Some lecture notes for Math 5281 - Partial Differential Equations, Spring 2019

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1 Variational methods

We discuss an example of the use of variational methods in obtaining existence of solutions.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded open set, with $n \geq 3$. Let $1 < p < \frac{n+2}{n-2}$. Then there exists a positive function $u \in C^3(\Omega)$ satisfies*

$$\begin{cases} -\Delta u = u^p \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

Proof. For $u \in H_0^1(\Omega)$, define the energy functional

$$I[u] = \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}$$

and

$$m = \inf_{u \in H_0^1(\Omega)} I[u].$$

Clearly, $m \geq 0$.

We will show in the below that m is attained by some function in $H_0^1(\Omega)$, which will be called a minimizer. Moreover, we will show that this minimizer will be a desired solution of (1).

Let $\{u_k\}$ be a sequence in $H_0^1(\Omega)$ such that

$$I[u_k] \to m \quad \text{as } k \to \infty.$$

By a normalization, we may assume that

$$\int_\Omega |u_k|^{p+1} = 1.$$

Consequently, u_k is a bounded sequence in $H_0^1(\Omega)$. Therefore, there exists $u \in H_0^1(\Omega)$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega).$$

Consequently,

$$\int_\Omega |\nabla u|^2 \le \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 \le m.$$

Meanwhile, since $p+1 < \frac{2n}{n-2}$, it follows from the Rellich compactness theorem that

$$u_k \to u \quad \text{strongly in } L^{p+1}(\Omega).$$

So,

$$\int_\Omega |u|^{p+1} = 1.$$

Therefore,

$$I[u] \le m,$$

and thus,

$$I[u] = m.$$

Since $I[|u|] \le I[u]$, we have that $u \ge 0$ or $u \le 0$. So we assume that $u \ge 0$. Since $\int_\Omega |u|^{p+1} = 1$, $u \not\equiv 0$.

So we have found a minimizer u for m. This implies that for every $\varphi \in H_0^1(\Omega)$,

$$0 = \frac{d}{dt}\bigg|_{t=0} I[u + t\varphi].$$

A calculation yield

$$0 = 2\int_\Omega \nabla u \cdot \nabla\varphi - 2m \int_\Omega u^p \varphi,$$

that is

$$\int_\Omega \nabla u \cdot \nabla\varphi - m \int_\Omega u^p \varphi = 0$$

So u is weak solution of (1) (in the distribution sense) after a scaling ($\tilde{u} = cu$ by some proper positive constant c).

Two regularity theory:

1. $W^{2,p}$ theory, that is for $q \in (1,\infty)$, if $-\Delta u = f$, where $f \in L^q$, then $u \in W^{2,q}_{loc}$.
2. Schauder theory, that is for $\alpha \in (0,1)$, if $f \in C^\alpha$, then $u \in C^{2,\alpha}_{loc}$.

These two theories plus Sobolev embedding implies that the solutions of (1) $u \in C^3$. This is called bootstrap arguments. Note that such bootstrap arguments will NOT work when $p = \frac{n+2}{n-2}$.

Finally, u is positive in Ω by the strong maximum principle. □

Next, we will show that the equation (1) does not have non-trivial solutions when $p \ge \frac{n+2}{n-2}$ and when the domain Ω is strictly star-shaped with respect to zero, that is $x \cdot \nu > 0$ everywhere on $\partial\Omega$. Here, ν is the outer normal of Ω

Theorem 1.2 (Pohozaev). *Let $p > \frac{n+2}{n-2}$ and Ω is strictly star-shaped with respect to zero. Suppose $u \in C^2(\overline{\Omega})$ is a solution of*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Then

$$u \equiv 0 \quad \text{in } \Omega.$$

Proof. Multiplying $x \cdot \nabla u$ on the both sides of (2) and doing some calculations, we have

$$\left(\frac{n-2}{2}\right)\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2(\nu\cdot x) = \frac{n}{p+1}\int_\Omega |u|^{p+1}.$$

This is usually called the *Pohozaev identity*.

Multiplying u on the both sides of (2) and integrating by parts, we have

$$\int_\Omega |\nabla u|^2 = \int_\Omega |u|^{p+1}.$$

Thus,

$$0 \le \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2(\nu\cdot x) = \left(\frac{n}{p+1} - \frac{n-2}{2}\right)\int_\Omega |u|^{p+1} \le 0, \tag{3}$$

where we used the fact that $p > \frac{n+2}{n-2}$ in the last inequality. Therefore, $u \equiv 0$. □

Remark 1.3. *Theorem 1.2 also holds for $p = \frac{n+2}{n-2}$. In this case, it follows from (3) that $\nabla u = 0$ on $\partial\Omega$, since $x\cdot\nu > 0$. Then by a unique continuation property, $u \equiv 0$, which is slightly complicated. However, if one additionally assumes that $u \ge 0$ in Ω, then*

$$0 = \int_\Omega -\Delta u = \int_\Omega u^p,$$

from which $u \equiv 0$ follows.

Remark 1.4. *The assumption that Ω is star shaped is necessary in Theorem 1.2. For example, if Ω is an annulus, then there exists a positive radial solution of*

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

2 Method of subsolutions and supersolutions

We will investigate the boundary -value problem for the nonlinear Poisson equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and $\|f'\|_{L^\infty(\mathbb{R})} \le C$ for some constant C.

