MULTISCALE TOTAL VARIATIONAL ALGGORITHMS FOR DENOISING NATURAL COLOR PHOTOS IN DIGITAL PHOTOGRAPHY

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Abstract. As digital photography rapidly replacing the traditional film photography as the photography of choice for all but a few devoted professionals, post image processing of natural color photos such as denoising becomes increasingly an integral part of digital photography. Although many denoising schemes have been designed, almost none specifically target natural color photos. Noise in natural color photos have special characteristics that are substantially different from those that have been added artificially.

In this paper we propose the multiscale total variational method (MTV) for denoising. Standing alone the MTV method is effective in denoising monochromatic images. However, it demonstrates outstanding denoising capabilities for natural color images. Key to the success is the understanding of the characteristics of digital noise in natural color images as well as a non-traditional color space we have introduced specifically for the purpose. An automatic stopping criterion is applied to each channel to prevent over processing.

1. Noise In Natural Color Photos

With the surging popularity of digital cameras, digital photography is rapidly replacing the traditional film photography as the photography of choice for virtually all but a few devoted professionals. In digital photography, post image processing is an integral part for obtaining better images even for the casual picture takers. Post image processing is especially important for people who are willing to go beyond point-and-shoot, and one of the key steps in image processing is denoising.

All digital cameras today take color photos. (Some cameras allow for black-and-white images, but these are converted from color images using in-camera firmwares.) Noise is present in virtually all digital photos, and there are several sources for it. When light (photons) strike the image sensor, electrons are produced. These "photoelectrons" give rise to analog signals which are then converted into digital pixels by an Analog to Digital

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(A/D) Converter. The random nature of photons striking the image sensor is an important source for noise. This type of noise, known as *photon shot noise*, is roughly proportional to the square root of the signal level as a result of the Central Limit Theorem. Thus the lower the signal is the higher the noise becomes relative to the signal. As a result noise in color images can be very pronounced in images shot under low light conditions because the signals must be amplified more. In general, noise level is very low for photos shot outdoor using low ISO (ISO 100 or less). But with most consumer compact cameras noise becomes visible at ISO 200, and it becomes unacceptable at ISO 400 or higher. With more advanced and expensive digital SLRs noise remains low even at ISO 400, and becomes unacceptable at ISO 1600 or higher. Noise is in general much worse under artificial lighting, especially under fluorescent lighting. One of the important characteristics of digital noise is that they are not uniform across all channels. For many digital cameras noise is concentrated in the blue channel while the green and the red channels are relatively clean. For some, noise may be more spread out. For most cameras, the green channel is usually less noisy than the blue and the red channels . For photos taken under artificial lighting (without a flash), the blue channel can be so noisy that it is often unrecognizable, see Figures 1 and 2.

Another significant source of noise is the so called *leakage current*. Semiconductor image sensors work by converting energy from photons into electrical energy, in the form of a current or voltage signal. Unfortunately thermal energy present in the semiconductor can also generate an electrical signal that is indistinguishable from the optical signal. As temperature increases, so does leakage current in the circuit. The effects of leakage current are most apparent in long exposures in which the light signal is very low.

Modeling noise in digital color photos can be a difficult task. The photon shot noise is clearly signal dependent and thus not uniform from pixel to pixel. Nearly all digital cameras[1] today use the so-called Bayer Pattern in their photo sensors, where half of their pixels are used to capture the green channel and the other half are divided evenly to capture the red and the blue channels. These partial data are then interpolated to complete the RGB channels of a color photo. So unless we access the raw data (most consumer digital cameras do not have this feature) it is clear that noise is not independent from pixel to pixel in any channel. Most digital photos are in JPEG format, which degrade images through quantization and artifacts. Furthermore, all cameras employ proprietary in-camera

[1]Sigma digital SLRs such as SD9 and SD10, which employ sensors by Feveon, are notable exceptions.

FIGURE 1. The original natural color image without artificial noise. Noise is not obvious due to limitation on the size of the display.

sharpening, denoising and anti-aliasing. These factors combine to make effective modeling of noise, at least in the images taken by consumer cameras exported in JPEG format, virtually impossible. For this reason, any noise model assuming independent and identically distributed noise from pixel to pixel can be unrealistic. Visual inspections clearly show that noise in natural images appears far more "blotchy" than the usual Gaussian i.i.d. type of artificially added noise.

