TILING $\mathbb{Z}^2$ BY A SET OF FOUR ELEMENTS

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1. Introduction

Let $\mathcal{D}$ be a finite subset of $\mathbb{Z}^d$. $\mathcal{D}$ *tiles* $\mathbb{Z}^d$ if an only if $\mathbb{Z}^d$ can be written as a disjoint union of translates of $\mathcal{D}$, i.e., there is a set $\mathcal{C} \in \mathbb{Z}^d$ such that every point $\mathbf{v} \in \mathbb{Z}^d$ can be expressed uniquely as $\mathbf{x}+\mathbf{y}$ with $\mathbf{x} \in \mathcal{D}$ and $\mathbf{y} \in \mathcal{C}$. In symbols, $\mathcal{D} \oplus \mathcal{C} = \mathbb{Z}^d$. $\mathcal{D}$ is called a *tile* and $\mathcal{C}$ *the translation set*.

In this note we give a sufficient and necessary condition for a subset $\mathcal{D}$ of $\mathbb{Z}^2$ with cardinality 4 to tile $\mathbb{Z}^2$. We may assume that $\mathcal{D}$ is not contained in a straight line. If $\mathcal{D}$ is contained in a line, then $\mathcal{D}$ can tile $\mathbb{Z}^2$ if and only if $\mathcal{D}$ can tile the set of integral points on that line, and such a sufficient and necessary condition for $\mathcal{D}$ was first given by Newman in [1]. In this not we prove the following result:

Theorem 1.1. *Let $\mathcal{D}$ be a subset of $\mathbb{Z}^2$ with cardinality 4. Assume that $\mathcal{D}$ is not contained in a line, and furthermore $\mathbf{0} \in \mathcal{D}$. Then a sufficient and necessary condition for $\mathcal{D}$ can not tile $\mathbb{Z}^2$ is that there exists a 2×2 matrix G so that*

$$G\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ p/q \end{pmatrix} \right\}$$

with $p, q \in \mathbb{Z} \setminus \{0\}$ and $p+q \in 2\mathbb{Z}+1$; In another words, $\mathcal{D}$ can not tile $\mathbb{Z}^2$ if and only if $\mathcal{D} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ such that $\mathbf{v}_2 - \mathbf{v}_1 = \frac{p}{q}(\mathbf{v}_4 - \mathbf{v}_3)$ for some $p \in 2\mathbb{Z} \setminus \{0\}$ and $q \in 2\mathbb{Z}+1$.

2. Proof of Theorem 1.1

Proposition 2.1. *Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^d$. Then the following statements are equivalent:*

(i) *There exists $\mathcal{B} \subset \mathbb{Z}^d$ such that $\mathcal{A} \oplus \mathcal{B} = \mathbb{Z}^d$.*

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(ii) *There exist a non-singular $d \times d$ matrix G with rational entries and $\mathcal{D} \subset \mathbb{Q}^d$ such that $(G\mathcal{A}) \oplus \mathcal{D}$ is a lattice in $\mathbb{R}^d$.*

Proof. The direction (i) $\Longrightarrow$ (ii) is clear. Now we show the opposite direction. Assume that $(G\mathcal{A}) \oplus \mathcal{D}$ is a lattice in $\mathbb{R}^d$, i.e., $(G\mathcal{A}) \oplus \mathcal{D} = H\mathbb{Z}^2$ for some $d \times d$ matrix H. Clearly H is rational. We may assume that H is non-singular (otherwise there exists $\mathcal{C} \subset \mathbb{Q}^d$ so that $H\mathbb{Z}^d \oplus \mathcal{C} = \tilde{H}\mathbb{Z}^d$ for some non-singular $d \times d$ raional matrix $\tilde{H}$ and $(G\mathcal{A}) \oplus \mathcal{D} \oplus \mathcal{C} = \tilde{H}\mathbb{Z}^2$). Then $(H^{-1}G\mathcal{A}) \oplus (H^{-1}\mathcal{D}) = \mathbb{Z}^d$. Choose an integer p so that $E := pH^{-1}G$ is an integral matrix. Note that $(E\mathcal{A}) \oplus (pH^{-1}\mathcal{D}) = p\mathbb{Z}^d$ and $E\mathcal{A} \subset \mathbb{Z}^d$. It follows that $\Lambda := (pH^{-1}\mathcal{D}) \subset \mathbb{Z}^d$. Since p is an integer, there exists a finite set $V \subset \mathbb{Z}^d$ so that $(p\mathbb{Z}^d) \oplus V = \mathbb{Z}^d$. Therefore $(E\mathcal{A}) \oplus \Lambda \oplus V = \mathbb{Z}^d$. Note that $\mathbb{Z}^d = (E\mathbb{Z}^d) \oplus U$ for some finite set $U \subset \mathbb{Z}^d$ with $\mathbf{0} \in U$. We have $(E\mathcal{A}) \oplus \Lambda \oplus V = (E\mathbb{Z}^d) \oplus U$. Letting $\tilde{\Lambda} = (E\mathbb{Z}^d) \cap (\Lambda \oplus V)$, we obtain $(E\mathcal{A}) \oplus \tilde{\Lambda} = E\mathbb{Z}^d$. This implies $\mathcal{A} \oplus (E^{-1}\tilde{\Lambda}) = \mathbb{Z}^d$. ∎