Definition 2.1. *(i). We say that $\bar{u} \in H^1(\Omega)$ is a weak supersolution of (4) if*

$$\int_\Omega \nabla\bar{u}\cdot\nabla v \mathrm{d}x \ge \int_\Omega f(\bar{u})v\mathrm{d}x \quad \textit{for every } v \in H_0^1(\Omega),\ v \ge 0\ a.e.$$

(ii). Similarly, we say that $\underline{u} \in H^1(\Omega)$ is a weak subsolution of (4) if

$$\int_\Omega \nabla\bar{u}\cdot\nabla v \mathrm{d}x \le \int_\Omega f(\bar{u})v\mathrm{d}x \quad \textit{for every } v \in H_0^1(\Omega),\ v \ge 0\ a.e.$$

(ii). We say that $u \in H_0^1(\Omega)$ *is a weak solution of* (4) *if*

$$\int_\Omega \nabla \bar{u} \cdot \nabla v \mathrm{d}x = \int_\Omega f(\bar{u}) v \mathrm{d}x \quad \textit{for every } v \in H_0^1(\Omega).$$

Note that if $\bar{u}, \underline{u} \in C^2(\Omega)$, then we have

$$-\Delta \bar{u} \geq f(\bar{u}), \quad -\Delta \underline{u} \leq f(\underline{u}).$$

Theorem 2.2. *Assume there exist a weak supersolution* $\bar{u}$ *and a weak subsolution* $\underline{u}$ *of* (4) *satisfying*

$$\underline{u} \leq 0, \quad \bar{u} \geq 0 \textit{ on } \partial\Omega \textit{ in the trace sense }, \quad \underline{u} \leq \bar{u} \textit{ a.e. in } \Omega.$$

Then there exists a weak solution u *of* (4) *such that*

$$\underline{u} \leq u \leq \bar{u} \quad \textit{a.e. in } \Omega.$$

Proof. Since $\|f'\|_{L^\infty(\mathbb{R})} \leq C$ for some constant C, we can choose a large $\lambda > 0$ such that

$$g(z) := f(z) + \lambda z$$

is an increasing function.

Now we denote $u_0 = \underline{u}$. Given u_k, $k = 0, 1, \cdots$,, we are going to inductively define u_{k+1} be the unique weak solution of the following liner boundary-value problem

$$\begin{cases} -\Delta u_{k+1} + \lambda u_{k+1} = f(u_k) + \lambda u_k & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We claim that

$$\underline{u} = u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq \bar{u}. \tag{5}$$

To prove this claim, we first note that for $k = 0$,

$$\int_\Omega \nabla(u_0 - u_1) \cdot \nabla v + \lambda(u_0 - u_1)v \leq 0 \quad \text{for every } v \in H_0^1(\Omega),\ v \geq 0 \; a.e.$$

Choose $v = (u_0 - u_1)^+$, we obtain

$$\int_\Omega \nabla(u_0 - u_1) \cdot \nabla(u_0 - u_1)^+ + \lambda(u_0 - u_1)(u_0 - u_1)^+ \leq 0$$

This implies that $(u_0 - u_1)^+ = 0$, that is, $u_0 \leq u_1$. Now we assume inductively that

$$u_{k-1} \leq u_k.$$

Then we have

$$\begin{aligned} &\int_\Omega \nabla(u_k - u_{k+1}) \cdot \nabla v + \lambda(u_k - u_{k+1})v \\ &= (f(u_{k-1} + \lambda u_{k-1} - f(u_k) - \lambda u_k))v \quad \text{for every } v \in H_0^1(\Omega),\ v \geq 0 \; a.e. \end{aligned}$$

Choosing $v = (u_k - u_{k+1})^+$ will lead to that $(u_k - u_{k+1})^+ = 0$, that is $u_k \le u_{k+1}$. This proves the claim on the monotonicity of the sequence $\{u_k\}$.

Then we show that $u_k \le \bar{u}$ for all k. This is true for $k = 0$. Assume the induction that $u_k \le \bar{u}$, then we have

$$\begin{aligned}&\int_\Omega \nabla(u_{k+1} - \bar{u}) \cdot \nabla v + \lambda(u_{k+1} - \bar{u})v \\ &= (f(u_k + \lambda u_k - f(\bar{u}) - \lambda\bar{u}))v \quad \text{for every } v \in H_0^1(\Omega),\ v \ge 0\ a.e.\end{aligned}$$

and let $v = (u_{k+1} - \bar{u})^+$, we have $(u_{k+1} - \bar{u})^+ = 0$, that is $u_{k+1} \le \bar{u}$.

Now, let

$$u = \lim_{k\to\infty} u_k.$$

By dominated convergence theorem, $u_k \to u$ in $L^2(\Omega)$. Since $\|f(u_k)\|_{L^2(\Omega)} \le C(\|u_k\|_{L^2} + 1) \le C(\|\bar{u}\|_{L^2} + 1)$, we have that

$$\sup_k \|u_k\|_{H_0^1(\Omega)} < \infty.$$

Therefore, subject to a subsequence which is still denoted as $\{u_k\}$,we have $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$. To verify that u is a weak solution of (4), we notice that

$$\int_\Omega \nabla\bar{u}_{k+1} \cdot \nabla v + \lambda u_{k+1} v \mathrm{d}x = \int_\Omega (f(u_k) + \lambda u_k) v \mathrm{d}x \quad \text{for every } v \in H_0^1(\Omega).$$

Sending $k \to \infty$ and cancelling the term with λ, we have the confirm

$$\int_\Omega \nabla\bar{u} \cdot \nabla v \mathrm{d}x = \int_\Omega f(u) v \mathrm{d}x \quad \text{for every } v \in H_0^1(\Omega).$$

Note that the monotonicity (5) can also be proved by maximum principle if $\bar{u}, \underline{u}$ are smooth. Then by the Schauder estimate, we have that every u_k is smooth. Then

$$-\Delta(u_k - u_{k+1}) + \lambda(u_k - u_{k+1}) = f(u_{k-1}) + \lambda u_{k-1} - f(u_k) - \lambda u_k \le 0$$

in the classical sense. Since $u_k \le u_{k+1}$ on $\partial\Omega$, we have $u_k \le u_{k+1}$ in Ω. □

3 The Dirichlet problem: Perron's method

In this section, we will discuss the solvability of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose $\partial\Omega \in C^2$. Let φ be a continuous function on $\partial\Omega$. Then there exists a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of* (6).

