LETTER TO THE EDITOR

THE REGULARITY OF REFINABLE FUNCTIONS

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ABSTRACT. The regularity of refinable functions has been studied extensively in the past. A classical result by Daubechies and Lagarias [6] states that a compactly supported refinable function in $\mathbb{R}$ of finite mask with integer dilation and translations cannot be in C^∞. A bound on the regularity based on the eigenvalues of certain matrices associated with the refinement equation is also given. Surprisingly this fundamental classical result has not been proved in the more general settings, such as in higher dimensions or when the dilation is not an integer. In this paper we extend this classical result to the most general setting for arbitrary dimension, dilation and translations.

1. INTRODUCTION

A *refinement equation* is a functional equation of the form

$$f(x) = \sum_{d\in\mathcal{D}} c_d f(Ax - d) \tag{1.1}$$

where $\mathcal{D} \subset \mathbb{R}^n$ is a finite set, $c_d \neq 0$ for any $d \in \mathcal{D}$ and $A \in \mathbb{M}_n(\mathbb{R})$ is an $n \times n$ expanding matrix, i.e. all eigenvalues of A have $|\lambda| > 1$. Since there are only finitely many nonzero coefficients c_d, (1.1) is often referred to as a refinement equation with a *finite mask*. Here we shall refer to a nontrivial function f satisfying (1.1) a *refinable function with dilation matrix A and translations $\mathcal{D}$*. In this paper, as in the vast majority of studies in the literature, the focus is on compactly supported refinable functions.

Refinement equations with finite masks play a fundamental role in many applications such as the construction of compactly supported wavelets and in the study of subdivision schemes in CAGD. The regularity of refinable functions is of great significance in those studies both in theory and in applications. It has been studied extensively, including the seminal work by Daubechies [5] which constructs compactly supported refinable functions with orthogonal integer translates of arbitrary regularity, leading to the fundamental class of Daubechies wavelets. A more general study by Daubechies and Lagarias [6] establishes a

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classical result on the regularity of a compactly supported refinable function in $\mathbb{R}$ of finite mask with integer dilation and translations. It states that such a function cannot be in C^∞, and it gives bound on the regularity based on the eigenvalues of certain matrices from the mask. The results in [6] have later been extended by several authors to obtain more refined regularity estimations. In addition, using the same matrix eigenvalue technique, one can extend the Daubechies-Lagarias result to refinable functions in $\mathbb{R}^n$ where $A \in \mathbb{M}_n(\mathbb{Z})$ and $\mathcal{D} \subset \mathbb{Z}^n$ (see Cabrelli, Heil and Molter [1]).

Quite surprisingly, there have been very few results in terms of extending the classic Daubechies-Lagarias result to the more general settings. This is perhaps due to the fact that the matrix technique that has been effective for the integral case can no longer be applied. Using techniques from number theory and harmonic analysis, Dubickas and Xu [8] prove that a refinable function in $\mathbb{R}$ with an arbitrary dilation λ and *integer* translations cannot be in C^∞. This appears to be the only generalization in this direction. There have been other studies on the regularity of refinable functions with non-integral dilations, e.g. in Dai, Feng and Wang [2] on the decay rate of the Fourier transform of a compactly supported refinable function with arbitrary dilation and translations, and in [3] by the same authors on refinable splines. There is also an extensive literature on the absolute continuity of self-similar measures, which are somewhat related to the study of regularity of refinable functions. Nevertheless none of these studies directly address the extension of the Daubechies-Lagarias result.

Many researchers in the community may have *assumed* that the Daubechies-Lagarias result is valid in the general setting while in reality other than those aforementioned special cases it has never been proved. The general result turns out to be rather nontrivial to be established. Our goal in this paper is to provide a short proof, thus establishing this important classical result under the most general settings. Our main theorem is:

Theorem 1.1. *Let f be a compactly supported refinable function of finite mask in $\mathbb{R}^n$. Then f is not in $C^\infty(\mathbb{R}^n)$.*

We shall prove the theorem in Section 2. In Section 3 we establish some upper bounds on the regularity of compactly supported refinable functions.

