Homework for Math 5281: PDEs, Spring 2019

Tianling Jin

March 26, 2019

Deadline: 10 May, 2019.

Set 2

In this homework set, we always assume the coefficients of the various PDEs are smooth and satisfy the uniform ellipticity condition. Also, $\Omega \subset \mathbb{R}^n$ is always an open, bounded set with smooth boundary $\partial\Omega$.

Almost all the problems below are from Evans' book.

1. Consider the Laplacian equation with potential function $c(x)$:

$$-\Delta u + cu = 0, \tag{1}$$

and the equation in divergence form

$$-\operatorname{div}(a\nabla u) = 0, \tag{2}$$

where the function $a(x)$ is positive.
(a): Show that if u solves (1) and $w > 0$ also solves (1), then $v := u/w$ solves (2) for $a := w^2$.
(b): Conversely, show that if v solves (2), then $u := va^{1/2}$ solves (1) for some potential c.

2.A function $u \in H_0^2(\Omega)$ is a weak solution of this boundary value problem for the biharmonic equation

$$\begin{cases} \Delta^2 u & = f \quad \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3}$$

provided

$$\int_\Omega \Delta u \Delta v dx = \int_\Omega f v dx \quad \text{for all } v \in H_0^2(\Omega).$$

Given $f \in L^2(\Omega)$, prove that there exists a unique weak solution of (3).

3. Assume Ω is connected. A function $u \in H^1(\Omega)$ is a weak solution of the Neumann's problem

$$\begin{cases} \Delta u & = f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{4}$$

if

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \text{for all } v \in H^1(\Omega).$$

Suppose $f \in L^2(\Omega)$. Prove that (4) has a weak solution if and only if

$$\int_\Omega f dx = 0.$$

4. Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$.

5. Let u be a smooth solution of $Lu := -\sum_{i,j=1}^n a^{ij} u_{ij} = 0$ in Ω. Assume all the coefficients a_{ij} are smooth and have bounded derivatives. Set $v := |\nabla u|^2 + \lambda u^2$. Show that $Lv \leq 0$ in Ω if λ is large enough. Then prove that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C(\|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}).$$

6. Assume Ω is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary value problem

$$\begin{cases} \Delta u & = 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \end{cases}$$

are constant functions.

7. Assume $u \in H^1(\Omega)$ is a bounded weak solution of

$$-\sum_{i,j=1}^n (a^{ij} u_i)_j = 0 \quad \text{in } \Omega.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex and smooth function. Set $w = \phi(u)$. Show that w is a weak subsolution, that is,

$$B[w, v] \leq 0 \quad \text{for all } v \in H^1(\Omega), v \geq 0.$$

8. We say that the uniformly elliptic operator

$$Lu := -\sum_{i,j=1}^n a^{ij} u_{ij} + b^i u_i + cu$$

satisfies the weak maximum principle if for all $u \in C^2(\Omega) \cap C(\overline{\Omega})$

$$\begin{cases} Lu & \leq 0 \quad \text{in } \Omega \\ u & \leq 0 \quad \text{on } \partial\Omega \end{cases}$$

implies that $u \leq 0$ in Ω. Suppose that there exists a function $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lv \geq 0$ in Ω and $v > 0$ in Ω. Show that L satisfies the weak maximum principle. Note that we do NOT have sign assumption on c.

Hint: Find an elliptic operator M with no zeroth order term such that $w := u/v$ satisfies $Mw \leq 0$ in the region $\{u > 0\}$. To do this, first compute $(v^2 w_i)_j$. See also the first problem here.

9. Fix $\alpha > 0$ and let $\Omega = B(0,1)$ the unit ball centered at the origin. Show that there exists a constant C depending only on n, α such that

$$\int_\Omega u^2 \mathrm{d}x \leq C \int_\Omega |\nabla u|^2 \mathrm{d}x$$

for all those $u \in H^1(\Omega)$ satisfying

$$|x \in \Omega : u(x) = 0| \geq \alpha.$$

10. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $\Delta u = 0$ in Ω. Assume that $u = \frac{\partial u}{\partial \nu} = 0$ on an open, smooth portion of $\partial\Omega$. Prove that u is identically zero.

Set 1

1. Prove that Laplacian equation $\Delta u = 0$ is rotational invariant, that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) = u(Ox) \quad x \in \mathbb{R}^n$$

then $\Delta v = 0$.

2. Let B be the unit ball centered at the origin in $\mathbb{R}^n$. Let u be a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B \\ u = g & \text{on } \partial B. \end{cases}$$

Prove that there exists a positive constant C, which depends *only* on n, such that

$$\max_B |u| \leq C(\max_{\partial B} |g| + \max_B |f|).$$

3. Let B^+ denote the open half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$. Assume that $u \in C(\overline{B^+})$ is harmonic in B^+ and $u = 0$ on $\partial B^+ \cap \{x_n = 0\}$. For every $x \in B$, set

$$v(x) := \begin{cases} u(x) & \text{if } x_n > 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Prove that v is harmonic in B.

Note that the above 3 problems are from the main reference book: PDEs by L.C. Evans.

4. Prove that every positive harmonic function in the whole space $\mathbb{R}^n$ has to be a constant function.

5. Let u be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$ with $n \geq 3$. Let $\xi \in \mathbb{R}^n$ and $\lambda > 0$. Define

$$u_{\xi,\lambda}(x) := \left(\frac{\lambda}{|x-\xi|}\right)^{n-2} u\left(\xi + \frac{\lambda^2(x-\xi)}{|x-\xi|^2}\right).$$

This $u_{\xi,\lambda}$ is called the *Kelvin transform* of u. Prove that $u_{\xi,\lambda}$ is also harmonic in its domain.

6. Let B be the unit ball in $\mathbb{R}^n$ centered at the origin. Let u be a positive harmonic function in $B \setminus \{0\}$. Prove that there exist a harmonic function v in B and a constant $c \geq 0$ such that

$$u(x) = \begin{cases} c|x|^{2-n} + v(x), & \text{when } n \geq 3 \\ c|\log|x|| + v(x), & \text{when } n = 2 \end{cases} \qquad \text{for all } x \in B \setminus \{0\}.$$

This theorem can be stated as: Every positive harmonic function in the punctured ball with an ***isolated singularity*** *has to be a fundamental solution plus a harmonic function in the whole ball.*

7. Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $k \geq 1$ be an integer, and $1 \leq p < \infty$. Let $u \in W^{k,p}(\Omega) \cap L^\infty(\Omega)$, and $\Phi \in C^k(\mathbb{R})$. Prove that the composition function $\Phi \circ u \in W^{k,p}(\Omega)$.