SPECTRAL TETRIS FUSION FRAME CONSTRUCTIONS

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Abstract. Spectral tetris is a flexible and elementary method to derive unit norm frames with a given frame operator having all of its eigenvalues ≥ 2. One important application of this method is to construct fusion frames. We will give necessary and sufficient conditions for a spectral tetris construction to give a fusion frame with prescribed eigenvalues for its fusion frame operator and with prescribed dimensions for its subspaces. This answers one of the major open problems in this area. We then generalize spectral tetris to use building blocks of size larger than 2×2 to construct unit norm tight frames of redundancy smaller then 2 and use it to derive non-equidimensional tight fusion frames having all eigenvalues of the fusion frame operator equal to $\lambda \in [1, 2)$.

1. Introduction

A fusion frame is a sequence of subspaces of a Hilbert space and a sequence of weights so that the sequence of weighted orthogonal projections onto these subspaces sums to an invertible operator on the space. Fusion frames were introduced in [5] (and refined in [7]) and quickly turned into an industry. The interest in fusion frames comes from their broad application to problems in distributed processing, sensor networks and a host of other directions. Fusion frames provide resilience to noise and erasures silience to noise and erasures due to, for instance, sensor failures or buffer overflows [1, 6, 8, 9]. which may be caused by sensor failures or buffer overflows, as well as robustness to subspace perturbations [7] which can happen because of imprecise knowledge of sensor network topology. For fusion frame applications, we generally need extra structure on the fusion frame such as prescribing the fusion frame operator or the dimensions of the subspaces - or both.

In this paper we address the question of how to efficiently construct fusion frames with prescribed dimensions of the subspaces and prescribed eigenvalues of the fusion frame operator. After reviewing some preliminaries we will in section 3 characterize for which sequences of eigenvalues $(\lambda_n)_{n=1}^N \subseteq [2, \infty)$ and dimensions we can use the elementary spectral tetris method to construct a fusion frame having those eigenvalues for its fusion frame operator and having those dimensions for its subspaces and how to construct it. In section 4 we extend spectral tetris to construct unit norm tight frames of redundancy smaller then 2, i.e. having all eigenvalues of the frame operator equal to $\lambda \in [1, 2)$. We then use this construction

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to derive non-equidimensional tight fusion frames with all eigenvalues of the fusion frame operator equal to $\lambda \in [1,2)$, provided the dimension of the subspaces is bounded by some constant dependent on the dimension of the ambient space and the sum of the dimensions of the subspaces.

2. Preliminaries

2.1. **Fusion Frames.** The *synthesis operator* of a finite sequence $\{f_m\}_{m=1}^M \subseteq \mathbb{C}^N$ is $F\colon \mathbb{C}^M \to \mathbb{C}^N$ given by

$$Fg = \sum_{m=1}^{M} g(m) f_m,$$

i.e. F is the $N \times M$ matrix whose m^{th}-column is f_m. The sequence $\{f_m\}_{m=1}^M$ is a *frame* if its *frame operator* $S = FF^*$ satisfies $AI \le S \le BI$ for some positive constants A, B where I is the identity on $\mathbb{C}^N$. It is a *tight frame* if $A = B$. In the case of a tight frame this constant equals M/N and is also called the *tight frame bound* or the *redundancy* of the frame. A *unit norm tight frame* is a tight frame $\{f_m\}_{m=1}^M$ for which $\|f_m\| = 1$ for all $m = 1, \ldots, M$. Unit norm tight frames provide Parseval-like decompositions in terms of nonorthogonal vectors of unit norm. If $\{f_m\}_{m=1}^M$ is unit norm, the operators $f \mapsto \langle f, f_m \rangle f_m$ arising in the frame operator

$$Sf = \sum_{m=1}^{M} \langle f, f_m \rangle f_m$$

are rank-one orthogonal projections. Fusion frame theory is the study of sums of projections with weights and of arbitrary rank. In particular, a sequence $\{W_k, v_k\}_{k=1}^K$ of subspaces of $\mathbb{C}^N$ is a *fusion frame* if the sequence $\{P_k\}_{k=1}^K$ of orthogonal projections onto those subspaces satisfies

$$AI \le \sum_{k=1}^{K} v_k^2 P_k \le BI$$

for some positive constants A, B. It is a *tight fusion frame* if $A = B$. Here we will restrict ourselfs to the case where all weights are equal to one. In this case, the *fusion frame operator* is $S = \sum_{k=1}^K P_k$. If $\{f_{k,d}\}_{d=1}^{D_k}$ is an orthonormal basis of the range of P_k then

$$Sf = \sum_{k=1}^{K} P_k f = \sum_{k=1}^{K} \sum_{d=1}^{D_k} \langle f, f_{k,d} \rangle f_{k,d}$$

for all $f \in \mathbb{C}^N$. This shows that every fusion frame arises from a traditional frame that satisfies additional orthogonality requirements. To be precise, a sequence $\{f_{k,d}\}_{k=1,d=1}^{K\ \ D_k} \subseteq \mathbb{C}^N$ generates a fusion frame $\{W_k\}_{k=1}^K$ with $\dim W_k = D_k$ for $k = 1 \ldots, K$ if $\{f_{k,d}\}_{k=1,d=1}^{K\ \ D_k}$ is a frame for $\mathbb{C}^N$ and $\{f_{k,d}\}_{d=1}^{D_k}$ is orthonormal for $k = 1 \ldots, K$.