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2. Proof of Main Theorem

Here in this section we prove our main theorem. We shall first investigate the support of f by examining the attractor of an iterated functions system (IFS) associated with a refinement equation and its convex hull. The IFS $\{\phi_d(x) = A^{-1}(x+d) : \ d \in \mathcal{D}\}$ is referred to as the *IFS associated with the refinement equation (1.1)*. By a well-known result of Hutchinson [10] there is a unique compact set T satisfying $T = \bigcup_{d\in\mathcal{D}} \phi_d(T)$. The set T is called the *attractor* of the IFS $\{\phi_d : \ d \in \mathcal{D}\}$. Let $\Phi(S) := \bigcup_{d\in\mathcal{D}} \phi_d(S)$ for any compact $S \subset \mathbb{R}^n$. Then $T = \lim_{k\to\infty} \Phi^k(S_0)$ in the Hausdorff metric for any nonempty compact S_0. We shall let $\Omega := \mathrm{supp}\,(f)$ denote the support of f. It follows from the refinement equation

(1.1) that $\Omega \subseteq \Phi(\Omega)$. By iterating it we obtain $\Omega \subset T$, where T is the attractor of the IFS $\{\phi_d :\ d \in \mathcal{D}\}$.

A key part of our tools involve the investigation of the convex hulls of various sets. A point z^* in a compact set S is called the *extremal point* of S if there is a unit vector $u \in \mathbb{R}^n$ such that z^* is the unique maximizer of $\langle u, x\rangle$ for $x \in S$. In this case we shall call z^* the *extremal point of S for the vector u.* These extremal points form the extremal points of the convex hull of S. Let $\mathcal{D}_e \subseteq \mathcal{D}$ be the set of extremal points of the convex hull of $\mathcal{D}$. Somewhat related to this paper are that the set of extremal points of T has been explicitly characterized in Strichartz and Wang [11], and furthermore it is shown in Dai and Wang [4] to be identical to the set of extremal points of Ω.

Before proceeding further we first introduce some notations. For any $m \geq 1$ we define the map $\pi_m : \mathcal{D}^m \longrightarrow \mathbb{R}^n$ by

$$\pi_m\big([d_0, \dots, d_{m-1}]\big) := \sum_{j=0}^{m-1} A^j d_j. \tag{2.1}$$

We let $\mathcal{D}_m := \pi_m(\mathcal{D}^m)$, which is

$$\mathcal{D}_m := \Big\{\sum_{j=0}^{m-1} A^j d_j :\ [d_0, \dots, d_{m-1}] \in \mathcal{D}^m\Big\}.$$

In general π_m is not one-to-one. If $0 \in \mathcal{D}$ then $\mathcal{D}_m \subseteq \mathcal{D}_{m+1}$. We shall frequently consider the extremal points of $\mathcal{D}_m$ in this paper, and to this end it is useful to introduce the set $\mathcal{U}$ of unit vectors defined by

$$\mathcal{U} := \Big\{u \in \mathbb{R}^n :\ \|u\| = 1 \text{ and } \langle u, d\rangle \neq \langle u, e\rangle \text{ for any distinct } d, e \in \bigcup_{m=1}^{\infty} \mathcal{D}_m\Big\}.$$

Note that $\bigcup_{m=1}^{\infty} \mathcal{D}_m$ is a countable set so $\mathcal{U}$ is the whole unit sphere in $\mathbb{R}^n$ minus a measure zero subset.

Now iterating the refinement equation (1.1) we obtain

$$f(x) = \sum_{v \in \mathcal{D}^m} c_v f\big(A^m x - \pi_m(v)\big) = \sum_{d \in \mathcal{D}_m} \tilde{c}_d f\big(A^m x - d\big), \tag{2.2}$$

where $c_v = \prod_{j=0}^{m-1} c_{d_j}$ for $v = [d_0, \dots, d_{m-1}]$ and $\tilde{c}_d = \sum_{v \in \mathcal{D}^m, \pi_m(v) = d} c_v$. Note that the support of the term $f\big(A^m x - d\big)$ is $A^{-m}(\Omega + d)$.

