100 MATHEMATICAL PROBLEMS

COMPILED BY YANG WANG

The following problems are compiled from various sources, particularly from

- D. J. Newman, *A Problem Seminar*, Springer-Verlag, 1982.
- Kenneth S. Williams, with Kenneth Hardy, *The Red Book of Mathematical Problems*, Dover Publications, 1997.
- Kenneth Hardy,Kenneth S. Williams, *The Green Book of Mathematical Problems*, Dover Publications, 1997.
- Problems from the past William Lowell Putnam Mathematical Competitions.

Most of the problems listed here require no advanced mathematical background to solve, and they range from fairly easy to moderately difficult. I have deliberately avoided including very easy and very difficult problems. Nevertheless if you have not had experience solving mathematical problems you may find many of them challenging. The problems have *NOT* being sorted according to the degree of difficulty.

Problem 1 Prove that the equation

$$y^2 = x^3 + 23$$

has no integer solutions.

Problem 2 Evaluate the limit

$$L = \lim_{n\to\infty} \frac{n}{2^n} \sum_{k=1}^{n} \frac{2^k}{k}.$$

Problem 3 Evaluate the integral

$$I = \int_0^1 \ln x \ln(1-x)\ dx.$$

Problem 4 Solve the recurrence relation

$$\sum_{k=1}^{n} \binom{n}{k} a(k) = \frac{n}{n+1}, \ n = 1, 2, \dots\ .$$

Problem 5 Let

$$a_n = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}, \ n = 0, 1, 2, \dots\ .$$

Does the series $\sum_{n=0}^{\infty} a_n$ converge, and if so, what is its sum?

Problem 6 Let $a_1, \dots, a_m$ be m (≥ 2) real numbers. Set

$$A_n = a_1 + a_2 + \cdots + a_n,\ n = 1, 2, \dots, m.$$

Prove that

$$\sum_{n=2}^{m} \left(\frac{A_n}{n}\right)^2 \leq 12 \sum_{n=1}^{m} a_n^2.$$

Problem 7 Let a and b be coprime positive integers. For k a positive integer, let $N(k)$ denote the number of integral solutions to the equation

$$ax + by = k,\ x \geq 0,\ y \geq 0.$$

Evaluate the limit

$$L = \lim_{k\to\infty} \frac{N(k)}{k}.$$

Problem 8 Let $a_1, \dots, a_m$ be m (≥ 1) real numbers which are such that $\sum_{n=1}^{m} a_n \neq 0$. Prove the inequality

$$\Big(\sum_{n=1}^{m} na_n^2\Big)^2 / \Big(\sum_{n=1}^{m} a_n^2\Big)^2 > \frac{1}{2\sqrt{m}}.$$

Problem 9 Evaluate the infinite series

$$\sum_{n=1}^{\infty} \tan^{-1}\Big(\frac{2}{n^2}\Big).$$

Problem 10 Let $p_1, \dots, p_n$ denote n distinct integers and let $f_n(x)$ be a polynomial of degree n given by

$$f_n(x) = (x - p_1)(x - p_2)\cdots(x - p_n).$$

Prove that the polynomial $g_n(x) = f_n^2(x) + 1$ cannot be expressed as the product of two non-constant polynomials with integral coefficients.

Problem 11 Let $f(x)$ be a monic polynomial of degree $n \geq 1$ with complex coefficients. Let $x_1, \dots, x_n$ be the n complex roots of $f(x)$. The discriminant $D(f)$ of $f(x)$ is the complex number

$$D(f) = \Pi_{1\leq i<j\leq n}(x_i - x_j)^2.$$

Express the discriminant of $f(x^2)$ in terms of $D(f)$.

Problem 12 Prove that for each positive integer n there exists a circle in the xy-plane which contains exactly n lattice points inside.

Problem 13 Let n be a given non-negative integer. Determine the number of solutions of the equation

$$x + 2y + 2z = n$$

in non-negative integers x, y, z.