2.2. **Spectral Tetris.** The term Spectral Tetris refers to the first systematic construction of unit norm tight frames. This construction was introduced in [4] to generate unit norm tight frames in $\mathbb{R}^N$ for any dimension N and any number of frame vectors M provided that $M \geq 2N$. The paper [4] provides a complete characterization of triples (N, K, d) for which equal-dimensional tight fusion frames in $\mathbb{R}^N$ exist. Here K is the number of fusion frame subspaces and d is their dimension. For most of the triples (N, K, d) the authors developed an elegant and simple algorithm to produce such tight fusion frames.

An extension to the construction of unit norm frames having a desired frame operator with eigenvalues $(\lambda_n)_{n=1}^N \in [2, \infty)$ satisfying $\sum_{n=1}^N \lambda_n = M$ was then introduced in [2]. For convinience we review this construction in figure 2.2 and will refer to it as the *spectral tetris construction* (STC). In STC, as in the rest of this paper, $\{e_n\}_{n=1}^N$ denotes the sequence of standard unit vectors of $\mathbb{C}^N$. A construction for equi-dimensional fusion frames having those eigenvalues for their fusion frame operator is given in [2]. The sufficient condition for this construction to work is that the dimension d of the subspaces satisfies $\sum_{n=1}^N \lambda_n = dK$ where K is the number of subspaces and that the sequence of eigenvalues is bounded by $K - 3$.

In this paper we want to use spectral tetris to construct fusion frames with given fusion frame operator and subspaces of not necessarily equal dimensions. We will say that a (fusion) frame has certain eigenvalues if its (fusion) frame operator has these eigenvalues.

Definition 2.1. *A frame constructed via the spectral tetris construction STC is called a* **spectral tetris frame**. *A fusion frame $\{W_k\}_{k=1}^K$ is called a spectral tetris fusion frame if there is a partition of a spectral tetris frame $\{f_{k,d}\}_{k=1,d=1}^{K\ \ D_k}$ such that $\{f_{k,d}\}_{d=1}^{D_k}$ is an orthonormal basis for W_k for every $k = 1, \ldots, K$.*

Aside from the fact that spectral tetris frames are easy to construct, their major advantage for applications is the sparsity of their synthesis matrices in terms of the number of non zero entries. This sparsity is dependend on the ordering of the given sequence of eigenvalues for which STC is performed. Note that the original form of the algorithm in [2] assumes the sequence of eigenvalues to be in decreasing order. This assumption, however, was made only for classification reasons, and it is easily seen that it can be dropped. The sparsest synthesis matrices are achieved if the sequence of eigenvalues $(\lambda_n)_{n=1}^N$ is ordered *blockwise*, i.e. if for any permutation π of $\{1, \ldots, N\}$ the set of partial sums $\{\sum_{j=1}^s \lambda_j : s = 1, \ldots, N\}$ contains at least as many integers as the set $\{\sum_{j=1}^s \lambda_{\pi(j)} : s = 1, \ldots, N\}$. It has been shown in [3] that spectral tetris frames are optimally sparse in the sense that given $M \geq 2N$ and a sequence of eigenvalues $(\lambda_n)_{n=1}^N \subseteq [2, \infty)$, the synthesis matrix of the spectral tetris frame having these parameters is sparsest in the class of all synthesis matrices of unit norm frames that have these parameters, provided STC is run for the sequence $(\lambda_n)_{n=1}^N$ rearranged to be ordered blockwise. Note that for unit norm tight frames all eigenvalues are equal so questions of rearranging the order of the eigenvalues does not arise.

3. Fusion frames with prescribed eigenvalues ≥ 2 and prescribed dimensions

Let $M \geq N$ be natural numbers and $(\lambda_n)_{n=1}^N \subseteq [2, \infty)$ such that $\sum_{n=1}^N \lambda_n = M$. Given a sequence of dimensions we ask the question of whether and how we can find a fusion frame for $\mathbb{R}^N$ whose subspaces have those prescribed dimensions and whose fusion frame operator

STC: Spectral Tetris Construction

Parameters:

- Dimension $N \in \mathbb{N}$.
- Number of frame elements $M \in \mathbb{N}$.
- Eigenvalues $(\lambda_n)_{n=1}^N \geq 2$ such that $\sum_{n=1}^N \lambda_n = M$.

Algorithm:

Set $k := 1$.
For $j = 1, \ldots, N$ do
    Repeat
        If $\lambda_j < 1$ then
            $f_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
            $f_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
            $k := k + 2$.
            $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
            $\lambda_j := 0$.
        else
            $f_k := e_j$.
            $k := k + 1$.
            $\lambda_j := \lambda_j - 1$.
        end.
    until $\lambda_j = 0$.
end.

Output:

- Frame $\{f_k\}_{k=1}^M$.