Lemma 2.1. *For any $u \in \mathcal{U}$ there is a unique $[d_0, d_1, \dots, d_{m-1}] \in \mathcal{D}^m$ such that*

$$\pi_m([d_0, d_1, \dots, d_{m-1}]) = \operatorname{argmax}_{d \in \mathcal{D}_m} \langle u, d\rangle.$$

Denote $d_m^u := \pi_m([d_0, d_1, \dots, d_{m-1}])$. Consequently $\tilde{c}_{d_m^u} = \prod_{j=0}^{m-1} c_{d_j}$.

Proof. The proof uses the same argument as in [11]. We have $\sum_{j=0}^{m-1} A^j d_j = \operatorname{argmax}_{d \in \mathcal{D}_m} \langle u, d\rangle$. Therefore for each j we must have $A^j d_j = \operatorname{argmax}_{d \in \mathcal{D}} \langle u, A^j d\rangle$. Furthermore, because $u \in \mathcal{U}$ such d_j must be unique, proving the lemma. Of course in this case $\tilde{c}_{d_m^u} = \prod_{j=0}^{m-1} c_{d_j}$. ∎

Definition 2.2. Let $\mathcal{E}$ be a subset of $\mathbb{R}^n$. Let $r_0 > 0$ and u be a unit vector in $\mathbb{R}^n$. We say $\mathcal{E}$ has (u, r_0)-*isolated extremal point* if there exists a $z_0 \in \mathcal{E}$ such that $\langle u, z_0\rangle > \langle u, z\rangle + r_0$ for all $z \in \mathcal{E}, z \neq z_0$.

Intuitively if $\mathcal{E}$ has an (u, r_0)-isolated extremal point then the maximal point of its projection onto the direction of u is more than r_0 separated from the other points in the projection. Our objective is to examine the sets $\mathcal{D}_m$ and investigate whether it has certain isolation property. A key result is the following:

Lemma 2.3. *Assume that there is an $r_0 > 0$ such that there exist infinitely many $m > 0$ such that $\mathcal{D}_m$ has (u_m, r_0)-isolated extremal point where $u_m \in \mathcal{U}$. Then a nontrivial compactly supported refinable function f satisfying (1.1) cannot be in C^∞.*

Proof. Iterating the refinement equation leads to the new refinement equation (2.2). Now substitute $A^{-m}x$ for x we obtain

$$f(A^{-m}x) = \sum_{v\in\mathcal{D}^m} c_v f\big(x - \pi_m(v)\big) = \sum_{d\in\mathcal{D}_m} \tilde{c}_d f\big(x-d\big). \tag{2.3}$$

Suppose $\mathcal{D}_m$ has (u_m, r_0)-isolated extremal point. Let $z_m \in \Omega$ such that $\langle u_m, z_m\rangle = \sup_{x\in\Omega}\langle u_m, x\rangle$. Clearly $z_m \in \partial\Omega$. By the assumption that $\mathcal{D}_m$ has (u_m, r_0)-isolated extremal point and $u_m \in \mathcal{U}$ we know that

$$\langle u_m, d_m^{u_m}\rangle > \langle u_m, d\rangle + r_0 \tag{2.4}$$

for all other $d \neq d_m^{u_m}$ in $\mathcal{D}_m$. It follows that $B_{r_0}(z_m) + d_m^{u_m}$ intersects the support of $f(x - d_m^{u_m})$ but is disjoint from the support of all other $f(x-d)$ with $d \neq d_m^{u_m}$ in $\mathcal{D}_m$.

Now by assumption we have infinitely many $z_m \in \Omega$. Let z^* be a cluster point. Clearly there are infinitely many $m > 0$ such that for $r_1 = r_0/2$, $B_{r_1}(z^*) + d_m^{u_m}$ intersects the support of $f(x - d_m^{u_m})$ but is disjoint from the support of all other $f(x-d)$ where $d_m^{u_m} \neq d \in \mathcal{D}_m$. For any such m, setting $x = z + d_m^{u_m}$ in (2.3) for $z \in B_{r_1}(z^*)$ yields