This uneven distribution of noise poses some challenges. Excessively denoising the blue channel can easily lead to color artifacts (such as color bleeding). A good denoising scheme must take this into consideration. One viable solution to this problem is to work in another color space rather than the RGB color space. A standard practice is to work in a color space that separates the luminance and chrominance. Commonly used color spaces are CIELAB, CIELUV and YCrCb. The YCrCb color space has the advange for being linear. It is the color space used for JPEG and JPEG 2000. One of the innovations in this paper is to design a new color space that effectively takes into account the distribution of noise. This new color space offers superior performance in our tests.

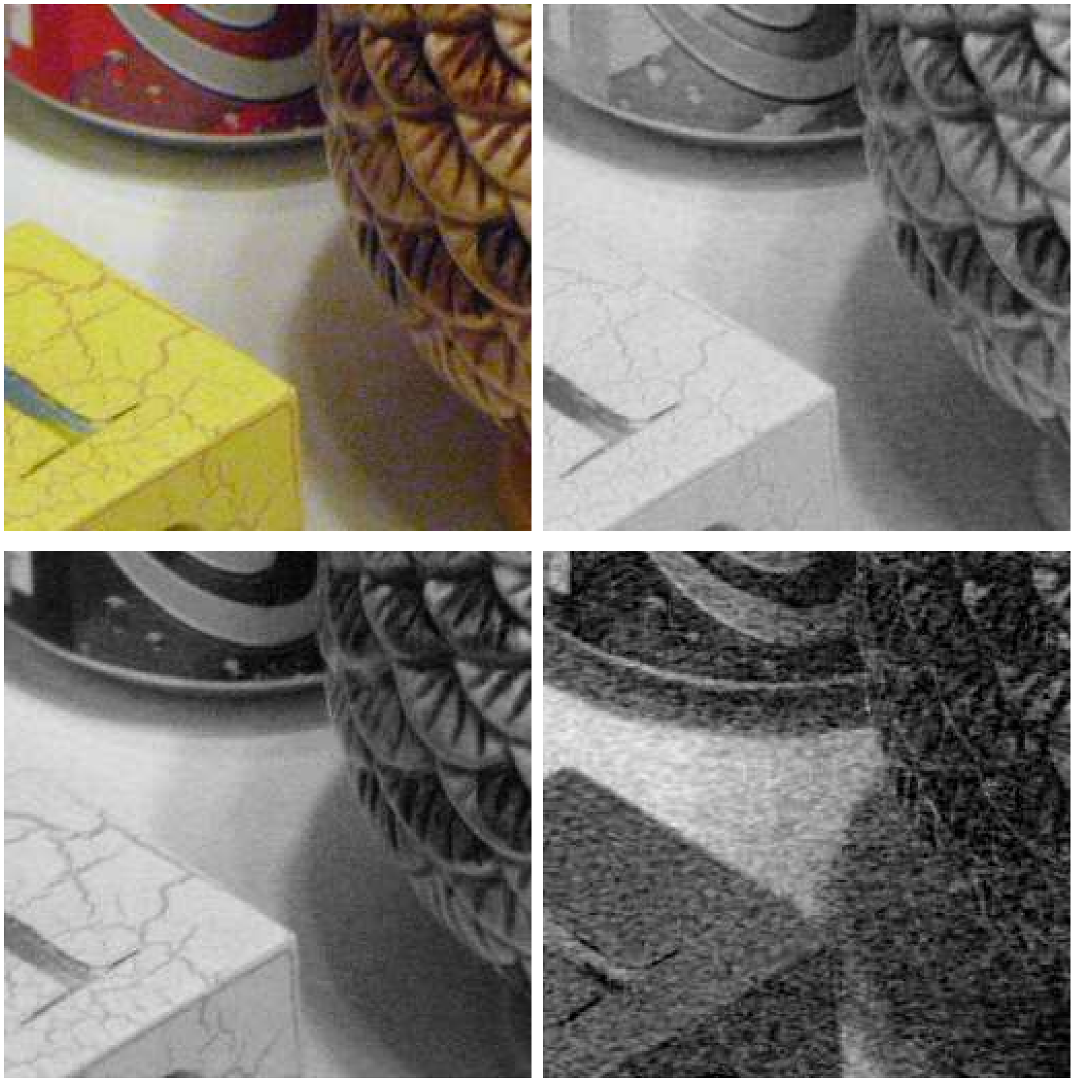

Figure 2. A zoom-in (upper-left) of the color image in Figure 1 and its RGB channels. Color noise is more evident. The red (upper-right) and green (lower-left) channels are much cleaner than the blue (lower-right) channel.