Figure 1. The STC algorithm for constructing a frame with a desired frame operator.

has the eigenvalues $(\lambda_n)_{n=1}^N$. We will characterize whether there is a spectral tetris fusion frame having these eigenvalues and dimensions.

To get started, consider the following example of integer eigenvalues. Let $(\lambda_n)_{n=1}^7 = (4, 3, 3, 3, 2, 1, 1)$. Given this sequence of eigenvalues the spectral tetris frame in $\mathbb{C}^7$ consists only of standard unit vectors:

$$S = \{e_1, e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, e_4, e_4, e_4, e_5, e_5, e_6, e_7\}.$$

The question we are asking above now takes the following form. We want to partition S into sets of pairwise orthonormal vectors, i.e. each set of the partition should not have more than one copy of any standard unit vector. What sizes can these sets have? We start by considering the partition $S = \bigcup_{n=1}^{4} P_n$, where

$$\begin{aligned} P_1 &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}, \\ P_2 &= \{e_1, e_2, e_3, e_4, e_5\}, \\ P_3 &= \{e_1, e_2, e_3, e_4\}, \\ P_4 &= \{e_1\}. \end{aligned}$$

The sets of this partition have the sizes $7, 5, 4$ and 1. To get a different partition we can not take any vector out of P_i and put it into P_j if $i > j$ as this would destroy the orthonormality of the sets. But we can take certain vectors out of P_i and put them into P_j if $i < j$ without destroying the orthonormality of the sets. Doing so we can for example easily find a partition into orthonormal sets of the sizes $5, 5, 4$ and 2. But it is not possible to find a partition into orthonormal sets of the sizes $7, 6, 3$ and 1. The sequence $7, 5, 4, 1$ majorizes the sequences of sizes of orthonormal sets which we can partition S into. Let us recall the notion of majorization. Given $a = (a_n)_{n=1}^N \in \mathbb{R}^N$, denote by $a^{\downarrow} \in \mathbb{R}^N$ the vector obtained by rearranging the coordinates of a in decreasing order. If $(a_n)_{n=1}^N, (b_n)_{n=1}^N \in \mathbb{R}^N$, we say $(a_n)_{n=1}^N$ majorizes $(b_n)_{n=1}^N$, denoted by $(a_n)_{n=1}^N \succeq (b_n)_{n=1}^M$, if $\sum_{n=1}^m a_n^{\downarrow} \geq \sum_{n=1}^m b_n^{\downarrow}$ for all $m = 1, \ldots, N-1$ and $\sum_{n=1}^N a_n = \sum_{n=1}^N b_n$.

We can use the idea of the above example to construct spectral tetris fusion frames in the general case of real eigenvalues as the spectral tetris frames for real eigenvalues consisting only of standard unit vectors or linear combinations of two standard unit vectors. As above we will determine a sequence of numbers depending on the given eigenvalues $(\lambda_n)_{n=1}^N$ and check whether or not this sequence majorizes the given sequence of dimensions. As in the example the sequence we are going to determine will be the sequence of dimensions of a certain fusion frame for $\mathbb{R}^N$ having the eigenvalues $(\lambda_n)_{n=1}^N$. We now introduce this fusion frame.

Definition 3.1. *Let $M \geq N$ be natural numbers and $(\lambda_n)_{n=1}^N \subseteq [2, \infty)$ such that $\sum_{n=1}^N \lambda_n = M$. The fusion frame constructed by the algorithm RFF presented in figure 2 is called the reference fusion frame for the eigenvalues $(\lambda_n)_{n=1}^N$.*

Note that if $(V_i)_{i=1}^t$ is the reference fusion frame for $(\lambda_n)_{n=1}^N$ and $(f_i)_{i=1}^M$ is the frame RFF constructs to span the subspaces of $(V_i)_{i=1}^t$, then by construction the following holds:

$$\forall 1 \leq i \leq j \leq t \quad \forall f_k \in V_i \quad \exists f_l \in V_j \colon \operatorname{supp} f_k \cap \operatorname{supp} f_l \neq \emptyset. \tag{1}$$

Further note that if $(\lambda_n)_{n=1}^N$ is ordered blockwise and has reference fusion frame $(V_n)_{n=1}^t$ then

$$(\dim V_n)_{n=1}^t \succeq (\dim U_n)_{n=1}^s, \tag{2}$$

where $(U_n)_{n=1}^s$ is the reference fusion frame for $(\lambda_{\pi(n)})_{n=1}^N$ with π being a permutation of $\{1, \ldots, N\}$ and the shorter tuple of dimensions is filled up with zeros to have tuples of the same length.

We will now use the reference fusion frame for $(\lambda_n)_{n=1}^N$ to decide whether or not a fusion frame for $\mathbb{R}^N$ with certain fusion frame operator and certain dimensions of the subspaces is

RFF: REFERENCE FUSION FRAME

Parameters:

- Eigenvalues $(\lambda_n)_{n=1}^N \subseteq [2,\infty)$ such that $\sum_{n=1}^N \lambda_n = M \in \mathbb{N}$.