Corollary 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two finite subsets of $\mathbb{Z}^d$. If $\mathcal{A} = G\mathcal{B}$ for some non-singular $d \times d$ rational matrix G, then $\mathcal{A}$ can tile $\mathbb{Z}^d$ if and only if $\mathcal{B}$ can tile $\mathbb{Z}^d$.*

Lemma 2.3. *Let $\mathcal{C} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} p_1 \\ \frac{1}{2} + p_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + p_3 \\ t + p_4 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + p_5 \\ \frac{1}{2} + t + p_6 \end{pmatrix} \right\}$ for some $t \in \mathbb{Q}$ and $p_j \in \mathbb{Z}$ ($j = 1, \ldots, 6$). Then there exists $\mathcal{D} \subset \mathbb{Q}^2$ such that $\mathcal{C} \oplus \mathcal{D}$ is a lattice in $\mathbb{R}^2$.*

Proof. Let $t = \frac{m}{n}$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\gcd(m, n) = 1$. Define

$$\mathcal{D} = \left\{ \begin{pmatrix} u \\ v + \frac{j}{2n} \end{pmatrix}, \ u, v \in \mathbb{Z}, \ j = 0, 1, \ldots, n-1 \right\}.$$

Then one can check that

$$\mathcal{C} \oplus \mathcal{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2n} \end{pmatrix} \mathbb{Z}^2.$$

This finishes the proof. ∎

Corollary 2.4. *Let $\mathcal{A}$ be a subset of $\mathbb{Z}^2$ of cardinality 4. Assume $\mathbf{0} \in \mathcal{A}$. Then $\mathcal{A}$ can tile $\mathbb{Z}^2$ if $\mathcal{A} = G\mathcal{C}$ for some 2×2 non-singular rational matrix G and some $\mathcal{C} \subset \mathbb{R}^2$ which has the form as in Lemma 2.3.*

Remark 2.5. *I doubt that the above "if" can be replaced by "iff".*

Lemma 2.6. *For any $u, v \in \mathbb{Q}$, one of the following three equations has a solution $(x, y, z) \in \mathbb{Z}^3$:*

(i) $(\frac{1}{2}+x)u+(\frac{1}{2}+y)v=z$.
(ii) $xu+(\frac{1}{2}+y)v=\frac{1}{2}+z$.
(iii) $(\frac{1}{2}+x)u+yv=\frac{1}{2}+z$.

Proof. It is easily check that if one of the above three equations has an integral solution, then that equation also has an integral solution when we change u,v to $\tilde{u}$ and $\tilde{v}$ so that $\tilde{u}/u, \tilde{v}/v \in \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$. Thus to prove the lemma, we can assume without loss of generality that $u=2^m$ and $v=2^n$ for $m,n\in\mathbb{Z}$. Then it is a routine to check one of the above three equations must have an integral solution. ∎

Proposition 2.7. *Let*

$$\mathcal{D}=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}u\\v\end{pmatrix}\right\}$$

where $u,v\in\mathbb{Q}$. Then there exists a non-singular 2×2 rational matrix G such that $G\mathcal{D}$ has the same form as $\mathcal{C}$ in Lemma 2.3 if u,v do not satisfy anyone of the following conditions:

(i) $u=1$ *and* $v\notin\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.
(ii) $v=1$ *and* $u\notin\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.
(iii) $u=-v$ *and* $u\notin\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.

Proof. We will prove the existence of G in each of the following scenarios:

(1) $u=1$ and $v\in\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.
(2) $v=1$ and $u\in\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.
(3) $u=-v$ and $u\in\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.
(4) $u\neq 1$, $v\neq 1$ and $u\neq -v$.

For scenario (1), let $v=\frac{2q+1}{2p+1}$, where $p,q\in\mathbb{Z}$. We may take $G=\begin{pmatrix}0 & p+\frac{1}{2}\\ \frac{1}{2} & 2p+1\end{pmatrix}$. Then $G\mathcal{D}=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}\frac{1}{2}+p\\2p+1\end{pmatrix},\begin{pmatrix}\frac{1}{2}+q\\\frac{1}{2}+2q+1\end{pmatrix}\right\}$.