Remark: That $\partial\Omega$ satisfies the exterior ball condition at every point on $\partial\Omega$ would be sufficient. That is, for every $\xi \in \partial\Omega$, there exists a ball $B_r(y)$ such that $\overline{B}_r(y) \cap \overline{\Omega} = \{\xi\}$.

We need some generalized subharmonic functions first.

Definition 3.2. *A $C^0(\Omega)$ function u is called subharmonic (or superharmonic) in Ω if for every ball $B \subset\subset \Omega$ and every harmonic function h in B satisfying $u \leq (\geq) h$ on ∂B, we also have $u \leq (\geq) h$ in B.*

Such subharmonic functions have several useful properties:

1. If u is subharmonic in a connected domain in Ω, then it satisfies the strong maximum principle in Ω. That is, if v is super harmonic in Ω with $v \geq u$ on $\partial\Omega$, then either $u > v$ in Ω or $v \equiv u$. To prove this, we suppose the contrary that at some point $x_0 \in \Omega$, we have

$$(u - v)(x_0) = \sup_{\Omega}(u - v) = M \geq 0,$$

but there is a ball $B = B_r(x_0)$ such that $u - v \not\equiv M$ on ∂B. Let $\bar{u}, \bar{v}$ be the harmonic functions respectively equal to u, v on ∂B (this is can be achieved by Green's representation). Then one sees that

$$M \geq \sup_{\partial B}(\bar{u} - \bar{v}) \geq (\bar{u} - \bar{v})(x_0) \geq (u - v)(x_0) = M.$$

Therefore, every inequality in the above has to be an equality. By the strong maximum principle for harmonic functions, it follows that $\bar{u} - \bar{v} \equiv M$. Thus $u - v \equiv M$, which is a contradiction.

2. Let u be subharmonic in Ω and B is a ball strictly contained in Ω. Denote $\bar{u}$ as the harmonic function in B satisfying $\bar{u} = u$ on ∂B. We define in Ω he harmonic lifting of u by

$$U(x) = \begin{cases} \bar{u}(x), & x \in B, \\ u(x), & x \in \Omega \setminus B. \end{cases}$$

Then the function U is also subharmonic in Ω. This can be proved as follows. Let $B' \subset \Omega$ be an arbitrary ball. Let h be harmonic in B such that $h = U$ on $\partial B'$. Then $h \geq u$ on $\partial B'$, and thus, $h \geq u$ in B'. So $h \geq U$ in $B' \setminus B$. Since U is harmonic in B, by maximum principle, we have $h \geq U$ in $B \cap B'$. Hence, $U \leq h$ in B', and thus, U is subharmonic in Ω.

3. Let $u_1, u_2, \cdots, u_k$ be subharmonic in Ω. Then the function $u(x) = \max(u_1, \cdots, u_k)$ is also subharmonic which is a trivial consequence of the definition of subharmonic functions.

Now let us prove Theorem 3.1. A $C^0(\overline{\Omega})$ function u is called a subsolution of (6) if u is subharmonic, and $u \leq \varphi$ on $\partial\Omega$. Similarly, a $C^0(\overline{\Omega})$ function v is called a supersolution of (6) if v is subharmonic, and $v \geq \varphi$ on $\partial\Omega$. Denote S be the set of all subsolutions of (6). $S \neq \emptyset$ since the constant function $\inf_{\partial\Omega}\varphi$ is a subsolution.

Proposition 3.3. *The function $u(x) = \sup_{v \in S} y(x)$ is harmonic in Ω.*

Proof. Since $\sup_{\partial\Omega}\varphi$ is a super solution, we know that $v \leq \sup_{\partial\Omega}\varphi$ for every $v \in S$. Thus, u is well-defined.

Let $y \in \Omega$ be a fixed point. There exists $v_k \in S$ such that

$$v_k(y) \to u(y).$$

By replacing v_k with $\max(v_k, \inf\varphi)$, we may assume that the sequence $\{v_k\}$ is bounded. Choose $R > 0$ such that $B = B_R(y) \subset\subset \Omega$, and define V_k as the harmonic lifting of v_k in B. Then $V_k \in$

S. By Harnack inequality and gradient estimate, there exists a subsequence $\{V_{n_k}\}$ converging uniformly in every ball $B_\rho(y)$ with $\rho < R$ to a harmonic function in B. Clearly, $v \le u$ in B and $v(y) = u(y)$.

We are going to show that $v = u$ in B. Suppose $v(z) < u(z)$ for some $z \in B$. Then there exists $w \in S$ such that $v(z) < w(z)$. Let $w_k = \max(V_{n_k}, w)$ and W_k be its harmonic lifting in B. As before, by Harnack and gradient estimate, a subsequence of W_k converges to a harmonic function $\overline{w}$ in B satisfying $v \le \overline{w} \le u$. Since $v(y) = u(y)$, we have $v(y) = \overline{w}(y)$, and by strong maximum principle. $v \equiv \overline{w}$, which contradicts with $v(z) < w(z) \le \overline{w}(z)$. □

Therefore, the u obtained in the above proposition is a candidate solution of (6). In the below, we will show that this u indeed satisfies the boundary condition $u = \varphi$ on $\partial\Omega$, which will finish the proof of Theorem 3.1. As along as the boundary $\partial\Omega$ satisfies the exterior boundary, we can construct some barrier functions for your purpose.

