ell^2 + k^2)(\ell^2 + \beta^2)^2}, \qquad \ell = \sqrt{\rho} \in [k, \sqrt{12}), \quad \beta = 7/4.$$

For fixed ℓ and $C = \ell^2 + \beta^2$,

$$\partial_k \frac{Ck + 2\ell k^2}{k^2 - 1/4} = -\frac{C(k^2 + 1/4) + \ell k}{(k^2 - 1/4)^2} < 0, \quad \partial_k \frac{1}{\ell^2 + k^2} = -\frac{2k}{(\ell^2 + k^2)^2} < 0.$$

It suffices to take $k = 1$ and prove

$$Q(\ell) = \frac{3}{4}(\ell^2 + 1)(\ell^2 + 49/16)^2 - 2\ell^3(\ell^2 + 2\ell + 49/16) > 0.$$

For $1 \le \ell \le 2$, set $x = \ell - 1 \in [0, 1]$. Then

$$1024Q(1 + x) = 768x^6 + 2560x^5 + 2656x^4 - 5888x^3 - 7613x^2 + 4870x + 12934.$$

Since $x \ge x^2 \ge x^3$, the last four terms are at least $12934 - 8631x^2 \ge 4303$. For $\ell \ge 2$, set $x = \ell - 2 \ge 0$. Here

$$1024Q(2 + x) = 768x^6 + 7168x^5 + 26976x^4 + 45696x^3 + 27779x^2 + 12x + 10287 > 0.$$

Thus $Q > 0$ on the required range. At $k = 1$, the rational function bounding ϑ^2 is continuous and strictly less than 1 on $[1, \sqrt{12}]$, so its maximum is less than 1. Also $k - \alpha > 1/2$ and $\beta - k \ge 1/4$.

The bounds in these finitely many cases, together with $\beta - k > 0$, give $\delta_* > 0$ and $0 < \vartheta_* < 1$.

For the remaining bounds, write the paired terms of Λ/σ as

$$q_j = \frac{1}{\sigma}\left(\frac{1}{(N - \sigma + 2j)^2 + \rho} - \frac{1}{(N + \sigma + 2j)^2 + \rho}\right), \qquad j \ge 0.$$

Then

$$\Lambda(-\rho)/\sigma = \sum_{j \ge 0} q_j, \qquad F_\sigma(-\rho) \le d_\sigma b_*.$$

Each pair satisfies

$$\frac{1}{\sigma}\left(\frac{1}{(x - \sigma)^2 + \rho} - \frac{1}{(x + \sigma)^2 + \rho}\right) = \frac{1}{\sigma}\int_{x-\sigma}^{x+\sigma} \frac{2u}{(u^2 + \rho)^2}\,\mathrm{d}u. \tag{B.7}$$

If $2u/(u^2 + \rho)^2$ is decreasing on $[N + 2J - \sigma, \infty)$, comparison of its integrals over $[N + 2j - \sigma, N + 2j + \sigma]$ and $[N + 2j - \sigma, N + 2j + 2 - \sigma]$ gives

$$\frac{1}{\sigma}\sum_{j \ge J}\int_{N+2j-\sigma}^{N+2j+\sigma} \frac{2u}{(u^2 + \rho)^2}\,\mathrm{d}u \ge \int_{N+2J-\sigma}^{\infty} \frac{2u}{(u^2 + \rho)^2}\,\mathrm{d}u = \frac{1}{\rho + (N + 2J - \sigma)^2}. \tag{B.8}$$

Indeed, the average over the first interval is at least the average over the second, and the intervals $[N+2j-\sigma, N+2j+2-\sigma]$, $j \ge J$, partition $[N+2J-\sigma, \infty)$.

Proof of $k\rho\Lambda(-\rho)/\sigma \ge 5/4$. For $n \ge 3$, Lemma B.2 gives

$$(N+\sigma)^2 \le \rho < \rho_N^* := 4(N+1)^2 + 3/20.$$

Case $n = 3$. Here $N = 3/2$, $\rho \ge (N+\sigma)^2 \ge 9/4$, and $\rho < \rho_{3/2}^* = 503/20$. The bounds for q_0, q_1 and $\sum_{j\ge 2} q_j$ give

$$\Lambda(-\rho)/\sigma \ge L(\rho) := \frac{6}{(\rho+13/4)^2-9} + \frac{14}{(\rho+53/4)^2} + \frac{1}{\rho+121/4}.$$

The denominator of q_0 is $(\rho+9/4)^2 + 2(\rho-9/4)\sigma^2 + \sigma^4$. It increases in σ^2 and is at most $(\rho+13/4)^2 - 9$. For q_1, use $(\rho+49/4+\sigma^2)^2 - 49\sigma^2 \le (\rho+53/4)^2$. For the sum over $j \ge 2$, the function $2x/(x^2+\rho)^2$ decreases for $x \ge N+4-\sigma > 9/2$, because $\rho < 503/20 < 3(9/2)^2$. The comparison (B.8) with $J = 2$ gives

$$\sum_{j\ge 2} q_j \ge \frac{1}{\rho + (N+4-\sigma)^2} \ge \frac{1}{\rho+121/4}.$$

Let $P(\rho) = (4\rho+1)(4\rho+25)(4\rho+53)^2(4\rho+121) > 0$. For $\rho = 9/4 + y$, direct expansion gives

$$\begin{aligned} P(\rho)(6\rho L(\rho) - 5) &= 1024y^5 + 13312y^4 + 567808y^3 + 3715072y^2 + 25536832y + 2349760 \\ &> 0. \end{aligned}$$

Since $k \ge 3/2$,

$$\frac{\rho\Lambda(-\rho)}{\sigma} > \frac{5}{6}, \qquad \frac{k\rho\Lambda(-\rho)}{\sigma} > \frac{3}{2}\frac{5}{6} = \frac{5}{4}.$$

Case $n \ge 4$. For $x_j = N + 2j$,

$$q_j \ge \frac{4x_j}{(x_j^2+\rho+\sigma^2)^2}.$$

If $x_j \le u \le x_j + 2$, then

$$\frac{4x_j}{(x_j^2+\rho+\sigma^2)^2} \ge \frac{x_j}{x_j+2}\,\frac{4u}{(u^2+\rho+\sigma^2)^2}.$$

Keep $j = 0$, integrate over the intervals with $j \ge 1$, and divide by their length two. This gives

$$\frac{\Lambda(-\rho)}{\sigma} \ge \frac{4N}{(N^2+\rho+\sigma^2)^2} + \frac{N+2}{N+4}\frac{1}{(N+2)^2+\rho+\sigma^2}.$$

For fixed $\rho \ge (N+\sigma)^2$, multiplication of this lower bound by $k = N+\sigma$ gives an increasing function of σ. Indeed,

$$\begin{aligned} N^2+\rho+\sigma^2-4\sigma(N+\sigma) &\ge 2(N^2-N\sigma-\sigma^2) > 0, \\ (N+2)^2+\rho+\sigma^2-2\sigma(N+\sigma) &> 0 \qquad \text{for } N \ge 2. \end{aligned}$$

We may therefore replace σ by zero in this lower bound. With $x = \rho/N^2 \geq 1$, and using $N(N+2)/(N+4) \geq 4/3$ and $(1+2/N)^2 \leq 4$, we obtain

$$\frac{k\rho\Lambda(-\rho)}{\sigma} \geq \frac{4x}{(1+x)^2} + \frac{4x}{3(x+4)} > \frac{5}{4}.$$