Algorithm:

1) Run STC for $(\lambda_n)_{n=1}^N$ and get frame $(f_i)_{i=1}^M$.
2) $t = \max_{j=1,\dots,N} |\operatorname{supp}(f_i(j))_{i=1}^M|$
3) $S_i = \emptyset$ for $i = 1,\dots,t$
4) $k = 0$
5) Repeat
6) $\quad k = k+1$
7) $\quad j = \min\{1 \le r \le t\colon \operatorname{supp} f_k \cap \operatorname{supp} f_s = \emptyset \quad \forall f_s \in S_r\}$
8) $\quad S_j = S_j \cup \{f_k\}$
9) until $k = M$.

Output:

- Fusion frame $(V_i)_{i=1}^t$, where $V_i = \operatorname{span} S_i$.

FIGURE 2. The RFF algorithm for constructing the reference fusion frame.

constructible via spectral tetris. In case it is constructible the proof describes an algorithm to construct it.

Theorem 3.2. *Let $M \ge N$ be natural numbers, $(\lambda_n)_{n=1}^N \subseteq [2,\infty)$ be ordered blockwise and let $(d_i)_{i=1}^D \subseteq \mathbb{N}$ such that $\sum_{n=1}^N \lambda_n = \sum_{n=1}^D d_n = M$. Let $(V_n)_{n=1}^t$ be the reference fusion frame for $(\lambda_n)_{n=1}^N$. Then there exists a spectral tetris fusion frame $(W_n)_{n=1}^D$ for $\mathbb{R}^N$ with dim $W_n = d_n$ for $n = 1,\dots,D$ and eigenvalues $(\lambda_n)_{n=1}^N$ if and only if*

$$D \ge t \quad \text{and} \quad (\dim V_n)_{n=1}^D \succeq (d_n)_{n=1}^D, \tag{3}$$

where $(\dim V_n)_{n=1}^D = (\dim V_1,\dots,\dim V_t, 0,\dots,0)$.

Proof. Let $(\lambda_n)_{n=1}^N$ be in some fixed order, not necessarily ordered blockwise. We prove that in this case (3) characterizes whether or not a fusion frame with the eigenvalues $(\lambda_n)_{n=1}^N$ and dimensions $(d_n)_{n=1}^D$ is constructible from the spectral tetris frame for this ordering of the eigenvalues. The claim of the theorem then follows from the observation made in (2).

We first show how to iteratively construct the desired fusion frame $(W_n)_{n=1}^D$ in case (3) holds. For $i = 1,\dots,t$ let $W_i^0 = S_i$, where t and S_i for $i = 1,\dots,t$ are given by RFF for $(\lambda_n)_{n=1}^N$. We add empty sets if necessary to obtain a collection $(W_i^0)_{i=1}^D$ of D sets. If $\sum_{i=1}^D ||W_i^0| - d_i| = 0$ then the sets $(W_i^0)_{i=1}^D$ span the desired fusion frame. Otherwise, starting from $(W_i^0)_{i=1}^D$ we will construct the spanning sets of the desired fusion frame. Let

$$m = \max\left\{j \le D\colon d_j \ne |W_j^0|\right\}.$$

Note that $\sum_{i=1}^{m}|W_i^0| = \sum_{i=1}^{m} d_i$ by the choice of m and $\sum_{i=1}^{m-1}|W_i^0| > \sum_{i=1}^{m-1} d_i$ by assumption (3). Therefore $d_m > |W_m^0|$ and there exists

$$k = \max\left\{ j < m \colon |W_j^0| > d_j \right\}.$$

Notice that $|W_m^0| < d_m \leq d_k < |W_k^0|$ implies $|W_m^0| + 2 \leq |W_k^0|$. If there is some element $w \in W_k^0$ which has disjoint support from every element in W_m^0 define $\{W_i^1\}_{i=1}^D$ by

$$W_i^1 = \begin{cases} W_k^0 \setminus \{w\} & \text{if } i = k, \\ W_m^0 \cup \{w\} & \text{if } i = m, \\ W_i^0 & \text{else.} \end{cases} \tag{4}$$

Now suppose there is no such element in W_k^0. Pick any $w_1 \in W_k^0$. Next choose all the elements from W_m^0 whose support intersect the support of w_1. Next choose all elements from W_k^0 whose support intersect the support of some element chosen so far. Continue by choosing all elements from W_m^0 whose support intersect the support of some element chosen so far. Continue until you can not choose an element anymore. Let S_1 be the set of the chosen elements. No element of S_1 has a support which intersects the support of any element of $(W_k^0 \cup W_m^0) \setminus S_1$. As $|W_m^0| + 2 \leq |W_k^0|$, there exists some $w_2 \in (W_k^0 \cup W_m^0) \setminus S_1$. Construct S_2 by the same procedure as above with $w_2 \in (W_k^0 \cup W_m^0) \setminus S_1$ instead of $w_1 \in W_k^0$. If there is some element $w_3 \in (W_k^0 \cup W_m^0) \setminus (S_1 \cup S_2)$ continue to construct S_3 in the above fashion. In this way we construct sets $S_1, \ldots, S_r$, say, until we used up all the elements of $W_k^0 \cup W_m^0$. For $i = 1, \ldots, r$ the number of elements in S_i chosen from W_k^0 and W_m^0 differs by at most one, as the elements of W_k^0, respectively W_m^0, have pairwise disjoint supports of sizes 1 or 2. As $|W_m^0| + 2 \leq |W_k^0|$ there is a set S_j that contains one element more from W_k^0 then from W_m^0. Define $\{W_i^1\}_{i=1}^D$ by

$$W_i^1 = \begin{cases} (W_k^0 \cup S_j) \setminus (S_j \cap W_k^0) & \text{if } i = k, \\ (W_m^0 \cup S_j) \setminus (S_j \cap W_m^0) & \text{if } i = m, \\ W_i^0 & \text{else.} \end{cases} \tag{5}$$