Proof of Theorem 3.1. For $\xi \in \partial\Omega$, there exists a ball $B = B_R(y)$ such that $\overline{B} \cap \overline{\Omega} = \{\xi\}$. Define

$$w(x) := \begin{cases} R^{2-n} - |x-y|^{2-n}, & n \ge 3 \\ -\log R + \log|x-y|, & n = 2. \end{cases}$$

Note that $w(\xi) = 0$, $w(x) > 0$ in $\overline{\Omega} \setminus \{\xi\}$, and w is harmonic.

Let $M = \sup_\partial \varphi$. Since φ is a continuous function, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(\xi)| < \varepsilon, \quad |x - \xi| < \delta.$$

Choose κ large enough such that

$$\kappa w(x) \ge 2M \quad |x - \xi| \ge \delta.$$

Then $\varphi(\xi) + \varepsilon + \kappa w$ and $\varphi(\xi) - \varepsilon - \kappa w$ are respectively supersolution and subsolutions of (6). Therefore,

$$\varphi(\xi) - \varepsilon - \kappa w(x) \le u(x) \le \varphi(\xi) + \varepsilon + \kappa w(x).$$

Hence

$$|u(x) - \varphi(\xi)| \le \varepsilon + \kappa w(x).$$

Since $w(\xi) = 0$, we obtain that

$$u(x) \to \varphi(\xi) \quad \text{as } x \to \xi.$$

□

4 Schauder estimates

We first prove the following Schauder estimates for Possion's equation.

Theorem 4.1. *Let $\alpha \in (0,1)$, $f \in C^{\alpha}(B_1)$, and $u \in C^2(B_1)$ be a solution of*

$$-\Delta u = f \quad \text{in } B_1.$$

Then there exists $C > 0$ depending only on α and the dimension n such that

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)})$$

Proof. First of all, by multiplying some proper constant, we can assume that $\|u\|_{L^\infty(B_1)}+\|f\|_{C^\alpha(B_1)} \leq 1$. Let $\rho = 1/2$. For $k = 1, 2, 3, \cdots$, let v_k be the solution of

$$\begin{cases} -\Delta v_k = f(0) & \text{in } B_k, \\ v_k = u & \text{on } \partial B_k, \end{cases}$$

where $B_k = B_{\rho^k}$ centered at the origin.

Claim 1: $\|v_k - v\|_{L^\infty(B_k)} \leq C\rho^{(2+\alpha)k}$.

This claim can be proved as follows. Let $\tilde{v}_k(x) = \rho^{-2k}(v_k - u)(\rho^k x)$, where $x \in B_1$. Then

$$\begin{cases} -\Delta \tilde{v}_k = f(0) - f(\rho^k) & \text{in } B_1, \\ \tilde{v}_k = 0 & \text{on } \partial B_1, \end{cases}$$

By the maximum principle (one of the homework problem), we have

$$\|\tilde{v}_k\|_{L^\infty(B_1)} \leq C(\|u\|_{L^\infty(\partial B_1)} + \|f(0) - f(\rho^k x)\|_{L^\infty(B_1)}) \leq C\rho^{\alpha k},$$

from which the claim follows.

Claim 2: $\|v_k - v_{k+1}\|_{L^\infty(B_{k+1})} \leq C\rho^{(2+\alpha)k}$.

This is because $v_k - v_{k+1}$ is harmonic in B_{k+1}, and therefore,

$$\|v_k - v_{k+1}\|_{L^\infty(B_{k+1})} = \|v_k - v_{k+1}\|_{L^\infty(\partial B_{k+1})} \leq \|v_k - u\|_{L^\infty(\partial B_{k+1})} \leq C\rho^{(2+\alpha)k},$$

where in the last inequality, we use Claim 1.

Let $w_k = v_{k+1} - v_k, w_0 = v_1$. Then we know that from Claim 2 that, for every $x \in B_{k+2}$ we have

$$|\nabla^j w_k(x)| \leq C\rho^{(2+\alpha-j)k}.$$

for $\rho^{i+3} \leq |x| < \rho^{i+2}$,

$$\begin{aligned}
&|u(x) - \sum_{\ell=0}^{\infty} w_\ell(0) - \sum_{\ell=0}^{\infty} Dw_\ell(0) \cdot x - \sum_{\ell=0}^{\infty} \frac{1}{2} x^T D^2 w_\ell(0) x| \\
&\leq |u(x) - \sum_{\ell=0}^{i} w_\ell(x)| + |\sum_{\ell=0}^{i} w_\ell(x) - \sum_{\ell=0}^{i} w_\ell(0) - \sum_{\ell=0}^{i} Dw_\ell(0) \cdot x - \sum_{\ell=0}^{i} \frac{1}{2} x^T D^2 w_\ell(0) x| \\
&\quad + |\sum_{\ell=i+1}^{\infty} w_\ell(0)| + |\sum_{l=i+1}^{\infty} Dw_\ell(0) \cdot x| + \frac{1}{2}|\sum_{\ell=i+1}^{\infty} x^T D^2 w_\ell(0) x| \\
&\leq \rho^{(2+\alpha)(i+1)} + 2c_2|x|^3 \sum_{\ell=0}^{i} \rho^{(\alpha-1)\ell} + \sum_{\ell=i+1}^{\infty} \rho^{(2+\alpha)\ell} + |x| \sum_{\ell=i+1}^{\infty} c_2 \rho^{(1+\alpha)\ell} \\
&\quad + |x|^2 \sum_{\ell=i+1}^{\infty} c_2 \rho^{\alpha \ell} \\
&\leq C_3 |x|^{2+\alpha}.
\end{aligned}$$

So we have proved that there exists a second order polynomial P such that

$$|u(x) - P(x)| \leq C|x|^{2+\alpha}$$

where all the coefficients of the polynomial are universally bounded.

