

Some lecture notes for Math 6050E: PDEs, Fall 2016

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1 Variational methods

We discuss an example of the use of variational methods in obtaining existence of solutions.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded open set, with $n \geq 3$. Let $1 < p < \frac{n+2}{n-2}$. Then there exists a positive function $u \in C^3(\Omega)$ satisfies*

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

Proof. For $u \in H_0^1(\Omega)$, define the energy functional

$$I[u] = \frac{\int_{\Omega} |\nabla u|^2}{\left(\int_{\Omega} |u|^{p+1}\right)^{\frac{2}{p+1}}}$$

and

$$m = \inf_{u \in H_0^1(\Omega)} I[u].$$

Clearly, $m \geq 0$.

We will show in the below that m is attained by some function in $H_0^1(\Omega)$, which will be called a minimizer. Moreover, we will show that this minimizer will be a desired solution of (1).

Let $\{u_k\}$ be a sequence in $H_0^1(\Omega)$ such that

$$I[u_k] \rightarrow m \quad \text{as } k \rightarrow \infty.$$

By a normalization, we may assume that

$$\int_{\Omega} |u_k|^{p+1} = 1.$$

Consequently, u_k is a bounded sequence in $H_0^1(\Omega)$. Therefore, there exists $u \in H_0^1(\Omega)$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega).$$

Consequently,

$$\int_{\Omega} |\nabla u|^2 \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 \leq m.$$

Meanwhile, since $p + 1 < \frac{2n}{n-2}$, it follows from the Rellich compactness theorem that

$$u_k \rightarrow u \quad \text{strongly in } L^{p+1}(\Omega).$$

So,

$$\int_{\Omega} |u|^{p+1} = 1.$$

Therefore,

$$I[u] \leq m,$$

and thus,

$$I[u] = m.$$

Since $I[|u|] \leq I[u]$, we have that $u \geq 0$ or $u \leq 0$. So we assume that $u \geq 0$. Since $\int_{\Omega} |u|^{p+1} = 1$, $u \neq 0$.

So we have found a minimizer u for m . This implies that for every $\varphi \in H_0^1(\Omega)$,

$$0 = \left. \frac{d}{dt} \right|_{t=0} I[u + t\varphi].$$

A calculation yield

$$0 = 2 \int_{\Omega} \nabla u \cdot \nabla \varphi - 2m \int_{\Omega} u^p \varphi,$$

that is

$$\int_{\Omega} \nabla u \cdot \nabla \varphi - m \int_{\Omega} u^p \varphi = 0$$

So u is weak solution of (1) (in the distribution sense) after a scaling ($\tilde{u} = cu$ by some proper positive constant c).

Two regularity theory:

1. $W^{2,p}$ theory, that is for $q \in (1, \infty)$, if $-\Delta u = f$, where $f \in L^q$, then $u \in W_{loc}^{2,q}$.
2. Schauder theory, that is for $\alpha \in (0, 1)$, if $f \in C^\alpha$, then $u \in C_{loc}^{2,\alpha}$.

These two theories plus Sobolev embedding implies that the solutions of (1) $u \in C^3$. This is called bootstrap arguments. Note that such bootstrap arguments will NOT work when $p = \frac{n+2}{n-2}$.

Finally, u is positive in Ω by the strong maximum principle. \square

Next, we will show that the equation (1) does not have non-trivial solutions when $p \geq \frac{n+2}{n-2}$ and when the domain Ω is strictly star-shaped with respect to zero, that is $x \cdot \nu > 0$ everywhere on $\partial\Omega$. Here, ν is the outer normal of Ω

Theorem 1.2 (Pohozaev). *Let $p > \frac{n+2}{n-2}$ and Ω is strictly star-shaped with respect to zero. Suppose $u \in C^2(\bar{\Omega})$ is a solution of*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Then

$$u \equiv 0 \quad \text{in } \Omega.$$

Proof. Multiplying $x \cdot \nabla u$ on the both sides of (2) and doing some calculations, we have

$$\left(\frac{n-2}{2}\right) \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (\nu \cdot x) = \frac{n}{p+1} \int_{\Omega} |u|^{p+1}.$$

This is usually called the *Pohozaev identity*.

Multiplying u on the both sides of (2) and integrating by parts, we have

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} |u|^{p+1}.$$

Thus,

$$0 \leq \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (\nu \cdot x) = \left(\frac{n}{p+1} - \frac{n-2}{2}\right) \int_{\Omega} |u|^{p+1} \leq 0, \quad (3)$$

where we used the fact that $p > \frac{n+2}{n-2}$ in the last inequality. Therefore, $u \equiv 0$. \square

Remark 1.3. *Theorem 1.2 also holds for $p = \frac{n+2}{n-2}$. In this case, it follows from (3) that $\nabla u = 0$ on $\partial\Omega$, since $x \cdot \nu > 0$. Then by a unique continuation property, $u \equiv 0$, which is slightly complicated. However, if one additionally assumes that $u \geq 0$ in Ω , then*

$$0 = \int_{\Omega} -\Delta u = \int_{\Omega} u^p,$$

from which $u \equiv 0$ follows.

Remark 1.4. *The assumption that Ω is star shaped is necessary in Theorem 1.2. For example, if Ω is an annulus, then there exists a positive radial solution of*

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

2 Method of subsolutions and supersolutions

We will investigate the boundary -value problem for the nonlinear Poisson equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth and $\|f'\|_{L^\infty(\mathbb{R})} \leq C$ for some constant C .