2. Multiscale Total Variational Method for Denoising Monochromatic Images

Denoising methods for monochramatic images are numerous, which include neighborhood filters, frequency domain methods, variational PDE based methods and non-local methods. Much has been made about the pros and cons of each method. It is perhaps safe to say that virtually every method has its advantages and disadvantages, and for different images some methods may work better than others. In this study, we shall focus on a particular denoising algorithm called *Multiscale Total Variational* method (MTV). The MTV method is a variational PDE method using wavelet bases. It is an extension of the wavelet TV method proposed in [13] and [30]. A detailed study of the MTV method can be found in Chan, Wang and Zhou [11]. For self-containment we include a brief overview of variational PDE based methods and the MTV method in this section.

We begin with a standard noisy monochromatic image model

$$z(x) = u_0(x) + n(x), \tag{2.1}$$

where $z(x)$, $u_0(x)$ and $n(x)$ are real valued functions defined on $\Omega \subset \mathbb{R}^2$, where Ω is a finite domain such as a rectangle. The function $u_0(x)$ denotes the underlying image uncorrupted by noise, $z(x)$ the observed image, and $n(x)$ the noise. In our general model, we assume that $z(x)$, $u_0(x)$ and $n(x)$ are in some space of functions $\mathcal{F}$, such as $L^2(\Omega)$ or $C^1(\Omega)$. With a variational PDE based denoising method, the denoised image is the minimizer of certain energy functional $\mathbf{E}(u)$. Typically $\mathbf{E}(u)$ can be written as

$$\mathbf{E}(u) = D(u,z) + R(u), \tag{2.2}$$

where $D(u,z)$ is the fitting term that represents the "distance" between u and observed image z, and $R(u)$ is a regularization term that smoothes out the image. In most algorithms the fitting term $D(u,z)$ is taken to be the L^2 distance $D(u,z) = \int_\Omega (u-z)^2$. Earlier efforts focused on least square based functionals $R(u)$'s such as $\|\Delta u\|_2^2$, $\|\nabla u\|_2^2$ and others. While noise can be effectively removed, these regularization functionals penalize discontinuity, resulting in soft and smooth reconstructed images, with subtle details lost. This is not acceptable in digital photography, as photographers often place premium emphasis on sharpness. The innovation of the total variational (TV) scheme by Rudin, Osher and Fatemi [27] is to set $R(u)$ to be the total variation $\int_\Omega |\nabla u|$ of u. With the total variation regularizer, extensive studies have shown that it does not penalize edges in u, thus it allows for sharper reconstructions, see e.g. [1, 6, 10, 16]. Among all the variational PDE based techniques, the TV minimization scheme offers one of the better combinations of noise removal and feature preservation. Since $\mathbf{E}(u)$ is convex the minimizer can be obtained by gradient flow, which amounts to time marching equation

$$u_t = \nabla \cdot \Big(\frac{\nabla u}{|\nabla u|}\Big) - \lambda(u - z) = 0, \quad u(x,0) = z(x). \tag{2.3}$$

The end result $u(x,T)$, if T is large enough, will have noise removed or reduced. One of the problems is that when (2.3) is left running for too long a denoised image will tend to become a cartoon-like piecewise constant image, wiping out all subtle details [23, 25]. With a more gentle run, one may not remove enough noise. For optimal results it is important to have an automatic stopping criterion. This is difficult to do. Although there are some attempts in this direction [20, 24, 31], some properties of the noise (such as the variance) are assumed to be known, which is not entirely realistic for natural color photos.

A modified version of the TV denoising scheme based on wavelets was introduced in Chan and Zhou [13]. Using this scheme, Wang and Zhou [30] devised an automatic stopping criterion. This criterion works surprisingly well in experiments, see [30] for a detailed discussion. Let $\{\psi_j : \ j \in I\}$ be an orthonormal or biorthonormal wavelet basis for $L^2(\Omega)$ such as those found in [15, 28]. In practice, we have always used the biorthonormal 7-9 wavelet basis as our basis $\{\psi_j\}$. (The conventional notation uses two sub-indices to denote a wavelet basis. Here we use only one for brevity. There should not be any confusion.) Now expand the observed image function $z(x)$ using the basis $\{\psi_j(x)\}$,