This leads to the conclusion of the theorem. □

Theorem 4.2. *Let $\alpha \in (0,1)$, $f \in C^\alpha(B_1)$, $a_{ij}(x) \in C^\alpha(B_1)$, and $u \in C^2(B_1)$ be a solution of*

$$-a_{ij}(x)u_{ij}(x) = f \quad \text{in } B_1,$$

where $\lambda I \leq (a_{ij}(x)) \leq \lambda^{-1} I$ in B_1. Then there exists $C > 0$ depending only on α, λ and the dimension n such that

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}).$$

Proof. Let $\rho = 1/2$. For $k = 1, 2, 3, \cdots$, let v_k be the solution of

$$\begin{cases} -a_{ij}(0)\partial_{ij} v_k = f(0) & \text{in } B_k, \\ v_k = u & \text{on } \partial B_k, \end{cases}$$

where $B_k = B_{\rho^k}$ centered at the origin. Then for $\tilde{v}_k = v_k - u$, we have

$$-a_{ij}(0)\partial_{ij}\tilde{v}_k = f(0) - f(x) + (a_{ij}(0) - a_{ij}(x))u_{ij}(x) \quad \text{in } B_k,$$

Following the proof in Theorem 4.1, we can show that

$$[\nabla^2 u]_{C^\alpha(B_{1/4})} \leq C(\|f\|_{C^\alpha(B_{1/2})} + \|u\|_{L^\infty(B_{1/2})} + \|\nabla^2 u\|_{L^\infty(B_{1/2})}), \tag{7}$$

where the constant C depends only on α, n, λ. Now that the domain on the two sides of the inequality are DIFFERENT. Then the conclusion follows from the next iteration lemma. □

M. Giaquinta and E. Giusti, On the regularity of the minima of variational integrals. Acta Math. 148 (1982), 31–46. Lemma 1.1.

Lemma 4.3. *Let $h(t)$ be a nonnegative bounded function defined for $0 \leq T_0 \leq t \leq T_1$. Suppose that for $T_0 \leq t < s \leq T_1$ we have*

$$h(t) \leq A(s-t)^{-\alpha} + B + \theta h(s)$$

where A, B, α, θ are nonnegative constants, and $\theta < 1$. Then there exists a constant $C > 0$, depending only on α, θ such that for every ρ, R, $T_0 \leq \rho < R \leq T_1$, we have

$$h(\rho) \leq C(A(R-\rho)^{-\alpha} + B).$$

Proof. Consider the sequence $\{t_j\}$ defined by

$$t_0 = \rho, \;\; t_{j+1} - t_j = (1-\tau)\tau^j (R - \rho)$$

with $\tau \in (0,1)$. By iteration

$$h(t_0) \leq \theta^k h(t_k) + \left(\frac{A}{(1-\tau)^\alpha}(R-\rho)^{-\alpha} + B \right) \sum_{i=0}^{k-1} \theta^j \tau^{-j\alpha}.$$

We choose now τ such that $\tau^{-\alpha}\theta < 1$ and let $k \to \infty$. Then the conclusion follows. □

Proof of Theorem 4.2 continued. Let

$$h(t) = [\nabla^2 u]_{C^\alpha(B_t)}.$$

We will show that this h will satisfies Lemma 4.3. Let $0 \le t < s \le 1$. For every $z \in B_t$, choose $r = s - t$. For $x \in B_1$, let

$$u_r(x) = u(z + rx), \quad \tilde{a}_{ij}(x) = a_{ij}(z + rx), \quad f_r(x) = r^2 f(z + rx).$$

Then

$$-\tilde{a}_{ij}\partial_{ij}u_r(x) = f_r(x) \quad \text{in } B_1.$$

Therefore, the estimate (7) holds for u_r, i.e.,

$$[\nabla^2 u_r]_{C^\alpha(B_{1/4})} \le C(\|f_r\|_{C^\alpha(B_{1/2})} + \|u_r\|_{L^\infty(B_{1/2})} + \|\nabla^2 u_r\|_{L^\infty(B_{1/2})}),$$

Scaling back, we have

$$[\nabla^2 u]_{C^\alpha(B_{r/4}(z))} \le Cr^{-2-\alpha}(\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}) + Cr^{-\alpha}\|\nabla^2 u\|_{L^\infty(B_{r/2}(z))}.$$

By a covering, we have

$$[\nabla^2 u]_{C^\alpha(B_t)} \le Cr^{-3}(\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}) + Cr^{-1}\|\nabla^2 u\|_{L^\infty(B_{(s+t)/2})}.$$

(From Gilbarg-Trudinger's book)We have the following interpolation lemma (the following is not sharp): there exists $C > 0$ independent of s, t such that for all $\varepsilon > 0$,

$$\|\nabla^2 u\|_{L^\infty(B_{(s+t)/2})} \le \varepsilon r^{-2}[\nabla^2 u]_{C^\alpha(B_s)} + C\varepsilon^{-\frac{2+\alpha}{\alpha}} r^{-2}\|u\|_{L^\infty(B_s)}.$$

Choose $\varepsilon = r^3/2C$, we have for some $\beta > 0$

$$[\nabla^2 u]_{C^\alpha(B_t)} \le \frac{1}{2}[\nabla^2 u]_{C^\alpha(B_s)} + Cr^{-\beta}(\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}).$$

Then the conclusion from from Lemma 4.3. $\square$

5 De Giorgi estimates

Let $a_{ij} \in L^\infty(B_1)$ and uniformly elliptic, that is

$$\lambda I \le a_{ij}(x) \le \Lambda I \quad \text{in } B_1$$

for some $\lambda, \Lambda > 0$.