Case $n = 2$, $1/2 \leq \sigma < 1$. By (B.3),

$$\frac{1}{2} \leq \sigma < 1, \qquad k = 1+\sigma, \qquad \frac{9}{4} \leq k^2 \leq \rho < 12.$$

The first term is $q_0 = 4/[((1-\sigma)^2+\rho)((1+\sigma)^2+\rho)]$. For the sum over $j \geq 1$, $2x/(x^2+\rho)^2$ decreases on $[3-\sigma, \infty)$, since $\rho < 12 < 3(3-\sigma)^2$. Apply (B.8) with $N = 1$ and $J = 1$ to get

$$\frac{\Lambda(-\rho)}{\sigma} \geq \frac{4}{((1-\sigma)^2+\rho)((1+\sigma)^2+\rho)} + \frac{1}{(3-\sigma)^2+\rho}.$$

Moreover,

$$\rho(\rho+4) - (\rho+(1-\sigma)^2)(\rho+(1+\sigma)^2) = (1-\sigma^2)(2\rho-1+\sigma^2) > 0, \quad (3-\sigma)^2 \leq \frac{25}{4}.$$

Consequently

$$\Lambda(-\rho)/\sigma \geq \frac{4}{\rho(\rho+4)} + \frac{1}{\rho+25/4},$$

and

$$6\rho\Lambda(-\rho)/\sigma - 5 \geq \frac{\rho^2 - 13\rho/4 + 25}{(\rho+4)(\rho+25/4)} = \frac{(\rho-13/8)^2 + 1431/64}{(\rho+4)(\rho+25/4)} > 0.$$

Thus

$$k\frac{\rho\Lambda(-\rho)}{\sigma} > \frac{3}{2}\frac{5}{6} = \frac{5}{4}.$$

Proof of $\rho\Lambda(-\rho)/\sigma > 1$. Here $n = 2$ and $0 < \sigma < 1/2$. For $1 \leq \rho \leq 12$ set $y = \sigma^2$ and $x_j = 1 + 2j$. Then

$$q_j = \frac{4x_j}{(\rho + x_j^2 + y)^2 - 4x_j^2 y}.$$

The first two terms satisfy

$$q_0 \geq \frac{4}{(\rho+1/4)(\rho+9/4)}, \qquad q_1 \geq \frac{12}{(\rho+37/4)^2}.$$

The denominator of q_0 increases in $y \in [0, 1/4]$ when $\rho \geq 1$. For q_1, use $(\rho+9+y)^2 - 36y \leq (\rho+37/4)^2$. To estimate $\sum_{j\geq 2} q_j$, let $g(x) = 2x/(x^2+\rho)^2$. By (B.7),

$$q_j = \frac{1}{\sigma}\int_{x_j-\sigma}^{x_j+\sigma} g(x)\,\mathrm{d}x, \qquad \partial_x^2 g(x) = \frac{24x(x^2-\rho)}{(x^2+\rho)^4}.$$

All these intervals for $j \geq 2$ lie in $[4, \infty)$, where g is convex because $\rho \leq 12$. Jensen's inequality gives $q_j \geq 2g(x_j)$. The trapezoidal inequality $\int_5^\infty g \leq g(5) + 2\sum_{j\geq 3} g(1+2j)$ then gives

$$\sum_{j\geq 2} q_j \geq 2\sum_{j\geq 2} g(1+2j) \geq \int_5^\infty g + g(5) = \frac{\rho+35}{(\rho+25)^2}.$$

Thus $\Lambda(-\rho)/\sigma \geq L(\rho)$, where

$$L(\rho) = \frac{4}{(\rho+1/4)(\rho+9/4)} + \frac{12}{(\rho+37/4)^2} + \frac{\rho+35}{(\rho+25)^2}.$$

With $P(\rho) = (4\rho+1)(4\rho+9)(4\rho+37)^2(\rho+25)^2 > 0$, one has

$$\begin{aligned} P(\rho)(\rho L(\rho) - 1) &= 256\rho^5 + 1632\rho^3 + 854640 + (12-\rho)(9216\rho^3 + 934160\rho) \\ &\quad + 8555265(\rho - 1) \\ &> 0 \quad \text{for } 1 \leq \rho \leq 12. \end{aligned}$$

□

C Estimates for boundary comparison

We prove the estimates used in Lemma 2.12: positivity of D_N and the weighted trace bound in Lemma C.4, followed by $R_{\mathrm{bd}} < 1$ in Lemma C.5. We first estimate the principal and second angular eigenvalues. Throughout, $n \geq 2$, $0 < \sigma < 1$, $N = n/2$, and $0 \leq b \leq b_\sigma$. For $b > 0$, write for simplicity

$$\lambda = \lambda_0(b), \qquad E_j = e_j(b, \pi/2).$$

The gaps to the first Neumann and Dirichlet levels are, respectively,

$$d = p^2 - \lambda, \qquad c = k^2 - \lambda = d + 4N\sigma.$$

The endpoint $b = 0$ is treated by continuity.

C.1 Spectral and resolvent estimates

Recall that $N = n/2$.

Lemma C.1. *Let $0 \leq b \leq b_\sigma$.*

(i) *If $\lambda_0(b) = -\rho < 0$, then*

$$\rho < \begin{cases} 3, & N = 1, \\ 3N - 1/10, & N \geq 3/2. \end{cases}$$

In particular, $\rho < 3N$. For $N \geq 3$, $\lambda_0(b) > -k^2$.

(ii) *One has*

$$\lambda_1(b) > \begin{cases} 4, & n = 2, \\ 25/4, & n = 3, \\ k^2, & n \geq 4. \end{cases}$$

Proof. For $N \geq 3/2$, we estimate $\lambda_0(b)$ by the Bessel comparison in dimension n. Concavity of the digamma function and its elementary Binet bound give

$$C_\sigma^{1/\sigma} \leq e^{2\psi(N+1)} < (N+1)^2 e^{-1/(N+1)} < N^2 + N + \frac{1}{2}.$$

The last step uses $e^{-x} < 1 - x + x^2/2$ for $x > 0$. At $\rho = 3N - 1/10$, Lemma B.1 gives

$$F_{n,\sigma}(-\rho)^{1/\sigma} \geq N^2 + N + 1/2 > C_\sigma^{1/\sigma}.$$

The Robin equation and strict increase of $F_{n,\sigma}(-\rho)$ in ρ prove the claim. Finally $N \geq 3$ implies $3N - 1/10 < N^2 \leq k^2$.

For $N = 1$, use the dimension-six comparison (B.4) at $\rho = 3$ and the gamma quotient bound (B.2). Since $\psi(2) < \log 2 - 1/4$,

$$F_{2,\sigma}(-3)^{1/\sigma} \geq \frac{33}{5e}, \qquad C_\sigma^{1/\sigma} \leq e^{2\psi(2)} < 4e^{-1/2}.$$

To compare these two bounds, note that $\frac{33}{5e} > 4e^{-1/2}$ is equivalent to $(33/20)^2 > e$. The elementary bound

$$e < \sum_{j=0}^{4} \frac{1}{j!} + \frac{1/5!}{1 - 1/6} = \frac{1631}{600} < \frac{1089}{400} = \left(\frac{33}{20}\right)^2$$

therefore gives $F_{2,\sigma}(-3) > C_\sigma \geq d_\sigma b$. Strict increase of $F_{2,\sigma}(-\rho)$ in ρ proves the claim.