$$z(x) = \sum_{j\in I} \alpha_j \psi_j(x).$$

Let

$$u(x, \boldsymbol{\beta}) := \sum_{j\in I} \beta_j \psi_j(x) \tag{2.4}$$

where $\boldsymbol{\beta} = (\beta_j)$. In the wavelet TV scheme the fitting term $D(u,z)$ is set as

$$D(u,z) := \sum_{j\in I} \lambda(\beta_j - \alpha_j)^2,$$

where $\lambda > 0$ while the term $R(u)$ remains to be the total variation. In [11] the *Multiscale Total Variation* (MTV) method was introduced, where the fitting term is modified to be

$$D(u,z) := \sum_{j\in I} \lambda(\beta_j - \alpha_j)^2. \tag{2.5}$$

In this paper we propose a new method based on the wavelet TV denoising scheme. The key feature is that λ_j *decreases* as the scale becomes more localized. More precisely, we use smaller values λ_j for high frequency terms and larger values for lower frequency terms.

Figure 3. Some comparisons using artificial noise. The orginal image (upper-left) is added with Gaussian white noise (upper-right). The standard TV denoised image (lower-left) has more noisy residual than the wavelet TV denoised image (lower-right). Both are obtained with same number of iterations.

In recogniztion that the cartoon-like tendency in the standard TV method can be a result of excessive diffusion of low frequency features in an images, this will help concentrating the diffusion more on high frequency features where the noise resides. Our tests illustrate that the MTV method is highly effective, particularly for denoising natural color images. Comparing to the standard TV or wavelet TV method, the MTV method requires fewer number of iterations to obtain comparable or better results, see [11]. As in the classical TV case, the energy

$$\mathbf{E}_{MTV}(u,z) := \sum_{j\in I} \lambda_j(\beta_j - \alpha_j)^2 + \int_{\mathbb{R}^2} |\nabla_x u(x,\boldsymbol{\beta})|\, dx. \tag{2.6}$$

with $u = u(x,\boldsymbol{\beta})$ is also convex. The minimizer satisfies the Euler-Lagrange equation

$$-\int_{\mathbb{R}^2} \nabla_x \cdot \left(\frac{\nabla_x u}{|\nabla_x u|}\right) \psi_j(x)dx + 2\lambda_j(\beta_j - \alpha_j) = 0. \tag{2.7}$$

Thus its minimizer can be solved using gradient flow via setting $\boldsymbol{\beta} = \boldsymbol{\beta}(t) = (\beta_j(t))$ and solving the following time evolution equation

$$\frac{\partial \beta_j}{\partial t} = \int_{\mathbb{R}^2} \nabla_x \cdot \left(\frac{\nabla_x u}{|\nabla_x u|}\right) \psi_j(x)dx - 2\lambda_j(\beta_j - \alpha_j), \qquad (\beta_j(0)) = (\alpha_j). \tag{2.8}$$

The automatic stopping criterion in [30] for the wavelet TV method is easily applicable for the MTV method.

One may ask what is the advantage in using the MTV method. As we shall see, one of the innovations in this study is the creation of a new color space, which is adaptively modified from the standard YCrCb color space. To ensure quality output it is vital that the Y-channel is kept sharp. The MTV algorithm maintains sharpness in an image like the standard TV method, but has the additional advantage that it does not cartoonize an image, making it well suited for color images.

3. Color Space and Automatic Stopping Criterion

One may argue that color images are no different from three monochromatic images once we consider the three channels separately, and therefore to denoise a color photo one only needs to denoise the three monochromatic channels separately. This view, however, misses some important subtle characteristics in naturally captured color images that, when fully utilized, yield superior results. To denoise color photos we must first understand the nature of the noise in these images, and take full advantages of all available informations.