Definition 2.1. (i). We say that $\bar{u} \in H^1(\Omega)$ is a weak supersolution of (4) if

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla v dx \geq \int_{\Omega} f(\bar{u}) v dx \quad \text{for every } v \in H_0^1(\Omega), v \geq 0 \text{ a.e.}$$

(ii). Similarly, we say that $\underline{u} \in H^1(\Omega)$ is a weak subsolution of (4) if

$$\int_{\Omega} \nabla \underline{u} \cdot \nabla v dx \leq \int_{\Omega} f(\underline{u}) v dx \quad \text{for every } v \in H_0^1(\Omega), v \geq 0 \text{ a.e.}$$

(ii). We say that $u \in H_0^1(\Omega)$ is a weak solution of (4) if

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla v dx = \int_{\Omega} f(\bar{u})v dx \quad \text{for every } v \in H_0^1(\Omega).$$

Note that if $\bar{u}, \underline{u} \in C^2(\Omega)$, then we have

$$-\Delta \bar{u} \geq f(\bar{u}), \quad -\Delta \underline{u} \leq f(\underline{u}).$$

Theorem 2.2. Assume there exist a weak supersolution \bar{u} and a weak subsolution \underline{u} of (4) satisfying

$$\underline{u} \leq 0, \quad \bar{u} \geq 0 \text{ on } \partial\Omega \text{ in the trace sense, } \quad \underline{u} \leq \bar{u} \text{ a.e. in } \Omega.$$

Then there exists a weak solution u of (4) such that

$$\underline{u} \leq u \leq \bar{u} \quad \text{a.e. in } \Omega.$$

Proof. Since $\|f'\|_{L^\infty(\mathbb{R})} \leq C$ for some constant C , we can choose a large $\lambda > 0$ such that

$$g(z) := f(z) + \lambda z$$

is an increasing function.

Now we denote $u_0 = \underline{u}$. Given $u_k, k = 0, 1, \dots$, we are going to inductively define u_{k+1} be the unique weak solution of the following linear boundary-value problem

$$\begin{cases} -\Delta u_{k+1} + \lambda u_{k+1} = f(u_k) + \lambda u_k & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We claim that

$$\underline{u} = u_0 \leq u_1 \leq u_2 \leq \dots \leq u_k \leq u_{k+1} \leq \dots \leq \bar{u}. \quad (5)$$

To prove this claim, we first note that for $k = 0$,

$$\int_{\Omega} \nabla(u_0 - u_1) \cdot \nabla v + \lambda(u_0 - u_1)v \leq 0 \quad \text{for every } v \in H_0^1(\Omega), v \geq 0 \text{ a.e.}$$

Choose $v = (u_0 - u_1)^+$, we obtain

$$\int_{\Omega} \nabla(u_0 - u_1) \cdot \nabla(u_0 - u_1)^+ + \lambda(u_0 - u_1)(u_0 - u_1)^+ \leq 0$$

This implies that $(u_0 - u_1)^+ = 0$, that is, $u_0 \leq u_1$. Now we assume inductively that

$$u_{k-1} \leq u_k.$$

Then we have

$$\begin{aligned} & \int_{\Omega} \nabla(u_k - u_{k+1}) \cdot \nabla v + \lambda(u_k - u_{k+1})v \\ &= (f(u_{k-1} + \lambda u_{k-1}) - f(u_k) - \lambda u_k)v \quad \text{for every } v \in H_0^1(\Omega), v \geq 0 \text{ a.e.} \end{aligned}$$

Choosing $v = (u_k - u_{k+1})^+$ will lead to that $(u_k - u_{k+1})^+ = 0$, that is $u_k \leq u_{k+1}$. This proves the claim on the monotonicity of the sequence $\{u_k\}$.

Then we show that $u_k \leq \bar{u}$ for all k . This is true for $k = 0$. Assume the induction that $u_k \leq \bar{u}$, then we have

$$\begin{aligned} & \int_{\Omega} \nabla(u_{k+1} - \bar{u}) \cdot \nabla v + \lambda(u_{k+1} - \bar{u})v \\ &= (f(u_k + \lambda u_k - f(\bar{u}) - \lambda \bar{u}))v \quad \text{for every } v \in H_0^1(\Omega), v \geq 0 \text{ a.e.} \end{aligned}$$

and let $v = (u_{k+1} - \bar{u})^+$, we have $(u_{k+1} - \bar{u})^+ = 0$, that is $u_{k+1} \leq \bar{u}$.

Now, let

$$u = \lim_{k \rightarrow \infty} u_k.$$

By dominated convergence theorem, $u_k \rightarrow u$ in $L^2(\Omega)$. Since $\|f(u_k)\|_{L^2(\Omega)} \leq C(\|u_k\|_{L^2} + 1) \leq C(\|\bar{u}\|_{L^2} + 1)$, we have that

$$\sup_k \|u_k\|_{H_0^1(\Omega)} < \infty.$$

Therefore, subject to a subsequence which is still denoted as $\{u_k\}$, we have $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$. To verify that u is a weak solution of (4), we notice that

$$\int_{\Omega} \nabla \bar{u}_{k+1} \cdot \nabla v + \lambda u_{k+1} v dx = \int_{\Omega} (f(u_k) + \lambda u_k) v dx \quad \text{for every } v \in H_0^1(\Omega).$$

Sending $k \rightarrow \infty$ and cancelling the term with λ , we have the confirm

$$\int_{\Omega} \nabla \bar{u} \cdot \nabla v dx = \int_{\Omega} f(u) v dx \quad \text{for every } v \in H_0^1(\Omega).$$

Note that the monotonicity (5) can also be proved by maximum principle if \bar{u}, u are smooth. Then by the Schauder estimate, we have that every u_k is smooth. Then

$$-\Delta(u_k - u_{k+1}) + \lambda(u_k - u_{k+1}) = f(u_{k-1}) + \lambda u_{k-1} - f(u_k) - \lambda u_k \leq 0$$

in the classical sense. Since $u_k \leq u_{k+1}$ on $\partial\Omega$, we have $u_k \leq u_{k+1}$ in Ω . □

3 The Dirichlet problem: Perron's method

In this section, we will discuss the solvability of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \quad (6)$$

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose $\partial\Omega \in C^2$. Let φ be a continuous function on $\partial\Omega$. Then there exists a solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of (6).*

Remark: That $\partial\Omega$ satisfies the exterior ball condition at every point on $\partial\Omega$ would be sufficient. That is, for every $\xi \in \partial\Omega$, there exists a ball $B_r(y)$ such that $\bar{B}_r(y) \cap \bar{\Omega} = \{\xi\}$.