For the second eigenvalue, the case $n \geq 4$ follows from Lemma 2.5. For $n = 3$, the recurrence formula for Γ and duplication give

$$F_{3,\sigma}(25/4) = \frac{\Gamma(3+\sigma)}{\Gamma(3-\sigma)} > \frac{\Gamma(5/2+\sigma)}{\Gamma(5/2-\sigma)} = C_\sigma \geq d_\sigma b.$$

The strict comparison follows because $x \mapsto \log\Gamma(x+\sigma) - \log\Gamma(x-\sigma)$ has positive derivative $\psi(x+\sigma) - \psi(x-\sigma)$. Since $k < 5/2 < p+2$, we have $25/4 \in (k^2, (p+2)^2)$. On this interval $F_{3,\sigma}$ is strictly decreasing, and $F_{3,\sigma}(\lambda_1(b)) = d_\sigma b$. Hence $\lambda_1(b) > 25/4$; at $b = 0$, $\lambda_1(0) = (p+2)^2 > 25/4$.

For $n = 2$, write $F_\sigma = F_{2,\sigma}$ for the angular Weyl function. The recurrence formula for Γ and duplication give

$$F_\sigma(4) = \left(\frac{1+\sigma}{1-\sigma}\right)^2 F_\sigma(0), \qquad \frac{F_\sigma(4)}{C_\sigma} = \frac{1+\sigma}{1-\sigma}\frac{\Gamma((1+\sigma)/2)}{\Gamma((1-\sigma)/2)}\frac{\Gamma(1-\sigma/2)}{\Gamma(1+\sigma/2)}.$$

The logarithm of this ratio is analytic for $|\sigma| < 1$ and contains only odd powers. Its linear coefficient is

$$2 + \psi(1/2) - \psi(1) = 2 - 2\log 2 > 0.$$

For every odd integer $a \geq 3$, the coefficient of σ^a is

$$\frac{2}{a}\left[1 - \sum_{j\geq 0}(1+2j)^{-a} + \sum_{j\geq 0}(2+2j)^{-a}\right] = \frac{2}{a}\{1 - \eta(a)\} > 0,$$

where $\eta(a) = \sum_{j\geq 1}(-1)^{j-1}j^{-a} \in (0,1)$ by the elementary alternating-series estimate. Thus $F_\sigma(4) > C_\sigma \geq d_\sigma b$. Since $(1+\sigma)^2 < 4 < (3-\sigma)^2$ and F_σ is strictly decreasing on $((1+\sigma)^2, (3-\sigma)^2)$, the Robin equation $F_\sigma(\lambda_1(b)) = d_\sigma b$ gives $\lambda_1(b) > 4$ for $b > 0$. For $b = 0$, $\lambda_1(0) = (3-\sigma)^2 > 4$. □

The trace and moment estimates use the tail of Λ/σ beyond the first Neumann–Dirichlet pair:

$$\tau(\lambda) = \sum_{j\geq 1} \frac{4(N+2j)}{\{(N+2j-\sigma)^2 - \lambda\}\{(N+2j+\sigma)^2 - \lambda\}}.$$

Lemma C.2. *For $-3N < \lambda \leq p^2$, one has*

$$\sum_{j\geq 1} \frac{4(N+2j)}{\{(N+2j)^2 - \lambda\}^2} \leq \tau(\lambda) \leq \frac{1}{(N+1)^2 - \lambda}.$$

More explicitly, for every integer $J \geq 1$,

$$\tau(\lambda) \geq \sum_{j=1}^{J-1} \frac{4(N+2j)}{\{(N+2j)^2 - \lambda\}^2} + \frac{1}{(N+2J)^2 - \lambda} + \frac{2(N+2J)}{\{(N+2J)^2 - \lambda\}^2}. \tag{C.1}$$

For $N = 1$, we also have $\tau > 1/(5-\lambda)$.

Proof. For $x = N + 2j$, the derivative of a summand with respect to $v = \sigma^2$ is

$$\frac{\partial}{\partial v}\frac{4x}{(x^2+v-\lambda)^2 - 4x^2 v} = \frac{8x(x^2+\lambda-v)}{\{(x^2+v-\lambda)^2 - 4x^2 v\}^2} > 0 \qquad \text{for } 0 \leq v \leq 1.$$

Indeed, $x \geq N+2$ and $\lambda > -3N$ give $x^2 + \lambda - v > N^2 + N + 3 > 0$.

At $v = 0$ the summand equals $4x/(x^2-\lambda)^2$, which gives the lower comparison. At $v = 1$ it equals

$$\frac{1}{(x-1)^2 - \lambda} - \frac{1}{(x+1)^2 - \lambda}.$$

Summing over $j \geq 1$,

$$\sum_{j\geq 1}\left(\frac{1}{(N+2j-1)^2 - \lambda} - \frac{1}{(N+2j+1)^2 - \lambda}\right) = \frac{1}{(N+1)^2 - \lambda},$$

since the remaining endpoint term tends to zero. This proves the upper bound.

For the explicit lower bound, retain the terms before J. The function $f(x) = 4x/(x^2-\lambda)^2$ is positive, decreasing and convex for $x \geq N+2$, since

$$f'(x) = -\frac{4(3x^2+\lambda)}{(x^2-\lambda)^3} < 0, \qquad f''(x) = \frac{48x(x^2+\lambda)}{(x^2-\lambda)^4} > 0.$$

The Hermite–Hadamard inequality on $[N+2j, N+2j+2]$, summed over $j \geq J$, gives

$$\sum_{j\geq J} f(N+2j) \geq \frac{1}{2}\int_{N+2J}^{\infty} f(x)\,\mathrm{d}x + \frac{1}{2}f(N+2J).$$

Since $\int_{N+2J}^{\infty} f(x)\,\mathrm{d}x = 2/((N+2J)^2-\lambda)$, this is (C.1).

For $N=1$, we also need the following lower bound for the trace estimate in Lemma C.4. For $x = 1+2j$,

$$\frac{4x}{(x^2-\lambda)^2} > \frac{1}{(x-1)^2+1-\lambda} - \frac{1}{(x+1)^2+1-\lambda},$$

because the denominator on the right is $(x^2-\lambda)^2 + 4(1-\lambda) > (x^2-\lambda)^2$ (as $\lambda \leq p^2 < 1$). Summing over $j \geq 1$,

$$\sum_{j\geq 1} \frac{4(1+2j)}{\{(1+2j)^2-\lambda\}^2} > \sum_{j\geq 1}\left(\frac{1}{(2j)^2+1-\lambda} - \frac{1}{(2j+2)^2+1-\lambda}\right) = \frac{1}{5-\lambda}.$$

Since $\tau \geq \sum_{j\geq 1} 4(1+2j)/\{(1+2j)^2-\lambda\}^2$, this proves $\tau > 1/(5-\lambda)$. □

Write $g_N = N^2 - \lambda$ for the gap from λ to N^2. Set $s = \sigma c \tau$; we will prove $0 < s < 1$. Write

$$A = 4N + dc\tau, \qquad K = 4N - (2N-\sigma)s.$$

Then $K > 0$ once $0 < s < 1$.