$$f\big(A^{-m}z + A^{-m}d_m^{u_m}\big) = \tilde{c}_{d_m^{u_m}} f(z). \tag{2.5}$$

Write $d_m^{u_m} = d_0 + Ad_1 + \cdots + A^{m-1}d_{m-1}$ where each $d_j \in \mathcal{D}$. The equation (2.4) implies that $d_j \in \mathcal{D}_e \subset \mathcal{D}$. By Lemma 2.1 we have $\tilde{c}_{d_m^{u_m}} = \prod_{j=0}^{m-1} c_{d_j}$. Hence $|\tilde{c}_{d_m^{u_m}}| \geq b^m$ where $b = \min\{|c_d| : d \in \mathcal{D}_e\}$. Now fix $x^* \in B_{r_1}(z^*)$ such that $|f(x^*)| > 0$. Let

$$y_m = A^{-m}z_m + A^{-m}d_m^{u_m},\ x_m = A^{-m}x^* + A^{-m}d_m^{u_m}.$$

Since z_m is on the boundary of Ω, by (2.5) we have $f(z_m) = 0$. In fact, since $z_m + d_m^{u_m}$ is an extremal point in $\Omega + \mathcal{D}_m$ for the vector u_m, y_m must be an extremal point of $A^{-m}(\Omega + \mathcal{D}_m)$ for the vector $(A^T)^m u_m$. Hence y_m must be on the boundary of $A^{-m}(\Omega + \mathcal{D}_m)$. Now $A^{-m}(\Omega + \mathcal{D}_m) \supseteq \Omega$. It follows that y_m must be either on the boundary of Ω or not in the support of f at all. In either case if $f \in C^K(\mathbb{R}^n)$ then all k-th order derivatives of f must vanish at y_m whenever $k \leq K$. We also have $|f(x_m)| = |f(x_m) - f(y_m)| \geq b^m|f(x^*)|$.

We can derive a contradiction. Let $\tau < 1$ be the spectral radius of A^{-1}. Then $\|x_m - y_m\| = \|A^{-m}(x^* - z_m)\| \leq \tau^{-m}r_1$. Assume that f is in $C_0^\infty(\mathbb{R}^n)$. Then for any $N > 0$

and any $y^* \in \partial\Omega$ we must have $|f(y) - f(y^*)| = o(\|y - y^*\|^N)$ uniformly. In particular $|f(x_m) - f(y_m)| = o(\|x_m - y_m\|^N)$. However,

$$\frac{|f(x_m) - f(y_m)|}{\|x_m - y_m\|^N} \geq \frac{b^m |f(x^*)|}{\tau^{mN}\, r_1^N} \tag{2.6}$$

for infinitely many m. By taking N large enough so that $\tau^N < b$ the right hand side of (2.6) does not tend to 0, a contradiction. Thus f cannot be in $C_0^\infty(\mathbb{R}^n)$. ∎

Remark 1. The above proof actually gives a bound on the smoothness of f. A very crude bound that can be derived easily from (2.6) is that if $f \in C^K(\mathbb{R}^n)$ then $K < \log b / \log \tau$ where τ is the spectral radius of A^{-1} and $b = \min_{d \in \mathcal{D}_e} |c_d|$. The $\log b$ part can be improved. In fact, let $\mathcal{J}$ be an infinite subset of indices m such that $\mathcal{D}_m$ has (u_m, r_0)-isolated extremal point for each $m \in \mathcal{J}$. Then the proof shows that

$$K < \frac{\limsup_{m \in \mathcal{J}} \log |c_{d_m^{u_m}}|}{m \log \tau}. \tag{2.7}$$

In many cases it allows us to obtain sharper upper bounds for the regularity of f.

A key ingredient in our proof of the main theorem is the Borel-Cantelli Lemma, which states that if E_k is a sequence of events in some probability space and suppose that $\sum_k \Pr(E_k) < \infty$ then $\Pr(\limsup_{k\to\infty} E_k) = 0$. In other words, the probability that infinitely many of them occur is 0. We use the Borel-Cantelli Lemma to prove the following key result.