We need some generalized subharmonic functions first.

Definition 3.2. A $C^0(\Omega)$ function u is called subharmonic (or superharmonic) in Ω if for every ball $B \subset\subset \Omega$ and every harmonic function h in B satisfying $u \leq (\geq)h$ on ∂B , we also have $u \leq (\geq)h$ in B .

Such subharmonic functions have several useful properties:

1. If u is subharmonic in a connected domain in Ω , then it satisfies the strong maximum principle in Ω . That is, if v is super harmonic in Ω with $v \geq u$ on $\partial\Omega$, then either $u > v$ in Ω or $v \equiv u$. To prove this, we suppose the contrary that at some point $x_0 \in \Omega$, we have

$$(u - v)(x_0) = \sup_{\Omega}(u - v) = M \geq 0,$$

but there is a ball $B = B_r(x_0)$ such that $u - v \not\equiv M$ on ∂B . Let \bar{u}, \bar{v} be the harmonic functions respectively equal to u, v on ∂B (this is can be achieved by Green's representation). Then one sees that

$$M \geq \sup_{\partial B}(\bar{u} - \bar{v}) \geq (\bar{u} - \bar{v})(x_0) \geq (u - v)(x_0) = M.$$

Therefore, every inequality in the above has to be an equality. By the strong maximum principle for harmonic functions, it follows that $\bar{u} - \bar{v} \equiv M$. Thus $u - v \equiv M$, which is a contradiction.

2. Let u be subharmonic in Ω and B is a ball strictly contained in Ω . Denote \bar{u} as the harmonic function in B satisfying $\bar{u} = u$ on ∂B . We define in Ω the harmonic lifting of u by

$$U(x) = \begin{cases} \bar{u}(x), & x \in B, \\ u(x), & x \in \Omega \setminus B. \end{cases}$$

Then the function U is also subharmonic in Ω . This can be proved as follows. Let $B' \subset \Omega$ be an arbitrary ball. Let h be harmonic in B such that $h = U$ on $\partial B'$. Then $h \geq u$ on $\partial B'$, and thus, $h \geq u$ in B' . So $h \geq U$ in $B' \setminus B$. Since U is harmonic in B , by maximum principle, we have $h \geq U$ in $B \cap B'$. Hence, $U \leq h$ in B' , and thus, U is subharmonic in Ω .

3. Let u_1, u_2, \dots, u_k be subharmonic in Ω . Then the function $u(x) = \max(u_1, \dots, u_k)$ is also subharmonic which is a trivial consequence of the definition of subharmonic functions.

Now let us prove Theorem 3.1. A $C^0(\bar{\Omega})$ function u is called a subsolution of (6) if u is subharmonic, and $u \leq \varphi$ on $\partial\Omega$. Similarly, a $C^0(\bar{\Omega})$ function v is called a supersolution of (6) if v is superharmonic, and $v \geq \varphi$ on $\partial\Omega$. Denote S be the set of all subsolutions of (6). $S \neq \emptyset$ since the constant function $\inf_{\partial\Omega} \varphi$ is a subsolution.

Proposition 3.3. The function $u(x) = \sup_{v \in S} v(x)$ is harmonic in Ω .

Proof. Since $\sup_{\partial\Omega} \varphi$ is a super solution, we know that $v \leq \sup_{\partial\Omega} \varphi$ for every $v \in S$. Thus, u is well-defined.

Let $y \in \Omega$ be a fixed point. There exists $v_k \in S$ such that

$$v_k(y) \rightarrow u(y).$$

By replacing v_k with $\max(v_k, \inf \varphi)$, we may assume that the sequence $\{v_k\}$ is bounded. Choose $R > 0$ such that $B = B_R(y) \subset\subset \Omega$, and define V_k as the harmonic lifting of v_k in B . Then $V_k \in$

S. By Harnack inequality and gradient estimate, there exists a subsequence $\{V_{n_k}\}$ converging uniformly in every ball $B_\rho(y)$ with $\rho < R$ to a harmonic function in B . Clearly, $v \leq u$ in B and $v(y) = u(y)$.

We are going to show that $v = u$ in B . Suppose $v(z) < u(z)$ for some $z \in B$. Then there exists $w \in S$ such that $v(z) < w(z)$. Let $w_k = \max(V_{n_k}, w)$ and W_k be its harmonic lifting in B . As before, by Harnack and gradient estimate, a subsequence of W_k converges to a harmonic function \bar{w} in B satisfying $v \leq \bar{w} \leq u$. Since $v(y) = u(y)$, we have $v(y) = \bar{w}(y)$, and by strong maximum principle. $v \equiv \bar{w}$, which contradicts with $v(z) < w(z) \leq \bar{w}(z)$. \square

Therefore, the u obtained in the above proposition is a candidate solution of (6). In the below, we will show that this u indeed satisfies the boundary condition $u = \varphi$ on $\partial\Omega$, which will finish the proof of Theorem 3.1. As along as the boundary $\partial\Omega$ satisfies the exterior boundary, we can construct some barrier functions for your purpose.