The most commonly used color space is the RGB color space. In the RGB color space we denoise each of the three channels to complete the denoising of the color image. This approach yields unsatisfactory results, particularly for images taken under artificial lighting where the blue channel can be excessively noisy, or under low lighting using high ISO where both the blue and the red channels can be noisy. A better way is to separate the luminance

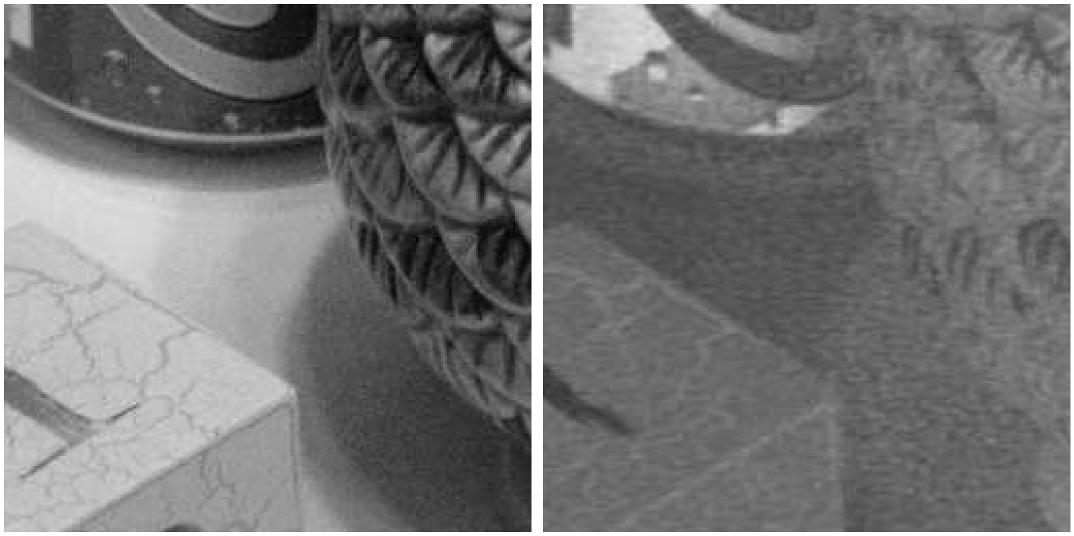

FIGURE 4. The Y (left) and Cr (right) channels of the natural color image shown in Figure 2

from the chrominance. There are a few ways one can achieve this. One way is to use the LAB color space, which is nonlinear against the RGB color space. Another choice is the YCrCb color space. Given that YCrCb is a linear transformation of RGB it is widely used in applications such as color video and JPEG compression of color images. The advantage of separating luminance from chrominance is that human vision is typically less sensitive to diffusions in chrominance. This is illustrated in Figures 4, 5 and 6. Figure 4 and the left on Figure 5 show the YCrCb channels of the natural color image displayed in Figure 2. We then performed a rather destructive wavelet thresholding on the chrominance channels Cr (on the right of Figure 5) and Cb (on the left of Figure 6). The right on Figure 6 shows the re-composed color image. As one can see, there is very little discernable difference between the two color images. This robustness against diffusion in the chrominance does not extend, however, to the luminance channel Y. In fact, even a tiny blurring in the luminance channel will be immediately visible in the re-composed color image. Given these characteristics of the luminance-chrominance decomposition, we would want to be more aggressive in denoising the chrominance channels while less so in denoising the luminance channel.

The problem with the standard luminance-chrominance decomposition, such as LAB and YCrCb, is that the luminance is "contaminated" by the blue and the red channels, where noise is most likely to concentrates as pointed out earlier. As a result the luminance channel