Define the scaled high resolvent and its mixed moment by

$$S_b(z) = \sigma b \sum_{j\geq 1} \frac{E_j^2}{\lambda_j(b) - z},$$
$$M_b = \frac{S_b(N^2) - S_b(\lambda)}{g_N} = \sigma b \sum_{j\geq 1} \frac{E_j^2}{(\lambda_j(b)-N^2)(\lambda_j(b)-\lambda)}.$$

The mixed moment M_b enters the trace bound in Lemma C.4. The sharper bound below retains one more Neumann–Dirichlet pair, with correction term

$$B_1 = 4(N+2)\left(\frac{g_N}{g_N+4N+4}\right)^3.$$

Lemma C.3. *For $0 < \sigma < 1$ and $0 < b \le b_\sigma$, one has $0 < s < 1$ and*

$$\sigma b E_0^2 = \frac{dc}{A}, \tag{C.2}$$

$$0 < S_b(N^2) \le \frac{4Nd(1-s)}{AK}, \tag{C.3}$$

$$0 < M_b \le \frac{4Ndc\tau(1-s)}{A^2K} < \frac{1}{16N}. \tag{C.4}$$

If $\lambda \le -k^2$, then

$$M_b \le \frac{dc\tau - B_1}{A^2} \le \frac{1}{4(4N+B_1)} < \frac{1}{16N}. \tag{C.5}$$

Proof. Remove the first Neumann zero and Dirichlet pole from F_σ by setting

$$G(z) = F_\sigma(z)\frac{k^2 - z}{p^2 - z}.$$

Then G is regular at $z = p^2$ and $z = k^2$, and its logarithmic derivative gives

$$\tau(\lambda) = -\frac{G'(\lambda)}{\sigma G(\lambda)}.$$

The recurrence formula for Γ gives $G = F_{n+4,\sigma}$. Applying Lemma 2.5 in dimension $n+4$, we see that G is positive, decreasing and concave below its first Neumann eigenvalue $(p+2)^2$. In particular,

$$\tau'(\lambda) - \sigma\tau^2 = -\frac{G''(\lambda)}{\sigma G(\lambda)} \ge 0.$$

The logarithmic derivative of F_σ is $\Lambda(\lambda) = \sigma(4N/(dc) + \tau)$. Hence (C.2) follows from $bE_0^2 = \Lambda(\lambda)^{-1}$, which was proved in Lemma 2.6.

The boundary resolvent identity in the proof of Lemma 2.6 and the Robin condition give

$$S_b(z) = \frac{\sigma F_\sigma(\lambda)}{F_\sigma(z) - F_\sigma(\lambda)} - \frac{\sigma b E_0^2}{\lambda - z}.$$

Since $F_\sigma(p^2) = 0$, (C.2) yields

$$S_b(p^2) = -\sigma + \frac{\sigma b E_0^2}{d} = \frac{d(1-s)}{A}.$$

Since $S_b(p^2) > 0$ and $A, d > 0$, we obtain $0 < s < 1$, hence $K = 4N - (2N - \sigma)s > 2N + \sigma > 0$.

Let

$$X = \frac{c}{g_N}\left(1 - \frac{G(N^2)}{G(\lambda)}\right).$$

Concavity gives $X \geq \sigma c\tau = s$. Substitution in the boundary resolvent yields

$$\frac{S_b(N^2) - S_b(p^2)}{d(N^2 - p^2)} = \frac{c\tau g_N(1 - X) - (2N + \sigma)(X - s)}{Ag_N\{4N - (2N - \sigma)X\}}.$$

The denominator is positive, since $F_\sigma(N^2) < 0 < F_\sigma(\lambda)$. The left side is positive, whence $s \leq X < 1$. The function $(1 - X)/(4N - (2N - \sigma)X)$ is decreasing on $[0, 1]$. Consequently

$$\frac{S_b(N^2) - S_b(p^2)}{N^2 - p^2} \leq \frac{dc\tau(1 - s)}{AK}.$$

Adding $S_b(p^2)$ and using $N^2 - p^2 = \sigma(2N - \sigma)$ proves (C.3).

Expanding the formula for $S_b(z)$ above at $z = \lambda$ and cancelling the ground-state pole gives

$$\begin{aligned} S_b(\lambda) &= -\frac{\sigma F_\sigma(\lambda) F''_\sigma(\lambda)}{2F'_\sigma(\lambda)^2} \\ &= \frac{d\{8N(1 - s) + dc^2(\tau' - \sigma\tau^2)\}}{2A^2} \\ &\geq \frac{4Nd(1 - s)}{A^2}, \end{aligned}$$

where the inequality follows from the concavity of G. Subtract this lower bound from (C.3). Since $A - K = c\tau g_N$, division by $g_N > 0$ gives the first bound in (C.4).

The last strict bound in (C.4) follows from

$$K - 4N(1 - s) = (2N + \sigma)s > 0, \qquad \frac{dc\tau}{A^2} = \frac{dc\tau}{(4N + dc\tau)^2} \leq \frac{1}{16N}.$$

For $\lambda = -\rho \leq -k^2$, write $v = dc\tau$ for simplicity. For the next zero-pole pair, set

$$d_1 = (N + 2 - \sigma)^2 + \rho, \qquad c_1 = (N + 2 + \sigma)^2 + \rho, \qquad \tau_1 = \frac{4(N + 2)}{d_1 c_1}.$$

Removing this pair from G gives $F_{n+8,\sigma}$. Its concavity implies

$$\tau' - \sigma\tau^2 \geq \partial_\lambda \tau_1 + \sigma\tau_1^2 - 2\sigma\tau_1\tau.$$

Retain $\tau' - \sigma\tau^2$ in the Laurent expansion above. Since $4N(1 - s)/K \leq 1 - s/2$, completing the square gives

$$\begin{aligned} (4N + v)^2 M_b &\leq \frac{4Nv(1 - s)}{K} - \frac{(dc)^2}{2g_N}(\tau' - \sigma\tau^2) \\ &\leq v - \frac{\sigma}{2d}\left(v - \frac{d^2 c\tau_1}{g_N}\right)^2 - \frac{(dc)^2}{2g_N}\left[\partial_\lambda\tau_1 + \frac{\sigma^2(2N - \sigma)}{g_N}\tau_1^2\right]. \end{aligned}$$

Here $g_N = N^2 + \rho \geq 2N^2$,

$$\begin{aligned} &dc(g_N + 4N + 4)^2 - g_N^2 d_1 c_1 \\ &= 8(N + 1)\sigma^2\Big[3(g_N - 2N^2)^2 + (8N^2 + 4N + 4)(g_N - 2N^2) + 4N^4 + (g_N + 2N + 2)\sigma^2\Big] \\ &> 0, \end{aligned}$$

and

$$(dc)^2\partial_\lambda\tau_1 = 8(N+2)\left(\frac{dc}{d_1c_1}\right)^2(g_N+4N+4+\sigma^2) \geq \frac{8(N+2)g_N^4}{(g_N+4N+4)^3}.$$

Dropping the square and the remaining positive term gives $M_b \leq (v-B_1)/(4N+v)^2$. Finally,

$$\frac{1}{4(4N+B_1)} - \frac{v-B_1}{(4N+v)^2} = \frac{(v-4N-2B_1)^2}{4(4N+B_1)(4N+v)^2} \geq 0,$$

which proves (C.5). □

C.2 Weighted trace inequality

The trace estimate in Lemma 2.12 requires a positive lower bound for $D_N(f,g)$ in terms of $\sigma b|f(\pi/2)|^2$. We fix a time and evaluate b and $\partial_t b$ there, using the form D_N from (2.26) with $e' = (\partial_t b)\,\partial_b e_0(b)$ and the scales in (2.25).