Proof of Theorem 3.1. For $\xi \in \partial\Omega$, there exists a ball $B = B_R(y)$ such that $\bar{B} \cap \bar{\Omega} = \{\xi\}$. Define

$$w(x) := \begin{cases} R^{2-n} - |x - y|^{2-n}, & n \geq 3 \\ -\log R + \log |x - y|, & n = 2. \end{cases}$$

Note that $w(\xi) = 0$, $w(x) > 0$ in $\bar{\Omega} \setminus \{\xi\}$, and w is harmonic.

Let $M = \sup_{\partial\Omega} \varphi$. Since φ is a continuous function, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(\xi)| < \varepsilon, \quad |x - \xi| < \delta.$$

Choose κ large enough such that

$$\kappa w(x) \geq 2M \quad |x - \xi| \geq \delta.$$

Then $\varphi(\xi) + \varepsilon + \kappa w$ and $\varphi(\xi) - \varepsilon - \kappa w$ are respectively supersolution and subsolutions of (6). Therefore,

$$\varphi(\xi) - \varepsilon - \kappa w(x) \leq u(x) \leq \varphi(\xi) + \varepsilon + \kappa w(x).$$

Hence

$$|u(x) - \varphi(\xi)| \leq \varepsilon + \kappa w(x).$$

Since $w(\xi) = 0$, we obtain that

$$u(x) \rightarrow \varphi(\xi) \quad \text{as } x \rightarrow \xi.$$

\square

4 Schauder estimates

We first prove the following Schauder estimates for Poisson's equation.

Theorem 4.1. Let $\alpha \in (0, 1)$, $f \in C^\alpha(B_1)$, and $u \in C^2(B_1)$ be a solution of

$$-\Delta u = f \quad \text{in } B_1.$$

Then there exists $C > 0$ depending only on α and the dimension n such that

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)})$$

Proof. First of all, by multiplying some proper constant, we can assume that $\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)} \leq 1$. Let $\rho = 1/2$. For $k = 1, 2, 3, \dots$, let v_k be the solution of

$$\begin{cases} -\Delta v_k = f(0) & \text{in } B_k, \\ v_k = u & \text{on } \partial B_k, \end{cases}$$

where $B_k = B_{\rho^k}$ centered at the origin.

Claim 1: $\|v_k - v\|_{L^\infty(B_k)} \leq C\rho^{(2+\alpha)k}$.

This claim can be proved as follows. Let $\tilde{v}_k(x) = \rho^{-2k}(v_k - u)(\rho^k x)$, where $x \in B_1$. Then

$$\begin{cases} -\Delta \tilde{v}_k = f(0) - f(\rho^k x) & \text{in } B_1, \\ \tilde{v}_k = 0 & \text{on } \partial B_1, \end{cases}$$

By the maximum principle (one of the homework problem), we have

$$\|\tilde{v}_k\|_{L^\infty(B_1)} \leq C(\|u\|_{L^\infty(\partial B_1)} + \|f(0) - f(\rho^k x)\|_{L^\infty(B_1)}) \leq C\rho^{\alpha k},$$

from which the claim follows.

Claim 2: $\|v_k - v_{k+1}\|_{L^\infty(B_{k+1})} \leq C\rho^{(2+\alpha)k}$.

This is because $v_k - v_{k+1}$ is harmonic in B_{k+1} , and therefore,

$$\|v_k - v_{k+1}\|_{L^\infty(B_{k+1})} = \|v_k - v_{k+1}\|_{L^\infty(\partial B_{k+1})} \leq \|v_k - u\|_{L^\infty(\partial B_{k+1})} \leq C\rho^{(2+\alpha)k},$$

where in the last inequality, we use Claim 1.

Let $w_k = v_{k+1} - v_k$, $w_0 = v_1$. Then we know that from Claim 2 that, for every $x \in B_{k+2}$ we have

$$|\nabla^j w_k(x)| \leq C\rho^{(2+\alpha-j)k}.$$

for $\rho^{i+3} \leq |x| < \rho^{i+2}$,

$$\begin{aligned} & |u(x) - \sum_{\ell=0}^{\infty} w_\ell(0) - \sum_{\ell=0}^{\infty} Dw_\ell(0) \cdot x - \sum_{\ell=0}^{\infty} \frac{1}{2} x^T D^2 w_\ell(0) x| \\ & \leq |u(x) - \sum_{\ell=0}^i w_\ell(x)| + \left| \sum_{\ell=0}^i w_\ell(x) - \sum_{\ell=0}^i w_\ell(0) - \sum_{\ell=0}^i Dw_\ell(0) \cdot x - \sum_{\ell=0}^i \frac{1}{2} x^T D^2 w_\ell(0) x \right| \\ & \quad + \left| \sum_{\ell=i+1}^{\infty} w_\ell(0) \right| + \left| \sum_{\ell=i+1}^{\infty} Dw_\ell(0) \cdot x \right| + \frac{1}{2} \left| \sum_{\ell=i+1}^{\infty} x^T D^2 w_\ell(0) x \right| \\ & \leq \rho^{(2+\alpha)(i+1)} + 2c_2 |x|^3 \sum_{\ell=0}^i \rho^{(\alpha-1)\ell} + \sum_{\ell=i+1}^{\infty} \rho^{(2+\alpha)\ell} + |x| \sum_{\ell=i+1}^{\infty} c_2 \rho^{(1+\alpha)\ell} \\ & \quad + |x|^2 \sum_{\ell=i+1}^{\infty} c_2 \rho^{\alpha\ell} \\ & \leq C_3 |x|^{2+\alpha}. \end{aligned}$$

So we have proved that there exists a second order polynomial P such that

$$|u(x) - P(x)| \leq C|x|^{2+\alpha}$$

where all the coefficients of the polynomial are universally bounded.