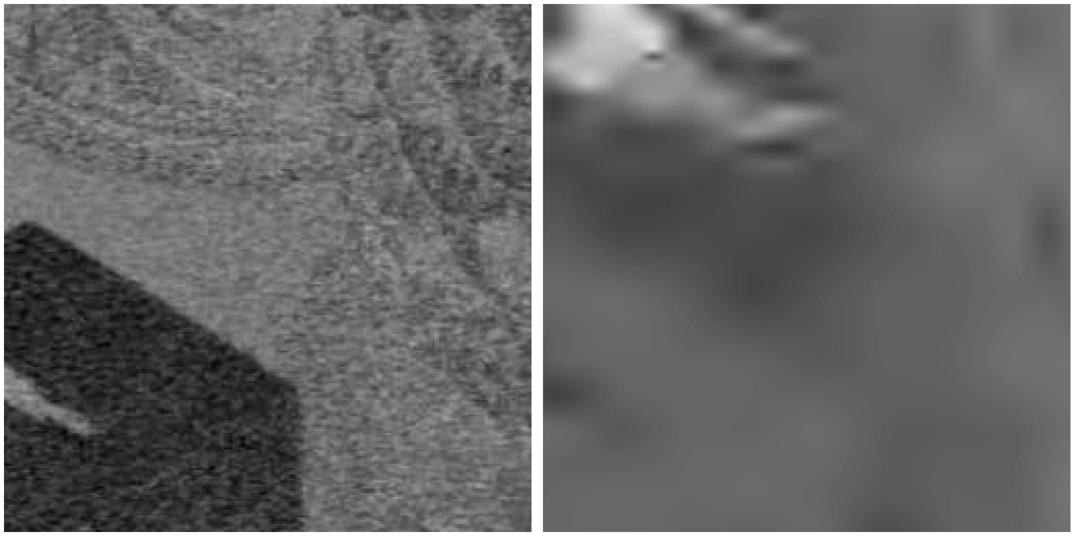

Figure 5. The Cb channel (left) of the natural color image shown in Figure 2. Using wavelet thresholding to severely blur the chrominance Cr channel (right), in which most of the details are removed.

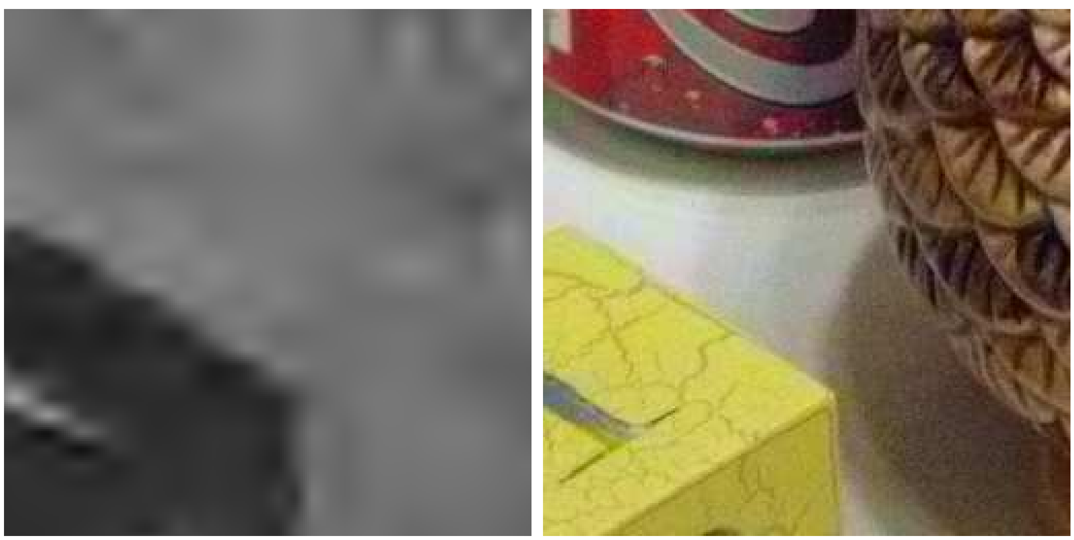

Figure 6. The other chrominance channel Cb (left) is also severely blurred by wavelet thresholding. While the re-composed color image (right) after blurring the CrCb channels seems to be a reasonable approximation to the original one.

can be somewhat noisy, and therefore substantial denoising will often have to be performed on it. This can adversely affect the quality of the denoised color image. To get around this problem we introduce a new color space, the *adaptive modified YCrCb* color space (m-YCrCb color space). In the m-YCrCb space, the "luminance" channel is also a linear combination of the RGB channels, but the weights are determined adaptively depending on how noisy the red and the blue channels are. More precisely, the m-YCrCb color space is obtained via the following linear transform from the RGB color space:

$$\begin{aligned} Y_m &= (1-\alpha_R-\alpha_B)G+\alpha_R R+\alpha_B B, \\ Cr_m &= (R-Y)/1.6, \\ Cb_m &= (B-Y)/2, \end{aligned}$$

where Y is still the "luminance" in the standard YCrCb color space, weights α_R, α_B are determined the noise level of the red and blue channels. We impose the condition that

$$\alpha_R+\alpha_B \le \frac{1}{3}$$

So that the "luminance" channel Y_m is primarily from the green channel as usual. However, if one or both of the red and blue channels are nosiy as measured using high frequency wavelet coefficients similar to the one used in the automatic stopping criterion, the weights will be adjusted to minimize the exposure of the Y_m channel to noise. In general, if noise is concentrated in the blue channel then taking $\alpha_R = 1/3$ and $\alpha_B = 0$ works very well.