In both of the above cases we have defined $\{W_i^1\}_{i=1}^D$ such that

$$\sum_{i=1}^{D} ||W_i^1| - d_i| < \sum_{i=1}^{D} ||W_i^0| - d_i|.$$

Note that $\{W_i^1\}_{i=1}^D$ satisfies (3) in the sense that $(|W_n^1|)_{n=1}^D \succeq (d_n)_{n=1}^D$. Thus if the sets of $\{W_i^1\}_{i=1}^D$ do not span the desired fusion frame, we can repeat the above procedure with $\{W_i^1\}_{i=1}^D$ instead of $\{W_i^0\}_{i=1}^D$ and get $\{W_i^2\}_{i=1}^D$ such that $\sum_{i=1}^D ||W_i^2| - d_i| < \sum_{i=1}^D ||W_i^1| - d_i|$. Continuing in this fashion we will, say after repeating the process l times, arrive at $\{W_i^l\}_{i=1}^D$ such that $\sum_{i=1}^D ||W_i^l| - d_i| = 0$, i.e. the sets of $\{W_i^l\}_{i=1}^D$ span the desired fusion frame $(W_n)_{n=1}^D$.

It remains to show that (3) is a necessary condition. Suppose there is a spectral tetris fusion frame $(W_n)_{n=1}^D$ such that $\dim W_n = d_n$ for $n = 1, \ldots, D$ and suppose $D \geq t$ but the majorization condition is not satisfied. Let

$$m = \min\left\{ k \colon \sum_{n=1}^{k} \dim V_n < \sum_{n=1}^{k} d_n \right\}.$$

Note that $\sum_{n=1}^{D} \dim V_n = \sum_{n=1}^{D} d_n$, thus $m < t$. Let S_n be the union of the supports of the frame vectors that span V_n. Then $S_i \subseteq S_j$ whenever $i \geq j$. By the choice of m we need all the frame vectors spanning $V_1, \ldots, V_m$ to span $W_1, \ldots, W_m$ but they do not suffice. At least one vector from $V_{m+1}, \ldots, V_t$ is needed. But as the sets S_n are in decreasing order, this contradicts the fact that $W_1, \ldots, W_m$ are orthonormal sets. For essentially the same reason we can not have $D < t$. □

Note that in the trivial case of integer eigenvalues we can run STC whenever all given eigenvalues are ≥ 1 to get the following corollary.

Corollary 3.3. *If $(\lambda_n)_{n=1}^{N}, (d_i)_{i=1}^{D} \subseteq \mathbb{N}$ such that $\sum_{n=1}^{N} \lambda_n = \sum_{n=1}^{D} d_n = M \in \mathbb{N}$, where $M \geq N$, then a spectral tertis fusion frame with eigenvalues $(\lambda_n)_{n=1}^{N}$ and dimensions $(d_n)_{n=1}^{D}$ exists if and only if $D \geq \max_{i=1,\ldots,N} \lambda_i$ and $(a_n)_{n=1}^{D} \succeq (d_n)_{n=1}^{D}$, where $a_n = \max\{r \colon \lambda_r \geq n\}$ for $n = 1, \ldots, \max_{i=1,\ldots,N} \lambda_i$ and $(a_n)_{n=1}^{D} = (a_1, \ldots, a_{\max_{i=1,\ldots,N} \lambda_i}, 0, \ldots, 0)$.*

Proof. Note that in this case RFF produces the output $t = \max_{i=1,\ldots,N} \lambda_i$ and $\dim V_i = a_i$ for $i = 1, \ldots, t$. □

4. Spectral tetris for unit norm tight fusion frames of redundancy < 2

Given $J \in \mathbb{N}$, let $\omega = \exp(\frac{2\pi i}{J})$. We define the discrete Fourier transform (DFT) matrix in $\mathbb{C}^{J \times J}$ by

$$D_J = \left(\omega^{ik}\right)_{i,k=0}^{J-1}.$$

Note that we do not normalize D_J by the factor $1/\sqrt{J}$ to make its rows and columns have norm one. Instead every entry of D_J has absolut value one. Further note that the rows of D_J are pairwise orthogonal vectors.

The algorithm DFTST introduced in figure 3 is a variation of spectral tetris. It uses scaled $J \times J$ DFT-matrices to construct M-element unit norm tight frames for $\mathbb{C}^N$ in the case $N < M < 2N$. Here J is the uniquely determined natural number for which $\frac{J}{J-1} < \frac{M}{N} \leq \frac{J-1}{J-2}$. Note that this implies $J \geq 3$. Before stating our main Theorem 4.2 about DFTST we will demonstrate in the following example what DFTST essentialy does.