Problem 14 Let n be a fixed integer ≥ 2. Determine all functions $f(x)$, which are boounded for $0 < x < a$, and which satisfy the functional equation

$$f(x) = \frac{1}{n^2}\Big(f(\frac{x}{n}) + f(\frac{x+a}{n}) + \cdots + f(\frac{x+(n-1)a}{n})\Big).$$

Problem 15 Evaluate the limit

$$L = \lim_{y \to 0} \frac{1}{y} \int_0^{\pi} \tan(y \sin x)\ dx.$$

Problem 16 Let $\epsilon \in (0,1)$. Prove that there are infinitely many integers n for which $\cos n \geq 1 - \epsilon$.

Problem 17 Determine all differentiable functions $f(x)$ such that

$$f(x) + f(y) = f\Big(\frac{x+y}{1-xy}\Big)$$

for all real x and y with $xy \neq 1$.

Problem 18 If x and y are rational numbers such that $\tan \pi x = y$, prove that $x = k/4$ for some integer k not congruent to 2 (mod4).

Problem 19 The sequence $x_0, x_1, x_2 \ldots$ is defined by the recurrence

$$x_0 = 0,\ x_1 = 1,\ x_{n+1} = \frac{x_n + n x_{n-1}}{n+1},\ n > 1.$$

Determine $L = \lim_{n \to \infty} x_n$.

Problem 20 Find the sum of the series

$$S = 1 - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{11} - \frac{1}{14} + \cdots\ .$$

Problem 21 Prove that

$$\frac{1}{n+1}\binom{2n}{n}$$

is an integer for all $n > 0$.

Problem 22 Let $x > 1$. Find the sum of the series

$$S = \sum_{n=0}^{\infty} \frac{2^n}{x^{2^n} + 1}.$$

Problem 23 Let k be an integer. Prove that the formal power series

$$\sqrt{1+kx} = 1 + a_1 x + a_2 x^2 + \cdots$$

has integral coefficients if and only if k is divisible by 4.

Problem 24 Find the sum of the series

$$\frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \cdots .$$

Problem 25 A cross-country racer runs a 10-mile race in 50 minutes. Prove that somewhere along the course the racer ran 2 miles in exactly 10 minutes.

Problem 26 Determine the inverse of the $n \times n$ matrix

$$S = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix},$$

where $n \geq 2$.

Problem 27 Determine 2 matrices B and C with integral entries such that

$$\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = B^3 + C^3.$$

Problem 28 Find two non-congruent similar triangles with sides of integral length having the lengths of two sides of one triangle equal to the lengths of two sides of the other.

Problem 29 Set $J_n = \{1, 2, \dots, n\}$. For each non-empty subset S of J_n define

$$w(S) = \max_{s \in S} s - \min_{s \in S} s.$$

Determine the average of $w(S)$ over all non-empty subsets S of J_n.

Problem 30 Prove that the number of odd binomial coefficients in each row of Pascal's triangle is a power of 2.

Problem 31 Prove that the polynomial

$$f(x) = x^n + x^3 + x^2 + x + 5$$

is irreducible over $\mathbb{Z}$ for $n \geq 4$.

Problem 32 Prove that $\tan \dfrac{3\pi}{11} + 4 \sin \dfrac{2\pi}{11} = \sqrt{11}$.

Problem 33 For $x > 1$ determine the sum of the series

$$\frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} + \frac{x^4}{(x+1)(x^2+1)(x^4+1)} + \cdots .$$

Problem 34 Let a_k be a sequence of positive numbers. Define $S_n = \sum_{k=1}^{n} a_k$. Prove that $\sum_{n=1}^{\infty} a_n/S_n^2$ converges. (In fact, $\sum_{n=1}^{\infty} a_n/S_n^{\alpha}$ converges for all $\alpha > 1$.)