Theorem 5.1 (De Giorgi, 1958)**.** *Let $u \in H^1(B_1)$ be a weak solution of*

$$-\partial_j(a_{ij}\partial_i u) = 0 \quad \text{in } B_1 \tag{8}$$

Then there exists $\alpha \in (0,1)$ and $C > 0$, both of which depends only on n, λ and Λ, such that

$$\|u\|_{C^\alpha(B_{1/2})} \le C\|u\|_{L^2(B_1)}.$$

The proof of this theorem consists of two steps: from L^2 to L^∞, and from L^∞ to C^α.

Proposition 5.2 (from L^2 to L^∞)**.** *Let $u \in H^1(B_1)$ be a weak subsolution of* (8)*, that is,*

$$-\partial_j(a_{ij}\partial_i u) \le 0 \quad \text{in } B_1.$$

Then there exists a constant $\delta = \delta(n, \lambda, \Lambda)$ such that if $\|u^+\|_{L^2(B_1)} < \delta$, then we have

$$\|u^+\|_{L^\infty(B_{1/2})} \le 1.$$

Applying this proposition to $(\sqrt{\delta}/\|u^+\|_{L^2(B_1)})u$, we obtain the following theorem:

Theorem 5.3 (from L^2 to L^∞)**.** *Let $u \in H^1(B_1)$ be a weak subsolution of* (8)*, that is,*

$$-\partial_j(a_{ij}\partial_i u) \le 0 \quad \text{in } B_1.$$

Then

$$\|u^+\|_{L^\infty(B_{1/2})} \le C(n, \lambda, \Lambda)\|u^+\|_{L^2(B_1)}$$

Consequently, if $u \in H^1(B_1)$ is a weak solution of (8)*, then*

$$\|u\|_{L^\infty(B_{1/2})} \le C(n, \lambda, \Lambda)\|u\|_{L^2(B_1)}$$

To prove Proposition 5.2, we shall use the following Caccioppoli inequality (or energy estimate)

Lemma 5.4 (Caccioppoli inequality)**.** *Let $u \in H^1(B_1)$ be a weak subsolution of* (8) *and $\varphi \in C_c^\infty(B_1)$, we have*

$$\int_{B_1} (\nabla(\varphi u^+))^2 \le C(n, \lambda, \Lambda) \int_{B_1} (u^+)^2 (\nabla\varphi)^2.$$

Proof. Since u is a subsolution, we have

$$\int_{B_1} a_{ij}\partial_i u \partial_j(\varphi^2 u^+) \le 0.$$

That is

$$\int_{B_1} a_{ij}\varphi^2 \partial_i u^+ \partial_j u^+ + 2\int_{B_1} a_{ij}\varphi u^+ \partial_i u^+ \partial_j \varphi \le 0.$$

Then

$$\begin{aligned}
\frac{1}{\lambda}\int_{B_1} (\nabla(\varphi u^+))^2 &\le \int_{B_1} a_{ij}\partial_i(\varphi u^+)\partial_j(\varphi u^+) \\
&\le C\int_{B_1} |\nabla\varphi|^2 (u^+)^2 + \int_{B_1} a_{ij}\varphi^2\partial_i u^+ \partial_j u^+ + \int_{B_1} a_{ij}\varphi u^+ \partial_i u^+ \partial_j\varphi \\
&\quad + \int_{B_1} a_{ij}\varphi u^+ \partial_i u \partial_j \varphi \\
&\le C\int_{B_1} |\nabla\varphi|^2 (u^+)^2 + 3(a_{ij} + a_{ji})\varphi u^+ \partial_i u^+ \partial_j \varphi \\
&\le C\int_{B_1} |\nabla\varphi|^2 (u^+)^2 + |u^+ \partial_i(u^+\varphi)\partial_j\varphi| \\
&\le C\int_{B_1} |\nabla\varphi|^2 (u^+)^2 + \frac{1}{2\lambda}(\nabla(\varphi u^+))^2,
\end{aligned}$$

where in the last inequality we have used Holder's inequality. □

Lemma 5.5. *Let $u \in H^1(B_1)$ be a weak solution of* (8)*. Then u^+ is a weak subsolution of* (8).

Proof. Exercise. □

Now we can prove Proposition 5.2.

Proof of Proposition 5.2. We will work on a family of ball

$$B_k = \{|x| \le 1/2 + 2^{-k}\}$$

and the family of truncated functions

$$u_k = (u - (1 - 2^{-k}))^+.$$

Define

$$U_k = \int_{B_k} u_k^2.$$

Let φ_k be a sequence of nonnegative shrinking cut-off functions: $\varphi_k = 1$ in B_k and $\varphi = 0$ in B_{k-1}^c. Also, $|\varphi_k| \le C2^k$.