This leads to the conclusion of the theorem. \square

Theorem 4.2. *Let $\alpha \in (0, 1)$, $f \in C^\alpha(B_1)$, $a_{ij}(x) \in C^\alpha(B_1)$, and $u \in C^2(B_1)$ be a solution of*

$$-a_{ij}(x)u_{ij}(x) = f \quad \text{in } B_1,$$

where $\lambda I \leq (a_{ij}(x)) \leq \lambda^{-1}I$ in B_1 . Then there exists $C > 0$ depending only on α, λ and the dimension n such that

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}).$$

Proof. Let $\rho = 1/2$. For $k = 1, 2, 3, \dots$, let v_k be the solution of

$$\begin{cases} -a_{ij}(0)\partial_{ij}v_k = f(0) & \text{in } B_k, \\ v_k = u & \text{on } \partial B_k, \end{cases}$$

where $B_k = B_{\rho^k}$ centered at the origin. Then for $\tilde{v}_k = v_k - u$, we have

$$-a_{ij}(0)\partial_{ij}\tilde{v}_k = f(0) - f(x) + (a_{ij}(0) - a_{ij}(x))u_{ij}(x) \quad \text{in } B_k,$$

Following the proof in Theorem 4.1, we can show that

$$[\nabla^2 u]_{C^\alpha(B_{1/4})} \leq C(\|f\|_{C^\alpha(B_{1/2})} + \|u\|_{L^\infty(B_{1/2})} + \|\nabla^2 u\|_{L^\infty(B_{1/2})}), \quad (7)$$

where the constant C depends only on α, n, λ . Now that the domain on the two sides of the inequality are DIFFERENT. Then the conclusion follows from the next iteration lemma. \square

M. Giaquinta and E. Giusti, On the regularity of the minima of variational integrals. Acta Math. 148 (1982), 31–46. Lemma 1.1.

Lemma 4.3. *Let $h(t)$ be a nonnegative bounded function defined for $0 \leq T_0 \leq t \leq T_1$. Suppose that for $T_0 \leq t < s \leq T_1$ we have*

$$h(t) \leq A(s - t)^{-\alpha} + B + \theta h(s)$$

where A, B, α, θ are nonnegative constants, and $\theta < 1$. Then there exists a constant $C > 0$, depending only on α, θ such that for every $\rho, R, T_0 \leq \rho < R \leq T_1$, we have

$$h(\rho) \leq C(A(R - \rho)^{-\alpha} + B).$$

Proof. Consider the sequence $\{t_j\}$ defined by

$$t_0 = \rho, \quad t_{j+1} - t_j = (1 - \tau)\tau^j(R - \rho)$$

with $\tau \in (0, 1)$. By iteration

$$h(t_0) \leq \theta^k h(t_k) + \left(\frac{A}{(1 - \tau)^\alpha} (R - \rho)^{-\alpha} + B \right) \sum_{i=0}^{k-1} \theta^i \tau^{-i\alpha}.$$

We choose now τ such that $\tau^{-\alpha}\theta < 1$ and let $k \rightarrow \infty$. Then the conclusion follows. \square

Proof of Theorem 4.2 continued. Let

$$h(t) = [\nabla^2 u]_{C^\alpha(B_t)}.$$

We will show that this h will satisfies Lemma 4.3. Let $0 \leq t < s \leq 1$. For every $z \in B_t$, choose $r = s - t$. For $x \in B_1$, let

$$u_r(x) = u(z + rx), \quad \tilde{a}_{ij}(x) = a_{ij}(z + rx), \quad f_r(x) = r^2 f(z + rx).$$

Then

$$-\tilde{a}_{ij} \partial_{ij} u_r(x) = f_r(x) \quad \text{in } B_1.$$

Therefore, the estimate (7) holds for u_r , i.e.,

$$[\nabla^2 u_r]_{C^\alpha(B_{1/4})} \leq C(\|f_r\|_{C^\alpha(B_{1/2})} + \|u_r\|_{L^\infty(B_{1/2})} + \|\nabla^2 u_r\|_{L^\infty(B_{1/2})}),$$

Scaling back, we have

$$[\nabla^2 u]_{C^\alpha(B_{r/4}(z))} \leq Cr^{-2-\alpha}(\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}) + Cr^{-\alpha} \|\nabla^2 u\|_{L^\infty(B_{r/2}(z))}.$$

By a covering, we have

$$[\nabla^2 u]_{C^\alpha(B_t)} \leq Cr^{-3}(\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}) + Cr^{-1} \|\nabla^2 u\|_{L^\infty(B_{(s+t)/2})}.$$

(From Gilbarg-Trudinger's book) We have the following interpolation lemma (the following is not sharp): there exists $C > 0$ independent of s, t such that for all $\varepsilon > 0$,

$$\|\nabla^2 u\|_{L^\infty(B_{(s+t)/2})} \leq \varepsilon r^{-2} [\nabla^2 u]_{C^\alpha(B_s)} + C\varepsilon^{-\frac{2+\alpha}{\alpha}} r^{-2} \|u\|_{L^\infty(B_s)}.$$

Choose $\varepsilon = r^3/2C$, we have for some $\beta > 0$

$$[\nabla^2 u]_{C^\alpha(B_t)} \leq \frac{1}{2} [\nabla^2 u]_{C^\alpha(B_s)} + Cr^{-\beta} (\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}).$$

Then the conclusion from Lemma 4.3. □

5 De Giorgi estimates

Let $a_{ij} \in L^\infty(B_1)$ and uniformly elliptic, that is

$$\lambda I \leq a_{ij}(x) \leq \Lambda I \quad \text{in } B_1$$

for some $\lambda, \Lambda > 0$.