Problem 35 Let $f(x)$ be a continuous function on $[0, a]$, where $a > 0$, such that $f(x) + f(a-x)$ does not vanish on $[0, a]$. Evaluate the integral

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)}\, dx.$$

Problem 36 For $\varepsilon > 0$ evaluate the limit

$$\lim_{x\to\infty} x^{1-\varepsilon} \int_x^{x+1} \sin(t^2)\, dt.$$

Problem 37 Prove that the equation

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24$$

has no integer solution.

Problem 38 Let $x_0 \geq 0$ be fixed, and a, b satisfy

$$\sqrt{b} < a < 2\sqrt{b}.$$

Define x_n recursively by

$$x_n = \frac{ax_{n-1} + b}{x_{n-1} + a},\ \ n = 1, 2, 3, \ldots .$$

Prove that $\lim_{n\to\infty} x_n$ exists and evaluate it.

Problem 39 Prove that for any integer $m \geq 0$ the sum

$$S_m(n) = \sum_{k=1}^{n} k^{2m+1}$$

is a polynomial in $n(n+1)$.

Problem 40 Determine a function $f(n)$ such that the n^{th} term of the sequence

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5, \ldots$$

is given by $\lfloor f(n) \rfloor$.

Problem 41 Let $a_1, a_2, \ldots, a_n$ be given. Find the least value of $x_1^2 + x_2^2 + \cdots + x_n^2$ given that

$$a_1x_1 + a_2 + x_2 + \cdots + a_nx_n = 1.$$

Problem 42 Evaluate the infinite series

$$S = 1 - \frac{2^3}{1!} + \frac{3^3}{2!} - \frac{4^3}{3!} + \cdots .$$

Problem 43 Let $F(x)$ be a differential function such that $F'(a-x) = F'(x)$ for all x in $[0, a]$. Evaluate $\int_0^a F(x)\, dx$ and give an example of such a function $F(x)$.

Problem 44 Determine the real function whose power series is

$$\frac{x^3}{3!} + \frac{x^9}{9!} + \frac{x^{15}}{15!} + \cdots .$$

Problem 45 Determine the value of the integral

$$I_n = \int_0^\pi \left(\frac{\sin nx}{\sin x}\right)^2 \, dx$$

for all positive integer n.

Problem 46 Let $f(x)$ be a continuous function on $[0, a]$, where $a > 0$, such that $f(x)f(a-x) = 1$. Prove that there are infinitely many such functions, and evaluate the integral

$$\int_0^a \frac{dx}{1 + f(x)}.$$

Problem 47 Let $p > 0$ be a real number and let $n \geq 0$ be an integer. Evaluate

$$u_n(p) = \int_0^\infty e^{-px} \sin^n x \, dx.$$

Problem 48 Evaluate $\displaystyle\sum_{k=0}^{n-2} 2^k \tan \frac{\pi}{2^{n-k}}$ for all $n \geq 2$.

Problem 49 Let $k \geq 2$ be a fixed integer. For $n \geq 1$ define

$$a_n = \begin{cases} 1, & \text{if } n \text{ is not a multiple of } k \\ -(k-1), & \text{if } n \text{ is a multiple of } k. \end{cases}$$

Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n}$.

Problem 50 The length of two altitudes of a triangle are h and k, where $h \neq k$. Find an upper and a lower bound for the length of the third altitude in terms of h and k.

Problem 51 Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}$.

Problem 52 Let $x > 2$ and $A_n = (a_{ij})$ be the $n \times n$ matrix where

$$a_{ij} = \begin{cases} x, & \text{if } i = j \\ 1, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate $D_n = \det(A_n)$.

Problem 53 Detremine a necessary and sufficient condition for the equation

$$\begin{cases} x + y + z &= A, \\ x^2 + y^2 + z^2 &= B, \\ x^3 + y^3 + z^3 &= C, \end{cases}$$

to have a solution with at least one of x, y, z equal to 0.

Problem 54 Let S be a set with n elements. Determine an explicit formula for the number $A(n)$ of subsets of S whose cardinality is a multiple of 3.