Example 4.1. *Let $N = 4$ and $M = 7$. Then $J = 3$ and DFTST will construct the synthesis matrix of a 7-element unit norm tight frame in $\mathbb{C}^4$ by using scaled copies of the DFT matrix D_3. We start to build the synthesis matrix from the upper left corner. In lines 3) to 7) of the algorithm we determine whether there still have to be built 3 more columns, which is the case. Thus $K = 3$, $\omega = \exp(\frac{2\pi i}{3})$ and we start by building the first 3 columns using lines 9) until 21) of the algorithm. Line 10) tells us to start by putting the first row of D_3, scaled so that this row square sums to the desired tight frame bound $7/4$.*

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix}.$$

Next, in the inner repeat loop of the algorithm, we check whether we can put the second row of D_3, scaled to square sum to $7/4$, below the first row whithout exceeding the column norm

DFT SPECTRAL TETRIS (DFTST)

Parameters:

- Dimension $N \in \mathbb{N}$.
- Number of frame elements $M \in \mathbb{N}$, where $N < M < 2N$.
- $J \in \mathbb{N}$ such that $\frac{J}{J-1} < \frac{M}{N} \le \frac{J-1}{J-2}$.

Algorithm:

$m = 0, n = 1, \lambda = 0$
Repeat
  If $J < M - m$ then
    $K = J$
  else
    $K = M - m$
  end.
  $w = \exp(\frac{2\pi i}{K})$
  For $r = m+1, \ldots, m+K$ do
    $f_r = \sqrt{\frac{M-N\lambda}{NK}} \cdot e_n$
    $k = 1$
    Repeat
      If $1 - \|f_r\|^2 \ge \frac{M}{NK}$ then
        $f_r = f_r + \sqrt{\frac{M}{NK}} \cdot \omega^{k(r-m-1)} \cdot e_n$
      else
        $f_r = f_r + \sqrt{1 - \|f_r\|^2} \cdot \omega^{k(r-m-1)} \cdot e_n$
      end.
      $k = k + 1$
      $n = n + 1$
    until $\|f_r\| = 1$.
  end.
  $m = m + K$
  $\lambda = \sum_{r=1}^{m+K} |f_r(n-1)|^2$
until $m = M$.

Output:

- Unit norm tight frame $\{f_i\}_{i=1}^M \in \mathbb{C}^N$.

FIGURE 3. The DFT Spectral Tetris algorithm for constructing a unit norm tight frame of redundancy < 2.

of 1 *for the desired synthesis matrix. Here this is not the case and so we scale the second row of* D_3 *such that the column norm of the first* 3 *columns of the synthesis matrix is* 1 *when setting the remaining entries of the first* 3 *columns to be zero. Note that scaling the rows of*

D_3 does not affect their orthogonality and that at this point the second row of the synthesis matrix square sums to less than the desired $7/4$.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & ? & ? & ? & ? \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? & ? & ? \end{bmatrix}.$$

Having constructed the first three columns, we run the outer repeat loop of the algorithm again, checking first whether we have 3 more columns to be constructed, which again is the case. The first row of the synthesis matrix already square sums to $7/4$ and therefore must have zeros as the remaining entries.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? & ? & ? \end{bmatrix}.$$

Line $10)$ of the algorithm tells us to scale the first row of D_3 so that, after putting it behind the last DFT row we used, this row of the synthesis matrix square sums to the desired $7/4$.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & ? \\ 0 & 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? & ? & ? \end{bmatrix}.$$

Lines $12)$ to $20)$ of the algorithm tell us again in which way to put scaled versions of the rows of D_3.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & ? \\ 0 & 0 & 0 & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}}\cdot\omega & \sqrt{\frac{7}{12}}\cdot\omega^2 & ? \\ 0 & 0 & 0 & \sqrt{\frac{3}{12}} & \sqrt{\frac{3}{12}}\cdot\omega^2 & \sqrt{\frac{3}{12}}\cdot\omega^4 & ? \end{bmatrix}.$$

Now rows 2 and 3 of the synthesis matrix are finished.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}}\cdot\omega & \sqrt{\frac{7}{12}}\cdot\omega^2 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{12}} & \sqrt{\frac{3}{12}}\cdot\omega^2 & \sqrt{\frac{3}{12}}\cdot\omega^4 & ? \end{bmatrix}.$$

We again check whether or not we have construct 3 more columns, which this time is not the case. Only one more column has to be constructed so we follow the above steps this time using D_1 instead of D_3 to obtain the synthesis matrix

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}}\cdot\omega & \sqrt{\frac{7}{12}}\cdot\omega^2 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{12}} & \sqrt{\frac{3}{12}}\cdot\omega^2 & \sqrt{\frac{3}{12}}\cdot\omega^4 & 1 \end{bmatrix}.$$

We can think of the structure of the synthesis matrix constructed via DFTST as consisting of blocks which arise from DFT-matrices whose rows have been scaled by appropriate factors. If M, N and J are as in DFTST and J divides M, then it is built from $\frac{M}{J}$ blocks of size $J \times J$. If J does not divide M, the synthesis matrix is built of $\lfloor \frac{M}{J} \rfloor$ blocks of size $J \times J$ and one block of size $(M - \lfloor \frac{M}{J} \rfloor J) \times (M - \lfloor \frac{M}{J} \rfloor J)$ in the lower right corner.