Lemma C.4. *Let $0 \leq b \leq b_\sigma$ and $|\partial_t b| \leq 2\sigma b$. Then the form D_N from* (2.26) *is positive definite on E_ω, and*

$$\sigma b|f(\pi/2)|^2 \leq C_n D_N(f,g) \qquad \textit{for } (f,g) \in E_\omega,$$

where

$$C_n = \begin{cases} 7, & n = 3, \\ 13/2, & n = 2 \textit{ or } n \geq 4. \end{cases} \tag{C.6}$$

For a nonzero pair the inequality is strict.

Proof. The case $b = 0$ follows directly from the uncoupled forms. Assume $b > 0$. We use the bounds from Lemma C.3 and Lemma C.2 throughout. The scalar bounds for $\lambda < -k^2$ are verified at the end of the proof.

The block inverse. Use the coordinates of (2.21). By Lemma 2.6,

$$\delta = \begin{cases} 0, & \lambda \geq -k^2, \\ (\partial_t b)E_0^2/(4(-\lambda)), & \lambda < -k^2. \end{cases}$$

For simplicity, set

$$u = \sigma b E_0^2 = \frac{dc}{A}, \qquad R = S_b(N^2), \qquad X = (\ell+N)M_b,$$

and write $Q_{\shortparallel} = Q_{\shortparallel}^N$, $Q_\perp = Q_\perp^N$ for the uncoupled forms in (2.26). $\lambda_1 > k^2 > N^2$ gives $Q_\perp > 0$, and $Q_{\shortparallel} > 0 \iff N^2 > a^2 + \delta^2$. The coupling functionals $\sqrt{\ell}(q_1 + \ell^{-1}q_2)$ and $\ell^{-1/2}\langle e', \ell\Phi_\perp + \Psi_\perp\rangle_\omega$ have squared dual norms

$$B_{\shortparallel} = \frac{2(N+a)}{N^2-a^2-\delta^2}, \qquad B_\perp = \frac{(\partial_t b)^2E_0^2}{\ell}\sum_{j\geq 1}\frac{E_j^2(\lambda_j+\ell^2+2N\ell)}{(\lambda_j-N^2)(\lambda_j-\lambda)^2}.$$

By Cauchy–Schwarz,

$$D_N \geq Q_{\shortparallel} + Q_{\perp} - \sqrt{B_{\shortparallel} B_{\perp} Q_{\shortparallel} Q_{\perp}} \geq \left(1 - \frac{\sqrt{B_{\shortparallel} B_{\perp}}}{2}\right)(Q_{\shortparallel} + Q_{\perp}).$$

Hence $B_{\shortparallel} B_{\perp} < 4$ gives positivity.

Under these conditions, the trace functional $\sqrt{\sigma b}\, f(\pi/2)$ has uncoupled squared dual norms

$$\frac{u(N+\delta)}{\ell(N^2 - a^2 - \delta^2)} \quad \text{and} \quad R.$$

Its pairings with the two coupling functionals are

$$T_{\shortparallel} = \sqrt{\frac{u}{\ell}}\, \frac{N + \delta + a}{N^2 - a^2 - \delta^2}, \qquad T_{\perp} = \frac{\partial_t b}{\sigma b} \sqrt{\frac{u}{\ell}}\, X.$$

Let C_{tr} be the best trace constant in $\sigma b |f(\pi/2)|^2 \leq C_{\mathrm{tr}} D_N(f, g)$. Inverting the two coupled blocks gives

$$C_{\mathrm{tr}} = R + \frac{u(N+\delta)}{\ell(N^2 - a^2 - \delta^2)} + \frac{B_{\perp} T_{\shortparallel}^2/4 + B_{\shortparallel} T_{\perp}^2/4 + T_{\shortparallel} T_{\perp}}{1 - B_{\shortparallel} B_{\perp}/4}.$$

The identities follow first for finite angular sums, then by convergence of the positive resolvent series.

For fixed b, the block formula gives

$$C_{\mathrm{tr}}(\partial_t b) \leq C_{\mathrm{tr}}(|\partial_t b|) \leq C_{\mathrm{tr}}(2\sigma b) \qquad (|\partial_t b| \leq 2\sigma b),$$

provided $N^2 > a^2 + \delta^2$ and $B_{\shortparallel} B_{\perp} < 4$ at $\partial_t b = 2\sigma b$. Hence take $\partial_t b = 2\sigma b$. Since $\lambda_j - \lambda > c$,

$$B_{\perp} \leq \frac{4u}{\ell}\left(1 + \frac{\ell^2 + 2N\ell + \lambda}{c}\right) M_b. \tag{C.7}$$

The range $\lambda \geq -k^2$. Here $\ell = k$, $\delta = 0$, and $0 \leq a \leq (N^2 + \sigma^2)/(N + \sigma) < N$. Let $Q = k(N - a)$ and $\Theta = 4kuX/(Qc)$. The bound $B_{\shortparallel} B_{\perp}/4 \leq \Theta$ ensures positivity when $\Theta < 1$, and then

$$C_{\mathrm{tr}} \leq R + \frac{u}{Q}\left\{\frac{N}{N+a} + \frac{\Theta/2 + 2X + 2X^2}{1 - \Theta}\right\}. \tag{C.8}$$

Indeed,

$$T_{\shortparallel}^2 = \frac{uk}{Q^2}, \qquad T_{\perp}^2 = \frac{4uX^2}{k}, \qquad T_{\shortparallel} T_{\perp} = \frac{2uX}{Q}, \qquad B_{\perp} \leq \frac{8uX}{c}.$$

Substitution in the block inverse gives (C.8).

Let $\zeta = 4N/A$. Lemma C.3 gives

$$R \leq \frac{d}{A}, \qquad M_b \leq \frac{1 - \zeta}{A}, \qquad X \leq \frac{k+N}{4N}\zeta(1-\zeta), \qquad \Theta \leq \frac{2k(k+N)}{27N^2}.$$

For the last inequality, use $Q = \sigma p + d/2 \geq d/2$ and $\zeta^2(1-\zeta) \leq 4/27$. To bound u/Q, use $u/Q \leq 2c/A$. Lemma C.2 implies

$$\tau \geq \frac{1}{d + 4N + 4 + 2N\sigma}.$$

For $N \geq 1$, set $y = d/N$ and $h = 4 + 2\sigma + 4/N$. After positive denominators are cleared, the assertion $c/A < 3/2$ reduces to

$$y^2 + (4 - 8/N)y + h(12 - 8\sigma) \geq y^2 - 4y + 24 > 0.$$

For $N \geq 2$, the stronger $c/A < 5/4$ follows by replacing h by its upper bound $6 + 2\sigma$. Five times the denominator minus four times the numerator then equals

$$(y - 2 - 2\sigma)^2 + 116 - 64\sigma - 36\sigma^2 \geq 16.$$

For $N = 1$, use $\tau > 1/(5 - \lambda)$ and set $h = 5 - p^2 = 4 + 2\sigma - \sigma^2$. The assertion $c/A < 9/8$ follows from

$$\begin{aligned} 9\{4(d+h) + d(d+4\sigma)\} - 8(d+4\sigma)(d+h) &= d^2 + (4 - 12\sigma + 8\sigma^2)d + 4h(9 - 8\sigma) \\ &> d^2 - d/2 + 16 \\ &> 0. \end{aligned}$$

For $N = 1$, the root bound gives $d < 4$, hence $R < 1$. The bounds used in (C.8) are listed in the following table. Entries are upper bounds; the last column is strict.

n	R	u/Q	X	Θ	C_{tr}
2	1	9/4	3/16	4/9	$15241/2560 < 6$
3	3/2	3	1/6	80/243	$1137/163 < 7$
$n \geq 4$	5/4	5/2	5/32	5/18	$36485/6656 < 11/2$

In particular, $\Theta < 1$, so (C.8) applies.