For scenario (2), let $u=\frac{2q+1}{2p+1}$, where $p,q\in\mathbb{Z}$. We may take $G=\begin{pmatrix}p+\frac{1}{2} & 0\\ 2p+1 & \frac{1}{2}\end{pmatrix}$. Then $G\mathcal{D}$ has the same expression as that in scenario (1).

For scenario (3), let $u=\frac{2q+1}{2p+1}$, where $p,q\in\mathbb{Z}$. We may take $G=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ 2p+1 & p+\frac{1}{2}\end{pmatrix}$. Then $G\mathcal{D}=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\frac{1}{2}\\2p+1\end{pmatrix},\begin{pmatrix}\frac{1}{2}\\p+\frac{1}{2}\end{pmatrix},\begin{pmatrix}0\\q+\frac{1}{2}\end{pmatrix}\right\}$.

Now let us turn to the scenario (4). By Lemma 2.6, one of the following equations has a integral solution (x, y, z):

(e1) $(\frac{1}{2}+x)u+(\frac{1}{2}+y)v=z$.
(e2) $xu+(\frac{1}{2}+y)v=\frac{1}{2}+z$.
(e3) $(\frac{1}{2}+x)u+yv=\frac{1}{2}+z$.

Assume at first that (e1) has an integral solution (x, y, z). Since $u \neq -v$, there exists $t \in \mathbb{Q}$ such that

$$(t+x)u+(\frac{1}{2}+t+y)v=\frac{1}{2}+z.$$

Take $G=\begin{pmatrix} \frac{1}{2}+x & \frac{1}{2}+y \\ t+x & \frac{1}{2}+t+y \end{pmatrix}$. Then $G\mathcal{D}=\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}+x \\ t+x \end{pmatrix}, \begin{pmatrix} \frac{1}{2}+y \\ \frac{1}{2}+t+y \end{pmatrix}, \begin{pmatrix} z \\ \frac{1}{2}+z \end{pmatrix}\right\}$. Now we assume (e2) has an integral solution (x, y, z). Since $v \neq 1$, there exists $t \in \mathbb{Q}$ so that $\frac{1}{2}u+tv=\frac{1}{2}+t$. Take $G=\begin{pmatrix} x & \frac{1}{2}+y \\ \frac{1}{2} & t \end{pmatrix}$. Then $G\mathcal{D}=\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2}+y \\ t \end{pmatrix}, \begin{pmatrix} \frac{1}{2}+z \\ \frac{1}{2}+t \end{pmatrix}\right\}$. If (e3) has an integral solution, we may construct G in a similar way as above. ∎

Proposition 2.8. *Let*

$$\mathcal{D}=\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right\}$$

where $u, v \in \mathbb{Q}$. Then there exists no $\mathcal{C}$ such that $\mathcal{D} \oplus \mathcal{C}$ is a lattice if u, v does satisfy one of the following conditions:

(i) *$u=1$ and $v \notin \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.*
(ii) *$v=1$ and $u \notin \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.*
(iii) *$u=-v$ and $u \notin \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.*

Proof. Without loss of generality we may only consider case (ii), since the sets $\mathcal{D}$ in cases (i) and (iii) differ from that in (ii) only by an affine map.

Assume $u=\frac{p}{q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $p+q \in 2\mathbb{Z}+1$. Take $G=\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$G\mathcal{D}=\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} q \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} p \\ 1 \end{pmatrix}\right\}$$

By Proposition 2.1, we only need to prove that $G\mathcal{D}$ can not tile $\mathbb{Z}^2$.

Assume on the contrary that $G\mathcal{D}$ can tile $\mathbb{Z}^2$, i.e., $(G\mathcal{D}) \oplus \Lambda=\mathbb{Z}^2$. Then any $\mathbf{x} \in \mathbb{Z}^2$ can be uniquely written as $\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2$ with $\mathbf{x}_1 \in G\mathcal{D}$ and $\mathbf{x}_2 \in \Lambda$. Define $\phi: \mathbb{Z}^2 \to G\mathcal{D}$ by

$\mathbf{x} \mapsto \mathbf{x}_1$. Let $\{a_n\}_{n\in\mathbb{Z}}$ be the sequence defined by

$$a_n = \begin{cases} 1 & \text{if } \phi(n,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 2 & \text{if } \phi(n,0) = \begin{pmatrix} q \\ 0 \end{pmatrix} \\ 3 & \text{if } \phi(n,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 4 & \text{if } \phi(n,0) = \begin{pmatrix} p \\ 1 \end{pmatrix} \end{cases}$$

We have the following observations:

(a) For any $n \in \mathbb{Z}$, $a_{n+p} \neq a_n$ and $a_{n+q} \neq a_n$.
(b) If $a_n = 1$ then $a_{n+q} = 2$. If $a_n = 2$ then $a_{n-q} = 1$. If $a_n = 3$ then $a_{n+p} = 4$. If $a_n = 4$ then $a_{n-p} = 3$.