Theorem 4.2. *Let $M, N \in \mathbb{N}$ such that $N < M < 2N$. Then DFTST constructs a unit norm tight frame $(f_m)_{m=1}^M \in \mathbb{C}^N$ with the property that $\langle f_m, f_{m'} \rangle = 0$ whenever $m - m' \geq 2J$, where $\frac{J}{J-1} < \frac{M}{N} \leq \frac{J-1}{J-2}$.*

Proof. We first show that while being in the case of line 3) of DFTST, i.e. while $K = J$, the columns that are constructed have maximum of their support, denoted by s, smaller or equal to N. This ensures the inner repeat loop terminates before exceeding the dimension of the columns. Indeed, assume $s > N$, and let n_0, respectively m_0, be the value of n, respectively m, at the beginning of the outer repeat loop that we are in, we get

$$\begin{aligned} \sum_{m=1}^{M}\sum_{n=1}^{N}|f_m(n)|^2 &= \sum_{m=1}^{m_0}\sum_{n=1}^{N}|f_m(n)|^2 + \sum_{m=m_0+1}^{m_0+J}\sum_{n=1}^{N}|f_m(n)|^2 = \sum_{m=1}^{m_0} 1 + \sum_{m=m_0+1}^{m_0+J}\sum_{n=1}^{N}|f_m(n)|^2 \\ &< m_0 + \sum_{m=m_0+1}^{m_0+J} 1 = m_0 + J < M \end{aligned}$$

while interchanging the order of summation yields

$$\begin{aligned} \sum_{n=1}^{N}\sum_{m=1}^{M}|f_m(n)|^2 &= \sum_{n=1}^{n_0-1}\sum_{m=1}^{M}|f_m(n)|^2 + \sum_{m=1}^{M}|f_m(n_0)|^2 + \sum_{n=n_0+1}^{N}\sum_{m=1}^{M}|f_m(n)|^2 \\ &= \sum_{n=1}^{n_0-1}\frac{M}{N} + \lambda + J\left(\frac{M - N\lambda}{NJ}\right) + \sum_{n=n_0+1}^{N} J\frac{M}{NJ} = \frac{NM}{N} = M, \end{aligned}$$

a contradiction.

We next show that, still being in the case of line 3), the inner repeat loop does not run more than $J - 1$ times, i.e. it terminates before or exactly when we have used up all the rows of D_J. For this it suffices to show that

$$1 - \frac{M - N\lambda}{NJ} - (J-2)\frac{M}{NJ} \leq \frac{M}{NJ}$$

and thus we have to show $(N-M)J + N\lambda < 0$. And indeed, using $\lambda < \frac{M}{N} \le \frac{J-1}{J-2}$ we have

$$\left(1-\frac{M}{N}\right)J + \lambda \le \left(1-\frac{M}{N}\right)J + \frac{M}{N} = J + \frac{M}{N}(1-J) \le J + \frac{(J-1)(1-J)}{J-2} < 0,$$

where we used the fact that $J \ge 3$.

Eventually DFTST will enter the case of line 5) and this will be the last time the outer repeat loop of the algorithm runs. In this last run the final $M-m_0$ columns $f_{m_0+1}, \dots, f_M$ of the synthesis matrix are constructed and we have to show that the maximum of the support of these vectors, which we denote by N_0, equals N, that their support has size of at most $M-m_0$ and that the last row of the synthesis matrix has norm M/N. We first show $N_0 = N$. Indeed, supposing $N_0 < N$ yields

$$M = \sum_{m=1}^{M} 1 = \sum_{m=1}^{M}\sum_{n=1}^{N} |f_m(n)|^2 = \sum_{n=1}^{N}\sum_{m=1}^{M} |f_m(n)|^2 = N_0 \frac{M}{N} < M,$$

a contradiction. Assuming $N_0 > N$ implies

$$M = N\frac{M}{N} = \sum_{n=1}^{N}\sum_{m=1}^{M} |f_m(n)|^2 = \sum_{m=1}^{M}\sum_{n=1}^{N} |f_m(n)|^2 < \sum_{m=1}^{M} 1 = M,$$

again a contradiction. Next we show that the last row of the constructed synthesis matrix has norm M/N. To see this, note that

$$\begin{aligned} M = \sum_{m=1}^{M} 1 &= \sum_{m=1}^{M}\sum_{n=1}^{N} |f_m(n)|^2 = \sum_{n=1}^{N}\sum_{m=1}^{M} |f_m(n)|^2 \\ &= \sum_{n=1}^{N-1}\sum_{m=1}^{M} |f_m(n)|^2 + \sum_{m=1}^{M} |f_m(N)|^2 = (N-1)\frac{M}{N} + \sum_{m=1}^{M} |f_m(N)|^2, \end{aligned}$$

and thus indeed

$$\sum_{m=1}^{M} |f_m(N)|^2 = \frac{M}{N}.$$

Finally suppose the support of f_M, which coinsides with the support of the other columns constructed in this run of the outer repeat loop, is of size $s > M - m_0$. Then

$$\sum_{n=1}^{N} |f_M(n)|^2 \ge s \cdot \frac{M}{N(M-m_0)} \ge \frac{M}{N} > 1$$

in contrdiction to the fact that f_M was constructed to have norm 1.