Theorem 2.4. *Let A be an $n \times n$ dilation matrix and let $\mathcal{D}$ be a finite set in $\mathbb{R}^n$. Then there exists an $r_0 > 0$ and unit vectors $u_m \in \mathcal{U}$ such that $\mathcal{D}_m$ has (u_m, r_0)-isolated extremal point for infinitely many $m > 0$.*

Proof. For $u \in \mathcal{U}$ let e_m^u denote the element in $\mathcal{D}_m$ that gives the second highest value of $\langle u, d\rangle$ for $d \in \mathcal{D}_m$, i.e.

$$e_m^u = \operatorname{argmax}_{d \in \mathcal{D}_M \setminus \{d_m^u\}} \langle u, d\rangle.$$

If there is an $r_0 > 0$ such that we can find infinitely many m such that $\langle u, d_m^u - e_m^u\rangle > r_0$ for some $u = u_m \in \mathcal{U}$ we are done. Set

$$g_m(u) = \langle u, d_m^u - e_m^u\rangle.$$

We have $g_m(u) > 0$ for all $m, u \in \mathcal{U}$. Assume the lemma is false. Then $\lim_m \sup_{u\in\mathcal{U}} g_m(u) = 0$. We shall derive a contradiction.

Assume that $d_m^u = \pi_m([d_0, d_1, \ldots, d_{m-1}]) = \sum_{j=0}^{m-1} A^j d_j$. We have already argued that each $d_j \in \mathcal{D}$ is the unique element in $\mathcal{D}$ satisfying $A^j d_j = \operatorname{argmax}_{d\in\mathcal{D}} \langle u, A^j d\rangle$. Now assume that $e_m^u = \sum_{j=0}^{m-1} A^j e_j$. We claim that $d_j \neq e_j$ for one and only one $0 \leq j < m$. Assume $d_k \neq e_k$ and $d_l \neq e_l$ where $l \neq k$. Then the element $\tilde{e}_m^u = \sum_{j=0}^{m-1} A^j \tilde{e}_j$ where $\tilde{e}_j = e_j$ for all $j \neq l$ and $\tilde{e}_l = d_l$ will have the property

$$\langle u, d_m^u\rangle > \langle u, \tilde{e}_m^u\rangle > \langle u, e_m^u\rangle,$$

contradicting the assumption that $\langle u, e_m^u\rangle$ is the second largest. Thus there exists a unique $0 \le k < m$ and $d_k \neq e_k$ in $\mathcal{D}$ such that $d_m^u - e_m^u = A^k(d_k - e_k)$. It follows that

$$g_m(u) = \langle u, A^k(d_k - e_k)\rangle.$$

We shall denote $p_m(u) := k$ and $v_m(u) := d_k - e_k$. Thus we can rewrite the above equation as

$$g_m(u) = \langle u, A^{p_m(u)} v_m(u)\rangle.$$

Note that $\lim_m \sup_{u\in\mathcal{U}} g_m(u) = 0$. So there exists a $C > 0$ such that $g_m(u) \le C$ for all m and $u \in \mathcal{U}$. Thus we have

$$\langle u, A^{p_m(u)} v_m(u)\rangle \le C. \tag{2.8}$$

Denote $\mathcal{E} := (\mathcal{D} - \mathcal{D}) \setminus \{0\}$, which is a finite set. Observe that $v_m(u) \in \mathcal{E}$. For any $k > 0$ define the set $E_k \subseteq \mathcal{U}$ by

$$E_k = \Big\{u \in \mathcal{U} : \min_{v\in\mathcal{E}} \langle u, A^k v\rangle \le C\Big\}.$$

Obviously, by (2.8) if we pick $k = p_m(u)$ then $u \in E_k$. In particular if $p_m(u)$ can take on infinitely many values then u will be in infinitely many E_k. We show that for almost all $u \in \mathcal{U}$, $p_m(u)$ can only take on finitely many values. To see this we claim:

Claim: *There exists a constant $M > 0$ such that $\mu(E_k) \le M\,\tau^k$, where μ is the normalized Hausdorff measure on the unit sphere in $\mathbb{R}^n$ and τ is the spectral radius of A^{-1}.*