Let us first prove (a). From $(G\mathcal{D}) \oplus \Lambda = \mathbb{Z}^2$ we obtain $(G\mathcal{D} - G\mathcal{D}) \cap (\Lambda - \Lambda) = \{\mathbf{0}\}$. Since $\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} q \\ 0 \end{pmatrix} \in G\mathcal{D} - G\mathcal{D}$, we have $\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} q \\ 0 \end{pmatrix} \notin \Lambda - \Lambda$. Now assume (a) is not true. Without loss of generality we assume $a_{n+p} = a_n$ for some n. Then

$$\begin{pmatrix} n \\ 0 \end{pmatrix} = \mathbf{y} + \lambda_{\mathbf{1}}, \qquad \begin{pmatrix} n+p \\ 0 \end{pmatrix} = \mathbf{y} + \lambda_{\mathbf{2}}$$

for some $\mathbf{y} \in G\mathcal{D}$ and $\lambda_1, \lambda_2 \in \Lambda$. It implies that $\begin{pmatrix} p \\ 0 \end{pmatrix} = \lambda_2 - \lambda_1 \in \Lambda - \Lambda$, which leads to a contradiction. This proves (a). To prove (b) without loss of generality we prove that $a_{n+q} = 2$ when $a_n = 1$. Since $a_n = 1$, we have $\begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda$ for some $\lambda \in \Lambda$. Therefore $\begin{pmatrix} n+q \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} + \lambda$, which implies $a_{n+q} = 2$. This finishes the proof of (b).

According to (a) and (b), we have the following claims:

(c1) Assume $p > 0$. If $a_n \in \{1, 3\}$, then $a_{n+p+q} \in \{1, 3\}$.
(c2) Assume $p < 0$. If $a_n \in \{1, 4\}$, then $a_{n-p+q} \in \{2, 3\}$.

Without loss of generality we only prove (c1). First assume $a_n = 1$. Then by (b) we have $a_{n+q} = 2$. Thus by (a) we have $a_{n+p+q} \neq 2$. In the same time by (b) we have $a_{n+p+q} \neq 4$ since otherwise $a_{n+q} = 3$. Therefore we always have $a_n \in \{1, 3\}$ when $a_n = 1$. Using an essentially identical argument, we can obtain that $a_n \in \{1, 3\}$ when $a_n = 3$. This finishes the proof of (c1).

Now assume $p > 0$. Then (c1) implies that the set $\{0, 1, \ldots, p+q\}$ can be partitioned into two set A and B such that there exists a large $N \in \mathbb{N}$ so that for $n > N$, $a_n \in \{1, 3\}$ if $n(\mod p+q) \in A$, and $a_n \in \{2, 4\}$ if $n(\mod p+q) \in B$. That means the density of those n with $a_n \in \{1, 3\}$ in $\mathbb{Z} \cap [N, \infty)$ is $\#A/(p+q)$, and the density of the rest is $\#B/(p+q)$. Since $p+q \in 2\mathbb{Z}+1$, these two densities are different. However from (b), these two densities must be the same. This leads to a contradiction.

A contradiction can be derived on the same line for the case $p < 0$. We omit the details. ∎

Proof of Theorem 1.1 Since $\mathcal{D}$ is not contained in a line, there exists a non-singular rational 2×2 matrix A so that $A\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\}$ with $u, v \in \mathbb{Q}$. Assume $\mathcal{D}$ can not tile $\mathbb{Z}^2$. Then by Proposition 2.1, There is no non-singular rational matrix G and $\mathcal{C} \subset \mathbb{Q}^2$ such that $G\mathcal{D} \oplus \mathcal{C}$ be a lattice. Therefore by Proposition 2.7 and Lemma 2.3, u, v do satisfy one of the following conditions:

(i) $u = 1$ and $v \notin \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.
(ii) $v = 1$ and $u \notin \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.
(iii) $u = -v$ and $u \notin \frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$.

Thus there exists a non-singular rational matrix B such that

$$BA\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ p/q \end{pmatrix} \right\}$$

with $p, q \in \mathbb{Z} \setminus \{0\}$ and $p + q \in 2\mathbb{Z} + 1$. This proves the necessity. The sufficiency is implied by Proposition 2.8. ∎

References

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