Note that where $u_{k+1} > 0$, then $u > 1 - 2^{-k-1} = 1 - 2^{-k} + 2^{-k-1}$, and thus, $u_k > 2^{-k-1}$. Therefore,

$$\{x : \varphi_{k+1} u_{k+1} > 0\} \subset \{x \in B_k : u_k > 2^{-k-1}\}.$$

We have from the Sobolev inequality with $p = \frac{2n}{n-2}$ (or any p if $n = 2$)

$$\left(\int |\varphi_{k+1} u_{k+1}|^p\right)^{2/p} \le C \int |\nabla(\varphi_{k+1} u_{k+1})|^2.$$

From Holder inequality

$$\int |\varphi_{k+1} u_{k+1}|^2 \le \left(\int |\varphi_{k+1} u_{k+1}|^p\right)^{2/p} \cdot |\{x : \varphi_{k+1} u_{k+1} > 0\}|^{\frac{2}{n}}.$$

Therefore,

$$U_{k+1} \le C \int |\nabla(\varphi_{k+1} u_{k+1})|^2 \cdot |\{x : \varphi_{k+1} u_{k+1} > 0\}|^{\frac{2}{n}}.$$

Using Lemma 5.4, we have

$$U_{k+1} \le C 2^{2k} \int_{\text{supp}(\varphi_{k+1})} (u_{k+1})^2 \cdot |\{x : \varphi_{k+1} u_{k+1} > 0\}|^{\frac{2}{n}}.$$

The support of φ_{k+1} in contained in B_k and $u_{k+1} \le u_k$, we have

$$U_{k+1} \le C 2^{2k} \int_{B_k} u_k^2 \cdot |\{x : \varphi_{k+1} u_{k+1} > 0\}|^{\frac{2}{n}}.$$

Since

$$\{x : \varphi_{k+1} u_{k+1} > 0\} \subset \{x \in B_k : u_k > 2^{-k-1}\}.$$

we have

$$|\{x : \varphi_{k+1}u_{k+1} > 0\}|^{\frac{2}{n}} \leq |\{x \in B_k : u_k > 2^{-k-1}\}| \leq 2^{\frac{4k}{n}} \left(\int_{B_k} u_k^2\right)^{\frac{2}{N}} = 2^{\frac{4k}{n}} U_k^{\frac{2}{n}}.$$

where the Chebyshev inequality was used in the last inequality.

Therefore, we have

$$U_{k+1} \leq C2^{4k}(U_k)^{1+\frac{2}{n}}.$$

The as long as $U_0 = \delta$ is small enough, we will have $U_k \to 0$ as $k \to \infty$. This proves that $u^+ \leq 1$ in $B_{1/2}$. In fact, U_k decays faster than geometric sequences. □

The next step is to prove the regularit from L^∞ to C^α. The kep step is this oscillation lemma.

Proposition 5.6. *Let $u \in H^1(B_2)$ be a weak subsolution of* (8) *in B_2. Suppose that $u \leq 1$ in B_2. Assume that $|B_1 \cap \{v \leq 0\}| \geq \mu(> 0)$. Then $u \leq 1 - \gamma$ in $B_{1/2}$, where γ depends only on λ, Λ, n, μ.*

In other words, if u is a subsolution and smaller than one, and is "far from 1" in a set of non-trivial measure, then u cannot get too close to 1 in $B_{1/2}$.

Let us postpone the proof to the end, we shall first use this proposition to prove the following Holder estimate.

Theorem 5.7. *Let $u \in H^1(B_3)$ be a weak solution of* (8) *in B_3. Then $u \in C^\alpha(B_{1/2})$. Moreover,*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^2(B_3)}.$$

Here, each of the constants α and C depend only on n, λ, Λ.

Proof. First of all, we know from Theorem 5.3 that

$$\|u\|_{L^\infty(B_2)} < C\|u\|_{L^2(B_3)}.$$

Denote

$$\text{osc}_{\,\Omega} u = \sup_\Omega u - \inf_\Omega u.$$

Consider the function

$$v(x) = \frac{2}{\text{osc}_{\,B_2} u}\left(u(x) - \frac{\sup_{B_2} u + \inf_{B_2} u}{2}\right),$$

We have that $-1 \leq v \leq 1$ in B_2.

Assume that $|B_1 \cap \{v \leq 0\}| \geq |B_1|/2$, then we can apply Proposition 5.6 on v to obtain $\text{osc}_{\,B_{1/2}} u \leq 2 - \gamma$. Hence, $\text{osc}_{\,B_{1/2}} u \leq (1 - \gamma/2)\text{osc}_{\,B_2} u$.

If $|B_1 \cap \{v \geq 0\}| \geq |B_1|/2$, then the same result holds by working with $-v$.

Therefore, we have proved that

$$\text{osc}_{\,B_{1/2}} u \leq \tilde{\gamma}\,\text{osc}_{\,B_2} u \tag{9}$$

with $\tilde{\gamma} = 1 - \gamma/2 < 1$ depends only on n, λ, Λ.

Take any x_0 in $B_{1/2}$, and introduce the rescaled functions

$$u_k(y) = u(x_0 + y/2^n), \quad a_{ij}^{(k)}(y) = a_{ij}(x_0 + y/2^n).$$

Then $a_{ij}^{(k)}$ satisfies the same assumptions as a_{ij}, and u_k is a weak solution of (8) with a_{ij} replaced by $a_{ij}^{(k)}$. We apply recursively (9) to u_k to obtain

$$\sup_{|x-x_0|\leq 2^{-k}} |u(x_0) - u(x)| \leq 2\|u\|_{L^\infty(B_2)}\tilde{\gamma}^k.$$

Then for y such that $2^{-k-1} \leq |x - x_0| < 2^{-k}$, we have

$$|u(x_0) - u(x)| \leq 2\|u\|_{L^\infty(B_2)}\tilde{\gamma}^k \leq \|u\|_{L^\infty(B_2)} 2 \cdot 2^{-(k+1)\alpha} 2^{(k+1)\alpha}\tilde{\gamma}^k \leq C\|u\|_{L^\infty(B_2)}|x - x_0|^\alpha,$$

as long as the α is chosen by $2^\alpha\tilde{\gamma} = 1$, that is,

$$\alpha = -\frac{\log\tilde{\gamma}}{\log 2} > 0.$$

Note that this estimate is independent of the choice of x_0. Then the conclusion follows. □