Theorem 5.1 (De Giorgi, 1958). *Let $u \in H^1(B_1)$ be a weak solution of*

$$-\partial_j(a_{ij} \partial_i u) = 0 \quad \text{in } B_1 \tag{8}$$

Then there exists $\alpha \in (0, 1)$ and $C > 0$, both of which depends only on n, λ and Λ , such that

$$\|u\|_{C^\alpha(B_{1/2})} \leq C \|u\|_{L^2(B_1)}.$$

The proof of this theorem consists of two steps: from L^2 to L^∞ , and from L^∞ to C^α .

Proposition 5.2 (from L^2 to L^∞). *Let $u \in H^1(B_1)$ be a weak subsolution of (8), that is,*

$$-\partial_j(a_{ij}\partial_i u) \leq 0 \quad \text{in } B_1.$$

Then there exists a constant $\delta = \delta(n, \lambda, \Lambda)$ such that if $\|u^+\|_{L^2(B_1)} < \delta$, then we have

$$\|u^+\|_{L^\infty(B_{1/2})} \leq 1.$$

Applying this proposition to $(\sqrt{\delta}/\|u^+\|_{L^2(B_1)})u$, we obtain the following theorem:

Theorem 5.3 (from L^2 to L^∞). *Let $u \in H^1(B_1)$ be a weak subsolution of (8), that is,*

$$-\partial_j(a_{ij}\partial_i u) \leq 0 \quad \text{in } B_1.$$

Then

$$\|u^+\|_{L^\infty(B_{1/2})} \leq C(n, \lambda, \Lambda)\|u^+\|_{L^2(B_1)}$$

Consequently, if $u \in H^1(B_1)$ is a weak solution of (8), then

$$\|u\|_{L^\infty(B_{1/2})} \leq C(n, \lambda, \Lambda)\|u\|_{L^2(B_1)}$$

To prove Proposition 5.2, we shall use the following Caccioppoli inequality (or energy estimate)

Lemma 5.4 (Caccioppoli inequality). *Let $u \in H^1(B_1)$ be a weak subsolution of (8) and $\varphi \in C_c^\infty(B_1)$, we have*

$$\int_{B_1} (\nabla(\varphi u^+))^2 \leq C(n, \lambda, \Lambda) \int_{B_1} (u^+)^2 (\nabla\varphi)^2.$$

Proof. Since u is a subsolution, we have

$$\int_{B_1} a_{ij}\partial_i u \partial_j(\varphi^2 u^+) \leq 0.$$

That is

$$\int_{B_1} a_{ij}\varphi^2 \partial_i u^+ \partial_j u^+ + 2 \int_{B_1} a_{ij}\varphi u^+ \partial_i u^+ \partial_j \varphi \leq 0.$$

Then

$$\begin{aligned} \frac{1}{\lambda} \int_{B_1} (\nabla(\varphi u^+))^2 &\leq \int_{B_1} a_{ij}\partial_i(\varphi u^+) \partial_j(\varphi u^+) \\ &\leq C \int_{B_1} |\nabla\varphi|^2 (u^+)^2 + \int_{B_1} a_{ij}\varphi^2 \partial_i u^+ \partial_j u^+ + \int_{B_1} a_{ij}\varphi u^+ \partial_i u^+ \partial_j \varphi \\ &\quad + \int_{B_1} a_{ij}\varphi u^+ \partial_i u \partial_j \varphi \\ &\leq C \int_{B_1} |\nabla\varphi|^2 (u^+)^2 + 3(a_{ij} + a_{ji})\varphi u^+ \partial_i u^+ \partial_j \varphi \\ &\leq C \int_{B_1} |\nabla\varphi|^2 (u^+)^2 + |u^+ \partial_i(u^+ \varphi) \partial_j \varphi| \\ &\leq C \int_{B_1} |\nabla\varphi|^2 (u^+)^2 + \frac{1}{2\lambda} (\nabla(\varphi u^+))^2, \end{aligned}$$

where in the last inequality we have used Holder's inequality. □

Lemma 5.5. *Let $u \in H^1(B_1)$ be a weak solution of (8). Then u^+ is a weak subsolution of (8).*

Proof. Exercise. □

Now we can prove Proposition 5.2.

Proof of Proposition 5.2. We will work on a family of ball

$$B_k = \{|x| \leq 1/2 + 2^{-k}\}$$

and the family of truncated functions

$$u_k = (u - (1 - 2^{-k}))^+.$$

Define

$$U_k = \int_{B_k} u_k^2.$$

Let φ_k be a sequence of nonnegative shrinking cut-off functions: $\varphi_k = 1$ in B_k and $\varphi = 0$ in B_{k-1}^c . Also, $|\varphi_k| \leq C2^k$.