To prove the claim, note that for each fixed unit vector v_0 the set $F = \{u \in \mathcal{U} : |\langle u, v_0\rangle| \le \varepsilon\}$ has measure $\mu(F) \le M_0\,\varepsilon$ for some constant M_0 depending only on the dimension n. Now for fixed k and any $v \in \mathcal{E}$ set $R_k(v) = \|A^k v\|$ and $w_k(v) = A^k v / R_k(v)$. It follows that the set

$$\{u \in \mathcal{U} : |\langle u, w_k(v)\rangle| \le \varepsilon\}$$

has measure bounded by $M_0\,\varepsilon$. Thus the set $F_{k,v} = \{u \in \mathcal{U} : |\langle u, A^k v\rangle| \le C\}$ has $\mu(F_{k,v}) \le M_0 C / R_m(v)$. Clearly, $\min_{v\in\mathcal{E}} 1/R_m(v) \le M_1 \tau^m$ for some constant $M_1 > 0$. Since E_k is the union of $F_{k,v}$ with where v runs through $\mathcal{E}$, it follows that

$$\mu(E_k) \le M_0\, C\, M_1\, L\, \tau^k$$

where L is the cardinality of $\mathcal{E}$. The claim follows by setting $M = M_0\, C\, M_1\, L$.

Since $0 < \tau < 1$ we have $\sum_k \mu(E_k) < \infty$. By the Borel-Cantelli Lemma, almost all $u \in \mathcal{U}$ belong to only finitely many E_k. In other words, for almost all $u \in \mathcal{U}$ the set $\{p_m(u)\}$ is a finite set. Thus taking any such $u \in \mathcal{U}$, the sequence $g_m(u) = \langle u, A^{p_m(u)} v_m(u)\rangle$ can take on only finitely many values. Furthermore we already know that $g_m(u) \neq 0$. This contradicts the assumption that $\lim_m g_m(u) = 0$. The lemma is now proved. ∎

Theorem 1.1 now follows easily from Lemma 2.3 and Theorem 2.4.

3. Regularity Upper Bounds

In proving our main theorem in the previous section we have in fact already established upper bounds for the regularity of refinable functions, or at least ways to estimate such bounds. We have already remarked in Section 2 that a very simple but crude bound for a compactly supported refinable function satisfying (1.1) is

$$K < \frac{\log b}{\log \tau}, \tag{3.1}$$

where $b = \min\{|c_d| : d \in \mathcal{D}_e\}$ and τ is the spectral radius of A^{-1}. We establish some more refined bounds here.

Theorem 3.1. *Let f be a compactly supported refinable function in $\mathbb{R}^n$ satisfying the refinement equation (1.1). Let $w \in \mathbb{R}^n$ be a unit eigenvector of A^T corresponding to a real eigenvalue λ of A^T. Assume that $\mathcal{D}$ has an extremal point $d_w \in \mathcal{D}$ for the vector w and $f \in C^K(\mathbb{R}^n)$, $K \geq 0$. Then*

$$K < \frac{\log |c_{d_w}|}{\log |\lambda|^{-1}}. \tag{3.2}$$

Proof. Since d_w is extremal in $\mathcal{D}$ for the vector w we can find an $r_0 > 0$ such that $r_0 < \langle w, d_w\rangle - \max_{\mathcal{D}\setminus\{d_w\}}\langle w, d\rangle$. With w being an eigenvector of A^T it is straightforward to see that $d_m^w = d_w + Ad_w + \cdots + A^{m-1}d_w$ is the extremal point of $\mathcal{D}_m$ for the vector w, and it gives $\mathcal{D}_m$ a (w, r_0)-isolated extremal point for each m. Let $z^* = \operatorname{argmax}_{x\in\Omega}\langle w, x\rangle$. Then for any $d \neq d_m^w$ in $\mathcal{D}_m$ and $x \in \Omega$ we must have

$$\langle w, z^* + d_m^w\rangle > \langle w, x + d\rangle + r_0. \tag{3.3}$$

Thus if $z \in \Omega$ such that $\langle w, z^* - z\rangle \leq r_0$ then $y = d_m^w + z$ is not in $d + \Omega$ for all $d \neq d_m^w$ in $\mathcal{D}_m$, and hence it satisfies $f(y - d) = 0$.