Now we are going to prove Proposition 5.6. We first note that if the set

$$|\{u \leq 0\} \cap B_1| \geq |B_1| - \delta/4$$

then

$$\|u^+\|_{L^2(B_1)} \leq \sqrt{\delta}/2,$$

and thus, by Proposition 5.2, we have

$$u \leq 1/2.$$

So we much bridge the gap between knowing that $|\{u \leq 0\} \cap B_1| \geq \mu$ and knowing $|\{u \leq 0\} \cap B_1| \geq |B_1| - \delta/4$. The main tool is the so-called De Giorgi isoperimetric inequality, which rough says that for an H^1 function u, it must pay in measure to increase from $u = 0$ to $u = 1$.

Lemma 5.8. *Consider a function w such that $\int_{B_1} |\nabla w^+|^2 \leq C_0$. Define*

$$\begin{aligned}
|A| &= |\{w \leq 0\} \cap B_1|, \\
|E| &= |\{w \geq 1\} \cap B_1|, \\
|D| &= |\{0 < w < 1\} \cap B_1|.
\end{aligned}$$

Then there exists a constant C_1 depends only on n such that

$$C_0|D| \geq C_1(|A||E|^{1-\frac{1}{n}})^2.$$

Proof. Consider $\bar{w} = \sup(0, \inf(w, 1))$. Note that $\nabla \bar{w} = \nabla w^+ \chi_{\{0 \le w \le 1\}}$.

For $x_0 \in E$ we integrate over lines in to $x \in A$

$$
\begin{aligned}
-1 = -\bar{w}(x_0) &= \int_0^1 \frac{d}{dt} \bar{w}(tx + (1-t)x_0) \\
&= \int_0^1 \nabla \bar{w}(tx + (1-t)x_0) \cdot (x - x_0)
\end{aligned}
$$

Integrating over A, we obtain that

$$
\begin{aligned}
|A| &\le |\int_0^1 dt \int_A \nabla \bar{w}(tx + (1-t)x_0) \cdot (x - x_0)| dx \\
&\le \int_0^1 dt \int_0^2 r^{n-1} dr \int_{\partial B_r(x_0) \cap B_1} |\nabla \bar{w}(tx + (1-t)x_0)| r dS(x) \\
&\le |\int_0^1 dt \int_0^2 r^n dr \int_{\partial B_{tr}(x_0) \cap B_1} |\nabla \bar{w}(z)| dS(z) \\
&\le |\int_0^2 r^{n-1} dr \int_0^r dt \int_{\partial B_t(x_0) \cap B_1} |\nabla \bar{w}(z)| dS(z) \\
&\le |\int_0^2 r^{n-1} dr \int_0^r t^{n-1} dt \int_{\partial B_t(x_0) \cap B_1} \frac{|\nabla \bar{w}(z)|}{|z - x_0|^{n-1}} dS(z)| \\
&\le C \int_{B_1} \frac{|\nabla \bar{w}(z)|}{|z - x_0|^{n-1}} dz \\
&\le C \int_D \frac{|\nabla \bar{w}(z)|}{|z - x_0|^{n-1}} dz
\end{aligned}
$$

Integrating $x_0 \in E$, we have

$$
|A||E| \le C \int_D |\nabla \bar{w}(z)| dz \int_E \frac{1}{|z - x_0|^{n-1}} dx_0
$$

Note that

$$
\int_E \frac{1}{|z - x_0|^{n-1}} dx_0 \le \int_B \frac{1}{|x|^{n-1}} dx = C|E|^{1/n},
$$

where B is the ball centered at the origin such that $|B| = |E|$.

Therefore, by Holder inequality,

$$
|A||E| \le C|E|^{1/n} |D|^{1/2} (\int_D |\nabla \bar{w}(z)|^2 dz)^{1/2}.
$$

□

Proof of Proposition 5.6. We consider the following new sequence of truncation

$$
w_k = 2^k (u - (1 - 2^{-k}))
$$

Note that $w_k \leq 1$ in B_2 and is also a subsolution. From the Caccippolli inequality, we have

$$\int_{B_1} |\nabla w^+|^2 \leq C_0.$$

We also have $|\{w_k \leq 0\} \cap B_1| \geq \mu$. We will apply Lemma 5.8 recursevely on $2w_k$ as long as

$$\int_{B_1} (w_{k+1}^+)^2 \geq \delta.$$

Note that

$$|\{2w_k \geq 1\} \cap B_1| = |\{w_{k+1} \geq 0\} \cap B_1| \geq \int_{B_1} (w_{k+1}^+)^2 \geq \delta.$$

From Lemma 5.8, there exists a positive constant β independent of k such that

$$|\{0 < w_k < 1/2\} \cap B_1| \geq \beta.$$

Therefore,

$$|\{w_{k+1} \leq 0\} \cap B_1| = |\{2w_k \leq 1\} \cap B_1| \geq |\{w_k \leq 0\} \cap B_1| + \beta \geq \cdots \geq \mu + k\alpha.$$

This clearly fails after a finite number of k. At this k_0, we have for sure that

$$\int_{B_1} (w_{k_0+1}^+)^2 \leq \delta.$$

Then Proposition 5.2 implies that

$$w_{k_0+1} \leq 1/2 \quad \text{in } B_{1/2}.$$

Rescaling back, we have

$$u \leq 1 - 2^{k_0+1} + 2^{k_0+2} = 1 - 2^{k_0+2} \quad \text{in } B_{1/2}.$$

□