Note that where $u_{k+1} > 0$, then $u > 1 - 2^{-k-1} = 1 - 2^{-k} + 2^{-k-1}$, and thus, $u_k > 2^{-k-1}$. Therefore,

$$\{x : \varphi_{k+1}u_{k+1} > 0\} \subset \{x \in B_k : u_k > 2^{-k-1}\}.$$

We have from the Sobolev inequality with $p = \frac{2n}{n-2}$ (or any p if $n = 2$)

$$\left(\int |\varphi_{k+1}u_{k+1}|^p \right)^{2/p} \leq C \int |\nabla(\varphi_{k+1}u_{k+1})|^2.$$

From Holder inequality

$$\int |\varphi_{k+1}u_{k+1}|^2 \leq \left(\int |\varphi_{k+1}u_{k+1}|^p \right)^{2/p} \cdot |\{x : \varphi_{k+1}u_{k+1} > 0\}|^{\frac{2}{n}}.$$

Therefore,

$$U_{k+1} \leq C \int |\nabla(\varphi_{k+1}u_{k+1})|^2 \cdot |\{x : \varphi_{k+1}u_{k+1} > 0\}|^{\frac{2}{n}}.$$

Using Lemma 5.4, we have

$$U_{k+1} \leq C2^{2k} \int_{\text{supp}(\varphi_{k+1})} (u_{k+1})^2 \cdot |\{x : \varphi_{k+1}u_{k+1} > 0\}|^{\frac{2}{n}}.$$

The support of φ_{k+1} in contained in B_k and $u_{k+1} \leq u_k$, we have

$$U_{k+1} \leq C2^{2k} \int_{B_k} u_k^2 \cdot |\{x : \varphi_{k+1}u_{k+1} > 0\}|^{\frac{2}{n}}.$$

Since

$$\{x : \varphi_{k+1}u_{k+1} > 0\} \subset \{x \in B_k : u_k > 2^{-k-1}\}.$$

we have

$$|\{x : \varphi_{k+1}u_{k+1} > 0\}|^{\frac{2}{n}} \leq |\{x \in B_k : u_k > 2^{-k-1}\}| \leq 2^{\frac{4k}{n}} \left(\int_{B_k} u_k^2 \right)^{\frac{2}{N}} = 2^{\frac{4k}{n}} U_k^{\frac{2}{n}}.$$

where the Chebyshev inequality was used in the last inequality.

Therefore, we have

$$U_{k+1} \leq C2^{4k}(U_k)^{1+\frac{2}{n}}.$$

The as long as $U_0 = \delta$ is small enough, we will have $U_k \rightarrow 0$ as $k \rightarrow \infty$. This proves that $u^+ \leq 1$ in $B_{1/2}$. In fact, U_k decays faster than geometric sequences. \square

The next step is to prove the regularity from L^∞ to C^α . The key step is this oscillation lemma.

Proposition 5.6. *Let $u \in H^1(B_2)$ be a weak subsolution of (8) in B_2 . Suppose that $u \leq 1$ in B_2 . Assume that $|B_1 \cap \{v \leq 0\}| \geq \mu (> 0)$. Then $u \leq 1 - \gamma$ in $B_{1/2}$, where γ depends only on λ, Λ, n, μ .*

In other words, if u is a subsolution and smaller than one, and is “far from 1” in a set of non-trivial measure, then u cannot get too close to 1 in $B_{1/2}$.

Let us postpone the proof to the end, we shall first use this proposition to prove the following Holder estimate.

Theorem 5.7. *Let $u \in H^1(B_3)$ be a weak solution of (8) in B_3 . Then $u \in C^\alpha(B_{1/2})$. Moreover,*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^2(B_3)}.$$

Here, each of the constants α and C depend only on n, λ, Λ .

Proof. First of all, we know from Theorem 5.3 that

$$\|u\|_{L^\infty(B_2)} < C\|u\|_{L^2(B_3)}.$$

Denote

$$\text{osc}_\Omega u = \sup_\Omega u - \inf_\Omega u.$$

Consider the function

$$v(x) = \frac{2}{\text{osc}_{B_2} u} \left(u(x) - \frac{\sup_{B_2} u + \inf_{B_2} u}{2} \right),$$

We have that $-1 \leq v \leq 1$ in B_2 .

Assume that $|B_1 \cap \{v \leq 0\}| \geq |B_1|/2$, then we can apply Proposition 5.6 on v to obtain $\text{osc}_{B_{1/2}} v \leq 2 - \gamma$. Hence, $\text{osc}_{B_{1/2}} u \leq (1 - \gamma/2)\text{osc}_{B_2} u$.

If $|B_1 \cap \{v \geq 0\}| \geq |B_1|/2$, then the same result holds by working with $-v$.

Therefore, we have proved that

$$\text{osc}_{B_{1/2}} u \leq \tilde{\gamma} \text{osc}_{B_2} u \tag{9}$$

with $\tilde{\gamma} = 1 - \gamma/2 < 1$ depends only on n, λ, Λ .

Take any x_0 in $B_{1/2}$, and introduce the rescaled functions

$$u_k(y) = u(x_0 + y/2^n), \quad a_{ij}^{(k)}(y) = a_{ij}(x_0 + y/2^n).$$

Then $a_{ij}^{(k)}$ satisfies the same assumptions as a_{ij} , and u_k is a weak solution of (8) with a_{ij} replaced by $a_{ij}^{(k)}$. We apply recursively (9) to u_k to obtain

$$\sup_{|x-x_0| \leq 2^{-k}} |u(x_0) - u(x)| \leq 2\|u\|_{L^\infty(B_2)} \tilde{\gamma}^k.$$

Then for y such that $2^{-k-1} \leq |x - x_0| < 2^{-k}$, we have

$$|u(x_0) - u(x)| \leq 2\|u\|_{L^\infty(B_2)} \tilde{\gamma}^k \leq \|u\|_{L^\infty(B_2)} 2 \cdot 2^{-(k+1)\alpha} 2^{(k+1)\alpha} \tilde{\gamma}^k \leq C\|u\|_{L^\infty(B_2)} |x - x_0|^\alpha,$$

as long as the α is chosen by $2^\alpha \tilde{\gamma} = 1$, that is,

$$\alpha = -\frac{\log \tilde{\gamma}}{\log 2} > 0.$$

Note that this estimate is independent of the choice of x_0 . Then the conclusion follows. \square