(SHALL WE ADD A COMMENT HERE TO COVER THE CASE THAT THE GREEN CHANNEL CONTAINS THE MOST NOISE, WE WILL COMPUTE THE LUMINANCE BY RED AND BLUE).

To denoise a color image we perform the MTV denoising scheme on each of the three channels Y_m, Cr_m, and Cb_m. An automatic stopping criterion is applied in the MTV scheme. Since the luminance channel in the m-YCrCb color space no longer contains any part of the blue channel it is usually much cleaner. The automatic stopping criterion, which we describe in details below, will stop the process for the Y_m channel after only a few iterations. The denoising of the Cr_m channel also takes only a few iterations for the very same reason. The process takes much longer in general for the Cb_m channel, in which noise is concentrated.

We now describe our automatic stopping criterion for the MTV denoising scheme with the wavelet basis $\{\psi_j : j \in I\}$. Again we remark that the conventional notation for wavelet bases use two or more indices, such as $\{\psi_{jk}\}$. In this paper we only use one index for conciseness, and there should not be any confusion. Like in the wavelet hard thresholding scheme, we first choose a threshold $\rho > 0$. Let $J_\rho = \{j \in I_D : \ |\beta_j(0)| = |\alpha_j| \leq \rho\}$, where $I_D \subset I$ is the index set corresponding to the diagonial portion of the highest frequency wavelet coefficients. Intuitively speaking, as in the wavelet hard thresholding scheme, the coefficients $\{\beta_j(0) : \ j \in J_\rho\}$ will indicate how noisy the image is. In a noise-free image these wavelet coefficients will mostly be very close to 0. But in a noisy image they will be mostly not close to zero. Define $\mu(t) = \frac{1}{|J_\rho|}\sum_{j\in J_\rho}|\beta_j(t)|$. So $\mu(t)$ measures the noise in the image at time t. The key idea is that an automatic stopping criterion of the time evolution can be designed by measuring the reduction in the value $\mu(t)$ from the orginal value $\mu(0)$.

In [30] we have described two different automatic stopping criteria, the *relative criterion* and the *absolute criterion*. Both of these can be adopted to the MTV scheme. For the relative automatic stopping criterion, we consider $\mu(t)/\mu(0)$. We will stop the time evolution whenever this value goes below a threshold b. For example, we may set $b = 0.1$. This threshold intuitively says that we stop the time evolution when we have reduced noise by 90%. For the absolute automatic stopping criterion, we stop the time evolution if $\mu(t)$ drops below a threshold c. Since in a noise-free image we expect $\mu(t)$ to be very close to zero, it is reasonable to set an absolute threshold for $\mu(t)$ to achieve a desired denoising effect.

In the actual implementation the value ρ does not seem to affect the automatic stopping time sensitively. We usually take $\rho = \frac{2}{|I_D|}\sum_{j\in I_D}|\alpha_j|$. Both the relative criterion and the absolute criterion work well, although we typically use the relative criterion. For an image with moderate noise we set the threshold b to be between 0.05 and 0.1. In more noisy cases such as the one in Figure 2, we use smaller threshold b around 0.03. It should be pointed out that we have tested the automatic stopping time criterion on a number of noisy color images as well as monochromatic medical images. The thresholds for optimal performance stayed remarkably consistent. This is an important attribute for batch processing medical images.

4. Examples

In this section we compare various denoising schemes using the noisy color image shown in Figure 2. These schemes include the wavelet hard thresholding, the wavelet soft thresholding, and MTV . We first compare the denoising on the RGB color space in Figures 7 and

FIGURE 7. The denoised images by wavelet hard (left) and soft (right) thresholdings in the RGB space. Either noticable noise still exists due to high noise in blue channel, or the image is excessively smeared.

8. As one can see, given the severity of the noise in the blue channel none of them has performed well, at least not well enough to be a serious tool practically in digital photography. We next show the denoising results on the m-YCrCb space (Figure 9). Clearly in the new color space there is a rather substantial improvement in perfromance. The reconstructed blue channel is perhaps the best indicator of the effectiveness of the denoising. The MTV method yields a superb reconstructed blue channel that is essentially noise free with all details maintained.

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FIGURE 8. The denoised image by MTV in the RGB space (left) and its blue channel (right). Noticable noise still exists due to high noise in blue channel even the recomposed image has most of the noise removed

FIGURE 9. The denoised image by MTV in m-YCrCb color space (left) and its blue channel (right) which is much cleaner than the original one.

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