The range $\lambda < -k^2$. Lemma C.1 leaves only $N = 1, 3/2, 2, 5/2$. For simplicity, write $\rho = -\lambda$ and $q = \sqrt{\rho}$. At $\partial_t b = 2\sigma b$, we have $\ell = q$, $a = 0$, and $\delta = u/(2q^2)$. The uncoupled principal trace contribution and the normalized coupling strength are given by

$$L = \frac{u}{q(N - \delta)}, \qquad \vartheta^2 = B_{\shortparallel} B_{\perp}/4.$$

If $\delta < N$ and $\vartheta < 1$, then

$$C_{\mathrm{tr}} \leq R + L\left[1 + \frac{\vartheta^2(1 + \delta/N)/2 + 2X + 2X^2/(1 + \delta/N)}{1 - \vartheta^2}\right]. \tag{C.9}$$

Indeed,

$$T_{\shortparallel}^2 = \frac{L}{N - \delta}, \qquad T_{\perp} = 2\sqrt{u/q}\, X, \qquad \frac{u(N + \delta)}{q(N^2 - \delta^2)} = L.$$

The block formula gives (C.9).

To bound the quantities in (C.9), we claim that $u < a_N \rho$ for $N = 3/2, 2, 5/2$, with

N	q upper bound	a_N
$3/2$	$21/10$	$6/5$
2	$5/2$	$3/2$
$5/2$	$11/4$	$7/4$

For $N = 1$, we also claim the sharper bounds

$$u < \frac{24}{25}\rho, \qquad M_b < \frac{1}{20}.$$

For $N = 3/2, 2, 5/2$, let a_N be the number in the table, set $r = q/N$, and use $M_b < 1/(16N)$. The common estimates are

$$\frac{\delta}{N} < \frac{a_N}{2N}, \quad L < \frac{a_N q}{N - a_N/2}, \quad R \le \frac{u}{q^2 + k^2} < 1, \quad X < \frac{r+1}{16},$$
$$\vartheta^2 < \frac{2a_N}{N} \frac{r(r+1)}{r^2+1} \frac{X}{1 - (\delta/N)^2},$$

For the last inequality, $(a_N - 1)q^2 < N^2$ in every row of the scalar parameter table. The coupling estimate follows from (C.7) and $q^2 + k^2 \ge q^2 + N^2$. The root bounds give $r < 7/5$ for $N = 3/2$ and $r < 5/4$ for $N = 2, 5/2$. Since $r(r+1)/(r^2+1)$ increases on these intervals, we obtain

N	δ/N	L	X	ϑ^2
$3/2$	$2/5$	3	$3/20$	$12/37 < 1/3$
$2, 5/2$	$3/8$	3	$3/20$	$648/2255 < 3/10$

Thus $\delta < N$ and $\vartheta < 1$. Also

$$\frac{r(r+1)}{r^2+1} \le \begin{cases} 42/37, & N = 3/2, \\ 45/41, & N = 2, 5/2, \end{cases} \qquad \frac{2X^2}{1 + \delta/N} \le 2X^2.$$

By (C.9),

$$C_{\mathrm{tr}} < \begin{cases} 2641/400 < 7, & N = 3/2, \\ 509/80 < 13/2, & N = 2, 5/2. \end{cases}$$

It remains to treat $N = 1$. The bound $\lambda_1 > 4$ gives

$$\vartheta^2 \le \frac{2u}{q(1-\delta^2)} \left(1 + \frac{2q}{q^2+4}\right) M_b.$$

For fixed q, M_b, $\delta = u/(2q^2)$, and

$$\partial_u L = \frac{1}{q(1-\delta)^2} > 0, \qquad \partial_u \frac{L}{1+\delta} = \frac{1+\delta^2}{q(1-\delta^2)^2} > 0.$$

Hence use $u = 24q^2/25$, $\delta = 12/25$ in (C.9). The two bounds claimed above and $q < 26/15$ give

$$R < 18/25, \qquad L < 16/5, \qquad X < 41/300, \qquad \vartheta^2 < 12/37.$$

For the last comparison use $1 + 2q/(q^2+4) \le 3/2$. Substitution in the same positive block formula gives

$$C_{\rm tr} < 181988/28125 = 13/2 - 1649/56250 < 13/2.$$

Thus D_N is positive definite and $C_{\rm tr} < C_n$ for all $|\partial_t b| \le 2\sigma b$.

Verification of the remaining scalar bounds. For $u < a_N\rho$ with $N = 3/2, 2, 5/2$, the indicated bounds on q follow from Lemma C.1. Since $\lambda < -k^2$, $\sigma < q - N$, and

$$dc = ((N-\sigma)^2 + q^2)((N+\sigma)^2 + q^2) \le 4q^2(q^2 - 2Nq + 2N^2).$$

The product increases with σ^2, since $q > N$. The first tail bound in Lemma C.2, with $J = 1$, therefore gives

$$\frac{1}{u} = \frac{4N}{dc} + \tau \ge \frac{N}{q^2(q^2-2Nq+2N^2)} + \frac{1}{q^2+(N+2)^2} + \frac{2(N+2)}{\{q^2+(N+2)^2\}^2}.$$

Multiplying by q^2 and discarding the last positive term gives

$$\frac{q^2}{u} > \frac{N}{N^2+(q-N)^2} + \frac{q^2}{q^2+(N+2)^2}.$$

For $q > N$, the first term decreases and the second increases with q. If $N = 2$, then $2 < q < 5/2$, so

$$\frac{q^2}{u} > \frac{8}{17} + \frac{1}{5} = \frac{57}{85} > \frac{2}{3}, \qquad u < \frac{3}{2}q^2.$$

If $N = 5/2$, then $5/2 < q < 11/4$, so

$$\frac{q^2}{u} > \frac{40}{101} + \frac{25}{106} > \frac{4}{7}, \qquad u < \frac{7}{4}q^2.$$

For $N = 3/2$, subtract $1/(a_N q^2)$ from the preceding lower bound for $1/u$. After multiplication by $a_N q^2(q^2 - 2Nq + 2N^2)\{q^2+(N+2)^2\}^2$, the numerator is

$$\begin{aligned} P_N(q) &= a_N N\{q^2+(N+2)^2\}^2 \\ &\quad + (q^2 - 2Nq + 2N^2)\left[a_N q^2\{q^2+(N+2)^2+2N+4\} - \{q^2+(N+2)^2\}^2\right]. \end{aligned}$$

Write $y = q - N$. For $N = 3/2$, we have

$$40P_{3/2}(3/2+y) = 1440 + 6129y - 3235y^2 + 480y^3 + 142y^4 + 48y^5 + 8y^6.$$

Here $0 \le y \le 3/5$, so $6129y - 3235y^2 \ge 4188y \ge 0$. Thus $u < a_N q^2$ in all three cases.