Now we are going to prove Proposition 5.6. We first note that if the set

$$|\{u \leq 0\} \cap B_1| \geq |B_1| - \delta/4$$

then

$$\|u^+\|_{L^2(B_1)} \leq \sqrt{\delta}/2,$$

and thus, by Proposition 5.2, we have

$$u \leq 1/2.$$

So we much bridge the gap between knowing that $|\{u \leq 0\} \cap B_1| \geq \mu$ and knowing $|\{u \leq 0\} \cap B_1| \geq |B_1| - \delta/4$. The main tool is the so-called De Giorgi isoperimetric inequality, which rough says that for an H^1 function u , it must pay in measure to increase from $u = 0$ to $u = 1$.

Lemma 5.8. *Consider a function w such that $\int_{B_1} |\nabla w^+|^2 \leq C_0$. Define*

$$\begin{aligned} |A| &= |\{w \leq 0\} \cap B_1|, \\ |E| &= |\{w \geq 1\} \cap B_1|, \\ |D| &= |\{0 < w < 1\} \cap B_1|. \end{aligned}$$

Then there exists a constant C_1 depends only on n such that

$$C_0|D| \geq C_1(|A||E|^{1-\frac{1}{n}})^2.$$

Proof. Consider $\bar{w} = \sup(0, \inf(w, 1))$. Note that $\nabla \bar{w} = \nabla w^+ \chi_{\{0 \leq w \leq 1\}}$.

For $x_0 \in E$ we integrate over lines in to $x \in A$

$$\begin{aligned} -1 = -\bar{w}(x_0) &= \int_0^1 \frac{d}{dt} \bar{w}(tx + (1-t)x_0) \\ &= \int_0^1 \nabla \bar{w}(tx + (1-t)x_0) \cdot (x - x_0) \end{aligned}$$

Integrating over A , we obtain that

$$\begin{aligned} |A| &\leq \left| \int_0^1 dt \int_A \nabla \bar{w}(tx + (1-t)x_0) \cdot (x - x_0) dx \right| \\ &\leq \int_0^1 dt \int_0^2 r^{n-1} dr \int_{\partial B_r(x_0) \cap B_1} |\nabla \bar{w}(tx + (1-t)x_0)| r dS(x) \\ &\leq \left| \int_0^1 dt \int_0^2 r^n dr \int_{\partial B_{tr}(x_0) \cap B_1} |\nabla \bar{w}(z)| dS(z) \right| \\ &\leq \left| \int_0^2 r^{n-1} dr \int_0^r dt \int_{\partial B_t(x_0) \cap B_1} |\nabla \bar{w}(z)| dS(z) \right| \\ &\leq \left| \int_0^2 r^{n-1} dr \int_0^r t^{n-1} dt \int_{\partial B_t(x_0) \cap B_1} \frac{|\nabla \bar{w}(z)|}{|z - x_0|^{n-1}} dS(z) \right| \\ &\leq C \int_{B_1} \frac{|\nabla \bar{w}(z)|}{|z - x_0|^{n-1}} dz \\ &\leq C \int_D \frac{|\nabla \bar{w}(z)|}{|z - x_0|^{n-1}} dz \end{aligned}$$

Integrating $x_0 \in E$, we have

$$|A||E| \leq C \int_D |\nabla \bar{w}(z)| dz \int_E \frac{1}{|z - x_0|^{n-1}} dx_0$$

Note that

$$\int_E \frac{1}{|z - x_0|^{n-1}} dx_0 \leq \int_B \frac{1}{|x|^{n-1}} dx = C|E|^{1/n},$$

where B is the ball centered at the origin such that $|B| = |E|$.

Therefore, by Holder inequality,

$$|A||E| \leq C|E|^{1/n}|D|^{1/2} \left(\int_D |\nabla \bar{w}(z)|^2 dz \right)^{1/2}.$$

□

Proof of Proposition 5.6. We consider the following new sequence of truncation

$$w_k = 2^k(u - (1 - 2^{-k}))$$

Note that $w_k \leq 1$ in B_2 and is also a subsolution. From the Caccioppoli inequality, we have

$$\int_{B_1} |\nabla w^+|^2 \leq C_0.$$

We also have $|\{w_k \leq 0\} \cap B_1| \geq \mu$. We will apply Lemma 5.8 recursively on $2w_k$ as long as

$$\int_{B_1} (w_{k+1}^+)^2 \geq \delta.$$

Note that

$$|\{2w_k \geq 1\} \cap B_1| = |\{w_{k+1} \geq 0\} \cap B_1| \geq \int_{B_1} (w_{k+1}^+)^2 \geq \delta.$$

From Lemma 5.8, there exists a positive constant β independent of k such that

$$|\{0 < w_k < 1/2\} \cap B_1| \geq \beta.$$

Therefore,

$$|\{w_{k+1} \leq 0\} \cap B_1| = |\{2w_k \leq 1\} \cap B_1| \geq |\{w_k \leq 0\} \cap B_1| + \beta \geq \dots \geq \mu + k\alpha.$$

This clearly fails after a finite number of k . At this k_0 , we have for sure that

$$\int_{B_1} (w_{k_0+1}^+)^2 \leq \delta.$$

Then Proposition 5.2 implies that

$$w_{k_0+1} \leq 1/2 \quad \text{in } B_{1/2}.$$

Rescaling back, we have

$$u \leq 1 - 2^{k_0+1} + 2^{k_0+2} = 1 - 2^{k_0+2} \quad \text{in } B_{1/2}.$$

□