Problem 55 Let S be the set of all composite positive odd integers less than 79.

(a) Show that S may be written as the union of three (not necessarily disjoint) arithmetic progressions.
(b) Show that S cannot be written as the union of two arithmetic progressions.

Problem 56 Let a, b be fixed positive integers. Find the general solution to the recurrence relation

$$x_0 = 0,\ x_{n+1} = x_n + a + \sqrt{b^2 + 4ax_n}.$$

Problem 57 Let $\varepsilon > 0$ Around every point of integer coordinates draw a circle of radius ε. Prove that every straight line through the origin must intersect an infinitely many circles.

Problem 58 Let p and q be distinct primes. Let S be the sequence

$$\{p^m q^n :\ m, n = 0, 1, 2, 3, \dots\}$$

arranged in increasing order. For any pair of nonnegative integers (a, b) give an explicit formula for the position of $p^a q^b$ in the sequence using a, b, p, q.

Problem 59 Let $f(x)$ be the unique differentiable real function satisfying

$$f^{2n+1}(x) + f(x) - x = 0.$$

Evaluate he integral

$$\int_0^x f(t)\ dt.$$

Problem 60 Evaluate the double integral

$$\int_0^1 \int_0^1 \frac{dxdy}{1-xy}.$$

Problem 61 Prove that the sum of two consecutive odd primes is the product of at least three (possibly repeated) prime factors.

Problem 62 If four distinct points lie in the plane such that any three of them can be covered by a disk of radius one, prove that the four points can be covered by a disk of radius one.

Problem 63 Let G be the group generated by a and b subject to the relation $aba = b^3$ and $b^5 = 1$. Prove that G is Abelian.

Problem 64 Let $u(x)$ be a non-trivial solution of the differential equation

$$u'' + pu = 0,$$

defined on the interval $[1, \infty)$, where $p = p(x)$ is continuous on I. Prove that u has only finitely many zeros in any interval $[a, b]$, $1 \leq a < b$.

Problem 65 Let M be a 3×3 matrix with entries chosen randomly from $\{0, 1\}$. What is the probability that $\det(M)$ is odd?

Problem 66 Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \dots)$?

Problem 67 Prove that any convex pentagon whose vertices (no three of which are colinear) have integer coordinates must have area at least $5/2$.

Problem 68 Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are need to remove every point?

Problem 69 If A, B are square matrices such that $ABAB = 0$, does it follow that $BABA = 0$?

Problem 70 Find all real-valued continuously differentiable fnctions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(x))^2 + (f'(x))^2]\, dt + 1990.$$

Problem 71 Let S be a set of 2×2 integer matrices whose entries a_{ij} (1) are all squares of integers and, (2) satisfy $a_{ij} \leq 200$. Show that if S has more than 50387 $(15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.

Problem 72 Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

Problem 73 Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2 B = B^2 A$, can $A^2 + B^2$ be invertible?

Problem 74 Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 < y < 1$.

Problem 75 For each integer $n > 0$ let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is the sequence eventually constant?

Problem 76 Suppose that f and g are non-constant and differentiable real functions defined on $(-\infty, \infty)$. For all x, y we have

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $f^2(x) + g^2(x) = 1$ for all x.

Problem 77 (Putnam 91) Does there exist a real number L such that, if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

Problem 78 (Putnam 92) Prove that $f(n) = 1 - n$ is the only integer valued function defined on the integers that satisfies the following conditions.

(1) $f(f(n)) = n$ for all integer n;
(2) $f(f(n+2)+2) = n$ for all integer n;
(3) $f(0) = 1$.

Problem 79 (Putnam 92) Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series of $(1+x)^{\alpha}$ about $x = 0$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

Problem 80 (Putnam 92) For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, such that

$$(x^2 + y^2)^m = (xy)^n.$$

Problem 81 (Putnam 92) Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

Problem 82 (Putnam 92) Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

Problem 83 (Putnam 65) At a party, assume that no boy dances with every girl but each girl dances with at least one boy. prove that there are two couples gb and $g'b'$ which dance whereas b doesn't dance with g' and b' doesn't dance with g.