Having shown the above about DFTST, it is now easy to check that the synthesis matrix it produces is that of a unit norm tight frame: It is clear by construction that each frame vector has norm one. The rows of the synthesis matrix are pairwise orthogonal since they are made of multiples of rows of DFT matrices, thus the frame operator of the constructed frame has zero entries off of its diagonal. The powers of ω coming up in the construction of the columns have absolut value 1. Therefore, rows of the synthesis matrix that are defined entirely within one run of the outer repeat loop have entries that square sum to $K \cdot \frac{M}{NK} = \frac{M}{N}$.

The same holds for rows that are defined using two consecutive runs of the outer repeat loop. Their entries square sum to

$$\lambda + K \cdot \frac{M - N\lambda}{NK} = \frac{M}{N}.$$

Thus the frame operator of the constructed frame is $\frac{M}{N}I$. To see the orthogonality property of the columns stated in the theorem we note the following. A column constructed in a certain run of the outer repeat loop has disjoint support from any column constructed in a run of the outer repeat loop that did not directly precede or follow it. In every run of the outer repeat loop, at most J vectors are constructed, which therefore yields the orthogonality statement of the theorem. □

We now show how to use the unit norm tight frames constructed via DFTST to construct non-equidimensional tight fusion frames. We can do this as long as the desired dimensions of the subspaces stay below a certain bound by making use of the fact that a lot of the frame vectors constructed via DFTST have disjoint support. We first need a technical lemma.

Lemma 4.3. *Let $M, L \in \mathbb{N}$ such that $L < M$. Then there exists a permutation π of $\{1, \ldots, M\}$ such that for all $k, l \in \{1, \ldots, M\}$ with $0 < |k - l| \leq \lfloor \frac{M}{L} \rfloor - 1$ we have $|\pi(k) - \pi(l)| \geq L$.*

Proof. Let $[1], \ldots, [L]$ be the partition of $(1, \ldots, M)$ into cosets modulo L. These cosets have either $\lfloor \frac{M}{L} \rfloor$ or $\lfloor \frac{M}{L} \rfloor + 1$ elements. Write each coset as an increasing sequence, say $[s] = (a_{s,1}, a_{s,2}, \ldots, a_{s,n_s})$, and let

$$(r_1, \ldots, r_M) := (a_{L,1}, \ldots, a_{L,n_L}, a_{L-1,1}, \ldots, a_{L-1,n_{L-1}}, \ldots, a_{1,1}, \ldots, a_{1,n_1}).$$

Define π by $\pi(i) = r_i$ for $i = 1, \ldots, M$. Let k, l as in the assumption of the lemma and assume as we may that $k < l$. If $\pi(k)$ and $\pi(l)$ belong to the same coset modulo L, then $|\pi(k) - \pi(l)| \geq L$. If not, there exists $m \in \{1, \ldots, L-1\}$ such that $\pi(k) \in [m+1]$, say $\pi(k) = a_{m+1,j}$ and $\pi(l) \in [m]$, say $\pi(l) = a_{m,i}$. As the cosets have at least $\lfloor \frac{M}{L} \rfloor$ elements, we have $i < j$ and thus

$$|\pi(k) - \pi(l)| = a_{m+1,j} - a_{m,i} > a_{m+1,j} - a_{m+1,i} \geq L.$$

□

Now we can state and prove the corollary.

Corollary 4.4. *If $N < M < 2N$ and $J \in \mathbb{N}$ such that $\frac{J}{J-1} < \frac{M}{N} \leq \frac{J-1}{J-2}$, then for all $(k_i)_{i=1}^K \subset \mathbb{N}$ with $\sum_{i=1}^K k_i = M$ and $k_i \leq \lfloor \frac{M}{2J} \rfloor$ for $i = 1, \ldots, K$, there exists a tight fusion frame $(V_l)_{l=1}^K$ with $\dim V_l = k_l$ for $l = 1, \ldots, K$.*

Proof. Let $\{f_m\}_{m=1}^M$ be the unit norm tight frame constructed in Theorem 4.2 and consider $\{f_{\pi(m)}\}_{m=1}^M$ where π is the permutation of $\{1, \ldots, M\}$ given in Lemma 4.3 for $L = 2J$. Let $R = \lfloor \frac{M}{2J} \rfloor$. Then for any $m = 1, \ldots, M - R + 1$ the vectors of the set $\{f_{\pi(m+i)}\}_{i=0}^{R-1}$ are pairwise orthonormal by Theorem 4.2. Note that this set contains R elements. Thus we can define

$$V_l = \operatorname{span}\left\{ f_{\pi(i + \sum_{j=1}^{l-1} k_j)} : i = 1, \ldots, k_l \right\}$$

for $l = 1, \ldots, K$. □

The new tools introduced in this paper should have broad application to other construction problems for frames and fusion frames. We are exploring these currently possibilities.

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