For $N = 1$, Lemma C.1 and $\lambda = -\rho < -k^2$ give $1 < \rho < 3$ and $0 < \sigma < \sqrt{\rho} - 1$. Since $\partial_{\sigma^2}(dc) = 2(\rho - 1 + \sigma^2) > 0$, we have

$$dc \le 4\rho(\rho - 2\sqrt{\rho} + 2).$$

For $1 \le \rho \le 3$,

$$121\rho - (4\rho + 7)^2 = (\rho - 1)(49 - 16\rho) \ge 0, \quad \sqrt{\rho} \ \ge \frac{4\rho + 7}{11}, \quad dc \le \frac{4\rho(3\rho + 8)}{11}.$$

The bound for u. Lemma C.2 with $J = 2$ gives

$$\tau \ge t_0(\rho) := \frac{12}{(\rho + 9)^2} + \frac{1}{\rho + 25} + \frac{10}{(\rho + 25)^2}.$$

Set $f(\rho) = \rho t_0(\rho)$. For $1 \le \rho \le 3$,

$$f''(\rho) = \frac{24(\rho - 18)}{(\rho + 9)^4} - \frac{50}{(\rho + 25)^3} + \frac{20(\rho - 50)}{(\rho + 25)^4} < 0.$$

Moreover,

$$f(1) = \frac{732}{4225}, \qquad f(3) = \frac{155}{392}, \qquad f(1) - \left(f(3) - \frac{2}{9}\right) = \frac{1021}{14905800} > 0.$$

Concavity gives

$$f(\rho) \ge \frac{3 - \rho}{2} f(1) + \frac{\rho - 1}{2} f(3) \ge \frac{155}{392} + \frac{\rho - 3}{9}.$$

Hence

$$\frac{\rho}{u} = \frac{4\rho}{dc} + \rho\tau \ge \frac{11}{3\rho + 8} + \frac{155}{392} + \frac{\rho - 3}{9}.$$

The right-hand side is decreasing, since

$$-\frac{33}{(3\rho + 8)^2} + \frac{1}{9} \le \frac{1}{9} - \frac{33}{289} = -\frac{8}{2601} < 0.$$

Therefore

$$\frac{\rho}{u} \ge \frac{11}{17} + \frac{155}{392} = \frac{25}{24} + \frac{2}{2499} > \frac{25}{24},$$

which proves $u < 24\rho/25$.

The bound for M_b. Set $v = dc\tau$ and $B(\rho) = 12((\rho + 1)/(\rho + 9))^3$. By (C.5),

$$M_b \le \frac{v - B(\rho)}{(4 + v)^2}.$$

The upper bound $\tau \le 1/(\rho + 4)$ in Lemma C.2 gives

$$v \le V(\rho) := \frac{4\rho(3\rho + 8)}{11(\rho + 4)} \le V(3) = \frac{204}{77} < 3,$$

since $V'(\rho) = \frac{4}{11}(3 - 16/(\rho+4)^2) > 0$. Also,

$$B(\rho) - \frac{2(\rho-1)}{9} = \frac{2(\rho-3)^2(87+22\rho-\rho^2)}{9(\rho+9)^3} \geq 0.$$

For $0 \leq v < 3$ and $B \geq 0$,

$$\partial_v \frac{v-B}{(4+v)^2} = \frac{4-v+2B}{(4+v)^3} > 0, \qquad \partial_B \frac{v-B}{(4+v)^2} = -\frac{1}{(4+v)^2} < 0.$$

Consequently,

$$M_b \leq F(\rho) := \frac{V(\rho) - 2(\rho-1)/9}{(4+V(\rho))^2} = \frac{11(\rho+4)(43\rho^2+111\rho+44)}{72(3\rho^2+19\rho+44)^2}.$$

Differentiating,

$$F'(\rho) = \frac{11(4-\rho)(129\rho^3+1397\rho^2+4304\rho+3696)}{72(3\rho^2+19\rho+44)^3} > 0 \qquad (1 \leq \rho \leq 3).$$

Thus

$$M_b \leq F(3) = \frac{14707}{294912} = \frac{1}{20} - \frac{193}{1474560} < \frac{1}{20}.$$

This proves the two remaining scalar bounds. □

C.3 The boundary comparison factor

We prove $R_{\mathrm{bd}} < 1$. Together with (2.38), this gives $\operatorname{sign} \Phi(0, \pi/2) = \operatorname{sign} q_1(0)$, since $E_0 > 0$.

For $b > 0$, write $r_j = \sqrt{\lambda_j(b)}$ for $j \geq 1$, and use ℓ from (2.25). For a trace constant $C > 0$, define

$$R_{\mathrm{bd}} = \frac{C\ell}{2c}\,\sigma b \sum_{j\geq 1} \frac{E_j^2(r_j+k)}{r_j^2(r_j-N)(r_j-p)}. \tag{C.10}$$

Lemma C.5. *Let* $0 < b \leq b_\sigma$. *The quantity* (C.10) *satisfies* $R_{\mathrm{bd}} < 1$ *for every* $0 < C \leq C_n$, *with* C_n *as in* (C.6).

Proof. It is enough to take $C = C_n$. Lemma C.1 allows the choices

$$\beta = \begin{cases} 2, & n = 2, \\ 5/2, & n = 3, \\ N, & n \geq 4. \end{cases}$$

Moment bounds. To estimate R_{bd}, use the weighted moments

$$U_j = \frac{\sigma b}{d} \sum_{i\geq 1} \frac{r_i^j E_i^2}{(r_i^2 - N^2)(r_i^2 - p^2)}, \quad j = 0, 2, -2.$$

The divided-resolvent identity and Lemma C.3 give

$$U_0 = \frac{S_b(N^2) - S_b(p^2)}{d(N^2 - p^2)} \le \frac{c\tau}{4NA}, \qquad U_2 - N^2 U_0 = \frac{S_b(p^2)}{d} = \frac{1 - \sigma c\tau}{A} \le \frac{1}{A}.$$

Also $U_{-2} \le U_0/\beta^2$, since $r_i > \beta$. For brevity, write $D = 4N\{(N+1)^2 - \lambda\} + dc$. Since $\tau \le \{(N+1)^2 - \lambda\}^{-1}$,

$$U_0 \le \frac{c}{4N} \frac{\tau}{4N + dc\tau} \le \frac{c}{4ND}.$$

For the numerator in (C.10), write $s_0 = \sqrt{3N^2 - \sigma^2}$ and use

$$\frac{(r+k)(r+N)(r+p)}{r^2} = r + 3N + \frac{s_0^2}{r} + \frac{kNp}{r^2},$$

and the square completion

$$\frac{r^2}{4s_0} + \frac{3}{2}s_0 + \frac{s_0^3}{4r^2} - r - \frac{s_0^2}{r} = \frac{s_0}{4}\left(\frac{r}{s_0} + \frac{s_0}{r} - 2\right)^2 \ge 0.$$

Thus, with

$$H_\beta = \frac{N^2}{4s_0} + 3N + \frac{3}{2}s_0 + \frac{s_0^3/4 + kNp}{\beta^2},$$

the definitions of U_j give

$$R_{\mathrm{bd}} \le \frac{C\ell d}{2c}\left\{\frac{U_2}{4s_0} + \left(3N + \frac{3}{2}s_0\right)U_0 + \left(\frac{s_0^3}{4} + kNp\right)U_{-2}\right\}.$$

Substituting the moment bounds and using $A \ge 4N$,

$$R_{\mathrm{bd}} \le \frac{C\ell d}{2c}\left(H_\beta U_0 + \frac{1}{4s_0 A}\right) \le \frac{C\ell}{8N}\left(\frac{dH_\beta}{D} + \frac{d}{4s_0 c}\right). \tag{C.11}$$

Finite sums suffice for these calculations; the positive sums converge to the displayed expressions.