Note that f is not identically 0 in any neighborhood of z^* since z^* is in the support of f. Pick an $x^* \in B_{r_0}(z^*)$ such that $f(x^*) \neq 0$. Now let $u \in \mathbb{R}^n$ be a unit λ-eigenvector of A such that $\langle w, u\rangle \geq 0$. Consider the set $E \subset \mathbb{R}^+$,

$$E = \big\{t \geq 0 : \; x^* + tu \in \Omega\big\}.$$

Let $t_0 = \max E$ and $z_0 = x^* + t_0 u$. Clearly $z_0 \in \partial\Omega$. Furthermore z_0 satisfying (3.2) since

$$\langle w, z^* - z_0\rangle = \langle w, z^* - x^*\rangle - t\langle w, u\rangle \leq \langle w, z^* - x^*\rangle \leq r_0.$$

Hence $d_m^w + z_0$ is not in $d + \Omega$ for all $d \neq d_m^w$ in $\mathcal{D}_m$. But $z_0 + d_m^w \in \partial(\Omega + d_m^w)$. It follows that $z_0 + d_m^w \in \partial(\Omega + \mathcal{D}_m)$. Set $y_m = A^{-m}(z_0 + d_m^w)$, which must be on the boundary of $A^{-m}(\Omega + \mathcal{D}_m)$. Now $A^{-m}(\Omega + \mathcal{D}_m) \supseteq \Omega$, which implies that y_m must be either on the boundary of Ω or not in the support of f at all. In either case since $f \in C^K(\mathbb{R}^n)$ all k-th order derivatives of f must vanish at y_m whenever $k \leq K$.

We can now bound the regularity K of f. Denote $x_m = A^{-m}(x^* + d_m^w)$. We have $f(x_m) = \tilde{c}_{d_m^w} = c_{d_w}^m f(x^*)$, and hence $|f(x_m) - f(y_m)| = |f(x_m)| \geq |c_{d_w}|^m |f(x^*)|$. Observe that $\|x_m - y_m\| = \|A^{-m}(x^* - z_0)\| = |\lambda|^{-m}\|x^* - z_0\|$ since $x^* - z_0 = t_0 u$ is a λ-eigenvector of

A. Since $f \in C_0^K(\mathbb{R}^n)$, for any $y^* \in \partial\Omega$ and $y \in \Omega$ we must have $|f(y)-f(y^*)| = o(\|y-y^*\|^K)$ uniformly. In particular $|f(x_m) - f(y_m)| = o(\|x_m - y_m\|^K)$. However,

$$\frac{|f(x_m) - f(y_m)|}{\|x_m - y_m\|^K} = \frac{|c_{d_w}|^m |f(x^*)|}{\|x^* - z_0\|^K \, |\lambda|^{-mK}}. \tag{3.4}$$

The right hand side of (3.4) will tend to 0 as $m \to \infty$ only if $K < \frac{\log |c_{d_w}|}{\log |\lambda|^{-1}}$. This proves the theorem. ∎

Remark 2. It is well known that without the compactly supported assumption f can in fact be C^∞. The simplest example is $f(x) = x$, which satisfies $f(x) = \frac{1}{2} f(2x)$. The study of refinable functions has largely imposed the additional condition $\sum_{d \in \mathcal{D}} c_d = |\det(A)|$, which stems from many applications such as wavelets. Under this condition, a refinement equation with finite mask has up to a scalar multiple a unique compactly supported distribution solution. If we restrict to only solutions in $L^1(\mathbb{R})$ then the uniqueness result also holds without the additional sum of coefficients condition in the one dimension, provided such a solution exists [6]. Surprisingly, just like the regularity result before this paper, this result has not been established for higher dimensions in the general setting.

Remark 3. With the additional condition $\sum_{d \in \mathcal{D}} c_d = |\det(A)|$ it is well known that a compactly supported refinable distribution f must have $\widehat{f}(0) \neq 0$. This fact can be combined with the projection method in [9] to yield a slightly less tedious proof of Theorem 3.1. Without the additional condition, however, one problem we cannot overcome is to show that the projection is nontrivial.

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