Problem 84 (Putnam 65) Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\Big\{\frac{\pi}{2n}(x_1 + x_2 + \cdots + x_n)\Big\} dx_1\, dx_2 \cdots dx_n.$$

Problem 85 (Putnam 65) Evaluate Prove that there are exactly three right-angled triangles whose sides are integers while the area is numerically equal to twice the perimeter.

Problem 86 (Putnam 66) Let a convex polygon P be contained in a square of side one. Show that the sum of the squares of the sides of P is less than or equal to 4.

Problem 87 (Putnam 66) Prove that among any ten consecutive integers at least one is relative prime to each of the others.

Problem 88 (Putnam 67) Let $f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$, where a_j are real numbers. Given that $|f(x)| \leq |\sin(x)|$ for all real x, prove that

$$|a_1 + 2a_2 + \cdots + na_n| \leq 1.$$

Problem 89 (Putnam 68) Assume that a 60^o angle cannot be trisected with ruler and compass alone. Prove that for any positive integer n on angle of $60^o/n$ can be trisected using ruler and compass alone.

Problem 90 Shuffle an ordinary deck of 52 playing cards. On the average, how far from the top will the first ace be?

Problem 91 Akira, Betul and Cleopatra fight a 3-way pistol duel. All three know that Akira's chance of hitting any target is 0.5, while Betul *never* misses, and Cleopatra has a 0.8 chance of hitting any target. The way the duel works is that each person is to fire at their choice of target. The order of firing is determined by a random drawing, and the firing proceeds cyclically in that order (unless someone is hit, in that case this person doesn't shoot and the turn goes to the next person). The duel ends when only one person is left unhit. What is the optimal strategy for each of them? Who is most likely to survive given that everyone adopts the optimal strategy? What is the probability that Akira will be the survivor?

Problem 92 Let a, b, c, d be non-negative real numbers. Prove that

$$\sqrt{a+b+c}+\sqrt{b+c+d}+\sqrt{c+a+b}+\sqrt{d+a+b}\geq 3\sqrt{a+b+c+d}.$$

Problem 93 Mr. and Mrs. Smith went to a party attended by 15 other couples. Various handshakes took place during the party. In the end, Mrs. Smith asked each person at the party how many handshakes did they have. To her surprise, each person gave a different answer. How many hand shakes did Mr. Smith have? (Here we assume that no person shakes hand with his/her spouse and of course, himself/herself.)

Problem 94 Suppose that $a+1/a\in\mathbb{Q}$. Prove that $a^n+1/a^n\in\mathbb{Q}$ for all integer $n>0$.

Problem 95 Suppose that the sequence of integers $\{a_n\}$ satisfies

$$a_0=0,\ a_1=a_2=1,\ \frac{a_{n+1}-3a_n+a_{n-1}}{2}=(-1)^n.$$

Prove that a_n is a perfect square.

Problem 96 24 chairs are evenly spaced around a circular table on which are name cards for 24 guests. The guests failed to notice these cards until they have sat down, and it turns out that no one is sitting in fornt of his/her own card. Prove that the table can be rotated so that at least two of these guests are simultaneously correctly seated. (A much harder questions is: Can the table be rotated so that at least 3 guests are simultaneously seated correctly?)

Problem 97 Let A be any set of 51 distinct integers chosen from $1,2,3,\ldots,100$. Prove that there must be two distinct integers in A such that one divides the other.

Problem 98 Given a positive integer n, show that there exists a positive integer containing only the digits 0 and 1 (in decimal notation), and which is divisible by n.

Problem 99 Let x_1, x_2, $\ldots,x_{20}$ be integers. Prove that some of them have sum divisible by 20.

Problem 100 Prove that from a set of 10 distinct two-digit numbers (in base 10), it is possible to select two disjoint subsets whose members have the same sum.