For fixed β, H_β decreases with σ, since

$$\frac{\partial_\sigma H_\beta}{\sigma} = \frac{N^2}{4s_0^3} - \frac{3}{2s_0} - \frac{3s_0}{4\beta^2} - \frac{2N}{\beta^2} < 0.$$

Here $H_\beta(0)$ denotes evaluation at $\sigma = 0$ with β fixed. The constants in (C.11) satisfy

N	$H_\beta(0)$ upper bound	s_0 lower bound
1	$19/3$	$7/5$
$3/2$	10	$9/4$
$N \ge 2$	$33N/4$	N

Indeed the first two values at zero are $13/4 + 85\sqrt{3}/48$ and $126/25 + 139\sqrt{3}/50$. For $\beta = N$, $H_N(0)/N = 4 + 9\sqrt{3}/4 + 1/(4\sqrt{3}) < 33/4$.

The range $\lambda \geq -k^2$. Here $\ell = k$, and

$$\frac{kd}{c} \leq \frac{N^2+\sigma^2}{k} < N,$$

because $d \leq p^2 + k^2$. The same denominator bound applies in every dimension:

$$\begin{aligned}\frac{D}{d} &= d + 4N(1+\sigma) + \frac{4N\{(N+1)^2 - p^2\}}{d} \\ &\geq 4N(1+\sigma) + 4\sqrt{N\{(N+1)^2-p^2\}} \\ &> 8k.\end{aligned}$$

For the last comparison, the two quantities being squared are positive, and their squared difference is

$$\begin{aligned}&N\{(N+1)^2 - p^2\} - \{N + (2-N)\sigma\}^2 \\ &\quad = 2(1-\sigma^2) + (N-1)(3+4\sigma+\sigma^2) + (N-1)^2(1+4\sigma-\sigma^2) \\ &\quad > 0.\end{aligned}$$

Thus (C.11) gives

$$R_{\mathrm{bd}} < \frac{C}{8N}\left(\frac{H_\beta(0)}{8} + \frac{N}{4s_0}\right) \leq \begin{cases} 2119/2688, & n=2, \\ 119/144, & n=3, \\ 481/512, & n \geq 4. \end{cases}$$

The range $\lambda = -\rho < -k^2$. For $n \geq 6$, Lemma C.1 gives $\lambda > -k^2$, so this range only occurs for $N = 1, 3/2, 2, 5/2$. Let $Z = (N+1)^2 + \rho$. Since $g_N = N^2 + \rho$, $d \leq g_N \leq c$ and $dc \geq g_N^2$. Keeping $1/A$ instead of replacing it by $1/(4N)$ gives

$$R_{\mathrm{bd}} \leq \frac{C\sqrt{\rho}}{8N}\frac{g_N\tau H_\beta + N/s_0}{4N + g_N^2\tau}.$$

Since $g_N < N^2 + 3N \leq 4N^2$, $H_\beta \geq 3N$, and $s_0 \geq N$,

$$\partial_\tau \frac{g_N\tau H_\beta + N/s_0}{4N + g_N^2\tau} = \frac{Ng_N(4H_\beta - g_N/s_0)}{(4N+g_N^2\tau)^2} > 0.$$

Thus $\tau \leq 1/Z$ gives

$$R_{\mathrm{bd}} \leq \frac{C\sqrt{\rho}}{8N}\frac{g_N H_\beta + ZN/s_0}{g_N^2 + 4NZ}.$$

With ρ and β fixed, the numerator decreases with σ, because

$$\begin{aligned}\frac{\partial_\sigma(g_N H_\beta + ZN/s_0)}{\sigma} &\leq \frac{N}{s_0^3}\left(Z - \frac{11N}{4}g_N\right) < 0, \\ Z - \frac{11N}{4}g_N &= 2N + 1 - \left(\frac{11N}{4} - 1\right)g_N \leq 3N - \frac{7}{2}N^3 < 0.\end{aligned}$$

Here $g_N \geq 2N^2$ and $s_0^2 \geq 2N^2$. Replacing σ by zero gives

$$R_{\mathrm{bd}} \leq \frac{C\sqrt{\rho}}{8N} \frac{(\rho + N^2)H_\beta(0) + \{\rho + (N+1)^2\}/\sqrt{3}}{(\rho + N^2)^2 + 4N\{\rho + (N+1)^2\}}. \tag{C.12}$$

The rational factor in (C.12) is bounded by

N	$H_\beta(0)$ upper bound	$1/\sqrt{3}$ upper bound	upper bound for the factor
1	$19/3$	$3/5$	$7/10$
$3/2$	10	$2/3$	$3/4$
$N \geq 2$	$33N/4$	$3/5$	$5/6$

For the first two rows, clearing the positive denominators reduces the comparisons to

$$21\rho^2 - 82\rho + 95 > 0, \qquad \frac{3}{4}\rho^2 - \frac{67}{24}\rho + \frac{1009}{192} > 0.$$

Their discriminants are -1256 and $-287/36$. For $N \geq 2$, the difference to be checked is

$$\frac{5}{6}(g_N^2 + 4NZ) - \frac{33}{4}Ng_N - \frac{3}{5}Z > 0.$$

Since $Z = g_N + 2N + 1$ and $g_N \geq 2N^2$, its derivative in g_N satisfies

$$\frac{5}{3}g_N - \frac{59}{12}N - \frac{3}{5} \geq \frac{10}{3}N^2 - \frac{59}{12}N - \frac{3}{5} > 0.$$

At $g_N = 2N^2$, with $y = N - 2 \geq 0$, the expression equals

$$\frac{100y^4 + 505y^3 + 794y^2 + 380y + 6}{30} > 0.$$

The root bounds give $\sqrt{\rho} < 7/4$ for $N = 1$, $\sqrt{\rho} < 15/7$ for $N = 3/2$, and $\sqrt{\rho}/N < 5/4$ for $N \geq 2$. Consequently the same formula gives

$$R_{\mathrm{bd}} < \begin{cases} 637/640, & n = 2, \\ 15/16, & n = 3, \\ 325/384, & n \geq 4. \end{cases}$$

□

AI usage statement: When applying the Hamiltonian method developed in [23] to the fractional setting in the present paper, the authors relied on GPT-6-Astra to carry out the highly involved computations. They also used it to edit the language and organize the exposition. The authors take full responsibility for all mathematical statements, arguments, citations, and conclusions in this paper.

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